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A Characterization of Besov-type Spaces and Applications to Hankel-type Operators

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Introduction

Let $\mathbb{D}$ be the unit disc of the complex plane. Given a real number β, let

$$dA_\beta(z) = (1+\beta)(1-|z|^2)^\beta\, dA(z),$$

where dA is the normalized area measure on $\mathbb{D}$. For $\beta > -1$ and $0 < p < \infty$, the Bergman space A^p_β consists of all analytic functions in $L^p(dA_\beta) := L^p(\mathbb{D}, dA_\beta)$ with norm

$$\|f\|^p_{A^p_\beta} = \int_{\mathbb{D}} |f(z)|^p\, dA_\beta(z).$$

For $1 < p < \infty$ and $\alpha \le 1/2$, let $B_p(\alpha)$ be the Besov-type space of those analytic functions on the unit disc $\mathbb{D}$ for which

$$\|f\|^p_{\alpha,p} = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p\, dA_{p,\alpha}(z) < \infty,$$

where

$$dA_{p,\alpha}(z) = (1-|z|^2)^{p-2+1-2\alpha}\, dA(z).$$

The space L^p_α is the space of smooth functions $u\colon \mathbb{D} \to \mathbb{C}$ for which

$$\|u\|^p_{\alpha,p} = |u(0)|^p + \int_{\mathbb{D}} |\nabla u(z)|^p\, dA_{p,\alpha}(z)$$

is finite. It is clear that $B_p(\alpha)$ is the subspace of all analytic functions in L^p_α. Note that the dual space of $B_p(\alpha)$ is isomorphic to $B_q(\alpha)$, where q is the conjugate exponent of p, under the pairing

$$\langle f, g\rangle_\alpha = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)}(1-|z|^2)^{1-2\alpha}\, dA(z),$$

defined for $f \in B_p(\alpha)$ and $g \in B_q(\alpha)$. Note that, by Hölder's inequality, if $f \in B_p(\alpha)$ then $f' \in A^1_{1-2\alpha}$. So, using the reproducing formula for the Bergman space

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and then integrating along the line segment joining 0 and z, we get the following reproducing formula for a function f in $B_p(\alpha)$ (see [8] for the case $p = 2$).

$$f(z) = f(0) + \int_{\mathbb{D}} f'(w)K(z,w)(1-|w|^2)^{1-2\alpha}\, dA(w), \tag{1}$$

where

$$K(z,w) = \frac{1-(1-\bar{w}z)^{2-2\alpha}}{\bar{w}(1-\bar{w}z)^{2-2\alpha}}.$$

Then the operator P_α given by

$$P_\alpha u(z) = u(0) + \int_{\mathbb{D}} \frac{\partial u}{\partial w}(w)K(z,w)(1-|w|^2)^{1-2\alpha}\, dA(w)$$

defines a projection from L^p_α to $B_p(\alpha)$. Let $\mathcal{P}$ denote the set of all polynomials on $\mathbb{D}$. Clearly $\mathcal{P}$ is dense in $B_p(\alpha)$. For a function $f \in L^p_\alpha$ we can define the (small) Hankel-type operator h^α_f with symbol f on $\mathcal{P}$ by

$$h^\alpha_f(g) = \overline{P_\alpha(f\bar{g})}.$$

When we say that h^α_f is bounded, we mean that there is a positive constant C such that

$$\|h^\alpha_f(g)\|_{\alpha,p} \le C\|g\|_{\alpha,p}$$

whenever $g \in \mathcal{P}$.

The purpose of this paper is to generalize the results given in [8] for the space $D_\alpha := B_2(\alpha)$ to all p with $1 < p < \infty$. We begin with a characterization of $B_p(\alpha)$ spaces that does not use derivatives by proving that an analytic function f is in $B_p(\alpha)$ if and only if the double integral

$$\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|f(z)-f(w)|^p}{|1-\bar{w}z|^{3+\sigma+\tau+2\alpha}}(1-|z|^2)^\sigma(1-|w|^2)^\tau\, dA(z)\, dA(w)$$

is finite. Here $\sigma, \tau > -1$ with $\min(\sigma,\tau) + 2\alpha > -1$. As in [8], we give some applications of the previous result. The first one is that (α, p)-Carleson measures are stable under transformation by some integral operators. Another application is the fact that the Hankel-type operator h^α_f is bounded if and only if the symbol f belongs to $W_{p,\alpha}$, the space of all analytic functions g such that $|g'(z)|^p\, dA_{p,\alpha}$ is an (α, p)-Carleson measure.

The paper is organized as follows. In Section 1 we give the preliminaries needed for the rest of the paper. In Section 2 we prove the characterization of $B_p(\alpha)$ spaces, a result that is applied in Section 3 to study (α, p)-Carleson measures. In Section 4 we study the Hankel-type operators h^α_f, and a decomposition-type theorem is given in Section 5.

Throughout the paper, we use the symbol C to denote a positive constant that can change at different occurrences but will not depend on the function or the measure that we deal with. We use the notation $a \lesssim b$ to indicate that there is a constant $C > 0$ with $a \le Cb$, and we use the symbol $\approx$ to mean "comparable to". Also, when $2\alpha < p$, we use the notation $A^p_{p,\alpha}$ for the Bergman space $A^p_{p-1-2\alpha}$.

1. Background and Preliminaries

The following standard lemma can be found in [10, Sec. 4.2].

LEMMA A. *Let $z \in \mathbb{D}$, $t > -1$, and $c > 0$. Then*

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^t}{|1-\bar{w}z|^{2+t+c}}\, dA(w) \approx (1-|z|^2)^{-c}.$$

The following useful inequality is from [5, Lemma 2.5]. A proof of this result can be found in [9].

LEMMA B. *Let $s > -1$, $r, t > 0$, and $r + t - s > 2$. If $t < s + 2 < r$, then*

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^s}{|1-\bar{w}z|^r |1-\bar{w}\zeta|^t}\, dA(w) \le C \frac{(1-|z|^2)^{2+s-r}}{|1-\bar{\zeta}z|^t}.$$

The following result is from [4, Thm. 1.9].

LEMMA C. *Let s and β be real numbers, and let T be the integral operator defined by*

$$Tg(z) = T_g(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)^s}{|1-\bar{w}z|^{2+s}} g(w)\, dA(w).$$

Let $1 \le p < \infty$. Then T is bounded on $L^p(dA_\beta)$ if and only if

$$0 < 1 + \beta < p(s+1).$$

The hyperbolic distance on $\mathbb{D}$ is defined by

$$d(z,w) = \log \frac{1 + \left|\dfrac{w-z}{1-\bar{w}z}\right|}{1 - \left|\dfrac{w-z}{1-\bar{w}z}\right|}.$$

A sequence $\{z_j\}$ in $\mathbb{D}$ is called a *d-lattice* if every point of $\mathbb{D}$ is within hyperbolic distance $5d$ of some z_j and no two points of this sequence are within hyperbolic distance $d/5$ of each other. The following result can be found in [6, Thm. 2.2].

THEOREM D. *Let $1 \le p < \infty$, $\beta > -1$, and $1+s > (1+\beta)/p$. Then there exists $d_0 > 0$ such that for any d-lattice $\{z_j\}$ in $\mathbb{D}$, $0 < d < d_0$, the following statements hold.*

(a) *If $f \in A^p_\beta$, then*

$$f(z) = \sum_{j=0}^{\infty} a_j \frac{(1-|z_j|^2)^{2+s-(2+\beta)/p}}{(1-\bar{z}_j z)^{2+s}}, \tag{2}$$

with

$$\sum_{j=0}^{\infty} |a_j|^p \le C \|f\|^p_{A^p_\beta}.$$

(b) *Conversely, if* $\{a_j\}$ *satisfies* $\sum_{j=0}^{\infty}|a_j|^p < \infty$ *then* f, *defined by* (2), *converges in* A^p_β *with*

$$\|f\|^p_{A^p_\beta} \le C\sum_{j=0}^{\infty}|a_j|^p.$$

2. A Derivative-free Characterization of $B_p(\alpha)$

We begin this section with a result that is of independent interest.

LEMMA 2.1. *Let* $1 < p < \infty$, *and let* $\sigma > -1$ *and* $b \ge 0$ *with* $b < 2+\sigma$. *Let* f *be analytic on* $\mathbb{D}$. *Then*

$$\int_{\mathbb{D}}|f(z)-f(0)|^p\frac{(1-|z|^2)^{\sigma}}{|1-\bar{\zeta}z|^b}\,dA(z) \lesssim \int_{\mathbb{D}}|f'(z)|^p\frac{(1-|z|^2)^{p+\sigma}}{|1-\bar{\zeta}z|^b}\,dA(z). \quad (3)$$

Proof. The case $b=0$ is proved in [4]. So assume that $b>0$. Choose $\varepsilon > 0$ with

$$\sigma - \varepsilon\max(1,p-1) > -1 \quad \text{and} \quad b+\varepsilon(p-1) < 2+\sigma.$$

Without loss of generality we may assume that the right-hand side of (3) is finite. Then, it follows from Hölder's inequality that $f' \in A^1_{1+\sigma}$. Hence the reproducing formula (1) gives

$$f(z)-f(0) = \int_{\mathbb{D}}\frac{1-(1-\bar{w}z)^{2+\sigma}}{\bar{w}(1-\bar{w}z)^{2+\sigma}}f'(w)(1-|w|^2)^{1+\sigma}\,dA(w).$$

Since

$$\sup_{z,w\in\mathbb{D}}\left|\frac{(1-\bar{w}z)^{2+\sigma}-1}{\bar{w}}\right| \le C,$$

by Hölder's inequality and Lemma A we have

$$\begin{aligned}
&|f(z)-f(0)|^p\\
&\lesssim \left(\int_{\mathbb{D}}\frac{|f'(w)|(1-|w|^2)^{1+\sigma}}{|1-\bar{w}z|^{2+\sigma}}\,dA(w)\right)^p\\
&\le \left(\int_{\mathbb{D}}|f'(w)|^p\frac{(1-|w|^2)^{(1+\varepsilon)p+\sigma-\varepsilon}}{|1-\bar{w}z|^{2+\sigma}}\,dA(w)\right)\left(\int_{\mathbb{D}}\frac{(1-|w|^2)^{\sigma-\varepsilon}}{|1-\bar{w}z|^{2+\sigma}}\,dA(w)\right)^{p-1}\\
&\lesssim \left(\int_{\mathbb{D}}|f'(w)|^p\frac{(1-|w|^2)^{(1+\varepsilon)p+\sigma-\varepsilon}}{|1-\bar{w}z|^{2+\sigma}}\,dA(w)\right)(1-|z|^2)^{-\varepsilon(p-1)}
\end{aligned}$$

since $\sigma-\varepsilon > -1$. Now, by Fubini's theorem and Lemma B we have

$$\begin{aligned}
&\int_{\mathbb{D}}|f(z)-f(0)|^p\frac{(1-|z|^2)^{\sigma}}{|1-\bar{\zeta}z|^b}\,dA(z)\\
&\lesssim \int_{\mathbb{D}}\left(\int_{\mathbb{D}}|f'(w)|^p\frac{(1-|w|^2)^{(1+\varepsilon)p+\sigma-\varepsilon}}{|1-\bar{w}z|^{2+\sigma}}\,dA(w)\right)\frac{(1-|z|^2)^{-\varepsilon(p-1)+\sigma}}{|1-\bar{\zeta}z|^b}\,dA(z)\\
&= \int_{\mathbb{D}}|f'(w)|^p(1-|w|^2)^{(1+\varepsilon)p+\sigma-\varepsilon}\left(\int_{\mathbb{D}}\frac{(1-|z|^2)^{-\varepsilon(p-1)+\sigma}}{|1-\bar{w}z|^{2+\sigma}|1-\bar{\zeta}z|^b}\,dA(z)\right)dA(w)\\
&\lesssim \int_{\mathbb{D}}(1-|w|^2)^p|f'(w)|^p\frac{(1-|w|^2)^{\sigma}}{|1-\bar{\zeta}w|^b}\,dA(w),
\end{aligned}$$

and this finishes the proof. □

The following derivative-free characterization of $B_p(\alpha)$ is a generalization of a result in [8], where the case $p = 2$ was proved. Our proof is quite different from [8], where Hilbert space techniques were used.

THEOREM 2.2. *Let* $1 < p < \infty$ *and let* f *be analytic on* $\mathbb{D}$. *Let* $\sigma, \tau > -1$ *and* $\alpha \le 1/2$ *such that* $\min(\sigma, \tau) + 2\alpha > -1$. *Then*

$$\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|f(z) - f(w)|^p}{|1 - \bar{w}z|^{3+\sigma+\tau+2\alpha}} (1 - |w|^2)^\sigma (1 - |z|^2)^\tau \, dA(z)\, dA(w)$$

is comparable to

$$\int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-1-2\alpha} \, dA(z).$$

Proof. We first prove the upper estimate. As in [8] we may assume that $\sigma = \tau$. Since

$$1 - |\varphi_w(z)|^2 = \frac{(1 - |w|^2)(1 - |z|^2)}{|1 - \bar{w}z|^2} \quad \text{and} \quad |\varphi_w'(z)| = \frac{1 - |\varphi_w(z)|^2}{1 - |z|^2},$$

a change of variables $\zeta = \varphi_w(z)$ and Lemma 2.1 (we can apply it since $\sigma + 2\alpha > -1$) gives

$$\begin{aligned}
&\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|f(z) - f(w)|^p}{|1 - \bar{w}z|^{3+2\sigma+2\alpha}} (1 - |w|^2)^\sigma (1 - |z|^2)^\sigma \, dA(z)\, dA(w) \\
&\quad = \int_{\mathbb{D}} (1 - |w|^2)^{-1-2\alpha} \int_{\mathbb{D}} |(f \circ \varphi_w)(\zeta) \\
&\qquad\qquad - (f \circ \varphi_w)(0)|^p \frac{(1 - |\zeta|^2)^\sigma}{|1 - \bar{w}\zeta|^{1-2\alpha}} \, dA(\zeta)\, dA(w) \\
&\quad \le C \int_{\mathbb{D}} (1 - |w|^2)^{-1-2\alpha} \int_{\mathbb{D}} |(f \circ \varphi_w)'(\zeta)|^p \frac{(1 - |\zeta|^2)^{p+\sigma}}{|1 - \bar{w}\zeta|^{1-2\alpha}} \, dA(\zeta)\, dA(w),
\end{aligned}$$

and, by the change of variables $z = \varphi_w(\zeta)$ and Fubini's theorem, this quantity equals

$$\begin{aligned}
&\int_{\mathbb{D}} (1 - |w|^2)^\sigma \int_{\mathbb{D}} |f'(z)|^p \frac{(1 - |z|^2)^{p+\sigma}}{|1 - \bar{w}z|^{3+2\sigma+2\alpha}} \, dA(z)\, dA(w) \\
&\quad = \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p+\sigma} \left(\int_{\mathbb{D}} \frac{(1 - |w|^2)^\sigma}{|1 - \bar{w}z|^{3+2\sigma+2\alpha}} \, dA(w) \right) dA(z) \\
&\quad \le C \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-1-2\alpha} \, dA(z)
\end{aligned}$$

after an application of Lemma A, and this proves the upper estimate.

Now, we are going to prove the lower estimate. First note that, using Cauchy's integral formula, it is easy to see that

$$|f'(0)| \lesssim \int_{|w|<1/2} |f(w) - f(0)| \, dA(w).$$

Therefore, for $1 < p < \infty$ and $\beta > 0$, we have

$$|f'(0)|^p \lesssim \int_{\mathbb{D}} |f(w) - f(0)|^p \, dA_\beta(w).$$

Replacing f by $f \circ \varphi_z$ we get

$$(1 - |z|^2)^p |f'(z)|^p \lesssim \int_{\mathbb{D}} |f \circ \varphi_z(w) - f(z)|^p \, dA_\beta(w). \tag{4}$$

Choose $\beta = 1 + \sigma$. Then, by (4) and the change of variables $w = \varphi_z(\zeta)$ we get

$$\begin{aligned}
&\int_{\mathbb{D}} (1 - |z|^2)^p |f'(z)|^p (1 - |z|^2)^{-1-2\alpha} \, dA(z) \\
&\quad \lesssim \int_{\mathbb{D}} \frac{dA(z)}{(1 - |z|^2)^{1+2\alpha}} \int_{\mathbb{D}} |f \circ \varphi_z(w) - f(z)|^p \, dA_\beta(w) \\
&\quad = \int_{\mathbb{D}} \frac{dA(z)}{(1 - |z|^2)^{1+2\alpha}} \int_{\mathbb{D}} |f(\zeta) - f(z)|^p |\varphi_z'(\zeta)|^2 (1 - |\varphi_z(\zeta)|^2)^\beta \, dA(\zeta) \\
&\quad = \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(\zeta) - f(z)|^p}{|1 - \bar{z}\zeta|^{3+2\sigma+2\alpha}} (1 - |z|^2)^\sigma (1 - |\zeta|^2)^\sigma \\
&\qquad \times \frac{(1 - |z|^2)^{2-2\alpha}(1 - |\zeta|^2)}{|1 - \bar{z}\zeta|^{3-2\alpha}} \, dA(z) \, dA(\zeta) \\
&\quad \lesssim \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(\zeta) - f(z)|^p}{|1 - \bar{z}\zeta|^{3+2\sigma+2\alpha}} (1 - |z|^2)^\sigma (1 - |\zeta|^2)^\sigma \, dA(z) \, dA(\zeta),
\end{aligned}$$

and this finishes the proof. □

3. Carleson Measures for $B_p(\alpha)$

A positive measure μ on $\mathbb{D}$ is an (α, p)-Carleson measure if

$$\int_{\mathbb{D}} |f|^p \, d\mu \le C \|f\|_{\alpha,p}^p$$

whenever f is in $B_p(\alpha)$. The best constant C, denoted by $\|\mu\|_{p,\alpha}$, is said to be the (α, p)-Carleson measure norm of μ.

The (α, p)-Carleson measures are described in [1], but for our purposes we need only the following simple result.

LEMMA 3.1. *Let* $1 < p < \infty$ *and* $\alpha \le 1/2$. *Let* μ *be an* (α, p)*-Carleson measure. Then for each* $\varepsilon > 0$,

$$\sup_{w \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |w|^2)^\varepsilon}{|1 - \bar{w}z|^{1+\varepsilon-2\alpha}} \, d\mu(z) < \infty.$$

Proof. Let

$$g_w(z) = \frac{(1 - |w|^2)^{\varepsilon/p}}{(1 - \bar{w}z)^{(1+\varepsilon-2\alpha)/p}}.$$

We have that $g_w \in B_p(\alpha)$ with $\|g_w\|_{\alpha,p}^p \le C$, where C is a positive constant independent of w. Therefore

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^\varepsilon}{|1-\bar{w}z|^{1+\varepsilon-2\alpha}}\, d\mu(z) = \int_{\mathbb{D}} |g_w(z)|^p \, d\mu(z) \le C\|g_w\|_{\alpha,p}^p \le C. \qquad \square$$

THEOREM 3.2. *Let $s > -1$ and $\alpha \le 1/2$, and let*

$$T_g(z) = \int_{\mathbb{D}} \frac{g(w)}{|1-\bar{w}z|^{2+s}}(1-|w|^2)^s \, dA(w).$$

(i) *Let $1 < p < 2$, and let $ps > -2\alpha(p-1)$ if $\alpha \ge 0$ and $s > -2\alpha$ if $\alpha < 0$. Suppose that*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|g(z)| < \infty. \tag{5}$$

If $|g(z)|^p(1-|z|^2)^{p-1-2\alpha}\, dA(z)$ is an (α, p)-Carleson measure, then the measure $|T_g(z)|^p(1-|z|^2)^{p-1-2\alpha}\, dA(z)$ is also an (α, p)-Carleson measure.

(ii) *Let $p \ge 2$, $\beta > -1$, $\beta + 2\alpha > -1$, and*

$$ps > 1 + \beta - p + \max(0, -2\alpha).$$

If $|g(z)|^p(1-|z|^2)^\beta \, dA(z)$ is an (α, p)-Carleson measure, then it follows that $|T_g(z)|^p(1-|z|^2)^\beta \, dA(z)$ is also an (α, p)-Carleson measure.

REMARKS. For $p = 2$, this result is proved by Rochberg and Wu in [8]. The case $\alpha = 1/2$ and $\beta = p-2$ is proved in [2] but only for the range $ps > 1$. Also, when $\alpha \ge 0$, the condition on s can be rewritten as

$$\max(p, q)s > \beta - (p-1).$$

For $p \ge 2$, we don't need condition (5). Also, for $1 < p < 2$, the result is obtained only for $\beta = p - 2 + (1-2\alpha)$. It would be interesting to extend the result for β in the same range as in the case $p \ge 2$.

Proof of Theorem 3.2. We must show that for all $f \in B_p(\alpha)$ we have

$$\int_{\mathbb{D}} |f(z)|^p |T_g(z)|^p (1-|z|^2)^\beta \, dA(z) \le C\|f\|_{\alpha,p}^p$$

for some positive constant C. Put $fT_g = (fT_g - T_{fg}) + T_{fg}$. By Lemma C and the fact that $|g(z)|^p(1-|z|^2)^\beta$ is an (α, p)-Carleson measure, we have

$$\int_{\mathbb{D}} |T_{fg}(z)|^p(1-|z|^2)^\beta \, dA(z) \lesssim \int_{\mathbb{D}} |f(z)|^p|g(z)|^p(1-|z|^2)^\beta \, dA(z) \le C\|f\|_{\alpha,p}^p.$$

On the other hand, we have

$$f(z)T_g(z) - T_{fg}(z) = \int_{\mathbb{D}} \frac{(f(z)-f(w))g(w)}{|1-\bar{w}z|^{2+s}}(1-|w|^2)^s \, dA(w). \tag{6}$$

Now we consider separately the cases $1 < p < 2$ and $p \ge 2$.

Case 1: $1 < p < 2$. In this case $\beta = p - 1 - 2\alpha$, and since $(1 - |z|^2)|g(z)| \leq C$ and $p < 2$ we have

$$|g(z)| = |g(z)|^{p-1}|g(z)|^{2-p} \leq C|g(z)|^{p-1}(1 - |z|^2)^{p-2}.$$

This gives

$$|f(z)T_g(z) - T_{fg}(z)|^p \lesssim \left(\int_{\mathbb{D}} \frac{|f(z) - f(w)|}{|1 - \bar{w}z|^{2+s}}|g(w)|^{p-1}(1 - |w|^2)^{p-2+s}\, dA(w)\right)^p.$$

If $\alpha = 1/2$, then by Hölder's inequality

$$\left(\int_{\mathbb{D}} \frac{|f(z) - f(w)|}{|1 - \bar{w}z|^{2+s}}|g(w)|^{p-1}(1 - |w|^2)^{p-2+s}\, dA(w)\right)^p \leq \|g\|^{p(p-1)}_{L^p(dA_\beta)} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^p}{|1 - \bar{w}z|^{(2+s)p}}(1 - |w|^2)^{ps+(p-2)}\, dA(w).$$

Hence, by (6) and Theorem 2.2 with $\sigma = ps + (p - 2)$ and $\tau = p - 2$,

$$\begin{aligned}\int_{\mathbb{D}} &|f(z)T_g(z) - T_{fg}(z)|^p(1 - |z|^2)^{p-1-2\alpha}\, dA(z)\\ &\leq C\|g\|^{p(p-1)}_{L^p(dA_\beta)} \int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|f(z) - f(w)|^p}{|1 - \bar{w}z|^{(2+s)p}}(1 - |z|^2)^{\sigma}(1 - |w|^2)^{\tau}\, dA(z)\, dA(w)\\ &\leq C\|g\|^{p(p-1)}_{L^p(dA_\beta)}\|f\|^p_{\alpha,p}.\end{aligned}$$

If $\alpha < 1/2$, we apply Hölder's inequality again, and then Lemma 3.1, to obtain

$$\begin{aligned}&\left(\int_{\mathbb{D}} \frac{|f(z) - f(w)|}{|1 - \bar{w}z|^{2+s}}|g(w)|^{p-1}(1 - |w|^2)^{p-2+s}\, dA(w)\right)^p\\ &\quad\leq \int_{\mathbb{D}} \frac{|f(z) - f(w)|^p}{|1 - \bar{w}z|^{p(1+s)+1}}(1 - |w|^2)^{ps-1+2\alpha(p-1)}\, dA(w)\\ &\quad\quad\times \left(\int_{\mathbb{D}} \frac{|g(w)|^p\, dA_{p,\alpha}(w)}{|1 - \bar{w}z|^{1+2\alpha-2\alpha}}\right)^{p-1}\\ &\quad\lesssim (1 - |z|^2)^{-2\alpha(p-1)} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^p}{|1 - \bar{w}z|^{p(1+s)+1}}(1 - |w|^2)^{ps-1+2\alpha(p-1)}\, dA(w).\end{aligned}$$

Therefore we have

$$\begin{aligned}\int_{\mathbb{D}} &|f(z)T_g(z) - T_{fg}(z)|^p(1 - |z|^2)^{p-1-2\alpha}\, dA(z)\\ &\lesssim \int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|f(z) - f(w)|^p}{|1 - \bar{w}z|^{p(1+s)+1}}(1 - |z|^2)^{\sigma}(1 - |w|^2)^{\tau}\, dA(z)\, dA(w),\end{aligned}$$

where $\sigma = (1 - 2\alpha)(p - 1) - 2\alpha$ and $\tau = ps - 1 + 2\alpha(p - 1)$.

Since $\alpha < 1/2$ we have $\sigma > -1$ and $\sigma + 2\alpha > -1$. Also, the conditions on s ensure that $\tau > -1$ and $\tau + 2\alpha > -1$. Since

$$3 + \sigma + \tau + 2\alpha = p(1 + s) + 1,$$

we can apply Theorem 2.2 to obtain

$$\int_{\mathbb{D}} |f(z)T_g(z) - T_{fg}(z)|^p(1 - |z|^2)^{p-1-2\alpha}\, dA(z) \lesssim \|f\|^p_{\alpha,p}.$$

Case 2: $p \geq 2$. Choose $\varepsilon > 0$ such that $\beta > -1 + \varepsilon(p-1) + \max(0, -2\alpha)$ and

$$ps > 1 + \beta - p + \varepsilon(p-2) + \max(0, -2\alpha).$$

Let q be the conjugate exponent of p. By Hölder's inequality we have

$$\begin{aligned}|f(z)T_g(z) - T_{fg}(z)|^p &\leq \left(\int_{\mathbb{D}} \frac{|g(w)||f(z)-f(w)|}{|1-\bar{w}z|^{2+s}}(1-|w|^2)^s\, dA(w)\right)^p \\ &\leq \left(\int_{\mathbb{D}} |g(w)|^q \frac{(1-|w|^2)^\gamma}{|1-\bar{w}z|^{B/(p-1)}}\, dA(w)\right)^{p/q} \\ &\quad \times \int_{\mathbb{D}} \frac{|f(z)-f(w)|^p}{|1-\bar{w}z|^C}(1-|w|^2)^\tau\, dA(w),\end{aligned}$$

where

$$\gamma = \frac{\beta - (1-\varepsilon)(p-2)}{p-1}, \qquad \tau = ps - \beta + (1-\varepsilon)(p-2),$$

$$A = 1 + 2\varepsilon, \qquad B = 1 + \varepsilon - 2\alpha + A(p-2),$$

$$C = (2+s)p - 1 - \varepsilon + 2\alpha - A(p-2).$$

Since $p \geq 2$, we can apply Hölder's inequality once again with exponent $p/q \geq 1$ and then apply Lemma 3.1 and Lemma A to obtain

$$\begin{aligned}&\left(\int_{\mathbb{D}} |g(w)|^q \frac{(1-|w|^2)^\gamma}{|1-\bar{w}z|^{B/(p-1)}}\, dA(w)\right)^{p/q} \\ &\quad \leq \left(\int_{\mathbb{D}} \frac{|g(w)|^p(1-|w|^2)^\beta}{|1-\bar{w}z|^{1+\varepsilon-2\alpha}}\, dA(w)\right)\left(\int_{\mathbb{D}} \frac{(1-|w|^2)^{-1+\varepsilon}}{|1-\bar{w}z|^{1+2\varepsilon}}\, dA(w)\right)^{p-2} \\ &\quad \lesssim (1-|z|^2)^{-\varepsilon}(1-|z|^2)^{-\varepsilon(p-2)}.\end{aligned}$$

Therefore, if $\sigma = \beta - \varepsilon(p-1)$, by Theorem 2.2 we have

$$\begin{aligned}&\int_{\mathbb{D}} |f(z)T_g(z) - T_{fg}(z)|^p(1-|z|^2)^\beta\, dA(z) \\ &\quad \lesssim \int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|f(z)-f(w)|^p}{|1-\bar{w}z|^C}(1-|z|^2)^\sigma(1-|w|^2)^\tau\, dA(z)\, dA(w) \\ &\quad \lesssim \|f\|_{\alpha,p}^p,\end{aligned}$$

since $3 + \sigma + \tau + 2\alpha = C$, and the choice of $\varepsilon > 0$ ensures that $\sigma, \tau > -1$ and $\min(\sigma, \tau) + 2\alpha > -1$. This finishes the proof. □

4. Hankel-type Operators

Let $s > -1$ and $1 \leq p < \infty$. Let

$$P_s f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-\bar{w}z)^{2+s}}\, dA_s(w).$$

From [4, Thm. 1.10], we have that $P_s \colon L^p(dA_\beta) \to A^p_\beta$ is a bounded projection (and onto) if and only if $\beta + 1 < (s+1)p$. For a function $f \in A^p_{p,\alpha}$ it is possible to define a (small) Hankel-type operator $h_{s,f}$ on $\mathcal{P}$ by

$$h_{s,f}(g) = \overline{P_s(f\bar{g})}, \quad g \in \mathcal{P}.$$

For $\alpha < 1$, define the space $W_{p,\alpha}$ to be the space of all analytic functions f on $\mathbb{D}$ for which

$$\|f\|_{W_{p,\alpha}} = \sup_{\|g\|_{\alpha,p} \le 1} \left(\int_{\mathbb{D}} |g(z)|^p |f'(z)|^p (1-|z|^2)^{p-1-2\alpha} \, dA(z) \right)^{1/p} < \infty.$$

LEMMA 4.1. *Let $1 < p < \infty$, $\alpha < p/2$, and $ps > -2\alpha$. Suppose that $u \in A^p_{p,\alpha}$ and that $h_{s,u}$ is bounded from $B_p(\alpha)$ to $L^p(dA_{p,\alpha})$. Then*

$$\sup_{a \in \mathbb{D}} (1-|a|^2)|u(a)| < \infty.$$

Proof. Choose $n \in \mathbb{N} \cup \{0\}$ with $np \ge -2\alpha$. For each $a \in \mathbb{D}$, consider the functions

$$f_a(z) = (1-|a|^2)^{(p-1+2\alpha+np)/p} \frac{z^{n+1}}{(1-\bar{a}z)^{n+1}},$$

$$a(z) = \frac{(1-|a|^2)^{(p+1-2\alpha)/p}(1-|z|^2)^{s-(p-1)+2\alpha}}{(1-\bar{a}z)^{2+s}}.$$

Since $p > 2\alpha$ and $ps > -2\alpha$, by Lemma A we have

$$\int_{\mathbb{D}} |f_a'(z)|^p \, dA_{p,\alpha}(z) = (n+1)(1-|a|^2)^{p-1+2\alpha+np} \int_{\mathbb{D}} \frac{(1-|z|^2)^{p-1-2\alpha}}{|1-\bar{a}z|^{(n+2)p}} \, dA(z) \approx 1$$

and, if q is the conjugate exponent of p,

$$\begin{aligned} &\int_{\mathbb{D}} |e_a(z)|^q \, dA_{p,\alpha}(z) \\ &\qquad = (1-|a|^2)^{(p+1-2\alpha)/(p-1)} \int_{\mathbb{D}} \frac{(1-|z|^2)^{qs-1+2\alpha(q-1)}}{|1-\bar{a}z|^{(2+s)q}} \, dA(z) \approx 1. \end{aligned}$$

Therefore, for any $a \in \mathbb{D}$, the function f_a is in $B_p(\alpha)$ with $\|f_a\|_{\alpha,p} \approx 1$ and the function e_a is in $L^q(dA_{p,\alpha})$ with $\|e_a\|_{L^q(dA_{p,\alpha})} \approx 1$. Since $ps > -2\alpha$, we have $u = P_s u$, and then

$$\begin{aligned} &(1-|a|^2)^{n+2} u^{(n+1)}(a) \\ &\quad = c_n (1-|a|^2)^{n+2} \int_{\mathbb{D}} \frac{\bar{z}^{n+1} u(z)}{(1-\bar{z}a)^{3+s+n}} (1-|z|^2)^s \, dA(z) \\ &\quad = c_n (1-|a|^2)^{n+2} \int_{\mathbb{D}} \frac{\bar{z}^{n+1} u(z)}{(1-\bar{z}a)^{n+1}} \left(\frac{1}{(1-\bar{z}a)^{2+s}} \right) (1-|z|^2)^s \, dA(z) \\ &\quad = c_n (1-|a|^2)^{n+2} \int_{\mathbb{D}} \frac{\bar{z}^{n+1} u(z)}{(1-\bar{z}a)^{n+1}} \left(\int_{\mathbb{D}} \frac{(1-|w|^2)^s \, dA(w)}{(1-\bar{z}w)^{2+s}(1-\bar{w}a)^{2+s}} \right) dA_s(z) \\ &\quad = c_n \int_{\mathbb{D}} \overline{h_{s,u}(f_a)(w) e_a(w)} \, dA_{p,\alpha}(w). \end{aligned}$$

Hence, by the boundedness of $h_{s,u}$,

$$\begin{aligned} (1-|a|^2)^{n+2} |u^{n+1}(a)| &\le C \|h_{s,u}(f_a)\|_{L^p(dA_{p,\alpha})} \|e_a\|_{L^q(dA_{p,\alpha})} \\ &\le C \|f_a\|_{\alpha,p} \|e_a\|_{L^q(dA_{p,\alpha})} \le C. \end{aligned}$$

Now, using the well-known fact that

$$\sup_{a\in\mathbb{D}}(1-|a|^2)|u(a)| \approx |u(0)| + \sum_{k=1}^{n}|u^{(k)}(0)| + \sup_{a\in\mathbb{D}}(1-|a|^2)^{n+2}|u^{(n+1)}(a)|,$$

the proof is complete. $\square$

THEOREM 4.2. *Let $1 < p < \infty$ and $\alpha \le 1/2$, and let s with $ps > -1$ if $\alpha = 1/2$ and $s > \max(0, -2\alpha)$ if $\alpha < 1/2$. Let u be analytic on $\mathbb{D}$. Then the operator $h_{s,u}$ is bounded from $B_p(\alpha)$ to $L^p(dA_{p,\alpha})$ if and only if the measure $|u(z)|^p\, dA_{p,\alpha}(z)$ is an (α, p)-Carleson measure.*

Proof. Suppose first that u is such that $|u(z)|^p\, dA_{p,\alpha}$ is an (α, p)-Carleson measure, and let $g \in B_p(\alpha)$. Then $u\bar{g} \in L^p(dA_{p,\alpha})$. By Lemma C with $\beta = p-1-2\alpha$ we have $h_{s,u}(g) \in L^p(dA_{p,\alpha})$, and

$$\|h_{s,u}(g)\|_{L^p(dA_{p,\alpha})} \le C\|u\bar{g}\|_{L^p(dA_{p,\alpha})} \le C\|g\|_{\alpha,p}.$$

This implies that $h_{s,u}$ is bounded from $B_p(\alpha)$ to $L^p(dA_{p,\alpha})$.

To prove the converse, let u be analytic on $\mathbb{D}$. We need to show

$$\|ug\|_{L^p(dA_{p,\alpha})} \le C\|g\|_{\alpha,p} \quad \forall g \in B_p(\alpha). \tag{7}$$

By the density of the polynomials in $B_p(\alpha)$, it is enough to prove (7) when g is in $\mathcal{P}$. Note that $\bar{u} = h_{s,u}(1) \in L^p(dA_{p,\alpha})$. Hence $u \in A^p_{p,\alpha}$, and the conditions on s imply that $u = P_s u$. Using the idea of the proof of Theorem 3.2, we study the difference

$$u(z)\overline{g(z)} - \overline{h_{s,u}(g)(z)} = \int_{\mathbb{D}} \frac{u(w)(\overline{g(z)-g(w)})}{(1-\bar{w}z)^{2+s}}(1-|w|^2)^s\, dA(w).$$

Since $h_{s,u}$ is bounded, we only need to show that the $L^p(dA_{p,\alpha})$ norm of this difference is dominated by the $B_p(\alpha)$ norm of g. From now, we use the notation $B(u)$ to mean the quantity $\sup_{z\in\mathbb{D}}(1-|z|^2)|u(z)|$.

If $\alpha = 1/2$, then $dA_{p,\alpha}(z) = (1-|z|^2)^{p-2}\, dA(z)$. If $1 < p < 2$, then by Hölder's inequality

$$\begin{aligned}
&\left|\int_{\mathbb{D}} \frac{u(w)(\overline{g(z)-g(w)})}{(1-\bar{w}z)^{2+s}}(1-|w|^2)^s\, dA(w)\right|^p \\
&\quad \le CB(u)^{(2-p)/p}\left(\int_{\mathbb{D}} \frac{|u(w)|^{p-1}|g(z)-g(w)|}{|1-\bar{w}z|^{2+s}}(1-|w|^2)^{p-2+s}\, dA(w)\right)^p \\
&\quad \le CB(u)^{(2-p)/p}\left(\int_{\mathbb{D}} |u(w)|^p(1-|w|^2)^{p-2}\, dA(z)\right)^{p-1} \\
&\qquad \times \int_{\mathbb{D}} \frac{|g(z)-g(w)|^p}{|1-\bar{w}z|^{(2+s)p}}(1-|w|^2)^{p-2+ps}\, dA(w).
\end{aligned}$$

Hence, by Theorem 2.2 with $\sigma = p-2$ and $\tau = ps+p-2$, we have

$$
\begin{aligned}
&\|u\bar{g} - \overline{h_{s,u}(g)}\|^p_{L^p(dA_{p,\alpha})} \\
&\qquad \le CB(u)^{(2-p)/p}\|u\|^{p(p-1)}_{L^p(dA_{p,\alpha})} \\
&\qquad\qquad \times \int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|g(z)-g(w)|^p}{|1-\bar{w}z|^{(2+s)p}}(1-|w|^2)^{ps+p-2}\,dA_{p,\alpha}(z)\,dA(w) \\
&\qquad \le CB(u)^{(2-p)/p}\|u\|^{p(p-1)}_{L^p(dA_{p,\alpha})}\|g\|^p_{\alpha,p}.
\end{aligned}
$$

When $\alpha = 1/2$ and $p \ge 2$ we apply Hölder's inequality two times and then Lemma A to see that

$$
\begin{aligned}
&\left|\int_{\mathbb{D}} \frac{u(w)(\overline{g(z)-g(w)})}{(1-\bar{w}z)^{2+s}}(1-|w|^2)^s\,dA(w)\right|^p \\
&\quad \le \left(\int_{\mathbb{D}} \frac{|u(w)|^q(1-|w|^2)^{\varepsilon q(p-2)/p}\,dA(w)}{|1-\bar{w}z|^{(q/p)(1+2\varepsilon)(p-2)}}\right)^{p/q} \\
&\qquad \times \int_{\mathbb{D}} |g(z)-g(w)|^p \frac{(1-|w|^2)^{ps-\varepsilon(p-2)}}{|1-\bar{w}z|^{p+2+ps-2\varepsilon(p-2)}}\,dA(w) \\
&\quad \le \|u\|^p_{A^p_{p,\alpha}}\left(\int_{\mathbb{D}} \frac{(1-|w|^2)^{-1+\varepsilon}}{|1-\bar{w}z|^{1+2\varepsilon}}\,dA(w)\right)^{p-2} \\
&\qquad \times \int_{\mathbb{D}} |g(z)-g(w)|^p \frac{(1-|w|^2)^{ps-\varepsilon(p-2)}}{|1-\bar{w}z|^{p+2+ps-2\varepsilon(p-2)}}\,dA(w) \\
&\quad \lesssim \|u\|^p_{A^p_{p,\alpha}}(1-|z|^2)^{-\varepsilon(p-2)}\int_{\mathbb{D}} |g(z)-g(w)|^p \frac{(1-|w|^2)^{ps-\varepsilon(p-2)}}{|1-\bar{w}z|^{p+2+ps-2\varepsilon(p-2)}}\,dA(w).
\end{aligned}
$$

Hence, by Theorem 2.2 with $\sigma = (1-\varepsilon)(p-2)$ and $\tau = ps - \varepsilon(p-2)$, we have

$$
\begin{aligned}
&\|u\bar{g} - \overline{h_{s,u}(g)}\|^p_{L^p(dA_{p,\alpha})} \\
&\quad \lesssim \|u\|^p_{A^p_{p,\alpha}} \\
&\qquad \times \int_{\mathbb{D}}\int_{\mathbb{D}} |g(z)-g(w)|^p \frac{(1-|w|^2)^{ps-\varepsilon(p-2)}(1-|z|^2)^{(1-\varepsilon)(p-2)}}{|1-\bar{w}z|^{p+2+ps-2\varepsilon(p-2)}}\,dA(w)\,dA(z) \\
&\quad \lesssim \|u\|^p_{A^p_{p,\alpha}}\|g\|^p_{\alpha,p}.
\end{aligned}
$$

If $\alpha < 1/2$, then applying again Hölder's inequality and then Lemma A yields

$$
\begin{aligned}
&\left|\int_{\mathbb{D}} \frac{u(w)(\overline{g(z)-g(w)})}{(1-\bar{w}z)^{2+s}}(1-|w|^2)^s\,dA(w)\right|^p \\
&\qquad \le \left(\int_{\mathbb{D}} \frac{|u(w)|^q}{|1-\bar{w}z|^{2+s}}(1-|w|^2)^{s+q-1}\,dA(w)\right)^{p-1} \\
&\qquad\quad \times \int_{\mathbb{D}} \frac{|g(z)-g(w)|^p}{|1-\bar{w}z|^{2+s}}(1-|w|^2)^{s-1}\,dA(w) \le
\end{aligned}
$$

$$\le CB(u)^p\bigg(\int_{\mathbb{D}}\frac{(1-|w|^2)^{s-1}}{|1-\bar{w}z|^{2+s}}\,dA(w)\bigg)^{p-1}$$
$$\times\int_{\mathbb{D}}\frac{|g(z)-g(w)|^p}{|1-\bar{w}z|^{2+s}}(1-|w|^2)^{s-1}\,dA(w)$$
$$\le CB(u)^p(1-|z|^2)^{-(p-1)}\int_{\mathbb{D}}\frac{|g(z)-g(w)|^p}{|1-\bar{w}z|^{2+s}}(1-|w|^2)^{s-1}\,dA(w).$$

Therefore

$$\|u\bar{g}-\overline{h_{s,u}(g)}\|^p_{L^p(dA_{p,\alpha})}$$
$$\le CB(u)^p\int_{\mathbb{D}}\int_{\mathbb{D}}\frac{|g(z)-g(w)|^p}{|1-\bar{w}z|^{2+s}}(1-|w|^2)^{s-1}(1-|z|^2)^{-2\alpha}\,dA(z)\,dA(w)$$
$$\le CB(u)^p\|g\|^p_{\alpha,p}.$$

The last inequality is obtained from Theorem 2.2 with $\sigma = s-1$ and $\tau = -2\alpha$. It follows from Lemma 4.1 that $B(u)$ is finite. Thus the proof is complete. □

COROLLARY 4.3. *Let $1 < p < \infty$ and $\alpha \le 1/2$. Let f be analytic on $\mathbb{D}$. Then h^α_f is bounded if and only if $f \in W_{p,\alpha}$.*

Proof. Let $s = 1-2\alpha$ and $u = f'$. Then we have

$$\frac{\partial}{\partial z}(h^\alpha_f(g))(z) = 0$$

and

$$\frac{\partial}{\partial\bar{z}}(h^\alpha_f(g))(z) = \overline{(2-2\alpha)\int_{\mathbb{D}}\frac{f'(w)\overline{g(w)}}{(1-\bar{w}z)^{3-2\alpha}}(1-|w|^2)^{1-2\alpha}\,dA(w)} = h_{s,u}(g)(z).$$

Hence h^α_f is bounded if and only if $h_{s,u}$ is bounded from $B_p(\alpha)$ to $L^p(dA_{p,\alpha})$, and by Theorem 4.2 this holds if and only if $f \in W_{p,\alpha}$. □

5. Atomic Decomposition

THEOREM 5.1 (Decomposition theorem). *Let $1 < p < \infty$ and $\alpha \le 1/2$, and let s with $s > 0$ if $\alpha < 1/2$ and $\max(p,q)s > -1$ if $\alpha = 1/2$. Then there exists $d_0 > 0$ such that for any d-lattice $\{z_j\}$ in $\mathbb{D}$, $0 < d < d_0$, the following statements hold.*

(a) *If f is analytic in $\mathbb{D}$ and $|f(z)|^p\,dA_{p,\alpha}(z)$ is an (α,p)-Carleson measure, then*

$$f(z) = \sum_{j=0}^{\infty} a_j\frac{(1-|z_j|^2)^{1+s-(1-2\alpha)/p}}{(1-\bar{z}_jz)^{2+s}} \tag{8}$$

with

$$\bigg\|\sum_{j=0}^{\infty}|a_j|^p\delta_{z_j}\bigg\|_{p,\alpha} \le C\||f|^p\,dA_{p,\alpha}\|_{p,\alpha}.$$

(b) *If* $\{a_j\}$ *satisfies*

$$\left\| \sum_{j=0}^{\infty} |a_j|^p \delta_{z_j} \right\|_{p,\alpha} < \infty,$$

then f, *defined by* (8), *is in* $A^p_{p,\alpha}$ *and* $|f(z)|^p \, dA_{p,\alpha}(z)$ *is an* (α, p)*-Carleson measure with*

$$\||f|^p \, dA_{p,\alpha}\|_{p,\alpha} \le C \left\| \sum_{j=0}^{\infty} |a_j|^p \delta_{z_j} \right\|_{p,\alpha}.$$

Proof. Without loss of generality, we may assume that $s > \max(0, -2\alpha)$ if $\alpha < 1/2$. Indeed, for $\alpha < 0$, it is easy to check that $|f|^p \, dA_{p,\alpha}$ is an (α, p)-Carleson measure if and only if $B(f) = \sup_{z \in \mathbb{D}} (1 - |z|^2)|f(z)| < \infty$. To see that, suppose first that $|f|^p \, dA_{p,\alpha}$ is an (α, p)-Carleson measure. Then by Theorem 4.2 we have that $h_{s,f}$ is bounded, and therefore it follows from Lemma 4.1 that $B(f)$ is finite. Conversely, if $B(f) < \infty$ and $g \in B_p(\alpha)$, then by Lemma 2.1 we have

$$\begin{aligned} \int_{\mathbb{D}} |g(z)|^p |f(z)|^p \, dA_{p,\alpha}(z) &\le B(f)^p \int_{\mathbb{D}} |g(z)|^p (1 - |z|^2)^{-1-2\alpha} \, dA(z) \\ &\le CB(f)^p \int_{\mathbb{D}} |g'(z)|^p \, dA_{p,\alpha}(z) \le CB(f)^p \|g\|^p_{\alpha,p}, \end{aligned}$$

and so $|f|^p \, dA_{p,\alpha}$ is an (α, p)-Carleson measure. Pick $\alpha' < 0$ so that $s > -2\alpha'$. Hence $|f|^p \, dA_{p,\alpha}$ is an (α, p)-Carleson measure if and only if $|f|^p \, dA_{p,\alpha'}$ is an (α', p)-Carleson measure.

We prove part (b) first. By Theorem 4.2, it is enough to show that the operator $h_{s,f}$ is bounded from $B_p(\alpha)$ to $L^p(dA_{p,\alpha})$. The assumption on the sequence $\{a_j\}$ implies that $\{a_j\}$ is p-summable. Hence, by Theorem D, the sum (8) converges in $A^p_{p,\alpha}$ and then f, defined by (8), is in $A^p_{p,\alpha}$. Let $g \in B_p(\alpha)$, and for each $z \in \mathbb{D}$ consider the function $h_z(\zeta) = g(\zeta)/(1 - \bar{z}\zeta)^{2+s}$. Fix $z \in \mathbb{D}$; then it is easy to check that $h_z \in A^p_{p,\alpha}$ with $\|h_z\|_{A^p_{p,\alpha}} \lesssim \|g\|_{\alpha,p} (1 - |z|^2)^{-(1+s)}$. Hence $h_z = P_s h_z$ and we have

$$\begin{aligned} \overline{h_{s,f}(g)(z)} &= \int_{\mathbb{D}} f(w) \frac{\overline{g(w)}}{(1 - \bar{w}z)^{2+s}} \, dA_s(w) \\ &= \sum_{j=0}^{\infty} a_j (1 - |z_j|^2)^{1+s-(1-2\alpha)/p} \, \overline{P_s h_z(z_j)} \\ &= \sum_{j=0}^{\infty} a_j \frac{(1 - |z_j|^2)^{1+s-(1-2\alpha)/p}}{(1 - \bar{z}_j z)^{2+s}} \overline{g(z_j)}. \end{aligned}$$

Then, by Theorem D(b) with $\beta = p - 1 - 2\alpha$, we have

$$\|h_{s,f}(g)\|^p_{A^p_{p,\alpha}} \le C \sum_{j=0}^{\infty} |a_j g(z_j)|^p \le C \left\| \sum_{j=0}^{\infty} |a_j|^p \delta_{z_j} \right\|_{p,\alpha} \|g\|^p_{\alpha,p}.$$

So (b) is proved.

Now we prove part (a). Let $g \in B_p(\alpha)$ and let $\{z_j\}$ be a d-lattice in $\mathbb{D}$. By [7, Lemma 2.2] (see also [8, Lemma B]), there is a disjoint decomposition $\{D_j\}$ of $\mathbb{D}$ $\big($i.e., $\mathbb{D} = \bigcup_j D_j\big)$ such that $|D_j| \approx (1-|z_j|^2)^2$, $z_j \in D_j$, and

$$|f(z) - (Af)(z)| \le C\, dT_{|f|}(z), \tag{9}$$

where $|D_j|$ is the area of D_j and

$$A(f)(z) = C \sum_{j=0}^{\infty} f(z_j)|D_j| \frac{(1-|z_j|^2)^s}{(1-\bar{z}_j z)^{2+s}}.$$

The assumption on f implies that $fg \in A^p_{p,\alpha}$, and the discrete version of this is that the sequence

$$\{f(z_j)g(z_j)(1-|z_j|^2)^{(p+1-2\alpha)/p}\}$$

is p-summable (see also [3] or [6]). Using the fact that $|fg|^p$ is subharmonic and the area mean value property, we can see that the measure

$$\sum_{j=0}^{\infty} |f(z_j)(1-|z_j|^2)^{(1-p-2\alpha)/p}|D_j||^p \delta_{z_j}$$

is an (α, p)-Carleson measure and

$$\left\| \sum_{j=0}^{\infty} |f(z_j)(1-|z_j|^2)^{(1-p-2\alpha)/p}|D_j||^p \delta_{z_j} \right\|_{p,\alpha} \le C \| |f|^p\, dA_{p,\alpha} \|_{p,\alpha}. \tag{10}$$

Then, by part (b), we have that $|A(f)(z)|^p\, dA_{p,\alpha}(z)$ is an (α, p)-Carleson measure.

Since A is an operator in the space

$$\{f \in A^p_{p,\alpha} : |f(z)|^p\, dA_{p,\alpha}(z) \text{ is an } (\alpha, p)\text{-Carleson measure}\},$$

we have, by (9),

$$|(I-A)(f)(z)| \le C\, dT_{|f|}(z).$$

Then, if we take d small enough, by Theorem 3.2 (with $\beta = p-1-2\alpha$) we have the operator norm estimate

$$\|I - A\| \le 1/2. \tag{11}$$

Note that, in order to apply Theorem 3.2 when $1 < p < 2$, we need to check that $B(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)|f(z)| < \infty$. Since $|f|^p\, dA_{p,\alpha}$ is an (α, p)-Carleson measure, it follows from Theorem 4.2 and Lemma 4.1 that $B(f)$ is finite.

Hence, by (11), A^{-1} exists and

$$\|A^{-1}\| \le \sum_{j=0}^{\infty} \|(I-A)^j\| \le 2.$$

Now we have

$$
\begin{aligned}
f(z) &= (AA^{-1}f)(z) \\
&= C\sum_{j=0}^{\infty}(A^{-1}f)(z_j)|D_j|\frac{(1-|z_j|^2)^s}{(1-\bar{z}_j z)^{2+s}} \\
&= C\sum_{j=0}^{\infty}(A^{-1}f)(z_j)|D_j|(1-|z_j|^2)^{(1-p-2\alpha)/p}\frac{(1-|z_j|^2)^{1+s-(1-2\alpha)/p}}{(1-\bar{z}_j z)^{2+s}}.
\end{aligned}
$$

By the inequality (10) and the boundeness of A^{-1}, we get

$$
\begin{aligned}
&\left\|\sum_{j=0}^{\infty}|(A^{-1}f)(z_j)(1-|z_j|^2)^{(1-p-2\alpha)/p}|D_j||^p\delta_{z_j}\right\|_{p,\alpha} \\
&\qquad\le C\||A^{-1}f|^p\, dA_{p,\alpha}\|_{p,\alpha} \le C\|A^{-1}\|\cdot\||f|^p\, dA_{p,\alpha}\|_{p,\alpha}.
\end{aligned}
$$

Thus the choice of $a_j = (A^{-1}f)(z_j)|D_j|(1-|z_j|^2)^{(1-p-2\alpha)/p}$ completes the proof. □

Now, as an immediate consequence of Theorem 5.1 we obtain the following decomposition of $W_{p,\alpha}$.

COROLLARY 5.2. *Let $1 < p < \infty$ and $\alpha \le 1/2$, and let s with $s > 0$ if $\alpha < 1/2$ and $\max(p,q)s > -1$ if $\alpha = 1/2$. Then there exists $d_0 > 0$ such that for any d-lattice $\{z_j\}$ in $\mathbb{D}$, $0 < d < d_0$, the following statements hold.*

(a) *If $f \in W_{p,\alpha}$, then*

$$
f(z) = \sum_{j=0}^{\infty} a_j \frac{(1-|z_j|^2)^{1+s-(1-2\alpha)/p}}{(1-\bar{z}_j z)^{1+s}} \tag{12}
$$

with

$$
\left\|\sum_{j=0}^{\infty}|a_j|^p\delta_{z_j}\right\|_{p,\alpha} \le C\|f\|_{W_{p,\alpha}}^p.
$$

(b) *If $\{a_j\}$ satisfies*

$$
\left\|\sum_{j=0}^{\infty}|a_j|^p\delta_{z_j}\right\|_{p,\alpha} < \infty,
$$

then f, defined by (12), converges in $B_p(\alpha)$ with

$$
\|f\|_{W_{p,\alpha}}^p \le C\left\|\sum_{j=0}^{\infty}|a_j|^p\delta_{z_j}\right\|_{p,\alpha}.
$$

Proof. The corollary follows from Theorem 5.1 via term-by-term integration. □

References

[1] N. Arcozzi, R. Rochberg, and E. Sawyer, *Carleson measures for analytic Besov spaces,* Rev. Mat. Iberoamericana 18 (2002), 443–510.

[2] B. Böe, *Interpolating sequences for Besov spaces,* J. Funct. Anal. 192 (2002), 319–341.
[3] R. Coifman and R. Rochberg, *Representation theorems for holomorphic and harmonic functions in L^p*, Astérisque 77 (1980), 11–66.
[4] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman spaces,* Grad. Texts in Math., 199, Springer-Verlag, New York, 2000.
[5] J. M. Ortega and J. Fàbrega, *Pointwise multipliers and corona type decomposition in BMOA,* Ann. Inst. Fourier (Grenoble) 46 (1996), 111–137.
[6] R. Rochberg, *Decomposition theorems for Bergman spaces and their applications,* Operators and function theory (Lancaster, 1984), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 153, pp. 225–277, Reidel, Dordrecht, 1985.
[7] R. Rochberg and S. Semmes, *A decomposition theorem for BMO and applications,* J. Funct. Anal. 67 (1986), 228–263.
[8] R. Rochberg and Z. Wu, *A new characterization of Dirichlet type spaces and applications,* Illinois J. Math. 37 (1993), 101–122.
[9] R. Zhao, *Distances from Bloch functions to some Möbius invariant spaces,* Ann. Acad. Sci. Fenn. Ser. A I Math. 33 (2008), 303–313.
[10] K. Zhu, *Operator theory in function spaces,* Monogr. Textbooks Pure Appl. Math., 139, Dekker, New York, 1990.

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