

A Minimal Brieskorn 5-Sphere in the Gromoll–Meyer Sphere and Its Applications

CARLOS DURÁN & THOMAS PÜTTMANN

1. Introduction

This paper links two previously more or less unrelated important examples at the intersection of the fields of transformation groups, exotic spheres, and non-negative curvature. These two examples are the exotic Gromoll–Meyer sphere Σ^7 and the Brieskorn sphere W_3^5 . Several applications are drawn from the interplay between the Riemannian geometry of a 2-parameter family of metrics on Σ^7 and the equivariant geometry of W_3^5 , which, surprisingly, determines the equivariant geometry of Σ^7 much more than the equivariant geometry of the exotic Brieskorn spheres $W_{6j-1,3}^7$, although the latter contain W_3^5 in a much more obvious way.

In 1974, Gromoll and Meyer [GrMy] constructed Σ^7 as a biquotient of the compact group $\mathrm{Sp}(2)$ and thereby the first exotic sphere with nonnegative sectional curvature. Note that Σ^7 can be regarded as the basic example of a biquotient in Riemannian geometry and, simultaneously, as the basic example of an exotic sphere. It generates the group $\Theta_7 \approx \mathbb{Z}_{28}$ of homotopy spheres in dimension 7, the first dimension except possibly 4 where exotic spheres can occur. Recently, it was shown that Σ^7 is actually the only exotic sphere that can be modeled by a biquotient of a compact Lie group [KaZ; To].

Because of this exceptional status of the Gromoll–Meyer sphere, it seems natural to study the geometry of Σ^7 in detail. Papers that do this from various viewpoints are [D; EK; PaSp; Y; Wi], for example. Here we investigate Σ^7 through the interaction between symmetry arguments, submanifold stratifications, and geodesic constructions. It is important, however, to note that we consider not only the Gromoll–Meyer metric on Σ^7 but also the entire 2-parameter family of metrics $\langle \cdot, \cdot \rangle_{\mu, \nu}$ that are $\mathrm{O}(2) \times \mathrm{SO}(3)$ invariant by construction. This family includes the Gromoll–Meyer metric ($\mu = \nu = \frac{1}{2}$) and the pointed wiedersehen metric constructed in [D] ($\mu = \nu = 1$) but not the metrics of almost positive sectional curvature obtained in [EK] and [Wi]. Extending the constructions of [D] and [ADPR], we obtain the following structural information.

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THEOREM 1.1. *The Gromoll–Meyer sphere Σ^7 is the join of a simple closed geodesic Σ^1 and a minimal subsphere Σ^5 that is $(\mathrm{O}(2) \times \mathrm{SO}(3))$ -equivariantly diffeomorphic to the Brieskorn sphere W_3^5 . For $\mu = 1$ and $\nu > 0$, the distance between Σ^1 and Σ^5 is a constant $\pi/2$, and the join structure is realized by distance-minimizing geodesics from Σ^1 to Σ^5 .*

Recall that the Brieskorn sphere W_3^5 is the submanifold of \mathbb{C}^4 defined by

$$\begin{aligned} z_0^3 + z_1^2 + z_2^2 + z_3^2 &= 0, \\ |z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^3 &= 1 \end{aligned}$$

and that there is a natural action of $\mathrm{O}(2) \times \mathrm{SO}(3)$ on W_3^5 (see Section 4). On the one hand, it is perhaps not surprising that W_3^5 plays a central role for the geometry of Σ^7 if one recalls that Σ^7 is diffeomorphic to $W_{6j-1,3}^7$ for any $j \in \{1, 9, 17, \dots\}$ (see [Bk]). Here, $W_{6j-1,3}^7 \subset \mathbb{C} \oplus \mathbb{C}^4$ is defined by the equations

$$\begin{aligned} u^{6j-1} + z_0^3 + z_1^2 + z_2^2 + z_3^2 &= 0, \\ |u|^2 + |z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^3 &= 1. \end{aligned}$$

On the other hand, whereas Σ^5 and W_3^5 are $(\mathrm{O}(2) \times \mathrm{SO}(3))$ -equivariantly diffeomorphic, the ambient spaces Σ^7 and $W_{6j-1,3}^7$ are *never* even $\mathrm{SO}(3)$ -equivariantly homeomorphic (see Corollary 7.6); in particular, none of them is $\mathrm{SO}(3)$ -equivariantly homeomorphic to a join of a circle and W_3^5 .

It is important to note that W_3^5 is not equivariantly diffeomorphic to \mathbb{S}^5 with any linear $(\mathrm{O}(2) \times \mathrm{SO}(3))$ -action, and this holds true if one restricts from $\mathrm{O}(2) \times \mathrm{SO}(3)$ to the subgroup $\mathrm{O}(3) = \{\pm 1\} \times \mathrm{SO}(3)$. This follows from the classification theorems of Jänich and of Hsiang and Hsiang (see [Bd2; HzMa]). On the other hand, these theorems imply that there exist $\mathrm{SO}(3)$ -equivariant diffeomorphisms $\mathbb{S}^5 \rightarrow W_3^5$ where $\mathrm{SO}(3)$ acts diagonally on $\mathbb{S}^5 \subset \mathbb{R}^3 \times \mathbb{R}^3$.

This brings us to the first application of Theorem 1.1. Using the geodesic join structure, we derive an explicit formula for an $\mathrm{SO}(3)$ -equivariant diffeomorphism $\mathbb{S}^5 \rightarrow W_3^5$. This nontrivial formula can be verified by a straightforward computation and immediately carries over to dimension 13.

THEOREM 1.2. *Formula (9) in Section 5 provides an $\mathrm{SO}(3)$ -equivariant diffeomorphism $\mathbb{S}^5 \rightarrow W_3^5$ and a G_2 -equivariant diffeomorphism $\mathbb{S}^{13} \rightarrow W_3^{13}$.*

As far as we know, this is the first explicit formula for diffeomorphisms between standard spheres and Brieskorn spheres W_d^{2n-1} with $n > 2$ and odd $d > 1$.

Related to Theorem 1.2 is Theorem 6.3, where we provide nonlinear actions of $\mathrm{O}(2) \times \mathrm{SO}(3)$ and $\mathrm{O}(2) \times G_2$ on the Euclidean spheres \mathbb{S}^5 and \mathbb{S}^{13} that are equivalent to the $(\mathrm{O}(2) \times \mathrm{SO}(3))$ -action on W_3^5 and to the $(\mathrm{O}(2) \times G_2)$ -action on W_3^{13} , respectively. Various models existed for these actions previously (see [Bd1]) but only on manifolds that were inexplicitly diffeomorphic to \mathbb{S}^5 and \mathbb{S}^{13} .

In order to explain the second application of Theorem 1.1, we have to digress briefly into the history of exotic involutions of spheres. A fixed point-free involution of the standard sphere is called *exotic* if the quotient is not diffeomorphic to the

real projective space. The first examples of such involutions were given by Hirsch and Milnor [HMi]. They considered the exotic Milnor sphere $\mathbb{S}^3 \cdots \Sigma_{2,-1}^7 \rightarrow \mathbb{S}^4$ and the involution of $\Sigma_{2,-1}^7$ induced by the antipodal map on the \mathbb{S}^3 -fibers, detected invariant subspheres of dimensions 5 and 6 in $\Sigma_{2,-1}^7$, and proved that the restrictions of the involution of $\Sigma_{2,-1}^7$ yield exotic involutions of these subspheres. The next example of an exotic involution was given by Calabi (unpublished; see [Bd1]), who showed that the involution of W_3^5 given by the map $(z_0, z') \mapsto (z_0, -z')$ is exotic. In [HzMa], the Calabi involution was identified with an involution constructed by Bredon, and this in turn was identified with the Hirsch–Milnor involution by Yang [Ya]. The latter identification, however, turned out to be incorrect (see a footnote in [SSi]), so an explicit identification between the Hirsch–Milnor involution and the Calabi involution was still missing.

In [ADPR] it was shown that the Hirsch–Milnor involutions are induced by the action of $-\mathbb{1} \in \text{Sp}(2)$ on $\Sigma^7 = \Sigma_{2,-1}^7$ and that the invariant subspheres of Hirsch and Milnor are precisely the sphere $\Sigma^5 \subset \Sigma^7$ of Theorem 1.1 and the intermediate equators $\Sigma^5 \subset \Sigma_{\pm A}^6 \subset \Sigma^7$ (see Section 2). In combination with explicit diffeomorphisms $\mathbb{S}^5 \rightarrow \Sigma^5$ and $\mathbb{S}^6 \rightarrow \Sigma_{\pm A}^6$ provided by the geodesic geometry of Σ^7 , this was used to derive explicit formulas for exotic involutions of the Euclidean spheres $\mathbb{S}^5, \mathbb{S}^6, \mathbb{S}^{13}$, and \mathbb{S}^{14} . As a consequence of Theorem 1.1, we now obtain the following result.

COROLLARY 1.3. *The equivariant diffeomorphism $\Sigma^5 \rightarrow W_3^5$ identifies the Hirsch–Milnor involution in dimension 5 with the Calabi involution of W_3^5 .*

Since $\Sigma^5/\{\pm \mathbb{1}\}$ and $\Sigma_{\pm A}^6/\{\pm \mathbb{1}\}$ are not diffeomorphic to real projective spaces, it is natural to investigate the metrics on Σ^5 and $\Sigma_{\pm A}^6$ induced by the metrics $\langle \cdot, \cdot \rangle_{\mu, \nu}$ on Σ^7 . We will show that Σ^5 or $\Sigma_{\pm A}^6$ is totally geodesic in Σ^7 for none of these metrics. Moreover, the induced metrics always have some negative sectional curvatures.

The third application of Theorem 1.1 concerns the full isometry group of Σ^7 . As mentioned already in [GrMy], Hsiang showed that the maximum dimension of the isometry group of any metric on Σ^7 is 4 (see [St] for a proof). Thus the identity component of the isometry group of $\langle \cdot, \cdot \rangle_{\mu, \nu}$ is the group $\text{SO}(2) \times \text{SO}(3)$. It remains a nontrivial problem to determine the other components. Of particular interest is to see which finite groups can act freely on Σ^7 . Recent papers [Ba; GSh; Sh1; Sh2] dealt with the analogous problem for the homogeneous and cohomogeneity-1 manifolds of positive sectional curvature. Surprisingly, it turned out that sometimes noncyclic groups can act freely on these spaces. For the cohomogeneity-3 metrics $\langle \cdot, \cdot \rangle_{\mu, \nu}$ on Σ^7 , the structural information of Theorem 1.1 can be used to reduce the problem to the corresponding problem for the induced cohomogeneity-1 metrics on Σ^5 . This latter problem can be solved with the help of some curvature computations.

THEOREM 1.4. *The group $\text{O}(2) \times \text{SO}(3)$ is the full isometry group of the metrics $\langle \cdot, \cdot \rangle_{\mu, \nu}$ on Σ^7 and on Σ^5 . Any subgroup that acts freely on either Σ^7 or Σ^5 is a finite cyclic group. Conversely, for any $m \in \mathbb{Z}$ the group \mathbb{Z}_m acts freely and isometrically on Σ^7 and on Σ^5 , even in several nonconjugate ways for a fixed $m > 2$.*

All the \mathbb{Z}_m -quotients of Σ^7 inherit nonnegative sectional curvature from the Gromoll–Meyer metric $\langle \cdot, \cdot \rangle_{1/2, 1/2}$. (It is interesting, however, to note that for $m > 2$ the known metrics with almost positive sectional curvature on Σ^7 [EK; Wi] are not invariant with respect to the \mathbb{Z}_m -actions.) In the case of Σ^5 , none of the metrics $\langle \cdot, \cdot \rangle_{\mu, \nu}$ has nonnegative curvature, but by the Grove–Ziller construction [GZ] there exist $(O(2) \times SO(3))$ -invariant metrics on Σ^5 with $K \geq 0$.

COROLLARY 1.5. *For any m not divisible by 6 and for any two integers p, q with $p \neq 0$, $3p - q \neq 0$, and $3p + q \neq 0$ such that m is relatively prime to p , $3p - q$, and $3p + q$, there is a 7-manifold with $K \geq 0$ that is homotopy equivalent to the lens space $L_m^7(p, p, 3p - q, 3p + q)$ but not diffeomorphic to any standard lens space. If m is even then there also exists a 5-manifold with $K \geq 0$ that is homotopy equivalent to the lens space $L_m^5(p, 3p - q, 3p + q)$ but not diffeomorphic to any standard lens space.*

In dimension 5 the case $m = 2$ was already covered in [GZ]. Apart from this case, these seem to be the first known exotic homotopy lens spaces with $K \geq 0$. Exotic lens spaces with positive Ricci and almost nonnegative sectional curvature in higher dimensions were found by Schwachhöfer and Tuschmann [ScT].

The nontrivial part of Corollary 1.5 is to determine the homotopy type of the free \mathbb{Z}_m -quotients of Σ^7 and Σ^5 . It is well known that the orbit spaces of free \mathbb{Z}_m -actions on homotopy spheres are homotopy equivalent to lens spaces (see [Bw]). For a concretely given action on a homotopy sphere, however, there is no canonical tractable way to determine the homotopy type of the quotient. In our case we will follow an idea of Orlik [Or] and construct branched coverings $\Sigma^5 \rightarrow \mathbb{S}^5$ that can be extended by the join structure of Theorem 1.1 to (continuous) branched coverings $\Sigma^7 \rightarrow \mathbb{S}^7$. The essential property of these branched coverings is that they are $(O(2) \times SO(3))$ -equivariant where $O(2) \times SO(3)$ acts linearly on \mathbb{S}^7 and \mathbb{S}^5 . The target spaces of the induced maps $\Sigma^7/\mathbb{Z}_m \rightarrow \mathbb{S}^7/\mathbb{Z}_m$ and $\Sigma^5/\mathbb{Z}_m \rightarrow \mathbb{S}^5/\mathbb{Z}_m$ are thus standard lens spaces, and this allows us to determine the homotopy type of Σ^7/\mathbb{Z}_m and Σ^5/\mathbb{Z}_m .

The fourth application of Theorem 1.1 resides in determining the structure of fixed point sets of isometries of Σ^7 . Fixed point sets of isometries are useful to understand the geometry of Riemannian manifolds because each connected component is a totally geodesic submanifold. In particular, they provide significant curvature information because the extrinsic and intrinsic sectional curvatures of a plane tangent to a totally geodesic submanifold are equal. In a general biquotient it is fairly difficult if not impossible to determine the structure of all fixed point sets. In Σ^7 , however, we can employ Theorem 1.1 to determine the metric structure of all fixed point sets in a very geometric way (see Section 7). It turns out that all fixed point sets with dimension > 1 are congruent to one of three 3-spheres $\Sigma_0^3, \Sigma_1^3, \Sigma_2^3$ to a real projective space P^3 or to a lens space $L^3 \approx \mathbb{S}^3/\mathbb{Z}_3$. It is interesting to see how the biquotient structure of Σ^7 causes Σ_0^3 and L^3 to have more intrinsic than extrinsic isometries: Both are intrinsically homogeneous although they inherit only a cohomogeneity-1 action from the $(O(2) \times SO(3))$ -action on

Σ^7 . The fact that L^3 and P^3 are fixed point sets with nontrivial fundamental group shows how much the geometry of Σ^7 differs from the geometry of the standard sphere \mathbb{S}^7 with constant curvature. The induced metrics $\langle \cdot, \cdot \rangle_{\mu, \nu}$ on Σ^3_2 are properly of cohomogeneity 2. They are remarkable in that the curvature tensor looks like the metrics would be of cohomogeneity 1 and there is no obvious deformation to the constant curvature metric through metrics with this property.

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2. The Geodesic Join Description of the Gromoll–Meyer Sphere

In this section we review and extend some of the constructions of [D] and [ADPR]. In particular, we use a 1-parameter family of metrics on the Gromoll–Meyer sphere Σ^7 with the pointed wiedersehen property along a circle Σ^1 to recognize Σ^7 as the geodesic join of Σ^1 and a minimal subsphere $\Sigma^5 \subset \Sigma^7$.

Let \mathbb{S}^3 denote the unit sphere in the quaternions \mathbb{H} and let $\text{Sp}(2)$ denote the group of 2×2 quaternionic matrices A such that $\bar{A}^t \cdot A = \mathbb{1}$. On $\text{Sp}(2)$ we consider the class of left-invariant and $\text{Sp}(1) \times \text{Sp}(1)$ right-invariant Riemannian metrics. After rescaling, these metrics correspond precisely to the $\text{Ad}_{\text{Sp}(1) \times \text{Sp}(1)}$ -invariant inner products

$$\left\langle \begin{bmatrix} x_1 & -\bar{y}_1 \\ y_1 & z_1 \end{bmatrix}, \begin{bmatrix} x_2 & -\bar{y}_2 \\ y_2 & z_2 \end{bmatrix} \right\rangle_{\mu, \nu} = \text{Re}(\mu \bar{x}_1 x_2 + \bar{y}_1 y_2 + \nu \bar{z}_1 z_2)$$

on the Lie algebra $\mathfrak{sp}(2)$. The standard biinvariant metric on $\text{Sp}(2)$ is $\langle \cdot, \cdot \rangle_{1/2, 1/2}$. This metric has nonnegative sectional curvature, and it follows from Cheeger’s construction [Ch] that all metrics $\langle \cdot, \cdot \rangle_{\mu, \nu}$ with $\mu, \nu \leq \frac{1}{2}$ have nonnegative sectional curvature as well.

Two free isometric actions of \mathbb{S}^3 on $\text{Sp}(2)$ play a central role in the rest of the paper: the standard action

$$\mathbb{S}^3 \times \text{Sp}(2) \rightarrow \text{Sp}(2), \quad q \bullet A = A \cdot \begin{bmatrix} 1 & 0 \\ 0 & \bar{q} \end{bmatrix}$$

and the Gromoll–Meyer action [GrMy]

$$\mathbb{S}^3 \times \text{Sp}(2) \rightarrow \text{Sp}(2), \quad q \star A = q \cdot A \cdot \begin{bmatrix} \bar{q} & 0 \\ 0 & 1 \end{bmatrix}.$$

Both these actions foliate $\text{Sp}(2)$ by \mathbb{S}^3 -orbits in two different ways. The orbit space of the standard action \bullet can be naturally identified with $\mathbb{S}^7 \subset \mathbb{H}^2$ by restricting a matrix in $\text{Sp}(2)$ to its first column. The orbit space Σ^7 of the Gromoll–Meyer action \star is diffeomorphic to the exotic Milnor sphere $\Sigma^7_{2, -1}$. The corresponding projection maps are denoted by $\pi_{\mathbb{S}^7}: \text{Sp}(2) \rightarrow \mathbb{S}^7$ and $\pi_{\Sigma^7}: \text{Sp}(2) \rightarrow \Sigma^7$. Throughout this paper both orbit spaces, \mathbb{S}^7 and Σ^7 , are presumed to carry metrics induced from $\langle \cdot, \cdot \rangle_{\mu, \nu}$ on $\text{Sp}(2)$ by Riemannian submersion. The metrics on \mathbb{S}^7 and Σ^7 will

also be denoted by $\langle \cdot, \cdot \rangle_{\mu, \nu}$. Since Riemannian submersions are curvature non-decreasing, it is clear that the sectional curvature of $(\Sigma^7, \langle \cdot, \cdot \rangle_{\mu, \nu})$ is nonnegative for $\mu, \nu \leq \frac{1}{2}$.

The starting point for our subsequent geometric constructions and considerations is the following elementary fact: The \bullet -orbit and the \star -orbit through any real matrix $A \in O(2) \subset Sp(2)$ are equal as sets because A commutes with all $q \in S^3$. A geodesic in $Sp(2)$ that passes perpendicularly through the common orbit

$$S^3 \bullet A = S^3 \star A = \left\{ A \cdot \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix} \mid q \in S^3 \right\}$$

is perpendicular to all \bullet -orbits and all \star -orbits and hence projects to geodesics in both orbit spaces, S^7 and Σ^7 . (Recall that the inner product between the velocity vector field and a Killing field is constant along a geodesic.) Now, fixing a matrix $A' \in S^3 \bullet A = S^3 \star A$, we get an identification between the geodesics in S^7 that start at the point $S^3 \bullet A$ and the geodesics in Σ^7 that start at the point $S^3 \star A$: Two such geodesics γ_{S^7} and γ_{Σ^7} correspond to each other if and only if there is a common horizontal lift through A' —that is, a geodesic $\tilde{\gamma}$ in $Sp(2)$ that starts at A' perpendicularly to $S^3 \bullet A = S^3 \star A$ such that $\gamma_{S^7} = \pi_{S^7} \circ \tilde{\gamma}$ and $\gamma_{\Sigma^7} = \pi_{\Sigma^7} \circ \tilde{\gamma}$. See Figure 1. Since $S^3 \bullet A = S^3 \star A$ are equal only as sets, this identification depends on the choice of A' . There is, however, a canonical choice for A' because $S^3 \bullet A = S^3 \star A$ intersects $O(2)$ precisely in the set $\{A, A \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\}$ and hence contains a unique element $A' \in SO(2)$.

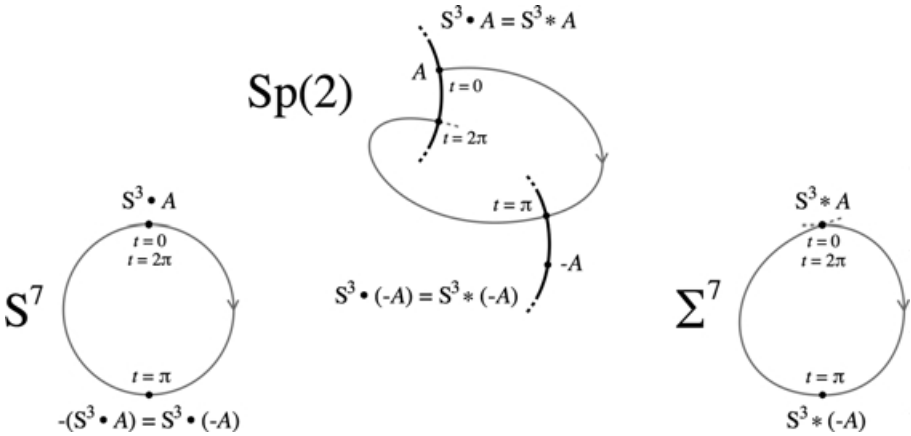


Figure 1 For $A \in SO(2) \subset Sp(2)$, a geodesic in S^7 through $S^3 \bullet A$ corresponds precisely to one geodesic in Σ^7 through $S^3 \star A$ via a common horizontal lift through A

This correspondence has an immediate application in the case $\mu = 1$ where each left-invariant metric $\langle \cdot, \cdot \rangle_{1, \nu}$ on $Sp(2)$ induces the standard metric on S^7 : All unit-speed geodesics of S^7 pass through their antipode after time π and return to their starting point after time 2π . This holds in particular for the geodesics that

start at a point $\mathbb{S}^3 \bullet A$ with $A \in O(2)$. Since the antipode of $\mathbb{S}^3 \bullet A$ is the orbit $\mathbb{S}^3 \bullet (-A) = \mathbb{S}^3 \star (-A)$, the geodesic correspondence just described implies the following recurrency behavior.

THEOREM 2.1 (see [D] for the case where $\nu = 1$). *The unit-speed geodesics of $(\Sigma^7, \langle \cdot, \cdot \rangle_{1,\nu})$ that start at a point $\mathbb{S}^3 \star A$ with $A \in O(2)$ all pass through $\mathbb{S}^3 \star (-A)$ after time π and return to $\mathbb{S}^3 \star A$ after time 2π (but do not close smoothly in general). These geodesics are length minimizing until time π .*

In accordance with the literature (see e.g. [Bs]), the points of the circle

$$\Sigma^1 := \{\mathbb{S}^3 \star A \mid A \in O(2)\} \subset \Sigma^7$$

will be called *wiedersehen* points. The wiedersehen property allows us to define natural subspheres of Σ^7 as follows. For $A \in O(2)$, the bisector

$$\Sigma_{\pm A}^6 := \{x \in \Sigma^7 \mid \text{dist}(x, \mathbb{S}^3 \star A) = \text{dist}(x, \mathbb{S}^3 \star (-A)) = \pi/2\}$$

is given by the midpoints of the geodesics that start at $\mathbb{S}^3 \star A$ and end at $\mathbb{S}^3 \star (-A)$. The intersection of all the bisectors $\Sigma_{\pm A}^6$ in Σ^7 is the set

$$\Sigma^5 := \bigcap_{A \in O(2)} \Sigma_{\pm A}^6 = \{x \in \Sigma^7 \mid \text{dist}(x, \Sigma^1) = \pi/2\}.$$

Recall that the join $X * Y$ of two spaces X, Y is the quotient of $X \times Y \times [0, 1]/\sim$, where $(x, y, 0) \sim (x, y', 0)$ and $(x, y, 1) \sim (x', y, 1)$ for all $x \in X$ and all $y \in Y$. For our purposes it is convenient to replace $[0, 1]$ with $[0, \pi/2]$.

COROLLARY 2.2. *For $\mu = 1$, the Gromoll–Meyer sphere Σ^7 is the geodesic join of the circle Σ^1 and the subsphere Σ^5 that have constant distance $\pi/2$. In other words, the map $\Sigma^1 * \Sigma^5 \rightarrow \Sigma^7$ that maps (x, y, t) to $\gamma(t)$, where $\gamma: [0, \pi/2] \rightarrow \Sigma^7$ is the unique unit-speed geodesic segment from x to y , is a homeomorphism.*

The identification of geodesics in Σ^7 that start at $\mathbb{S}^3 \bullet A$ with the geodesics of Σ^7 that start at a point $\mathbb{S}^3 \star A$ provides an $SO(3)$ -equivariant homeomorphism between \mathbb{S}^7 and Σ^7 that restricts to a diffeomorphism between $\mathbb{S}^7 \setminus (\mathbb{S}^3 \bullet (-A))$ and $\Sigma^7 \setminus (\mathbb{S}^3 \star (-A))$.

This diffeomorphism further restricts to diffeomorphisms $S_{\pm A}^6 \rightarrow \Sigma_{\pm A}^6$ and $S^5 \rightarrow \Sigma^5$, where

$$\begin{aligned} S_{\pm A}^6 &= \left\{ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathbb{S}^7 \subset \mathbb{H}^2 \mid \text{dist}\left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}\right) = \text{dist}\left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, -\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}\right) = \frac{\pi}{2} \right\} \\ &= \left\{ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathbb{S}^7 \subset \mathbb{H}^2 \mid \text{Re}(a_{11}w_1 + a_{21}w_2) = 0 \right\} \\ &= \left\{ A \cdot \begin{bmatrix} p \\ w \end{bmatrix} \in \mathbb{S}^7 \subset \mathbb{H}^2 \mid p \in \text{Im } \mathbb{H}, w \in \mathbb{H} \right\} \end{aligned}$$

and

$$S^5 = \left\{ \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \mid p_1, p_2 \in \text{Im } \mathbb{H}, |p_1|^2 + |p_2|^2 = 1 \right\}.$$

Note that

$$\begin{aligned} \pi_{\Sigma^7}^{-1}(\Sigma_{\pm A}^6) &= \pi_{\mathbb{S}^7}^{-1}(S_{\pm A}^6) = \left\{ A \cdot \begin{bmatrix} p & * \\ w & * \end{bmatrix} \in \text{Sp}(2) \mid p \in \text{Im } \mathbb{H}, w \in \mathbb{H} \right\}, \\ \pi_{\Sigma^7}^{-1}(\Sigma^5) &= \pi_{\mathbb{S}^7}^{-1}(S^5) = \left\{ \begin{bmatrix} p_1 & * \\ p_2 & * \end{bmatrix} \in \text{Sp}(2) \mid p_1, p_2 \in \text{Im } \mathbb{H} \right\}, \end{aligned} \tag{1}$$

since the two sets on the right-hand side are invariant under the \star -action.

There are explicit formulas for the horizontal lifts of the relevant geodesics in \mathbb{S}^7 and hence for the diffeomorphisms $S_{\pm A}^6 \rightarrow \Sigma_{\pm A}^6$ and $S^5 \rightarrow \Sigma^5$. Consider the geodesic

$$\gamma_{\left[\begin{smallmatrix} p \\ w \end{smallmatrix}\right]}(t) = \cos t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin t \begin{bmatrix} p \\ w \end{bmatrix}$$

in $\mathbb{S}^7 \subset \mathbb{H}^2$ that emanates from the north pole with initial velocity $\left[\begin{smallmatrix} p \\ w \end{smallmatrix}\right] \in \mathbb{S}^6 \subset \text{Im } \mathbb{H} \times \mathbb{H}$. The unique horizontal lift $\tilde{\gamma}_{\left[\begin{smallmatrix} p \\ w \end{smallmatrix}\right]}$ of $\gamma_{\left[\begin{smallmatrix} p \\ w \end{smallmatrix}\right]}$ to $\text{Sp}(2)$ with $\tilde{\gamma}_{\left[\begin{smallmatrix} p \\ w \end{smallmatrix}\right]}(0) = \mathbb{1}$ is given by

$$\tilde{\gamma}_{\left[\begin{smallmatrix} p \\ w \end{smallmatrix}\right]}(t) = \cos t \begin{bmatrix} 1 & 0 \\ 0 & \frac{w}{|w|} e^{tp} \frac{\bar{w}}{|w|} \end{bmatrix} + \sin t \begin{bmatrix} p & -e^{tp} \bar{w} \\ w & -\frac{p}{|w|} e^{tp} \frac{\bar{w}}{|w|} \end{bmatrix}, \tag{2}$$

where $e^p = \cos|p| + (p/|p|) \sin|p|$ denotes the exponential map of $\mathbb{S}^3 \subset \mathbb{H}$ at 1. Note that, for $w = 0$, equation (2) simply becomes $\tilde{\gamma}_{\left[\begin{smallmatrix} p \\ 0 \end{smallmatrix}\right]}(t) = \begin{bmatrix} e^{tp} & 0 \\ 0 & 1 \end{bmatrix}$. Now the curve $\pi_{\Sigma^7} \circ \tilde{\gamma}_{\left[\begin{smallmatrix} p \\ w \end{smallmatrix}\right]}$ is a geodesic of Σ^7 for all metrics $\langle \cdot, \cdot \rangle_{1,\nu}$ and

$$\mathbb{S}^6 \rightarrow \Sigma_{\pm \mathbb{1}}^6, \quad \begin{bmatrix} p \\ w \end{bmatrix} \mapsto \pi_{\Sigma^7} \circ \tilde{\gamma}_{\left[\begin{smallmatrix} p \\ w \end{smallmatrix}\right]} \left(\frac{\pi}{2} \right) \tag{3}$$

is an analytic diffeomorphism. This diffeomorphism restricts to an analytic diffeomorphism $\mathbb{S}^5 \rightarrow \Sigma^5$ for $\text{Re } w = 0$.

In [ADPR] it was shown that $\Sigma^5/\{\pm \mathbb{1}\}$ and $\Sigma_{\pm A}^6/\{\pm \mathbb{1}\}$ are homotopy equivalent but not diffeomorphic to $\mathbb{R}\mathbb{P}^5$ and $\mathbb{R}\mathbb{P}^6$, respectively. We conclude this section with the following observation.

LEMMA 2.3. *Let $A_0 := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. For any $A \in \text{O}(2)$, the bisector $\Sigma_{\pm A \cdot A_0}^6$ in $(\Sigma^7, \langle \cdot, \cdot \rangle_{1,\nu})$ is geodesic at the two points $\mathbb{S}^3 \star A$ and $\mathbb{S}^3 \star (-A)$; that is, any geodesic of $(\Sigma^7, \langle \cdot, \cdot \rangle_{1,\nu})$ that starts at one of these points tangentially to $\Sigma_{\pm A \cdot A_0}^6$ is completely contained in $\Sigma_{\pm A \cdot A_0}^6$.*

Proof. It suffices to consider the case $A = \mathbb{1}$. By (1) we have

$$\Sigma_{\pm A_0}^6 = \pi_{\Sigma^7} \left(\left\{ \begin{bmatrix} w' & * \\ p' & * \end{bmatrix} \in \text{Sp}(2) \mid p' \in \text{Im } \mathbb{H}, w' \in \mathbb{H} \right\} \right).$$

From (2) it is now evident that all the geodesics $\pi_{\Sigma^7} \circ \tilde{\gamma}_{\left[\begin{smallmatrix} p \\ w \end{smallmatrix}\right]}$ with $\text{Re } w = 0$ are contained in $\Sigma_{\pm A_0}^6$. □

COROLLARY 2.4. *The exotic projective space $\Sigma_{\pm A}^6/\{\pm \mathbb{1}\}$ inherits from Σ^7 a 1-parameter family of metrics that are Blaschke at one point.*

3. Isometries of the Gromoll–Meyer Sphere and the Cohomogeneity-1 Action on Σ^5

The \bullet -action of \mathbb{S}^3 on $\text{Sp}(2)$ of the previous section extends to the action

$$\text{O}(2) \times \mathbb{S}^3 \times \text{Sp}(2) \rightarrow \text{Sp}(2), \quad (A, q) \bullet B = A \cdot B \cdot \begin{bmatrix} 1 & 0 \\ 0 & \bar{q} \end{bmatrix}.$$

This action is isometric for all metrics $\langle \cdot, \cdot \rangle_{\mu, \nu}$ on $\text{Sp}(2)$ and commutes with the Gromoll–Meyer action \star . Hence, it induces an effective isometric action

$$\text{O}(2) \times \text{SO}(3) \times \Sigma^7 \rightarrow \Sigma^7, \quad \text{SO}(3) = \mathbb{S}^3 / \{\pm 1\},$$

on Σ^7 , which will again be denoted by \bullet . This action already appeared in the original paper of Gromoll and Meyer [GrMy]. At the end of Section 8 we will show that $\text{O}(2) \times \text{SO}(3)$ is the full isometry group for all the metrics $\langle \cdot, \cdot \rangle_{\mu, \nu}$ on Σ^7 . The following simple fact is fundamental for the rest of the paper. It allows us to investigate geometric properties of the metrics that Σ^5 inherits from Σ^7 , it yields an equivariant diffeomorphism between Σ^5 and the Brieskorn sphere W_3^5 , and it is the key to determining which isometries act freely on the Gromoll–Meyer sphere.

LEMMA 3.1. *The \bullet -action of $\text{O}(2) \times \text{SO}(3)$ on Σ^7 leaves Σ^1 and Σ^5 invariant. The induced \bullet -action on Σ^5 is of cohomogeneity 1.*

Note that the \bullet -action of $\text{O}(2) \times \text{SO}(3)$ does not leave any of the $\Sigma_{\pm A}^6$ invariant. The largest action that preserves $\Sigma_{\pm 1}^6$ (and also $\Sigma_{\pm A_0}^6$ with $A_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$) is the restriction of the \bullet -action to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \text{SO}(3)$, where $\mathbb{Z}_2 \times \mathbb{Z}_2$ is the group of diagonal matrices in the $\text{O}(2)$ -factor.

COROLLARY 3.2. *The Gromoll–Meyer sphere Σ^7 is $(\text{O}(2) \times \text{SO}(3))$ -equivariantly homeomorphic to the join $\Sigma^1 \star \Sigma^5$.*

We now study the cohomogeneity-1 action \bullet on Σ^5 in detail. The essential technical step is to find a normal geodesic—that is, a geodesic that crosses all \bullet -orbits perpendicularly. This normal geodesic and the information on the isotropy groups along it will be used throughout the paper. Recall from (1) that

$$\Sigma^5 = \pi_{\Sigma^7} \left(\left\{ \begin{bmatrix} p_1 & * \\ p_2 & * \end{bmatrix} \in \text{Sp}(2) \mid p_1, p_2 \in \text{Im } \mathbb{H} \right\} \right).$$

We will show that the curve $\alpha(s) = \pi_{\Sigma^7}(\tilde{\alpha}(s))$ with

$$\tilde{\alpha}(s) = \begin{bmatrix} j \cos s & k \sin s \\ k \sin s & j \cos s \end{bmatrix} \tag{4}$$

is such a normal geodesic and then compute the isotropy groups along this geodesic and the induced Riemannian metrics on the principal orbits. Finally, we will show that $\text{O}(2) \times \text{SO}(2)$ is the full isometry group of $(\Sigma^5, \langle \cdot, \cdot \rangle_{\mu, \nu})$.

LEMMA 3.3. *The curve $\tilde{\alpha}$ intersects all \star -orbits in $\text{Sp}(2)$ perpendicularly. In other words, $\tilde{\alpha}$ is horizontal with respect to the submersion $\pi_{\Sigma^7}: \text{Sp}(2) \rightarrow \Sigma^7$.*

Proof. For all the metrics $\langle \cdot, \cdot \rangle_{\mu, \nu}$ the tangent vector of $\tilde{\alpha}$,

$$\tilde{\alpha}'(s) = \tilde{\alpha}(s) \cdot \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix},$$

is perpendicular to the vertical space at $\tilde{\alpha}(s)$, which is spanned by the three vectors

$$\begin{aligned} \xi_1(s) &:= \frac{d}{d\tau} \left(e^{i\tau} \cdot \tilde{\alpha}(s) \cdot \begin{bmatrix} e^{-i\tau} & 0 \\ 0 & 1 \end{bmatrix} \right) \Big|_{\tau=0} = \tilde{\alpha}(s) \cdot \begin{bmatrix} -2i & 0 \\ 0 & -i \end{bmatrix}, \\ \xi_2(s) &:= \frac{d}{d\tau} \left(e^{j\tau} \cdot \tilde{\alpha}(s) \cdot \begin{bmatrix} e^{-j\tau} & 0 \\ 0 & 1 \end{bmatrix} \right) \Big|_{\tau=0} = \tilde{\alpha}(s) \cdot \begin{bmatrix} j(\cos 2s - 1) & k \sin 2s \\ k \sin 2s & j \cos 2s \end{bmatrix}, \\ \xi_3(s) &:= \frac{d}{d\tau} \left(e^{k\tau} \cdot \tilde{\alpha}(s) \cdot \begin{bmatrix} e^{-k\tau} & 0 \\ 0 & 1 \end{bmatrix} \right) \Big|_{\tau=0} = \tilde{\alpha}(s) \cdot \begin{bmatrix} -k(\cos 2s + 1) & j \sin 2s \\ j \sin 2s & -k \cos 2s \end{bmatrix}. \end{aligned}$$

□

LEMMA 3.4. *The curve $\tilde{\alpha}$ is a geodesic for any of the metrics $\langle \cdot, \cdot \rangle_{\mu, \nu}$ on $\mathfrak{Sp}(2)$.*

Proof. Since $\tilde{\alpha}$ is an integral curve of the left-invariant vector field v given by $\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \in \mathfrak{sp}(2)$, it suffices to compute $\nabla_v v$ at the identity matrix. For an arbitrary left-invariant vector field w , the Kozul formula for the Levi-Civita connection yields

$$\langle \nabla_v v, w \rangle_{\mu, \nu} = -\langle v, [v, w] \rangle_{\mu, \nu}.$$

Using the special value of v at the identity matrix and the fact that ad_v is skew symmetric with respect to the biinvariant metric $\langle \cdot, \cdot \rangle_{1/2, 1/2}$, one gets

$$\langle \nabla_v v, w \rangle_{\mu, \nu} = -\langle v, [v, w] \rangle_{\mu, \nu} = -\langle v, [v, w] \rangle_{1/2, 1/2} = -\langle [v, v], w \rangle_{1/2, 1/2} = 0$$

at the identity matrix. □

COROLLARY 3.5. *The curve $\alpha := \pi_{\Sigma^7} \circ \tilde{\alpha}$ is a geodesic in Σ^7 , and this geodesic is contained in the 5-sphere $\Sigma^5 \subset \Sigma^7$.*

LEMMA 3.6. *The geodesic α in $\Sigma^5 \subset \Sigma^7$ intersects all \bullet -orbits perpendicularly.*

Proof. The tangent space to the \bullet -orbit through $\tilde{\alpha}(s)$ is spanned by

$$\begin{aligned} \hat{v}_0(s) &:= \frac{d}{d\theta} \left(\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \cdot \tilde{\alpha}(s) \right) \Big|_{\theta=0} = \tilde{\alpha}(s) \cdot \begin{bmatrix} i \sin 2s & -\cos 2s \\ \cos 2s & -i \sin 2s \end{bmatrix}, \\ \hat{v}_1(s) &:= \frac{d}{d\tau} \left(\tilde{\alpha}(s) \cdot \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\tau} \end{bmatrix} \right) \Big|_{\tau=0} = \tilde{\alpha}(s) \cdot \begin{bmatrix} 0 & 0 \\ 0 & -i \end{bmatrix}, \\ \hat{v}_2(s) &:= \frac{d}{d\tau} \left(\tilde{\alpha}(s) \cdot \begin{bmatrix} 1 & 0 \\ 0 & e^{-j\tau} \end{bmatrix} \right) \Big|_{\tau=0} = \tilde{\alpha}(s) \cdot \begin{bmatrix} 0 & 0 \\ 0 & -j \end{bmatrix}, \\ \hat{v}_3(s) &:= \frac{d}{d\tau} \left(\tilde{\alpha}(s) \cdot \begin{bmatrix} 1 & 0 \\ 0 & e^{-k\tau} \end{bmatrix} \right) \Big|_{\tau=0} = \tilde{\alpha}(s) \cdot \begin{bmatrix} 0 & 0 \\ 0 & -k \end{bmatrix}. \end{aligned} \tag{5}$$

All four vectors are perpendicular to the horizontal vector $\tilde{\alpha}'(s)$. □

The isotropy groups of the \bullet -action along the geodesic α are regular for $s \notin \pi/4 \cdot \mathbb{Z}$ and are denoted by H . The singular isotropy groups at $s = 0$ and $s = \pi/4$ are denoted by K_- and K_+ , respectively. Straightforward computations yield the following result.

LEMMA 3.7. *The isotropy groups along the normal geodesic α are given by*

$$\begin{aligned}
 H &= \left\{ (\mathbf{1}, \pm 1), (-\mathbf{1}, \pm i), \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \pm j \right), \left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \pm k \right) \right\} \\
 &\approx \mathbb{Z}_2 \times \mathbb{Z}_2, \\
 K_- &= \{(\mathbf{1}, \pm e^{j\tau})\} \cup \{(-\mathbf{1}, \pm ie^{j\tau})\} \\
 &\quad \cup \left\{ \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \pm e^{j\tau} \right) \right\} \cup \left\{ \left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \pm ie^{j\tau} \right) \right\} \\
 &\approx \mathbb{Z}_2 \times \mathrm{O}(2), \\
 K_+ &= \left\{ \left(\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \pm e^{-(3/2)i\theta} \right) \right\} \\
 &\quad \cup \left\{ \left(\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \pm e^{-(3/2)i\theta} j \right) \right\} \\
 &\approx \mathrm{O}(2).
 \end{aligned}$$

Note that K_+ is isomorphic to an $\mathrm{O}(2)$ that projects surjectively onto the $\mathrm{O}(2)$ factor in the definition of the \bullet -action while K_- is isomorphic to $\mathbb{Z}_2 \times \mathrm{O}(2)$ where the $\mathrm{O}(2)$ -factor corresponds to $\{(\mathbf{1}, \pm e^{j\tau})\} \cup \{(-\mathbf{1}, \pm ie^{j\tau})\}$, which is contained in the identity component of the acting group $\mathrm{O}(2) \times \mathrm{SO}(3)$. The singular orbit at $s = 0$ is diffeomorphic to $\mathbb{S}^2 \times_{\mathbb{Z}_2} \mathbb{S}^1$ and the singular orbit at $s = \pi/4$ is diffeomorphic to $\mathrm{SO}(3)$.

The normal geodesic α can be used to express the metric $\langle \cdot, \cdot \rangle_{\mu, \nu}$ on Σ^5 as $ds^2 + \omega_{\mu, \nu}(s)$, where $\omega_{\mu, \nu}(s)$ is the metric on principal orbits. We now compute $\omega_{\mu, \nu}(s)$. This computation will be used in Section 7 to discuss the geometric properties of the metrics on Σ^5 and $\Sigma_{\pm A}^6$ and to determine the full isometry group of the Gromoll–Meyer sphere.

We need to compute the inner products of four linearly independent Killing fields along the normal geodesic α in Σ^5 . Such Killing fields $v_0(s), \dots, v_3(s)$ are given by the horizontal parts $\tilde{v}_0(s), \dots, \tilde{v}_3(s)$ of the Killing fields $\hat{v}_0(s), \dots, \hat{v}_3(s)$ along $\tilde{\alpha}$ given in (5). Straightforward computations using the orthogonal basis $\xi_1(s), \xi_2(s)$, and $\xi_3(s)$ of the vertical space at $\tilde{\alpha}(s)$ given in (3) show that

$$\begin{aligned}
 \tilde{v}_0(s) &= \tilde{\alpha}(s) \cdot \left(\frac{3 \sin 2s}{4\mu + \nu} \begin{bmatrix} i\nu & 0 \\ 0 & -2i\mu \end{bmatrix} + \cos 2s \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right), \\
 \tilde{v}_1(s) &= \tilde{\alpha}(s) \cdot \frac{2}{4\mu + \nu} \begin{bmatrix} i\nu & 0 \\ 0 & -2i\mu \end{bmatrix}, \\
 \tilde{v}_2(s) &= \tilde{\alpha}(s) \cdot \left(\begin{bmatrix} 0 & 0 \\ 0 & -j \end{bmatrix} + \frac{\cos 2s}{\nu \cos^2 2s + 4(1 - (1 - \mu) \sin^2 s) \sin^2 s} \right. \\
 &\quad \left. \cdot \begin{bmatrix} j(\cos 2s - 1) & k \sin 2s \\ k \sin 2s & j \cos 2s \end{bmatrix} \right), \\
 \tilde{v}_3(s) &= \tilde{\alpha}(s) \cdot \left(\begin{bmatrix} 0 & 0 \\ 0 & -k \end{bmatrix} + \frac{\cos 2s}{\nu \cos^2 2s + 4(1 - (1 - \mu) \cos^2 s) \cos^2 s} \right. \\
 &\quad \left. \cdot \begin{bmatrix} k(1 + \cos 2s) & -j \sin 2s \\ -j \sin 2s & k \cos 2s \end{bmatrix} \right).
 \end{aligned}$$

The action of the principal isotropy group H on these four Killing fields along α is given by the matrices

$$\begin{aligned} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \\ & \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{6}$$

Let

$$\begin{aligned} a(s) &= 1 - \left(1 - \frac{9\mu v}{4\mu + v}\right) \sin^2 2s, \\ b(s) &= \frac{6\mu v}{4\mu + v} \sin 2s, \\ c(s) &= v \frac{4(1 - (1 - \mu) \sin^2 s) \sin^2 s}{v \cos^2 2s + 4(1 - (1 - \mu) \sin^2 s) \sin^2 s}, \\ d(s) &= v \frac{4(1 - (1 - \mu) \cos^2 s) \cos^2 s}{v \cos^2 2s + 4(1 - (1 - \mu) \cos^2 s) \cos^2 s}. \end{aligned}$$

LEMMA 3.8. *The metric $\omega_{\mu,v}(s)$ on the principal orbit through $\alpha(s)$ is given by the matrix*

$$((\tilde{v}_j(s), \tilde{v}_k(s)))_{j,k=0,\dots,3} = \begin{bmatrix} a(s) & b(s) & 0 & 0 \\ b(s) & \frac{4\mu v}{4\mu + v} & 0 & 0 \\ 0 & 0 & c(s) & 0 \\ 0 & 0 & 0 & d(s) \end{bmatrix}.$$

Proof. The equality follows from straightforward computations. □

This matrix description of the cohomogeneity-1 metrics $\langle \cdot, \cdot \rangle_{\mu,v}$ on Σ^5 will be interpreted in Section 7 in terms of totally geodesic submanifolds L^3 , Σ^2 , and $\tilde{\Sigma}^2$, which intersect pairwise perpendicularly in the normal geodesic α . The upper left 2×2 block of the matrix in Lemma 3.8 describes cohomogeneity-1 metrics on the lens space $L^3 \approx \mathbb{S}^3/\mathbb{Z}_3$ (the block becomes singular at $s \in \pi/4 + (\pi/2)\mathbb{Z}$; the smoothness at these times can best be seen by passing from \tilde{v}_0 to $2\tilde{v}_0 - 3\tilde{v}_1$ and from s to $s + \pi/4$). The numbers $c(s)$ and $d(s)$ describe cohomogeneity-1 metrics on the 2-spheres Σ^2 and $\tilde{\Sigma}^2$.

4. Identification of Σ^5 with the Brieskorn Sphere W_3^5

We will now construct an $(O(2) \times SO(3))$ -equivariant diffeomorphism between the sphere $\Sigma^5 \subset \Sigma^7$ and the Brieskorn sphere W_3^5 given by the equations

$$\begin{aligned} \frac{8}{9}z_0^3 + z_1^2 + z_2^2 + z_3^2 &= 0, \\ \frac{4}{3}|z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 &= \frac{4}{9} \end{aligned}$$

in $\mathbb{C}^4 = \mathbb{C} \oplus \mathbb{C}^3$. It is crucial for Section 5 that we have modified the coefficients compared to the standard definition of W_3^5 . The advantage of our choice is that there exists an explicit formula for a unit-speed geodesic in W_3^5 that intersects all orbits of the action

$$\begin{aligned} \mathrm{O}(2) \times \mathrm{SO}(3) \times W_3^5 &\rightarrow W_3^5, \\ \left(\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, A \right) \cdot (z_0, z) &= (e^{2i\theta} z_0, e^{3i\theta} Az), \\ \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, A \right) \cdot (z_0, z) &= (\bar{z}_0, A\bar{z}) \end{aligned} \tag{7}$$

with $z \in \mathbb{C}^3$ perpendicularly. This action on W_3^5 was first considered by Calabi (see Bredon’s survey [Bd1]). In the literature, however, the subaction of the identity component $\mathrm{SO}(2) \times \mathrm{SO}(3)$ is used almost exclusively. The additional \mathbb{Z}_2 symmetry causes the fixed point set of the principal isotropy group to be 1-dimensional. Therefore, we prefer to use normal geodesics and hence canonical identifications between Σ^5 and W_3^5 (see Lemma 4.2).

Consider the curve

$$\beta(s) = \left(-\frac{1}{2} \cos 2s, \frac{1}{6} \begin{bmatrix} 0 \\ 3 \cos s - \cos 3s \\ 3i \sin s + i \sin 3s \end{bmatrix} \right)$$

in $W_3^5 \subset \mathbb{C} \oplus \mathbb{C}^3$. It is straightforward to check that β is parameterized by arc length and to compute the isotropy groups along β .

LEMMA 4.1. *The isotropy groups H at $\beta(s)$ for $s \notin (\pi/4)\mathbb{Z}$, K_- at $\beta(0)$, and K_+ at $\beta(\pi/4)$ are given by*

$$\begin{aligned} H &= \left\{ (\mathbb{1}, \mathbb{1}), \left(-\mathbb{1}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right), \right. \\ &\quad \left. \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right), \left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \right\} \\ &\approx \mathbb{Z}_2 \times \mathbb{Z}_2, \\ K_- &= \left\{ \left(\mathbb{1}, \begin{bmatrix} * & 0 & * \\ 0 & 1 & 0 \\ * & 0 & * \end{bmatrix} \right) \right\} \cup \left\{ \left(-\mathbb{1}, \begin{bmatrix} * & 0 & * \\ 0 & -1 & 0 \\ * & 0 & * \end{bmatrix} \right) \right\} \\ &\quad \cup \left\{ \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} * & 0 & * \\ 0 & 1 & 0 \\ * & 0 & * \end{bmatrix} \right) \right\} \cup \left\{ \left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} * & 0 & * \\ 0 & -1 & 0 \\ * & 0 & * \end{bmatrix} \right) \right\} \\ &\approx \mathbb{Z}_2 \times \mathrm{O}(2), \\ K_+ &= \left\{ (D(\theta), \begin{bmatrix} 1 & 0 \\ 0 & D(-3\theta) \end{bmatrix}) \right\} \\ &\quad \cup \left\{ \left(D(\theta) \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & D(-3\theta) \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right) \right\} \\ &\approx \mathrm{O}(2), \end{aligned}$$

where $D(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ denotes the rotation in \mathbb{R}^2 with angle θ .

LEMMA 4.2. *The curve β is a unit-speed geodesic in W_3^5 that intersects all orbits of the $(O(2) \times SO(3))$ -action perpendicularly.*

Proof. The fixed point set of the principal isotropy group H clearly contains a geodesic that intersects all orbits perpendicularly. This fixed point set is given by

$$\operatorname{Im} z_0 = 0, \quad z_1 = 0, \quad \operatorname{Im} z_2 = 0, \quad \operatorname{Re} z_3 = 0.$$

It is easy to check that this fixed point set is 1-dimensional and that β maps into the fixed point set of H . □

In the following theorem it is assumed that $SO(3) = \mathbb{S}^3/\{\pm 1\}$ is identified with the matrix group $SO(3)$ by the action of \mathbb{S}^3 on the imaginary quaternions by conjugation.

THEOREM 4.3. *The map*

$$\Sigma^5 \rightarrow W_3^5, \quad (A, \pm q) \bullet \alpha(s) \mapsto (A, \pm q) \cdot \beta(s)$$

is a well-defined $(O(2) \times SO(3))$ -equivariant diffeomorphism.

Proof. This follows from the isotropy groups in Lemma 3.7 and Lemma 4.1. □

COROLLARY 4.4. *There is an $(O(2) \times SO(3))$ -equivariant homeomorphism*

$$\mathbb{S}^1 * W_3^5 \rightarrow \Sigma^7.$$

Here, $O(2)$ acts on \mathbb{S}^1 in the canonical way.

Proof. This follows directly from Corollary 2.2 and Theorem 4.3. □

5. An Explicit Parameterization of Two Brieskorn Spheres

In this section we present an explicit formula for two diffeomorphisms between Euclidean spheres and Brieskorn spheres. The coefficients in this formula are rational functions of the coordinates of the sphere. They are simple enough that the entire formula fits into a few lines (just in one direction; the formula for the inverse is too long to be reproduced here) but complicated enough that they could never be guessed. The formula was obtained by combining the geodesic parameterization of $\Sigma^5 \subset \Sigma^7$ and the cohomogeneity-1 diffeomorphism between Σ^5 and the Brieskorn sphere W_3^5 . The steps of the computations behind this approach will be explained at the end of this section. The properties of the final formula, however, can be verified directly; this shows that the formula is also valid in dimension 13, where no geometric derivation is possible so far.

Analogously to the previous section, the Brieskorn sphere W_3^{2n-1} is defined by the equations

$$\begin{aligned} \frac{8}{9}z_0^3 + z_1^2 + z_2^2 + \cdots + z_n^2 &= 0, \\ \frac{4}{3}|z_0|^2 + |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 &= \frac{4}{9} \end{aligned}$$

for $(z_0, z) \in \mathbb{C} \oplus \mathbb{C}^n$. For odd $n \geq 3$, the Brieskorn sphere W_3^{2n-1} is diffeomorphic to the Kervaire sphere (see e.g. [HzMa]). Hence, by a result of Brouwer, W_3^{2n-1}

can be diffeomorphic to \mathbb{S}^{2n-1} only if $n = 2^m - 1$. Until now, this is known to hold only for $n \in \{3, 7, 15, 31\}$ (see [La]). For $n = 3, 7$, the classification theorems of Jänich and of Hsiang and Hsiang show that there exist $\text{SO}(3)$ -equivariant diffeomorphisms $\mathbb{S}^5 \rightarrow W_3^5$ and G_2 -equivariant diffeomorphisms $\mathbb{S}^{13} \rightarrow W_3^{13}$. We will construct the first explicit formulas for such diffeomorphisms here.

We decompose z_0 and z into their real and imaginary parts; that is, we set $z_0 = x_0 + iy_0$ and $z = x + iy$. This leads to the equivalent definition of the Brieskorn sphere W_3^{2n-1} by the three real equations

$$\begin{aligned} |x|^2 &= \frac{2}{9}(1 - 2x_0^3 + 6x_0y_0^2 - 3x_0^2 - 3y_0^2), \\ |y|^2 &= \frac{2}{9}(1 + 2x_0^3 - 6x_0y_0^2 - 3x_0^2 - 3y_0^2), \\ \langle x, y \rangle &= \frac{4}{9}y_0(y_0^2 - 3x_0^2) \end{aligned} \tag{8}$$

for $x_0, y_0 \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$. The natural $\text{SO}(n)$ -action on W_3^{2n-1} multiplies x and y by a matrix $A \in \text{SO}(n)$ and leaves x_0 and y_0 unchanged. Analogously to [HzMa, pp. 31–32], one can show that the orbit space of this action can be identified with the disc $D^2 = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$ by the projection map $W_3^{2n-1} \rightarrow D^2$, $(z_0, z) \mapsto 2z_0$. For $n = 7$, the action of $\text{G}_2 \subset \text{SO}(7)$ has the same orbits as the $\text{SO}(7)$ -action.

We now parameterize the standard spheres \mathbb{S}^5 and \mathbb{S}^{13} by two vectors $p, w \in \mathbb{R}^3$ (resp. \mathbb{R}^7) with $|p|^2 + |w|^2 = 1$ and set

$$\begin{aligned} x_0 &= \frac{1}{2}(|w|^2 - |p|^2), \\ y_0 &= -\langle p, w \rangle, \\ x &= \frac{1}{3(1 + |p|^2)^2} \left(((3 - 2|p|^2)(1 + |p|^2)^2 - 4(1 - |p|^2)\langle w, p \rangle^2)p \right. \\ &\quad \left. - 2(3 + 8|p|^2 + |p|^4 - 4\langle w, p \rangle^2)\langle p, w \rangle w \right. \\ &\quad \left. - 8|p|^2\langle p, w \rangle p \times w \right), \\ y &= \frac{1}{3(1 + |p|^2)^2} \left((-(1 + 2|p|^2)(1 - 6|p|^2 + |p|^4) - 4(1 + 3|p|^2)\langle w, p \rangle^2)w \right. \\ &\quad \left. + 2(1 - |p|^2)(1 + 3|p|^2)\langle w, p \rangle p \right. \\ &\quad \left. - 4(1 + 2|p|^2)(1 - |p|^2)p \times w \right). \end{aligned} \tag{9}$$

Here, we use the standard cross product on \mathbb{R}^3 and the cross product on \mathbb{R}^7 that comes from the imaginary part of the product of two imaginary octonions. It is straightforward but tedious to check that x_0, y_0, x, y satisfy the equations (8). If n is different from 3 and 7 then the formulas (9) do not work. What we need is to assign to $p, w \in \mathbb{R}^n$ a vector that is perpendicular to both and that is different from 0 if p and w are linearly independent. Such a cross product exists only in dimensions 3 and 7 (see [Ms]).

On $\mathbb{S}^5 \subset \mathbb{R}^3 \times \mathbb{R}^3$ and on $\mathbb{S}^{13} \subset \mathbb{R}^7 \times \mathbb{R}^7$ we consider the diagonal actions of $\text{SO}(3)$ and G_2 , respectively. The orbit spaces of these actions can again be identified with D^2 by the projection maps $\mathbb{S}^{2n-1} \rightarrow D^2$, $(p, w) \mapsto |w|^2 - |p|^2 - 2i\langle p, w \rangle$. Note

that the preimage of the boundary of D^2 consists precisely of the pairs (p, w) for which p and w are linearly dependent.

THEOREM 5.1. *The formulas (9) provide an $\text{SO}(3)$ -equivariant diffeomorphism $\mathbb{S}^5 \rightarrow W_3^5$ and a G_2 -equivariant diffeomorphism $\mathbb{S}^{13} \rightarrow W_3^{13}$.*

Proof. A complete geometric derivation of the formula for the diffeomorphism in the case $n = 3$ will be given shortly. This derivation is based on the geometry of the Gromoll–Meyer sphere, and there is no known way to extend it to the case $n = 7$. Hence we sketch an algebraic proof here that is completely self-contained and works for $n = 3$ and $n = 7$.

The maps $\psi: \mathbb{S}^{2n-1} \rightarrow W_3^{2n-1}$ defined by (9) are smooth, equivariant, and induce the identity between the orbit spaces D^2 . Hence they are homeomorphisms. Their inverses $\psi^{-1}: W_3^{2n-1} \rightarrow \mathbb{S}^{2n-1}$ can be computed explicitly, but the formulas are far too long to be reproduced here (any computer algebra system, however, can do this inversion easily). We just mention the key properties as follows. The coefficients in the equations for $x, y,$ and $x \times y$ as combinations of $p, w,$ and $p \times w$ are rational functions of x_0 and y_0 with nonzero denominators. The determinant of the coefficient matrix is a polynomial of degree 12 in x_0 and y_0 that can be seen to be always greater than or equal to $\frac{256}{9(3-2x_0)^8}$ if $x_0^2 + y_0^2 \leq \frac{1}{4}$. Hence, the coefficient matrix can be inverted even if p and w become linearly dependent, and p and w can be expressed as combinations of $x, y,$ and $x \times y$ with rational coefficients in x_0 and y_0 that do not have singularities for $x_0^2 + y_0^2 \leq \frac{1}{4}$. \square

In the rest of this section we will describe how (9) was obtained in the case $n = 3$. During this derivation we will introduce a simple formula for an injective map $\mathbb{S}^5 \setminus \{w = 0\} \rightarrow W_3^5$ that extends to all odd dimensions and thus yields injective maps $\mathbb{S}^{2n-1} \setminus \{w = 0\} \rightarrow W_3^{2n-1}$. These maps are given by replacing the expressions for x and y in (9) with

$$\begin{aligned} -3x &= \left(|p|^2 + 3|w|^2 - 4 \left\langle \frac{w}{|w|}, p \right\rangle^2 \right) p + 2|p|^2 \left\langle p, \frac{w}{|w|} \right\rangle \frac{w}{|w|}, \\ 3y &= -(3|p|^2 + |w|^2)w + 6\langle w, p \rangle p. \end{aligned}$$

The cross products in dimensions 3 and 7 are needed to twist these maps so that they extend to diffeomorphisms $\mathbb{S}^5 \rightarrow W_3^5$ and $\mathbb{S}^{13} \rightarrow W_3^{13}$.

Now we start with the geometric derivation of formula (9). The basic idea is to combine the explicit parameterization $\mathbb{S}^5 \rightarrow \Sigma^5 \subset \Sigma^7$ by the geodesics $\pi_{\Sigma^7} \circ \tilde{\gamma}_{[w]}^p$ from (2) at time $\pi/2$ with the equivariant diffeomorphism $\Sigma^5 \rightarrow W_3^5$ from Theorem 4.3. In order to make this composition of maps explicit, we describe the point $\pi_{\Sigma^7} \circ \tilde{\gamma}_{[w]}^p(\pi/2)$ in terms of the cohomogeneity-1 action of $\text{SO}(3) \times \text{SO}(2)$ on Σ^5 —that is, by the parameter s of the normal geodesic α in (4), an angle θ , and a unit quaternion $q \in \mathbb{S}^3$ (with several ambiguities). In other words, we solve the equation

$$q' \star \tilde{\gamma}_{[w]}^p\left(\frac{\pi}{2}\right) = \left(\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, q \right) \bullet \tilde{\alpha}(s) \tag{10}$$

(which holds for some $q' \in \mathbb{S}^3$) for s, θ , and $\pm q$ (not caring about any ambiguities). We then plug the results into the corresponding cohomogeneity-1 parameterization

$$\begin{aligned} & \left(\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \pm q \right) \cdot \beta(s) \\ &= \left(-\frac{1}{2} \cos 2\theta \cos 2s - \frac{i}{2} \sin 2\theta \cos 2s, x(s, \theta, q) + iy(s, \theta, q) \right) \end{aligned} \tag{11}$$

of W_3^5 , where $x(s, \theta, q), y(s, \theta, q) \in \mathbb{R}^3$ will be evaluated later. By Theorem 4.3 it is clear that this procedure yields a well-defined smooth diffeomorphism $\mathbb{S}^5 \rightarrow W_3^5$. The actual computations, however, are lengthy and not straightforward. It thus seems appropriate to indicate how they can be done efficiently. First, we identify \mathbb{R}^3 with the imaginary quaternions. The homomorphism $\mathbb{S}^3 \rightarrow \text{SO}(3)$ is then given by assigning to $\pm q$ the matrix $(qi\bar{q}, qj\bar{q}, qk\bar{q}) \in \text{SO}(\text{Im } \mathbb{H})$. With this identification, $x(s, \theta, q)$ and $y(s, \theta, q)$ can be evaluated as

$$\begin{aligned} -3x(s, \theta, q) &= 2(1 + \cos 2\theta \cos 2s)q(j \cos s \cos \theta - k \sin s \sin \theta)\bar{q} \\ &\quad - (4 \cos 2\theta + \cos 2s)q(j \cos s \cos \theta + k \sin s \sin \theta)\bar{q}, \\ 3y(\theta, s) &= 2(1 - \cos 2\theta \cos 2s)q(j \cos s \sin \theta + k \sin s \cos \theta)\bar{q} \\ &\quad + (4 \cos 2\theta - \cos 2s)q(j \cos s \sin \theta - k \sin s \cos \theta)\bar{q}. \end{aligned} \tag{12}$$

After a few computations, one sees from (10) that

$$\begin{aligned} \cos 2\theta &= \frac{|p|^2 - |w|^2}{\sqrt{(|p|^2 - |w|^2)^2 + 4\langle p, w \rangle^2}}, & \sin 2\theta &= \frac{2\langle p, w \rangle}{\sqrt{(|p|^2 - |w|^2)^2 + 4\langle p, w \rangle^2}}, \\ \cos 2s &= \sqrt{(|p|^2 - |w|^2)^2 + 4\langle p, w \rangle^2}, & \sin 2s &= 2\sqrt{(|p|^2|w|^2 - \langle p, w \rangle^2)}. \end{aligned}$$

Moreover, that (10) holds for some $q' \in \mathbb{S}^3$ is equivalent to

$$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \cdot \tilde{\gamma}_{\left[\frac{p}{w}\right]} \left(\frac{\pi}{2} \right) = \bar{q}' \star (q \bullet \tilde{\alpha}(s))$$

for some $\tilde{q}' \in \mathbb{S}^3$. Computing formally the “determinant” $ad - bc$ of the quaternionic 2×2 matrices $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ on both sides of the latter equation, one obtains

$$\bar{q}' \cos 2s = \left((|p|^2 - |w|^2) \frac{w}{|w|} - 2 \left\langle \frac{w}{|w|}, p \right\rangle p \right) e^{(\pi/2)p} \frac{\bar{w}}{|w|} q.$$

This identity can now be plugged into (10), and the result allows us to evaluate (12):

$$\begin{aligned} -3x &= \frac{w}{|w|} e^{-(\pi/2)p} \left(\left(|p|^2 + 3|w|^2 - 4 \left\langle \frac{w}{|w|}, p \right\rangle^2 \right) p \right. \\ &\quad \left. + 2|p|^2 \left\langle p, \frac{w}{|w|} \right\rangle \frac{w}{|w|} \right) e^{(\pi/2)p} \frac{\bar{w}}{|w|}, \\ 3y &= \frac{w}{|w|} e^{-(\pi/2)p} (-3|p|^2 + |w|^2)w + 6\langle w, p \rangle p e^{(\pi/2)p} \frac{\bar{w}}{|w|}. \end{aligned}$$

Expressing all quaternionic products by inner products and cross products, we obtain the formulas

$$\begin{aligned}
 3x &= \left(3 - 2|p|^2 - 2\frac{1 + \cos \pi|p|}{1 - |p|^2} \langle w, p \rangle^2 \right) p \\
 &\quad - 2 \left(3 + |p|^2 \frac{1 + \cos \pi|p|}{1 - |p|^2} - 2\frac{1 + \cos \pi|p|}{(1 - |p|^2)^2} \langle w, p \rangle^2 \right) \langle p, w \rangle w \\
 &\quad - 2|p|^2 \frac{\sin \pi|p|}{(1 - |p|^2)|p|} \langle w, p \rangle p \times w, \\
 3y &= -(1 + 2|p|^2) \cos \pi|p| \cdot w + 2\frac{-1 + 4|p|^2 + (1 + 2|p|^2) \cos \pi|p|}{|p|^2(1 - |p|^2)} \langle w, p \rangle^2 w \\
 &\quad - \frac{-1 + 4|p|^2 + (1 + 2|p|^2) \cos \pi|p|}{|p|^2(1 - |p|^2)} |w|^2 \langle w, p \rangle p \\
 &\quad - (1 + 2|p|^2) \frac{\sin \pi|p|}{|p|} p \times w,
 \end{aligned}$$

where all the fractions are real-analytic functions of $|p|$. This can now be seen as a final formula for the diffeomorphism $\mathbb{S}^5 \rightarrow W_3^5$. In formula (9) we passed to an isotopic rational version by replacing $\sin(\pi/2)|p|$ and $\cos(\pi/2)|p|$ with $2|p|/(1 + |p|^2)$ and $(1 - |p|^2)/(1 + |p|^2)$, respectively.

REMARK 5.2. Note that the diffeomorphisms of Theorem 5.1 equip W_3^5 and W_3^{13} with explicit $\text{SO}(3)$ - and G_2 -invariant metrics of constant curvature 1. Wilking (unpublished) proved that there do not exist $\text{SO}(n)$ -invariant metrics with positive sectional curvature on any of the W_d^{2n-1} with $n > 3$ and odd $d > 1$. Moreover, it has been shown [GVWZ] that there do not exist cohomogeneity-1 metrics with nonnegative sectional curvature on any of the W_d^{2n-1} with $n > 3$ and odd $d > 1$.

6. Nonlinear Cohomogeneity-1 Actions on Euclidean Spheres

In this section we present the first explicit formulas for cohomogeneity-1 actions of $\text{O}(2) \times \text{SO}(3)$ and $\text{O}(2) \times G_2$ on the Euclidean spheres \mathbb{S}^5 and \mathbb{S}^{13} that are equivalent to the standard cohomogeneity-1 actions on the Brieskorn spheres W_3^5 and W_3^{13} (see Section 4).

The essential parts of these actions are the nonlinear subactions of $\text{SO}(2) \subset \text{O}(2)$. It is convenient, however, to describe the linear parts first. Let p and w denote two imaginary quaternions (octonions) with $|p|^2 + |w|^2 = 1$. The action of $\text{SO}(3) = \mathbb{S}^3/\{\pm 1\}$ on \mathbb{S}^5 is given by

$$\text{SO}(3) \times \mathbb{S}^5 \rightarrow \mathbb{S}^5, \quad (\pm q) \cdot \begin{bmatrix} p \\ w \end{bmatrix} = q \begin{bmatrix} p \\ w \end{bmatrix} \bar{q} = \begin{bmatrix} qp\bar{q} \\ qw\bar{q} \end{bmatrix}.$$

The G_2 -action on \mathbb{S}^{13} is defined in the same diagonal way. (Recall that G_2 is the automorphism group of the octonions.) The $\text{O}(2)$ -actions on \mathbb{S}^5 and \mathbb{S}^{13} contain the linear \mathbb{Z}_2 -subactions

$$\mathbb{Z}_2 \times \mathbb{S}^5 \rightarrow \mathbb{S}^5, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} p \\ w \end{bmatrix} = \begin{bmatrix} p \\ -w \end{bmatrix}.$$

We will now turn to the nonlinear $SO(2)$ -actions. In order to write them down explicitly, we need some preparatory work. Let e^p denote the exponential map of $\mathbb{S}^3 \subset \mathbb{H}$ (or $\mathbb{S}^7 \subset \mathbb{O}$). For $\theta \in \mathbb{R}$, set

$$\begin{bmatrix} p_\theta \\ w_\theta \end{bmatrix} := \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \cdot \begin{bmatrix} p \\ w \end{bmatrix} = \begin{bmatrix} p \cos \theta - w \sin \theta \\ p \sin \theta + w \cos \theta \end{bmatrix}$$

and

$$Q\left(\begin{bmatrix} p \\ w \end{bmatrix}, \theta\right) = \frac{w}{|w|} e^{-(\pi/2)p} \frac{w_\theta}{|w_\theta|} \frac{\bar{w}}{|w|} e^{(\pi/2)p_\theta} \frac{\bar{w}_\theta}{|w_\theta|}.$$

At first glance one would not expect that this formula defines a smooth map.

LEMMA 6.1. Q extends to an analytic map $\mathbb{S}^5 \times \mathbb{S}^1 \rightarrow \mathbb{S}^3$ and $\mathbb{S}^{13} \times \mathbb{S}^1 \rightarrow \mathbb{S}^7$, respectively.

Proof. Expanding the exponential maps in the definition of Q and applying the two identities $p_\theta = w_{(\theta+\pi/2)}$ and $w_\theta w_\kappa w_\tau = w_\tau w_\kappa w_\theta$, one obtains

$$\begin{aligned} Q\left(\begin{bmatrix} p \\ w \end{bmatrix}, \theta\right) &= w w_\theta \bar{w} \bar{w}_\theta \cdot \frac{\cos \frac{\pi}{2}|p|}{1 - |p|^2} \cdot \frac{\cos \frac{\pi}{2}|p_\theta|}{1 - |p_\theta|^2} \\ &\quad - p_\theta p \cdot \frac{\sin \frac{\pi}{2}|p|}{|p|} \cdot \frac{\sin \frac{\pi}{2}|p_\theta|}{|p_\theta|} \\ &\quad + w p_\theta \bar{w} \cdot \frac{\cos \frac{\pi}{2}|p|}{1 - |p|^2} \cdot \frac{\sin \frac{\pi}{2}|p_\theta|}{|p_\theta|} \\ &\quad - w_\theta p \bar{w}_\theta \cdot \frac{\sin \frac{\pi}{2}|p|}{|p|} \cdot \frac{\cos \frac{\pi}{2}|p_\theta|}{1 - |p_\theta|^2}. \quad \square \end{aligned}$$

LEMMA 6.2. Q has the following property:

$$Q\left(\begin{bmatrix} p \\ w \end{bmatrix}, \theta\right) Q\left(\begin{bmatrix} p_\theta \\ w_\theta \end{bmatrix}, \tau\right) = Q\left(\begin{bmatrix} p \\ w \end{bmatrix}, \theta + \tau\right).$$

Proof. This property is based on the identity $w_\theta \bar{w} w_\tau = w_\tau \bar{w} w_\theta$. □

THEOREM 6.3. *The assignment*

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \bullet \begin{bmatrix} p \\ w \end{bmatrix} := Q\left(\begin{bmatrix} p \\ w \end{bmatrix}, \theta\right) \begin{bmatrix} p_\theta \\ w_\theta \end{bmatrix} \overline{Q\left(\begin{bmatrix} p \\ w \end{bmatrix}, \theta\right)}$$

defines nonlinear $SO(2)$ -actions on \mathbb{S}^5 and \mathbb{S}^{13} that extend to cohomogeneity-1 actions of $O(2) \times SO(3)$ and $O(2) \times G_2$, respectively. These latter actions are equivalent to the standard actions on the Brieskorn spheres W_3^5 and W_3^{13} .

Proof. The map $Q: \mathbb{S}^5 \times \mathbb{S}^1 \rightarrow \mathbb{S}^3$ is equivariant under conjugation with unit quaternions; that is,

$$Q\left(\begin{bmatrix} qp\bar{q} \\ qw\bar{q} \end{bmatrix}, \theta\right) = qQ\left(\begin{bmatrix} p \\ w \end{bmatrix}, \theta\right)\bar{q}.$$

In an analogous way, $Q: \mathbb{S}^{13} \times \mathbb{S}^1 \rightarrow \mathbb{S}^7$ is equivariant under G_2 . With Lemma 6.2 it is now straightforward to check that the assignment of Theorem 6.3 defines

SO(2)-actions on \mathbb{S}^5 and \mathbb{S}^{13} that commute with the SO(3)-action on \mathbb{S}^5 and the G_2 -action on \mathbb{S}^{13} . It can be proved in various ways that \mathbb{S}^5 and \mathbb{S}^{13} equipped with the full actions of $O(2) \times SO(3)$ and $O(2) \times G_2$ are equivariantly diffeomorphic to W_3^5 and W_3^{13} —for example, by computing the isotropy groups along the curve

$$s \mapsto \left[\begin{array}{c} j \cos s \\ (k \cos(\pi \cos s) - i \sin(\pi \cos s)) \sin s \end{array} \right],$$

which corresponds precisely to the geodesics α on Σ^5 and β on W_3^5 and W_3^{13} under the identifications established in the previous sections. (For the isotropy group computation, note that if p and w anticommute and have the same norm then $p_\theta = e^{-v(\theta/2)} p e^{v(\theta/2)}$, where $v = (p/|p|)(w/|w|)$ and a similar expression holds for w_θ .) □

REMARK 6.4. The formula of Theorem 6.3 was obtained by pulling back the \bullet -action on Σ^5 by the explicit diffeomorphism $\mathbb{S}^5 \rightarrow \Sigma^5$ given in (3).

REMARK 6.5. For $\theta = \pi$, the formula of Theorem 6.3 gives exotic involutions on \mathbb{S}^5 and \mathbb{S}^{13} . These are studied in [ADPR].

REMARK 6.6. If one replaces $\pi/2$ in the definition of Q with $(2m + 1)(\pi/2)$, then one obtains an action that is conjugate to the original action by σ^m , where σ is the restriction of the exotic diffeomorphism $\sigma : \mathbb{S}^6 \rightarrow \mathbb{S}^6$ to \mathbb{S}^5 (see [D]).

7. Fixed Point Sets of Isometries

We now study the geometric properties of the large fixed point sets in detail. This information will allow us to show that $O(2) \times SO(3)$ is the full isometry group of Σ^5 and Σ^7 .

Recall from Lemma 3.1 that the $(O(2) \times SO(3))$ -action leaves Σ^5 and Σ^1 invariant. Any element (or subgroup) of $O(2) \times SO(3)$ fixes either the entire circle Σ^1 , two antipodal points in Σ^1 , or no points in Σ^1 at all.

LEMMA 7.1. *If an element $\psi \in O(2) \times SO(3)$ does not have a fixed point in Σ^1 then its fixed point set $(\Sigma^7)^\psi$ is completely contained in Σ^5 .*

Proof. The statement obviously does not depend on any metric. In order to prove it, however, it is helpful to use any of the pointed wiedersehen metrics $\langle \cdot, \cdot \rangle_{1,v}$. By Lemma 3.1, the isometry ψ maps Σ^1 and Σ^5 to themselves. Through any point $p \in \Sigma^7$ outside $\Sigma^1 \cup \Sigma^5$ there is a unique geodesic segment from Σ^1 to Σ^5 with length $\pi/2$. If p is fixed by the isometry ψ , then this segment is fixed pointwise as well. □

COROLLARY 7.2. *For all the metrics $\langle \cdot, \cdot \rangle_{\mu,v}$ the 5-sphere Σ^5 is a minimal submanifold of Σ^7 and of each $\Sigma_{\pm A}^6$.*

Proof. All principal isotropy groups of the \bullet -action on $\Sigma^5 \subset \Sigma^7$ are conjugate to the subgroup $H \subset O(2) \times SO(3)$ determined in Lemma 3.7. The union $\Sigma_{(H)}^7$

of all orbits of type (H) in Σ^7 (i.e., the set of all points whose isotropy group is conjugate to H) is a possibly disconnected open minimal submanifold of Σ^7 (see [HsL]). Any subgroup conjugate to H contains an element of the form $(-\mathbb{1}, \pm q)$. All elements of this form act on Σ^1 by the antipodal map. Thus, Lemma 7.1 implies that $\Sigma^7_{(H)}$ is contained in Σ^5 . Now, Σ^5 is apparently the closure of $\Sigma^7_{(H)}$ and hence minimal. An analogous argument using the $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \text{SO}(3))$ -action shows that Σ^5 is minimal in $\Sigma^6_{\pm\mathbb{1}}$. \square

LEMMA 7.3. *If $\psi \in \text{O}(2) \times \text{SO}(3)$ fixes precisely two antipodal points $\mathbb{S}^3 \star (\pm A) \in \Sigma^1$ with $A \in \text{O}(2)$, then $(\Sigma^7)^\psi$ is contained in $\Sigma^6_{\pm A \cdot A_0}$ with $A_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and is also a suspension of $(\Sigma^5)^\psi$ from the two points $\mathbb{S}^3 \star (\pm A)$. In particular, $(\Sigma^7)^\psi$ and $(\Sigma^5)^\psi$ are both diffeomorphic to spheres.*

Proof. It suffices to consider any of the metrics $\langle \cdot, \cdot \rangle_{1,v}$. Between any two points $p, q \in \Sigma^7$ with $p \in \Sigma^1$ and $q \notin \Sigma^1$ there exists a unique minimizing geodesic segment from p to q . If ψ fixes p and q then it fixes the geodesic as well. Note that $(\Sigma^7)^\psi$ is either empty or odd-dimensional because ψ is orientation preserving. (As a generator of the group $\Theta_7 \approx \mathbb{Z}_{28}$ of homotopy spheres, Σ^7 does not admit orientation-reversing diffeomorphisms.) \square

LEMMA 7.4. *If $\psi \in \text{O}(2) \times \text{SO}(3)$ fixes all points in the circle Σ^1 , then either $(\Sigma^7)^\psi$ is equal to Σ^1 or $(\Sigma^7)^\psi$ is the join of Σ^1 and $(\Sigma^5)^\psi$ and thus the suspension of $(\Sigma^6_{\pm A})^\psi$ from any two antipodal points $\mathbb{S}^3 \star (\pm A) \in \Sigma^1$. In particular, $(\Sigma^7)^\psi$, $(\Sigma^6_{\pm A})^\psi$, and $(\Sigma^5)^\psi$ are diffeomorphic to spheres.*

Proof. Similar to the proofs of Lemma 7.1 and Lemma 7.3. \square

The preceding lemmas show that topologically interesting fixed point sets can arise only in the case where the isometry has no fixed points in Σ^1 —that is, only when the fixed point set is completely contained in Σ^5 .

We now discuss the fixed point sets in more detail. It is clear that conjugate elements of $\text{O}(2) \times \text{SO}(3)$ yield congruent fixed point sets. Every element in $\text{SO}(2) \times \text{SO}(3)$ is conjugate to an element in the maximal torus

$$\text{SO}(2) \times \text{SO}(2) = \left\{ \left(\begin{bmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{bmatrix}, \pm e^{i\theta_2} \right) \mid \theta_1, \theta_2 \in \mathbb{R} \right\}$$

of $\text{SO}(2) \times \text{SO}(3)$. All the other elements of $\text{O}(2) \times \text{SO}(3)$ are conjugate to an element of the form $\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \pm e^{i\theta} \right)$ and fix precisely two points in Σ^1 ; they will be discussed at the end of this section. We first inspect the fixed point sets of the elements in the maximal torus $\text{SO}(2) \times \text{SO}(2)$ of $\text{SO}(2) \times \text{SO}(3)$.

We first deal with the elements of $\text{SO}(2) \times \text{SO}(2)$ that fix Σ^1 pointwise. These isometries are clearly of the form $(\mathbb{1}, \pm e^{i\theta})$, and in Figure 2 this set is indicated by the two vertical lines. From Lemma 3.7 we see that the fixed point set in Σ^5 is a circle located in the singular orbit through $\alpha(0)$ if $e^{i\theta} \neq \pm 1$. Thus, by Lemma 7.4, the fixed point set in Σ^7 is the join of Σ^1 and this circle and hence is diffeomorphic to \mathbb{S}^3 . Although not all elements of the form $(\mathbb{1}, \pm e^{i\theta})$ are conjugate to $(\mathbb{1}, \pm i)$,

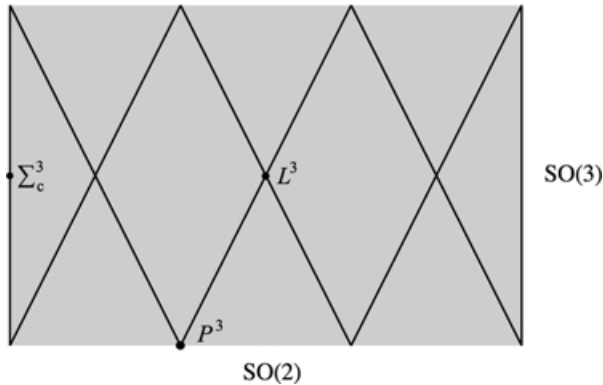


Figure 2 The elements in the maximal torus of $SO(2) \times SO(3)$ with fixed points are precisely illustrated by the two vertical lines

all their fixed point sets are congruent to the fixed point set of $(\mathbb{1}, \pm i)$, which will be denoted by Σ_0^3 .

LEMMA 7.5. *The fixed point set Σ_0^3 of $(\mathbb{1}, \pm i)$ on Σ^7 is isometric to a 3-sphere equipped with a Berger metric where the horizontal geodesics have length 2π and the Hopf circles have length $2\pi\sqrt{\mu}$.*

Proof. With the structural information just given, it is immediate that $\Sigma_0^3 = \pi_{\Sigma^7}(U(2))$; this is isometric to the homogeneous space $U(2)/U(1)$, where $U(1)$ is embedded into the right lower corner and $U(2)$ is equipped with the metric $\langle \cdot, \cdot \rangle_{\mu, \nu}$. □

COROLLARY 7.6. *There is no $SO(3)$ -equivariant homeomorphism between Σ^7 and any of the Brieskorn spheres $W_{6j-1,3}^7$.*

Proof. In $W_{6j-1,3}^7$, the fixed point set of $\pm i = \text{diag}(1, -1, -1) \in SO(3)$ is the integral homology sphere $W_{6j-1,3}^3 = W_{6j-1,3,2}^3$. For $j = 1$ this space is diffeomorphic to Poincare dodecahedral space, and for $j > 1$ the universal cover of $W_{6j-1,3,2}^3$ is $\widetilde{SL}(2, \mathbb{R})$ (see [Mi]). □

Note that this last argument also gives a simple reason for why there are no $SO(3)$ -invariant Riemannian metrics on $W_{6j-1,3}^7$ with $K > 0$ for $j > 1$.

LEMMA 7.7. *The circle Σ^1 is a closed geodesic for all $SO(3)$ -invariant Riemannian metrics on Σ^7 and in particular for all metrics $\langle \cdot, \cdot \rangle_{\mu, \nu}$.*

Proof. It is easy to check that Σ^1 is the intersection of all the fixed point sets $(\Sigma^7)^{(\mathbb{1}, \pm q)}$, that is, the common fixed point set of $SO(3)$. □

We now consider the elements in the maximal torus $SO(2) \times SO(2)$ of $SO(2) \times SO(3)$ that do not fix any point in Σ^1 . The elements with fixed points in Σ^7 correspond up to conjugation to the skew lines in Figure 2. It suffices to consider the left half of Figure 2, since one can conjugate by $\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \mathbb{1}\right) \in O(2) \times SO(3)$.

LEMMA 7.8. *The fixed point set L^3 of the isometry $(-\mathbf{1}, \pm i)$ on Σ^7 is diffeomorphic to the lens space $\mathbb{S}^3/\mathbb{Z}_3$ and totally geodesic in Σ^5 , $\Sigma_{\pm A}^6$, and Σ^7 . Moreover, L^3 is isometrically covered by a Berger metric on \mathbb{S}^3 where the horizontal geodesics have length 2π and the Hopf circles have length $2\pi\sqrt{9\mu\nu/(4\mu + \nu)}$. In particular, the extremal values of the sectional curvature of L^3 at any point are $9\mu\nu/(4\mu + \nu)$ and $4 - 27\mu\nu/(4\mu + \nu)$.*

Proof. By Theorem 4.3, Σ^5 and the Brieskorn sphere W_3^5 are equivariantly diffeomorphic. In W_3^5 the corresponding fixed point set is W_3^3 , which is well known to be diffeomorphic to $\mathbb{S}^3/\mathbb{Z}_3$ (see e.g. [HzMa]). In Σ^5 there exists a direct argument that allows a simple curvature computation: Straightforward computations show that the horizontal lift of $T_{\alpha(s)}L$ at $\tilde{\alpha}(s)$ is spanned by the horizontal vectors $\tilde{\alpha}'(s)$, $\tilde{v}_0(s)$, and $\tilde{v}_1(s)$. Thus L is 3-dimensional. Let $U(2)$ be the centralizer of $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ in $Sp(2)$. It is straightforward to see that $\pi_{\Sigma^7}(jU(2))$ embeds into the fixed point set. Now $\pi_{\Sigma^7}(jU(2))$ is isometric to the quotient of $jU(2) \subset Sp(2)$ by the $U(1)$ -action $(\lambda, jA) \mapsto \lambda jA \begin{bmatrix} \bar{\lambda} & 0 \\ 0 & 1 \end{bmatrix}$, where $U(2)$ carries the metric $\langle \cdot, \cdot \rangle_{\mu, \nu}$ induced from $Sp(2)$. Since i and j anticommute, this quotient is isometric the homogeneous space $U(2)/\{\begin{bmatrix} \bar{\lambda}^2 & 0 \\ 0 & \bar{\lambda} \end{bmatrix}\}$ and hence is diffeomorphic to $\mathbb{S}^3/\mathbb{Z}_3$. The vector $\tilde{\alpha}'(0) = j \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$ is horizontal with respect to the fibration

$$U(2)/\{\begin{bmatrix} \bar{\lambda}^2 & 0 \\ 0 & \bar{\lambda} \end{bmatrix}\} \rightarrow U(2)/(U(1) \times U(1)) = \mathbb{C}P^1$$

and the curve $\exp(t\tilde{\alpha}'(0))$ closes first after length 2π in $U(2)$. The vector

$$\tilde{v}_1(0) = j \frac{2}{4\mu + \nu} \begin{bmatrix} i\nu & 0 \\ 0 & -2i\mu \end{bmatrix}$$

is vertical. It is easy to check that the curve $\exp(t\tilde{v}_1(0))$ in $U(2)$ meets the circle $\{\begin{bmatrix} \bar{\lambda}^2 & 0 \\ 0 & \bar{\lambda} \end{bmatrix}\}$ first at time $T = \pi$. Hence, the length of the vertical circle in the lens space is $\pi|\tilde{v}_1(0)| = 2\pi\sqrt{\mu\nu/(4\mu + \nu)}$. In the universal cover \mathbb{S}^3 , the length of this Hopf circle is three times as long. \square

The fixed point sets of the remaining elements of the form $(\begin{bmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{bmatrix}, \pm e^{i\theta_2})$ are all contained in the singular orbit through $\alpha(\pi/4)$ by Lemma 3.7. Up to conjugation, only the following fixed point set P^3 is more than 1-dimensional.

LEMMA 7.9. *The fixed point set P^3 of the isometry $(\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \mathbf{1})$ with $\theta = 2\pi/3$ on Σ^7 is isometric to $\mathbb{R}P^3$ covered by a Berger \mathbb{S}^3 whose horizontal geodesics have length $2\pi\sqrt{\nu}$ and whose Hopf circles have length $2\pi\sqrt{4\mu\nu/(4\mu + \nu)}$.*

Proof. It is immediate from the isotropy groups along the normal geodesic α in Lemma 3.7 that P^3 is precisely the $(O(2) \times SO(3))$ -orbit through $\alpha(\pi/4)$. The subgroup $SO(3)$ acts simply transitively on this orbit, and the induced metric follows from Lemma 3.8. \square

We finally deal with the isometries that fix precisely two points in Σ^1 . It is clear that these isometries are not contained in $SO(2) \times O(3)$. By Lemma 3.7 and Lemma 7.3 they are conjugate to $(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \pm 1)$ or $(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \pm j)$ if the dimension of the fixed

point set is > 1 . From Lemma 7.3 it is also clear that the fixed point sets of these isometries are suspensions of subspheres of Σ^5 from the two points $\mathbb{S}^3 \star (\pm \mathbb{1})$.

LEMMA 7.10. *The fixed point set Σ_1^3 of the isometry $(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \pm 1)$ is diffeomorphic to \mathbb{S}^3 . The induced metric on $\Sigma_1^3 \setminus (\mathbb{S}^3 \star (\pm \mathbb{1}))$ is isometric to the metric*

$$\mu \left(dt^2 + \frac{v \sin^2 t}{v + 4\mu \sin^2 t} g_{\text{can}}^{\mathbb{S}^2} \right)$$

on $[0, \pi] \times \mathbb{S}^2$. Hence, the sectional curvatures vary between $v/\mu(4\mu + v)$ and $(12\mu + v)/\mu v$.

Proof. By Lemma 3.7, the fixed point set is the suspension of homogeneous 2-spheres from the two points $\mathbb{S}^3 \star (\pm \mathbb{1})$. This suspension is given by the geodesics $\pi_{\Sigma^7} \circ \tilde{\gamma}_{[0]}^p$ in (2). It is straightforward to compute the diameter of the $\text{SO}(3)$ -orbits through $\pi_{\Sigma^7} \circ \gamma_{[0]}^p(t)$. □

LEMMA 7.11. *The fixed point set Σ^2 of the isometry $(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \pm j)$ on Σ^5 and on $\Sigma_{\pm \mathbb{1}}^6$ is isometric to a 2-sphere equipped with the metric $ds^2 + \frac{1}{4}c(s)d\phi^2$, where $c(s)$ is the function $[0, \pi] \rightarrow \mathbb{R}$ defined in Lemma 3.8. The sectional curvature K of Σ^2 satisfies*

$$K|_{s=0} = \frac{12}{v} - 8 - 3\mu, \quad K|_{s=\pi/4} = \frac{4v}{1 + \mu}, \quad K|_{s=\pi/2} = -\frac{v(1 + 2\mu)}{\mu(4\mu + v)}.$$

Proof. It is easy to see that Σ^2 contains the normal geodesic α and that the tangent space to Σ^2 at $\alpha(s)$ is spanned by $\alpha'(s)$ and the Killing field $v_2(s)$ if $s \notin \pi\mathbb{Z}$. A straightforward computation shows that the circle that corresponds to v_2 and that acts effectively on Σ^2 inherits the length $\pi \cdot \sqrt{c(s)}$ at time s . The formula for $c(s)$ shows that Σ^2 is diffeomorphic to a sphere. The curvature computations are now straightforward. Finally, Lemma 7.1 ensures that the fixed point set of the isometry on $\Sigma_{\pm \mathbb{1}}^6$ is contained in Σ^5 . □

REMARK 7.12. Another way to see that Σ^2 is diffeomorphic to \mathbb{S}^2 is to pass from Σ^5 to the Euclidean sphere \mathbb{S}^5 with the nonlinear action obtained in Section 6. On $\mathbb{S}^5 \subset \text{Im } \mathbb{H} \times \mathbb{H}$ it is straightforward to check that the transformation $(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \pm j)$ fixes precisely the 2-sphere that consists of all unit vectors of the form $\begin{bmatrix} p \\ w \end{bmatrix}$ with $p \in j\mathbb{R}$ and $w \in \text{span}_{\mathbb{R}}\{i, k\}$.

COROLLARY 7.13. *The fixed point set Σ_2^3 of the isometry $(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \pm j)$ on Σ^7 is diffeomorphic to \mathbb{S}^3 .*

Proof. This follows immediately from Lemma 7.11 and Lemma 7.3. □

In order to describe the metric that Σ_2^3 inherits from Σ^7 , it is useful to note that the horizontal lift of the tangent space $T_{\mathbb{S}^3 \star \mathbb{1}} \Sigma_2^3$ at $\mathbb{1}$ is spanned by the three vectors

$$\begin{bmatrix} j & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix}.$$

Hence, Σ_2^3 can be parameterized by the horizontal geodesics $\tilde{\gamma}_{[w]}^p$ given in (2) with

$$p = j \cos \theta \quad \text{and} \quad w = i \sin \theta \cos \phi + k \sin \theta \sin \phi,$$

where $t \in [0, \pi]$, $\theta \in [0, \pi]$, and $\phi \in [0, 2\pi]$. Thus, Σ_2^3 corresponds to a maximal choice of anticommuting p and w . In the coordinates (t, θ, ϕ) , the metric on Σ_2^3 is given by

$$\begin{aligned} g_{11} &= 1 - \frac{1}{D}(1 - \mu)(4 \sin^2 t \sin^2 \theta + \nu(1 - 2 \sin^2 t \sin^2 \theta)^2) \cos^2 \theta, \\ g_{22} &= \sin^2 t + \frac{1}{D} \sin^2 t \sin^2 \theta (\nu \sin^2 \theta (2t - \sin 2t)^2 \\ &\quad - (1 - \mu)((\nu + 4 \sin^2 t \sin^2 \theta) \cos^2 t \\ &\quad + 2\nu \sin^2 \theta (2t \sin^2 t \sin^2 \theta - \sin 2t))), \\ g_{33} &= \frac{1}{D} \nu \sin^2 t \sin^2 \theta (1 - (1 - \mu) \sin^2 t \sin^2 \theta), \\ g_{23} &= \frac{1}{D} \nu \sin^2 t \sin^3 \theta \left(2t - \sin 2t + \frac{1 - \mu}{2} (\sin 2t - 4t \sin^2 t \sin^2 \theta) \right), \\ g_{13} &= -\frac{1}{D} \nu (1 - \mu) \sin^2 t \sin^2 \theta \cos \theta (1 - 2 \sin^2 t \sin^2 \theta), \\ g_{12} &= \frac{1 - \mu}{4D} \sin 2\theta (4 \sin 2t \sin^2 t \sin^2 \theta \\ &\quad - \nu(1 - 2 \sin^2 t \sin^2 \theta)(4t \sin^2 t \sin^2 \theta - \sin 2t)), \end{aligned}$$

where

$$D = 4(1 - (1 - \mu) \sin^2 t \sin^2 \theta) \sin^2 t \sin^2 \theta + \nu(1 - 2 \sin^2 t \sin^2 \theta)^2.$$

This specializes for $\mu = 1$ to

$$\begin{aligned} g &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin^2 t & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &\quad + \frac{\nu \sin^2 t \sin^2 \theta}{4 \sin^2 t \sin^2 \theta + \nu(1 - 2 \sin^2 t \sin^2 \theta)^2} \begin{bmatrix} 0 \\ (2t - \sin 2t) \sin \theta \\ 1 \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} 0 \\ (2t - \sin 2t) \sin \theta \\ 1 \end{bmatrix}^T. \end{aligned}$$

Note that Σ_2^3 is invariant under the isometry of Σ^7 induced by $-\mathbb{1} \in \text{Sp}(2)$. In our coordinates, this isometry is given by $(t, \theta, \phi) \mapsto (\pi - t, \theta + \pi, \phi - 2\pi \cos \theta)$. This coordinate change allows us to glue Σ_2^3 from two disks equipped with g .

Although the metric g is of cohomogeneity 2, the curvature of g behaves as if the action on Σ_2^3 were of cohomogeneity 1: The orbit space of the natural $\text{SO}(2)$ -action on Σ_2^3 (given in our coordinates by translation in ϕ) can easily be shown to be the hemisphere of constant curvature 1 for $\mu = 1$. In our coordinates, this hemisphere is given by $0 \leq t \leq \pi$, $0 \leq \theta \leq \pi$, and is represented by geodesics

from a point in the boundary. It is reasonable to switch to polar coordinates—that is, to make the coordinate change

$$\begin{bmatrix} \cos t \\ \cos \theta \sin t \\ \sin \theta \sin t \end{bmatrix} = \begin{bmatrix} \sin \omega \cos \psi \\ \sin \omega \sin \psi \\ \cos \omega \end{bmatrix}$$

with $0 \leq \omega \leq \pi/2$ and $0 \leq \psi \leq 2\pi$.

LEMMA 7.14. *The orbit space of the natural $SO(2)$ -action on $(\Sigma_2^3, \langle \cdot, \cdot \rangle_{\mu, \nu})$ is a hemisphere that inherits a rotationally invariant metric with curvature*

$$\mu \frac{1 + 2(1 - \mu) \cos^2 \omega}{(1 - (1 - \mu) \cos^2 \omega)^2}.$$

In particular, the curvature is constant if and only if $\mu = 1$.

The eigenvalues of the Einstein tensor (i.e., the critical values of the sectional curvature) of Σ_2^3 also turn out to be independent of ψ . The metric g itself, however, does not improve in the coordinates (ω, ψ, ϕ) , nor does the curvature computation become simpler. The next lemma gives some more detailed curvature information.

LEMMA 7.15. *The scalar curvature of $(\Sigma_2^3, \langle \cdot, \cdot \rangle_{1, \nu})$ is given by*

$$\frac{4(-12 + 4\nu + 9\nu^2 + 2(21\nu - 8) \cos 2\omega + (9\nu^2 + 16\nu - 4) \cos 4\omega + 2\nu \cos 6\omega)}{(4 + \nu + 4 \cos 2\omega + \nu \cos 4\omega)^2}.$$

For $\mu, \nu \leq 1$, the minimum of the sectional curvature is given by

$$\min K = \min \left\{ \frac{\mu\nu}{4\mu + \nu}, \frac{12 - 8(\mu + \nu) - 3\mu\nu}{4\mu + \nu} \right\}.$$

In the Gromoll–Meyer case where $\mu = \nu = \frac{1}{2}$, Σ_2^3 inherits a metric with $\frac{\min K}{\max K} = \frac{1}{145}$.

Note that by construction and Lemma 7.3, Σ_2^3 is totally geodesic in Σ^7 and in $\Sigma_{\pm A_0}^6$ with $A_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and Σ^2 is totally geodesic in Σ^5 and in $\Sigma_{\pm \mathbb{1}}^6$.

COROLLARY 7.16. *There is a point in Σ^2 that has negative curvature for all the metrics $\langle \cdot, \cdot \rangle_{\mu, \nu}$ on Σ^7 . Moreover, Σ^5 and $\Sigma_{\pm \mathbb{1}}^6$ are not totally geodesic in Σ^7 for any of these metrics.*

Proof. The intrinsic sectional curvature of Σ^2 at $\alpha(\pi/2)$ is $-v(1 + 2\mu)/\mu(4\mu + v) < 0$ by Lemma 7.11. The point $\alpha(\pi/2) \in \Sigma^2 \subset \Sigma_2^3$ corresponds to the coordinates $t = \theta = \phi = \pi/2$ on Σ_2^3 . The extrinsic sectional curvature of the tangent space of Σ^2 at this point can be computed as $\mu\nu/(4\mu + v) > 0$. \square

We would like to add some comments on these fixed point sets. First, the spheres Σ_0^3 , Σ_1^3 , and Σ_2^3 are indexed according to their intrinsic cohomogeneity. Second, the fixed point sets Σ_0^3 , L^3 , and Σ_2^3 yield necessary conditions for $(\Sigma^7, \langle \cdot, \cdot \rangle_{\mu, \nu})$ to have nonnegative sectional curvature: That Σ_0^3 inherits $K \geq 0$ implies $\mu \leq \frac{4}{3}$, that L^3 inherits $K \geq 0$ implies $4(4\mu + v) - 27\mu\nu \geq 0$, and that Σ_2^3 inherits $K \geq 0$ implies $12 - 8(\mu + v) - 3\mu\nu \geq 0$. The last inequality is the most restricting one.

Of particular interest is the question of whether the nice behavior of geodesics on Σ^7 for $\mu = 1$ can be combined with nonnegative sectional curvature. In this case, the inequalities just listed show that necessarily $\nu \leq 4/11$ (this is precisely the inequality that guarantees Σ^3_2 inherits $K \geq 0$). For $\mu = 1$ and any $\nu > 0$, there are always some negative sectional curvatures on $(\text{Sp}(2), \langle \cdot, \cdot \rangle_{\mu, \nu})$. The question of whether these disappear for small $\nu > 0$ when going down to Σ^7 is subtle.

A distinguished metric on Σ^7 is $\langle \cdot, \cdot \rangle_{1, 1/2}$. In this case Σ^3_0 and L^3 both have constant curvature 1.

The results on fixed point sets in this section allow us to compare the metrics $\langle \cdot, \cdot \rangle_{\mu, \nu}$ on Σ^5 (and hence on the exotic projective space $\Sigma^5/\{\pm 1\}$) briefly to those that come from the Grove–Ziller construction for cohomogeneity manifolds with codimension-2 singular orbits [GZ]. Our metrics are analytic and there are always points with negative sectional curvature. The Grove–Ziller metrics are merely smooth but have nonnegative sectional curvature; on the lens space L^3 they induce a proper cohomogeneity-1 metric with planes of zero sectional curvature over an open set of points.

Finally, we would like to point out that not all totally geodesic submanifolds of Σ^7 are fixed point sets of isometries, as the following lemma shows.

LEMMA 7.17. *For any metric $\langle \cdot, \cdot \rangle_{\mu, \nu}$, the rectangular 2-torus T^2 in $\text{Sp}(2)$ parameterized by*

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{j\beta} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\alpha} & ie^{j\beta} \\ ie^{i\alpha} & e^{j\beta} \end{bmatrix}$$

with $\alpha, \beta \in \mathbb{R}$ is totally geodesic and horizontal with respect to the submersion $\pi_{\Sigma^7}: \text{Sp}(2) \rightarrow \Sigma^7$. Its image is a totally geodesic rectangular 2-torus in Σ^7 covered twice by T^2 .

Proof. Consider the subgroup G of $\text{Sp}(1) \times \text{Sp}(1) \subset \text{Sp}(2)$ generated by the two elements $\begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & j \end{bmatrix}$. This group G acts by conjugation isometrically on $(\text{Sp}(2), \langle \cdot, \cdot \rangle_{\mu, \nu})$. The rectangular torus

$$\left\{ \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{j\beta} \end{bmatrix} \mid \alpha, \beta \in \mathbb{R} \right\} \subset \text{Sp}(2)$$

is the common fixed point set of G and hence is totally geodesic. Therefore, its left translated copy T^2 is totally geodesic, too. It is straightforward to show that T^2 is horizontal and that π_{Σ^7} restricted to T^2 induces an embedding of $T^2/\begin{bmatrix} 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$ into Σ^7 . □

This torus was already implicitly contained in [GrMy] and is also listed in [Wi]. The fundamental difference between the standard action \bullet of \mathbb{S}^3 on $\text{Sp}(2)$ and the Gromoll–Meyer action \star appears here very clearly: The torus T^2 is horizontal for the \star -action whereas only an \mathbb{S}^1 -factor is horizontal for the \bullet -action.

8. The Isometry Group of the Gromoll–Meyer Sphere

In Lemma 3.1 it was shown that all elements of $O(2) \times \text{SO}(3)$ map Σ^1 and Σ^5 to themselves. We show that the same is true for any isometry of $(\Sigma^7, \langle \cdot, \cdot \rangle_{\mu, \nu})$.

LEMMA 8.1. *Every isometry of $(\Sigma^7, \langle \cdot, \cdot \rangle_{\mu,v})$ maps Σ^1 and Σ^5 to themselves.*

Proof. The maximum dimension of any compact differentiable transformation group on Σ^7 is 4 (see [St]). The \bullet -action of $G = O(2) \times SO(3)$ on Σ^7 is effective. Hence, the subgroup $G_0 = SO(2) \times SO(3)$ of G is the identity component of the isometry group \tilde{G} of Σ^7 . Let $\psi \in \tilde{G}$ be any isometry of Σ^7 . Since G_0 is a normal subgroup of \tilde{G} , it follows that conjugation by ψ on \tilde{G} maps $SO(3)$ to itself. Hence, ψ maps the fixed point set of $SO(3)$ in Σ^7 to itself. This fixed point set is precisely the circle Σ^1 of wiedersehen points. Moreover, ψ also maps G_0 -orbits diffeomorphically to G_0 -orbits. It thus follows from the isotropy groups (as in the proof of Corollary 7.2) that ψ maps Σ^5 to itself. \square

LEMMA 8.2. *The isometry group of $(\Sigma^5, \langle \cdot, \cdot \rangle_{\mu,v})$ is the group $O(2) \times SO(3)$.*

Proof. It is obvious from the isotropy group computation in Lemma 3.7 that the \bullet -action of $G = O(2) \times SO(3)$ on Σ^5 is effective. The full isometry group \tilde{G} of Σ^5 cannot act transitively on Σ^5 . Otherwise all fixed point sets of isometries would be homogeneous, but Σ^2 from Lemma 7.11 is not homogeneous. Consider the normal geodesic α in Σ^5 given in formula (4). This geodesic intersects all G -orbits perpendicularly. Each principal G -orbit is contained in a \tilde{G} -orbit of the same dimension (since \tilde{G} does not act transitively) and hence α also intersects all \tilde{G} -orbits perpendicularly. Let \tilde{H} denote the common principal isotropy group along the geodesic α and let $\tilde{H}(s)$ denote the group of isomorphisms of \mathbb{R}^4 that preserve the symmetric bilinear form given by the matrix in Lemma 3.8. Clearly, \tilde{H} is isomorphic to a subgroup of the intersection of all $\tilde{H}(s)$. It is straightforward to see that the intersection of all $\tilde{H}(s)$ is the group of order 8 generated by the three involutions

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

If this entire group of order 8 were the isotropy group along α rather than just the index-2 subgroup given in (6), then the curvature tensor $\langle R(\alpha', v_1)v_2, v_3 \rangle_{\mu,v}$ of Σ^5 would vanish identically. However, computations show that

$$\langle R(\alpha', v_1)v_2, v_3 \rangle_{\mu,v} = r_{\mu,v}(\cos 2s) \sin 4s,$$

where $r_{\mu,v}$ is a rational function with $r_{\mu,v}(0) = -4\mu v^2 / ((1 + \mu)(4\mu + v))$. This implies that $\langle R(\alpha', v_1)v_2, v_3 \rangle_{\mu,v}$ does not vanish for s close to but not equal to $\pi/4$. It follows that the isotropy group H of the \bullet -action is the full principal isotropy group \tilde{H} of $(\Sigma^5, \langle \cdot, \cdot \rangle_{\mu,v})$. Because the principal orbits G/H are connected, G is the full subgroup of \tilde{G} that preserves the principal orbits. Moreover, \tilde{G} is a finite extension of G and $G \subset \tilde{G}$ is a normal subgroup. If G were a proper subgroup of \tilde{G} , then \tilde{G}/G would act nontrivially on the orbit space and the Weyl group of the cohomogeneity-1 action of \tilde{G} on Σ^5 would be larger than that of the action of G . This is impossible, as one can see from the isotropy groups in Lemma 3.7. \square

THEOREM 8.3. *The isometry group of $(\Sigma^7, \langle \cdot, \cdot \rangle_{\mu, \nu})$ is the group $O(2) \times SO(3)$.*

Proof. Lemma 8.2 shows that $O(2) \times SO(3)$ is the isometry group of $(\Sigma^5, \langle \cdot, \cdot \rangle_{\mu, \nu})$. By Lemma 8.1, every isometry of $(\Sigma^7, \langle \cdot, \cdot \rangle_{\mu, \nu})$ maps Σ^5 to itself. It suffices to show that the only isometry of Σ^7 that fixes Σ^5 pointwise is the identity. Let ψ be such an isometry. The fixed point set $(\Sigma^7)^\psi$ is the disjoint union of totally geodesic submanifolds. Let M be the component of $(\Sigma^7)^\psi$ that contains Σ^5 . Consider the fixed point set $\Sigma^2 = \Sigma^5 \cap \Sigma_2^3$ from Lemma 7.11. Obviously, $\Sigma^2 \subset M \cap \Sigma_2^3$. If Σ_2^3 were not contained in M then $M \cap \Sigma_2^3$ would be a proper totally geodesic submanifold of Σ_2^3 that contains Σ^2 and is thus equal to Σ^2 . We know from Lemma 7.15 that Σ^2 is not totally geodesic in Σ_2^3 . Hence, $\Sigma_2^3 \subset M$. Now consider the congruent copy $\tilde{\Sigma}_2^3$ of Σ_2^3 given by the fixed point set of the isometry $(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \pm k)$ in Σ^7 . We again have $\Sigma^2 = \Sigma^5 \cap \tilde{\Sigma}_2^3$. Hence, by the same argument as before, $\tilde{\Sigma}_2^3$ is also contained in M . The inclusion $\Sigma^5 \cup \Sigma_2^3 \cup \tilde{\Sigma}_2^3 \subset M$ implies that $\dim M = 7$ (inspect the tangent spaces along the normal geodesic) and hence that $\psi = \text{id}_{\Sigma^7}$. □

9. Free Actions on the Gromoll–Meyer Sphere

In this section we classify all closed subgroups of $O(2) \times SO(3)$ that act freely on Σ^7 and determine the homotopy type of the orbit spaces. Recall that $O(2) \times SO(3)$ is the full isometry group of $(\Sigma^7, \langle \cdot, \cdot \rangle_{\mu, \nu})$ and $(\Sigma^5, \langle \cdot, \cdot \rangle_{\mu, \nu})$ and that all elements in $O(2) \times SO(3)$ that are not contained in $SO(2) \times SO(3)$ have fixed points in $\Sigma^5 \subset \Sigma^7$ because they reverse the orientation of Σ^5 . In $SO(2) \times SO(3)$, it suffices to consider the elements of a maximal torus.

LEMMA 9.1. *An element in $SO(2) \times SO(3)$ has a fixed point in Σ^7 if and only if it has a fixed point in Σ^5 .*

Proof. It follows from the isotropy group K_- determined in Lemma 3.7 that any isometry $(\mathbb{1}, \pm q)$ has fixed points in Σ^5 . All other elements of $SO(2) \times SO(3)$ are covered by Lemma 7.1. □

LEMMA 9.2. *An element in $SO(2) \times SO(3)$ has fixed points on Σ^5 if and only if it is conjugate to an element of the subset of the maximal torus of $SO(2) \times SO(3)$ illustrated by all the lines in Figure 2.*

Proof. Whether an element of $SO(2) \times SO(3)$ has a fixed point in Σ^5 is a property that does not change under conjugation. Hence, it suffices to consider the elements of a fixed maximal torus of $SO(2) \times SO(3)$, that is, elements of the form

$$f(\theta_1, \theta_2) = \left\{ \left(\begin{bmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{bmatrix}, \pm e^{i\theta_2} \right) \right\}.$$

Since all isotropy groups in Σ^5 are conjugate to isotropy groups along the normal geodesic α , an element $f(\theta_1, \theta_2)$ has a fixed point in Σ^5 if and only if there exists some $q \in \mathbb{S}^3$ such that $(1, q) f(\theta_1, \theta_2) (1, \bar{q})$ is contained in one of the groups H, K_-, K_+ given in Lemma 3.7. This is precisely the case if $\theta_1 = 0$ and if θ_2 is

arbitrary (the two vertical lines in Figure 2 that represent one circle in the torus) or $\theta_2 = \pm \frac{3}{2}\theta_1$ (the diagonal lines in Figure 2). □

COROLLARY 9.3. *Every finite group that acts both freely and isometrically on $(\Sigma^5, \langle \cdot, \cdot \rangle_{\mu, \nu})$ and equivalently on $(\Sigma^7, \langle \cdot, \cdot \rangle_{\mu, \nu})$ is cyclic.*

Proof. Let G be a finite subgroup of $SO(2) \times SO(3)$. From Figure 2 we see that the kernel of the projection from G to the $SO(2)$ factor is trivial. □

COROLLARY 9.4. *All finite cyclic groups act both freely and isometrically on $(\Sigma^5, \langle \cdot, \cdot \rangle_{\mu, \nu})$ and hence also on the Gromoll–Meyer sphere $(\Sigma^7, \langle \cdot, \cdot \rangle_{\mu, \nu})$.*

Proof. This is evident from extending the pattern of Figure 2 periodically to all of \mathbb{R}^2 . □

COROLLARY 9.5. *For every $m \in \mathbb{N}$ there exist 7-dimensional exotic homotopy lens spaces with fundamental group \mathbb{Z}_m and nonnegative sectional curvature. For every even $m \in \mathbb{N}$ there exist 5-dimensional exotic homotopy lens spaces with fundamental group \mathbb{Z}_m and nonnegative sectional curvature.*

Proof. It is well-known that the quotient of a homotopy sphere by a cyclic group is homotopy equivalent to a lens space (cf. [Bw]). Since the Gromoll–Meyer sphere is not diffeomorphic to the standard sphere, its quotients by finite cyclic groups cannot be diffeomorphic to lens spaces. This completes the proof in the 7-dimensional case. In the 5-dimensional case, $\Sigma^5/\{\pm 1\}$ is an exotic projective space. All finite subgroups of $SO(2) \times SO(3)$ of even order that act freely on Σ^5 contain $\{\pm 1\}$. Hence, all the quotients of Σ^5 by such groups are not diffeomorphic to lens spaces. The metrics $\langle \cdot, \cdot \rangle_{\mu, \nu}$ on Σ^5 do not have nonnegative sectional curvature. However, by the Grove–Ziller construction [GZ] there are $(O(2) \times SO(3))$ -invariant metrics on $\Sigma^5 \approx W_3^5$ with $K \geq 0$. These metrics go down to the quotients and hence supply the desired family of nonnegatively curved exotic homotopy lens spaces. □

In the rest of this section we will determine to which lens spaces the orbit spaces of the \mathbb{Z}_m -actions on Σ^7 and Σ^5 are homotopy equivalent. Following an idea of Orlik [Or], we construct $(O(2) \times SO(3))$ -equivariant (continuous) branched coverings $\Sigma^5 \rightarrow \mathbb{S}^5$ and $\Sigma^7 \rightarrow \mathbb{S}^7$. Using these branched coverings, we obtain maps of degree $ml + 1$ for some positive integer l from Σ^7/\mathbb{Z}_m and Σ^5/\mathbb{Z}_m to standard lens spaces. By a theorem of Olum [O], the existence of such maps implies the existence of homotopy equivalences.

Let $D(\theta)$ denote the counterclockwise rotation in \mathbb{R}^2 by the angle θ . Then the subgroup $H_{m;p,q}$ of $SO(2) \times SO(3)$ generated by the element

$$\psi_{m;p,q} = \left(D\left(\frac{2\pi}{m}p\right), \pm e^{i(\pi/m)q} \right) = \left(D\left(\frac{2\pi}{m}p\right), \begin{bmatrix} 1 & 0 \\ 0 & D\left(\frac{2\pi}{m}q\right) \end{bmatrix} \right)$$

acts freely on Σ^7 if and only if $p \neq 0$, $3p - q \neq 0$, $3p + q \neq 0$, m and p are relatively prime, m and $3p - q$ are relatively prime, and m and $3p + q$ are relatively prime. This is precisely what Figure 2 expresses graphically.

LEMMA 9.6. *If m is not divisible by 6 then the quotient of W_3^5 by the free action of $H_{m;p,q}$ is homotopy equivalent to the lens space $L_m^5(p, 3p + q, 3p - q)$.*

Proof. Suppose first that m is not divisible by 3. The map

$$\varphi: W_3^5 \rightarrow \mathbb{S}^5, \quad (z_0, z_1, z_2, z_3) \mapsto \frac{1}{\sqrt{2(1 - |z_0|^2)}}(\sqrt{2}z_1, z_2 + iz_3, z_3 + iz_2)$$

is a three-to-one covering branched along the singular orbit of the $(\text{SO}(2) \times \text{SO}(3))$ -action on W_3^5 given by $z_0 = 0$. If we define a \mathbb{Z}_m -action on \mathbb{S}^5 by

$$\mathbb{Z}_m \times \mathbb{S}^5 \rightarrow \mathbb{S}^5,$$

$$(j + m\mathbb{Z}, (z_1, z_2, z_3)) \mapsto (e^{i(2\pi j/m)3p} z_1, e^{i(2\pi j/m)(3p+q)} z_2, e^{i(2\pi j/m)(3p-q)} z_3),$$

then φ is \mathbb{Z}_m -equivariant. The orbit space of this \mathbb{Z}_m -action on \mathbb{S}^5 is a lens space generally denoted by $L_m^5(3p, 3p - q, 3p + q)$. Since m is not divisible by 3, there exists a positive integer r such that $3r \equiv 1$ modulo m . Identify \mathbb{S}^5 homeomorphically with the join $\mathbb{S}^1 * \mathbb{S}^3$ and consider the continuous map $\rho: \mathbb{S}^5 \rightarrow \mathbb{S}^5$ of degree r induced by the map $\mathbb{S}^1 * \mathbb{S}^3 \rightarrow \mathbb{S}^1 * \mathbb{S}^3, (\lambda, w) \mapsto (\lambda^r, w)$. We now obtain the commutative diagram

$$\begin{array}{ccccc} W_3^5 & \xrightarrow{\varphi} & \mathbb{S}^5 & \xrightarrow{\rho} & \mathbb{S}^5 \\ \downarrow & & \downarrow & & \downarrow \\ W_3^5/H_{m;p,q} & \longrightarrow & L_m^5(3p, 3p - q, 3p + q) & \longrightarrow & L_m^5(3rp, 3p - q, 3p + q) \\ & & & & = L_m^5(p, 3p - q, 3p + q), \end{array} \tag{13}$$

which includes a map of degree $3r$ between $W_3^5/H_{m;p,q}$ and $L_m^5(p, 3p - q, 3p + q)$. By [O, Thm. 4], these two spaces are homotopy equivalent. In the case where m is not divisible by 2 we can proceed similarly by exchanging the role of z_0 and z_1 in the definition of φ . □

COROLLARY 9.7. *$W_3^5/H_{7;1,0}$ and $W_3^5/H_{7;1,1}$ are not homotopy equivalent.*

Proof. This follows from the homotopy classification of lens spaces; see [O]. □

COROLLARY 9.8. *If m is not divisible by 6, then the quotient of Σ^7 by the free action of $H_{m;p,q}$ is homotopy equivalent to the lens space $L_m^7(p, p, 3p - q, 3p + q)$.*

Proof. By Corollary 4.4, Σ^7 is equivariantly homeomorphic to the join $\mathbb{S}^1 * W_3^5$. We obtain the statement by joining all spaces in the diagram (13) with \mathbb{S}^1 (note that $\psi_{m;p,q}$ acts on the circle $\mathbb{S}^1 \subset \Sigma^7$ as the rotation $D(\frac{2\pi}{m} p)$). □

The condition “ m is not divisible by 6” seems to be a technical artifact.

COROLLARY 9.9. *$\Sigma^7/H_{5;1,0}$ and $\Sigma^7/H_{5;1,1}$ are not homotopy equivalent.*

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C. Durán
IMECC-UNICAMP
Cidade Universitária
13083-859 Campinas, SP
Brasil
cduran@ime.unicamp.br

T. Püttmann
Mathematisches Institut
Universität Bonn
D-53115 Bonn
Germany
puttmann@math.uni-bonn.de