

A Sharp Bound for the Slope of Double Cover Fibrations

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0. Introduction and Preliminaries

A *fibred surface*, or simply a *fibration*, is a proper surjective morphism with connected fibers f from a smooth surface X to a smooth complete curve B . Call F the general fiber of f . A fibration is said to be *relatively minimal* if the fibers contain no (-1) -curves. The genus g of F is called the *genus of the fibration*. We say that f is smooth if all the fibers are smooth, isotrivial if all the smooth fibers are mutually isomorphic, and *locally trivial* if it is smooth and isotrivial. A *hyperelliptic* (resp. *bielliptic*) *fibration* is a fibred surface whose general fiber is a hyperelliptic (resp. bielliptic) curve. A fibration is said to be *semistable* if all its fibers are reduced nodal curves that are moduli semistable (any rational smooth component meets the rest of the curve in at least two points).

RELATIVE INVARIANTS. As usual, the *relative dualizing sheaf* of a fibration $f: X \rightarrow B$ is the line bundle

$$\omega_f = \omega_X \otimes (f^*\omega_B)^{-1},$$

where ω_V is the canonical sheaf of V .

REMARK 0.1. A relatively minimal fibration is a fibration such that ω_f is f -nef. A semistable fibration is a relatively minimal fibration whose fibers are nodal and reduced.

The basic invariants for a relatively minimal fibration $f: X \rightarrow B$ are

$$(\omega_f \cdot \omega_f), \quad \deg f_*\omega_f \quad \text{and} \quad e_f := e(X) - e(B)e(F),$$

where e is the topological Euler number. These invariants are related by Noether's formula,

$$(\omega_f \cdot \omega_f) = 12 \deg f_*\omega_f - e_f.$$

It is well known that all these invariants are greater than or equal to 0; moreover, $\deg f_*\omega_f = 0$ if and only if f is locally trivial, $(\omega_f \cdot \omega_f) = 0$ implies that f is isotrivial, and $e_f = 0$ if and only if the fibration is smooth.

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SLOPE. Assuming that the fibration is not locally trivial, we can consider the ratio

$$s(f) = \frac{(\omega_f \cdot \omega_f)}{\deg f_* \omega_f},$$

which is itself an important invariant of the fibration called the *slope*. Noether's formula gives the upper bound $s(f) \leq 12$, which is achieved when all the fibers are smooth (e.g., for the Kodaira fibrations). The lower bound for relatively minimal fibrations of genus $g \geq 2$ is given by the so-called *slope inequality* (see [6; 9; 12]):

$$(\omega_f \cdot \omega_f) \geq \frac{4(g-1)}{g} \deg f_* \omega_f. \quad (0.1)$$

That is, $s(f) \geq 4(g-1)/g$. Observe that the slope inequality implies in particular that $(\omega_f \cdot \omega_f) = 0$ if and only if f is locally trivial. The inequality is sharp; equality holds for certain hyperelliptic fibrations (see [1; 6]).

One of the main problems in the study of fibered surfaces is to understand how properties of the general fiber influence the slope. It is significant that, if the bound $4(g-1)/g$ is reached, then F has a g_2^1 . As a matter of fact, the gonality (or the Clifford index) of the general fiber plays an important role: there are several results in this direction (mainly due to Konno), although an explicit sharp bound on the slope depending on the gonality is still not known and seems to be hard to find.

DOUBLE COVERS. Another direction in which the hyperelliptic case can be generalized is the study of fibrations whose general fiber F is a double cover of a smooth curve of genus $\gamma \geq 0$, which we will call double fibrations. As will be made clear in Section 1, it is necessary to work with a smaller family of fibrations—that is, fibrations that possess a *global* involution restricting to an involution of the general fibers (double cover fibrations). The bielliptic case ($\gamma = 1$) has been treated by Barja in [3]: 4 is the sharp lower bound for the slope and it is possible to give a characterization of the fibrations that reach the bound. About the general case, what is known at present is a result of Barja and Zucconi, who show that the slope of a double cover fibration with $g \geq 2\gamma + 1$ is again greater than 4 (cf. [4, Thm. 0.6]).

It is easy to construct examples of double cover fibrations with slope $4(g-1)/(g-\gamma)$, whereas examples with smaller slope are known only for $g < 4\gamma$ (see [2] and Examples 4.1 and 4.2). Moreover, for hyperelliptic and bielliptic fibrations the number $4(g-1)/(g-\gamma)$ gives exactly the sharp bound. It is therefore natural to conjecture, as Barja does in [2, Sec. 4.2], that this is the sharp bound for double cover fibrations with $g \geq 4\gamma + 1$.

Here we give an affirmative answer to this conjecture (Theorem 3.1 and Theorem 3.2) together with a characterization of the fibrations that reach the bound. Our results follow from an application of the slope inequality for fibered surfaces and of the algebraic index theorem, or signature theorem (see e.g. [5, Thm. IV.2.14] or [7]).

In Section 1 we discuss the problem of gluing involutions on the general fibers to a global involution. In Section 2 we recall a standard construction that allows

us to relate the invariants of a double cover fibration to the ones of a fibration that is a “true” double cover of a relatively minimal fibration of genus γ . Section 3 is devoted to the proof of the bound and to the characterization of the extremal case, and in Section 4 we present two examples that prove the sharpness of the bound.

1. Double Covers and Double Fibrations

The notion of double fibration is a natural extension of the one of hyperelliptic or bielliptic fibration.

DEFINITION 1.1. A *double fibration of type (g, γ)* is a relatively minimal genus g fibered surface $f: X \rightarrow B$ such that there is a degree-2 morphism from the general fiber of f to a smooth curve of genus γ .

The sheet interchange involution on the general fiber of a double fibration $f: X \rightarrow B$ does not necessarily come from a global involution on X . The problem, of course, is that the involution on the general fiber may not be unique and there may not exist a rational section of $\mathbf{Aut}_B(X) \rightarrow B$ that reduces to the given involution on the general fiber, where $\mathbf{Aut}_B(X)$ stands for the relative Hilbert scheme parameterizing automorphisms of fibers of $X \rightarrow B$. If such a section exists—for instance, if the involution on the general fiber of f is unique—then it gives a global rational involution on X , which is actually regular when $g \geq 2$ because of the relative minimality of $X \rightarrow B$. In this case we thus get what we call a double cover fibration.

DEFINITION 1.2. A *double cover fibration of type (g, γ)* is the datum of a genus- g fibration $f: X \rightarrow B$ together with a global involution on X that restricts, on the general fiber, to an involution with genus- γ quotient.

This definition of double cover fibration is slightly more restrictive than the one given in [4]; in the cases we shall be concerned with, however, the two definitions are equivalent. Because one can always produce sections of the scheme of relative automorphisms after a suitable finite base change $T \rightarrow B$, one might think that, to prove slope inequalities for general double fibrations of genus $g \geq 2$, it would suffice to prove them for double cover fibrations. Although this strategy works well if $X \rightarrow B$ is a semistable fibration, in the general case it runs into difficulties that at present appear insurmountable, since the slope behaves very badly under base change [10; 11].

We shall now show that, under the assumption $g > 4\gamma + 1$, the involution on a general fiber of a double cover fibration of type (g, γ) is indeed unique and that the same is true when $g = 4\gamma + 1$ except in a special case. The argument is due to Barja [2, Lemma 4.7] save for the discussion of the case $g = 4\gamma + 1$.

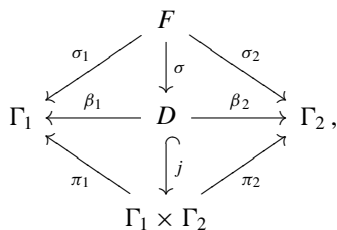
LEMMA 1.3. *Let F be a smooth curve of genus g , and let $\gamma \geq 1$ be an integer. If $g > 4\gamma + 1$, then F has at most one involution ι such that $\Gamma = F/\langle \iota \rangle$ has genus $\leq \gamma$. If instead $g = 4\gamma + 1$ and if there are distinct involutions ι_1, ι_2 of F such that the quotients $F/\langle \iota_1 \rangle = \Gamma_1$ and $F/\langle \iota_2 \rangle = \Gamma_2$ have genera γ_1, γ_2 both not exceeding*

γ , then $\gamma_1 = \gamma_2 = \gamma$, Γ_1 and Γ_2 are hyperelliptic, the natural map $F \rightarrow \Gamma_1 \times \Gamma_2$ is an embedding, and its image belongs to the linear system $|\pi_1^*(2q_1) + \pi_2^*(2q_2)|$, where q_i is a Weierstrass point on Γ_i and π_i denotes the projection $\Gamma_1 \times \Gamma_2 \rightarrow \Gamma_i$.

Proof. Suppose ι_1 and ι_2 are two involutions of F such that the quotients

$$F/\langle \iota_1 \rangle = \Gamma_1 \quad \text{and} \quad F/\langle \iota_2 \rangle = \Gamma_2$$

have genera γ_1 and γ_2 not greater than γ . Consider the commutative diagram



where the σ_i are the quotient morphisms, $D = \sigma_1 \times \sigma_2(F)$, $j \circ \sigma = \sigma_1 \times \sigma_2$, the π_i are the projections, and the β_i are their restrictions to D . Clearly, the degree of σ is either 1 or 2. If it is 2 then the β_i must be isomorphisms; therefore, σ_1 and σ_2 are the quotient maps of the *same* involution on F . Conversely, if the involutions ι_1 and ι_2 coincide, then the degree of σ must be 2.

Now suppose that $\text{deg } \sigma = 1$. Set $L_i = \pi_i^{-1}(p_i) \subseteq \Gamma_1 \times \Gamma_2$ with $p_i \in \Gamma_i$. The effective divisor $L_1 + L_2$ has self-intersection $2 > 0$. By the index theorem, the determinant of the intersection matrix of the pair $(D, L_1 + L_2)$ has to be nonpositive. In other words,

$$2(D \cdot D) - (D \cdot L_1 + L_2)^2 \leq 0.$$

Since $(D \cdot L_i) = 2$, we obtain that $(D \cdot D) \leq 8$. By the adjunction formula,

$$2g - 2 \leq 2p_a(D) - 2 = (K_{\Gamma_1 \times \Gamma_2} + D \cdot D) \leq 4(\gamma_1 + \gamma_2) \leq 8\gamma. \tag{1.1}$$

Thus $g \leq 4\gamma + 1$, and the first part of the lemma is proved.

If $g = 4\gamma + 1$, then all the preceding inequalities must necessarily be equalities. In particular, $\gamma_1 = \gamma_2 = \gamma$ and $p_a(D) = g$, so σ is an isomorphism. Furthermore, by the index theorem, D must be numerically equivalent to a rational multiple of $L_1 + L_2$. Intersecting with L_1 and L_2 , one sees that D is numerically equivalent to $2L_1 + 2L_2$. Hence D is linearly equivalent to a divisor of the form $\pi_1^*A_1 + \pi_2^*A_2$, where A_i is a divisor of degree 2 on Γ_i . Observe that

$$H^0(\Gamma_1 \times \Gamma_2, \mathcal{O}(\pi_1^*A_1 + \pi_2^*A_2)) = H^0(\Gamma_1, \mathcal{O}(A_1)) \otimes H^0(\Gamma_2, \mathcal{O}(A_2)). \tag{1.2}$$

It follows in particular that the dimension of $H^0(\Gamma_i, \mathcal{O}(A_i))$ must be strictly positive, since the linear system $|\pi_1^*A_1 + \pi_2^*A_2|$ is nonempty. If the dimension of $H^0(\Gamma_1, \mathcal{O}(A_1))$ were equal to 1, then (1.2) would imply that every divisor in $|\pi_1^*A_1 + \pi_2^*A_2|$ is of the form $\pi_1^*D_1 + \pi_2^*D_2$, where D_1 is the unique divisor in $|A_1|$ and D_2 belongs to $|A_2|$, and is therefore singular. Since D is smooth and

belongs to $|\pi_1^*A_1 + \pi_2^*A_2|$, it follows that the dimension of $H^0(\Gamma_1, \mathcal{O}(A_1))$ must be at least 2. Similar considerations apply to $H^0(\Gamma_2, \mathcal{O}(A_2))$. We therefore conclude that the Γ_i are hyperelliptic and that A_i is linearly equivalent to $2q_i$, where q_i is a Weierstrass point. \square

Lemma 1.3 implies in particular that, if $g = 4\gamma + 1$ and F has an involution ι such that $F/\langle \iota \rangle$ has genus γ , then the involution is unique provided that $F/\langle \iota \rangle$ is nonhyperelliptic.

If the general fibers of a double fibration are as described in the second part of Lemma 1.3, then it is possible that a nontrivial base change is needed in order to get a global involution, as the following example shows (see also [3] for an example in the bielliptic case).

EXAMPLE 1.4. Let Γ be a smooth hyperelliptic curve of genus γ . Let B be a smooth curve of positive genus, and let $\alpha: T \rightarrow B$ be an unramified degree-2 covering; thus B is the quotient of T modulo a basepoint-free involution σ . We set $G = \langle \sigma \rangle$ and denote by τ the “exchange of components” automorphism of $\Gamma \times \Gamma$. Let π_1 and π_2 denote the projections from $\Gamma \times \Gamma$ to the two factors. Let q be a Weierstrass point of Γ , and consider the linear system $|\pi_1^*(2q) + \pi_2^*(2q)|^\tau$ of effective τ -invariant divisors linearly equivalent to $\pi_1^*(2q) + \pi_2^*(2q)$; it is immediate to show that this system is basepoint-free and not composed with an involution. So, by Bertini’s theorem, a general member $F \in |\pi_1^*(2q) + \pi_2^*(2q)|^\tau$ is smooth and irreducible. The genus of F is $g = 4\gamma + 1$. Let G act on $\Gamma \times \Gamma \times T$ by

$$\sigma(f_1, f_2, t) = (f_2, f_1, \sigma(t)) = (\tau(f_1, f_2), \sigma(t)).$$

Clearly, the subvariety $F \times T \subseteq \Gamma \times \Gamma \times T$ is G -invariant, and the action of G on it is compatible with the action of G on T . Dividing by these two actions, we thus get from $F \times T \rightarrow T$ a new fibration $X \rightarrow B$, where $X = (F \times T)/G$, with fibers all isomorphic to F . The curve F carries two involutions, ι_1 and ι_2 , which correspond to the projections to the two factors of $\Gamma \times \Gamma$. Notice that ι_1 and ι_2 are conjugate under the involution τ (i.e., that $\tau\iota_1 = \iota_2\tau$). For each $b \in B$, the fiber X_b of $X \rightarrow B$ inherits from F two involutions with quotient of genus γ . In particular, $X \rightarrow B$ is a double fibration of type (g, γ) . We claim that, for any $b \in B$, the two involutions on X_b belong to the same component of the scheme $\mathbf{Aut}_B(X)$. In fact, $\mathbf{Aut}_B(X)$ is the quotient modulo G of $\mathbf{Aut}_T(F \times T) = \text{Aut}(F) \times T$, where σ acts by sending $(\alpha, t) \in \text{Aut}(F) \times T$ to $(\tau\alpha\tau, \sigma(t))$, while the irreducible components of $\mathbf{Aut}_B(X)$ are the images of the irreducible components of $\mathbf{Aut}_T(F \times T)$. On the other hand, the images of the components $\{\iota_1\} \times T$ and $\{\iota_2\} \times T$ coincide, as follows from the observation that $\sigma(\iota_1, t) = (\tau\iota_1\tau, \sigma(t)) = (\iota_2, \sigma(t))$. This proves our claim. A consequence is that, for any $b \in B$, there is no global involution ι in the group $\text{Aut}_B(X)$ of automorphisms of X over B that pulls back on X_b to one of the two involutions coming from ι_i and ι_2 . In fact, if such a ι existed, it would give an involution on every fiber of $X \rightarrow B$ and hence a section of $\mathbf{Aut}_B(X)$ over B . This section would be a component of $\mathbf{Aut}_B(X)$ containing only one of the two involutions on X_b , contrary to what we have just proved.

2. Reduction to Double Covers of Relatively Minimal Fibrations

Because this is crucial for our argument, we now sketch a well-known procedure that associates to a double cover fibration a double cover of a relatively minimal fibration. (See e.g. [1] or [4] for the precise construction.)

Let $f : X \rightarrow B$ be a double cover fibration. Let ι be the involution on X . If ι has a fixed locus of codimension 1, then the quotient $X/\langle \iota \rangle$ is a smooth surface. Otherwise, consider the blow-up \tilde{X} of X at the isolated fixed points of ι , and call $\tilde{\iota}$ the induced involution on it. The quotient $\tilde{X}/\langle \tilde{\iota} \rangle = \tilde{Y}$ is a smooth surface with a natural fibration $\tilde{\alpha}$ over B that is not necessarily relatively minimal. Let $\alpha : Y \rightarrow B$ be its minimal model. Then

$$\begin{array}{ccccc}
 \tilde{X} & \longrightarrow & \tilde{Y} & \longrightarrow & Y \\
 \downarrow & & \tilde{\alpha} \downarrow & \swarrow \alpha & \\
 X & \xrightarrow{f} & B & &
 \end{array}$$

The direct image R of the branch locus of the double cover $\tilde{X} \rightarrow \tilde{Y}$ induces a double cover $X' \rightarrow Y$ with X' normal but not necessarily smooth; but notice that, by construction, X' is a hypersurface in a threefold that is smooth over B , so $X' \rightarrow B$ admits an invertible relative dualizing sheaf. To obtain a smooth double cover we perform the *canonical resolution* (see [1, Sec. 2.2; 3, Sec. 2; 5, III.7])

$$\begin{array}{ccccccc}
 X_k & \xrightarrow{\sigma_k} & X_{k-1} & \longrightarrow & \cdots & X_1 & \xrightarrow{\sigma_1} & X_0 = X' \\
 \downarrow & & \downarrow & & & \downarrow & & \downarrow \\
 Y_k & \xrightarrow{\tau_k} & Y_{k-1} & \longrightarrow & \cdots & Y_1 & \xrightarrow{\tau_1} & Y_0 = Y,
 \end{array}$$

where the τ_j are successive blow-ups that resolve the singularities of R ; the morphism $X_j \rightarrow Y_j$ is the double cover with branch locus $R_j := \tau_j^* R_{j-1} - 2[m_{j-1}/2]E_j$, where E_j is the exceptional divisor of τ_j , m_{j-1} is the multiplicity of the blown-up point, and $[\cdot]$ stands for integral part. Let $f_j : X_j \rightarrow B$ and $f' : X' \rightarrow B$ be the induced fibrations. A computation shows that

$$(\omega_{f_k} \cdot \omega_{f_k}) = (\omega_{f'} \cdot \omega_{f'}) - 2 \sum_{i=1}^k \left(\left[\frac{m_i}{2} \right] - 1 \right)^2$$

and

$$\deg(f_{k*} \omega_{f_k}) = \deg f'_* \omega_{f'} - \frac{1}{2} \sum_{i=1}^k \left[\frac{m_i}{2} \right] \left(\left[\frac{m_i}{2} \right] - 1 \right).$$

Observe that, since X_k is smooth, by the relative minimality of $f : X \rightarrow B$ there is a morphism $\beta : X_k \rightarrow X$. Therefore

$$(\omega_f \cdot \omega_f) = (\omega_{f_k} \cdot \omega_{f_k}) + \varepsilon,$$

where ε is the number of blow-ups that make up β . Moreover, observe that $f_*\omega_f = f_{k*}\omega_{f_k}$. Hence we get the following fundamental identity:

$$\begin{aligned} (\omega_f \cdot \omega_f) - 4 \frac{g-1}{g-\gamma} \deg(f_*\omega_f) &= (\omega_{f'} \cdot \omega_{f'}) - 4 \frac{g-1}{g-\gamma} \deg(f'_*\omega_{f'}) \\ &+ 2 \sum_{i=1}^k \left(\left[\frac{m_i}{2} \right] - 1 \right) \left(\frac{\gamma-1}{g-\gamma} \left[\frac{m_i}{2} \right] + 1 \right) + \varepsilon. \end{aligned} \tag{2.1}$$

DEFINITION 2.1. In the preceding case, we say that the branch divisor $R \subset Y$ has *negligible singularities* if all the multiplicities m_i in the process just described equal 2 or 3 (cf. [8]).

3. The Bound

THEOREM 3.1. Let $f: X \rightarrow B$ be a double fibration of type (g, γ) . If $g > 4\gamma + 1$, then

$$(\omega_f \cdot \omega_f) \geq 4 \frac{g-1}{g-\gamma} \deg f_*\omega_f. \tag{3.1}$$

If $\gamma \geq 1$, then equality holds if and only if X is the minimal desingularization of a double cover $\pi: \tilde{X} \rightarrow Y$ of a locally trivial genus- γ fibration $\alpha: Y \rightarrow B$ such that the branch locus R of π has only negligible singularities and, in addition, when $\gamma > 1$, is numerically equivalent to a rational linear combination of ω_α and a fiber of α .

Proof. The case $\gamma = 0$ is the slope inequality for hyperelliptic fibrations. The case $\gamma = 1$ has been proved in [3]. We therefore assume that $\gamma > 1$. In view of Lemma 1.3, the assumptions about g and γ guarantee that we are in fact dealing with a double cover fibration. We adopt the notation introduced in Section 2. In view of (2.1), to prove (3.1) it suffices to prove its analogue for $f': X' \rightarrow B$. Recall that the double covering $\xi: X' \rightarrow Y$ corresponds to a line bundle \mathcal{L} on Y such that $\mathcal{L}^2 = \mathcal{O}(R)$, where R is the ramification divisor of ξ , and that

$$\xi_*\mathcal{O}_{X'} = \mathcal{O}_Y \oplus \mathcal{L}^{-1}, \quad \omega_{f'} = \xi^*(\omega_\alpha \otimes \mathcal{L}).$$

It follows that

$$(\omega_{f'} \cdot \omega_{f'}) = 2(\omega_\alpha \otimes \mathcal{L} \cdot \omega_\alpha \otimes \mathcal{L}) = 2(\omega_\alpha \cdot \omega_\alpha) + 4(\mathcal{L} \cdot \omega_\alpha) + 2(\mathcal{L} \cdot \mathcal{L})$$

and also, by the Riemann–Roch theorem, that

$$\deg f'_*\omega_{f'} = 2 \deg \alpha_*\omega_\alpha + \frac{(\mathcal{L} \cdot \mathcal{L})}{2} + \frac{(\mathcal{L} \cdot \omega_\alpha)}{2}.$$

Hence we may write

$$\begin{aligned} (\omega_{f'} \cdot \omega_{f'}) - 4 \frac{g-1}{g-\gamma} \deg f'_*\omega_{f'} &= 2 \left((\omega_\alpha \cdot \omega_\alpha) - 4 \frac{g-1}{g-\gamma} \deg \alpha_*\omega_\alpha \right) \\ &\quad - 2 \frac{\gamma-1}{g-\gamma} (\mathcal{L} \cdot \mathcal{L}) + 2 \frac{g-2\gamma+1}{g-\gamma} (\mathcal{L} \cdot \omega_\alpha). \end{aligned}$$

Using the slope inequality (0.1) for $\alpha: Y \rightarrow B$, we obtain that

$$(\omega_\alpha \cdot \omega_\alpha) - 4 \frac{g-1}{g-\gamma} \deg \alpha_* \omega_\alpha \geq \frac{-g-\gamma^2+2\gamma}{(\gamma-1)(g-\gamma)} (\omega_\alpha \cdot \omega_\alpha).$$

Therefore

$$\begin{aligned} & (\omega_{f'} \cdot \omega_{f'}) - 4 \frac{g-1}{g-\gamma} \deg f'_* \omega_{f'} \\ & \geq \frac{2}{g-\gamma} \left((g-2\gamma+1)(\omega_\alpha \cdot \mathcal{L}) - (\gamma-1)(\mathcal{L} \cdot \mathcal{L}) - \frac{\gamma^2+g-2\gamma}{\gamma-1} (\omega_\alpha \cdot \omega_\alpha) \right) \\ & = \frac{1}{g-\gamma} \left(2(\mathcal{L} \cdot \Gamma)(\omega_\alpha \cdot \mathcal{L}) - (\omega_\alpha \cdot \Gamma)(\mathcal{L} \cdot \mathcal{L}) - 4 \frac{\gamma^2+g-2\gamma}{(\omega_\alpha \cdot \Gamma)} (\omega_\alpha \cdot \omega_\alpha) \right), \end{aligned}$$

where Γ stands for a general fiber of α . Since $(\omega_\alpha \cdot \omega_\alpha) \geq 0$, it follows that the intersection matrix of ω_α , \mathcal{L} , and Γ cannot be negative definite. The index theorem then implies that its determinant is nonnegative—in other words, that

$$2(\mathcal{L} \cdot \Gamma)(\omega_\alpha \cdot \Gamma)(\omega_\alpha \cdot \mathcal{L}) - (\omega_\alpha \cdot \Gamma)^2(\mathcal{L} \cdot \mathcal{L}) \geq (\mathcal{L} \cdot \Gamma)^2(\omega_\alpha \cdot \omega_\alpha).$$

Combining this inequality with the ones obtained previously, we get

$$\begin{aligned} & (\omega_{f'} \cdot \omega_{f'}) - 4 \frac{g-1}{g-\gamma} \deg f'_* \omega_{f'} \\ & \geq \frac{1}{g-\gamma} \left(\frac{(\mathcal{L} \cdot \Gamma)^2}{(\omega_\alpha \cdot \Gamma)} - 4 \frac{\gamma^2+g-2\gamma}{(\omega_\alpha \cdot \Gamma)} \right) (\omega_\alpha \cdot \omega_\alpha), \end{aligned}$$

and so

$$(\omega_{f'} \cdot \omega_{f'}) - 4 \frac{g-1}{g-\gamma} \deg f'_* \omega_{f'} \geq \frac{(g-4\gamma-1)(g-1)}{2(g-\gamma)(\gamma-1)} (\omega_\alpha \cdot \omega_\alpha). \tag{3.2}$$

The expression on the right is clearly nonnegative as soon as $g \geq 4\gamma + 1$. Note that the argument so far applies to any double cover fibration.

To prove the characterization of the fibrations that reach the bound, observe first that the coefficient of $(\omega_\alpha \cdot \omega_\alpha)$ in (3.2) is not zero when $g > 4\gamma + 1$, so the local triviality of α is a necessary condition. Now recall that $\mathcal{O}(R) = \mathcal{L}^2$ and notice that, if (3.1) is an equality, then all the inequalities in the proof must be equalities and the terms $2 \sum_{i=1}^k \left(\left[\frac{m_i}{2} \right] - 1 \right) \left(\frac{\gamma-1}{g-\gamma} \left[\frac{m_i}{2} \right] + 1 \right)$ and ε in (2.1) must vanish. In particular, we get that

$$(g-2\gamma+1)(\omega_\alpha \cdot \mathcal{L}) - (\gamma-1)(\mathcal{L} \cdot \mathcal{L}) = 0,$$

which, in view of $(\omega_\alpha \cdot \omega_\alpha) = 0$ and of $\mathcal{O}(R) = \mathcal{L}^2$, is equivalent to the vanishing of the determinant of the intersection matrix of ω_α , Γ , and R —that is, to R being numerically equivalent to a rational linear combination of ω_α and Γ . □

The analogous result for $g = 4\gamma + 1$ can be stated as follows.

THEOREM 3.2. *Let $f: X \rightarrow B$ be a double fibration of type (g, γ) with $g = 4\gamma + 1$. Then inequality (3.1) holds, provided we are in one of the following cases:*

- (1) f is a double cover fibration;
- (2) f is a semistable fibration.

In particular, (3.1) is valid if a smooth fiber of f admits an involution whose quotient is a nonhyperelliptic curve of genus γ .

Moreover, a necessary condition for the slope to reach the bound is that the associated relatively minimal fibration of genus γ be either locally trivial or hyperelliptic with slope $4(\gamma - 1)/\gamma$.

Proof. Case (1) is covered by the argument used to prove Theorem 3.1. In case (2), let $\rho: T \rightarrow B$ be a finite map with T smooth. Then $X \times_B T$ has A_n singularities. Let X' be its minimal resolution and $f': X' \rightarrow T$ the natural projection; clearly, this is a minimal fibration. As is well known, the slope of f' is equal to that of f . In fact, in this case $\omega_{f'}$ is just the pullback to X' of ω_f , so $(\omega_{f'} \cdot \omega_{f'})$ and $\deg f'_* \omega_{f'}$ are both equal to $\deg \rho$ times the corresponding invariant of f . Since the base change ρ can be chosen so that $f': X' \rightarrow T$ is a double cover fibration, we are reduced to the previous case. It follows from Lemma 1.3 and the comment immediately following its proof that a sufficient condition for f to be a double cover fibration is that one of its smooth fibers admit an involution with nonhyperelliptic genus- γ quotient. The coefficient of $(\omega_\alpha \cdot \omega_\alpha)$ in (3.2) is 0 when $g = 4\gamma + 1$. Hence the local triviality of α is no longer a necessary condition for the fibration to reach the bound. If α is not locally trivial, it is instead necessary that α itself attain the bound given by the slope inequality, so we are done. □

Clearly, one could give necessary and sufficient conditions as in Theorem 3.1 by requiring that the inequalities in the proof be equalities. It is interesting to notice that, in this borderline case, the conditions change substantially because local triviality of the fibration of genus γ is no longer needed and, indeed, one can construct a fibered surface of arbitrary genus g that reaches the bound and is a double cover of a non-locally trivial fibration of genus $\gamma = (g - 1)/4$ (see Example 4.2).

4. Examples

We next present two examples, both due to Barja (cf. [2, Sec. 4.5]) showing that the bound given is indeed sharp. The first is an example of double cover fibration reaching the bound; in the second we construct a fibration with $g = 4\gamma + 1$ that reaches the bound and is a double cover of a hyperelliptic fibration, which in turn reaches the bound given by the slope inequality. The last construction also leads to counterexamples to the bound for $g < 4\gamma$.

EXAMPLE 4.1. This is a generalization of the examples constructed in [12] and in [6]. Let Γ and B be smooth curves. Call γ the genus of Γ . Let $p_1: B \times \Gamma \rightarrow B$ and $p_2: B \times \Gamma \rightarrow \Gamma$ be the two projections, and let H_1 and H_2 be their general fibers. For sufficiently large integers n and m , the linear system $|2nH_1 + 2mH_2|$ is basepoint-free. Hence, by Bertini's theorem there exists a smooth divisor $R \in |2nH_1 + 2mH_2|$. Since R is even, we can construct the double cover

$\rho: X \rightarrow B \times \Gamma$ ramified over R . Consider the fibration $f := p_1 \circ \rho: X \rightarrow B$; its general fiber is a double cover of Γ , and its genus is $g = 2\gamma + m - 1$. Observe that

$$\omega_f \cong \rho^*(\omega_{p_1}(nH_1 + mH_2)) \cong \rho^*(\mathcal{O}(nH_1 + (2\gamma - 2 + m)H_2))$$

and

$$\deg f_*\omega_f = \deg p_{1*}(\omega_{p_1}(nH_1 + mH_2)) = n(\gamma - 1 + m).$$

Therefore, the slope of f is exactly

$$s(f) = 4 \frac{2\gamma + m - 2}{\gamma + m - 1} = 4 \frac{g - 1}{g - \gamma}.$$

If we consider a general divisor $R \in |2nH_1 + 2mH_2|$, it has only simple ramification points over B and we obtain a semistable fibration. Notice that R is numerically equivalent to a linear combination of ω_{p_1} and Γ (as it should be) because

$$R \equiv 2nH_1 + 2mH_2 \equiv 2n\Gamma + \frac{m}{\gamma - 1}K_{p_1}.$$

EXAMPLE 4.2. Consider the fibration of Example 4.1 with $\Gamma = \mathbb{P}^1$, and set $f_i = p_i \circ \rho$. Call F_i the general fiber of f_i ; hence F_1 is hyperelliptic of genus $\gamma = m - 1$. By what we observed in Example 4.1,

$$\omega_{f_1} \cong \rho^*(\omega_{p_1}(nH_1 + mH_2)) \cong \mathcal{O}(nF_1 + (m - 2)F_2).$$

Let x and y be positive integers, and consider the linear system $|2xF_1 + 2yF_2|$. Applying Bertini's theorem again, for large enough x and y we can find a smooth even divisor Δ belonging to this system. Let $\pi: Y \rightarrow X$ be the double cover ramified over Δ . Call h the fibration $p_1 \circ \rho \circ \pi: Y \rightarrow B$; then the general fiber F of h is a double cover of F_1 . Its genus is $g = 2(m - 1) + 2y - 1$. Now $\omega_h \cong \pi^*(\omega_{f_1}(xF_1 + yF_2)) \cong \mathcal{O}((n + x)F + (m + y - 2)\pi^*F_2)$, so

$$(\omega_h \cdot \omega_h) = 8(x + n)(y + m - 2)$$

while

$$\begin{aligned} h_*\omega_h &= f_{1*}(\omega_{f_1}(xF_1 + yF_2)) \oplus f_{1*}(\omega_{f_1}) \\ &= p_{1*}(\rho_*(\rho^*(\mathcal{O}((n + x)H_1 + (m + y - 2)H_2)))) \\ &= p_{1*}(\omega_{p_1}((n + x)H_1 + (m + y)H_2)) \\ &\quad \oplus p_{1*}\mathcal{O}(xH_1 + (y - 2)H_2) \oplus f_{1*}(\omega_{f_1}). \end{aligned}$$

Therefore

$$\deg h_*\omega_h = (x + n)(y + m - 1) + x(y - 1) + n(m - 1).$$

If we choose (as we may) $m = y$, then we get exactly $g = 4m - 3 = 4\gamma + 1$ and slope

$$s(h) = 8 \frac{2m - 2}{3m - 2} = 4 \frac{g - 1}{g - \gamma}.$$

Notice moreover that by choosing $m > y$ we obtain fibrations with $g \leq 4\gamma - 1$ and slope strictly smaller than $4(g - 1)/(g - \gamma)$.

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