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The Chern Coefficients of Local Rings

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Dedicated to Professor Melvin Hochster on the occasion of his 65th birthday

1. Introduction

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$, and let I be an $\mathfrak{m}$-primary ideal. One of our goals is to study the set of I-good filtrations of R. More concretely, we will consider the set of multiplicative, decreasing filtrations of R ideals, $\mathcal{A} = \{I_n, I_0 = R, I_{n+1} = II_n, n \gg 0\}$, that is integral over the I-adic filtration and conveniently coded in the corresponding Rees algebra and its associated graded ring:

$$\mathcal{R}(\mathcal{A}) = \sum_{n\geq 0} I_n t^n, \qquad \mathrm{gr}_{\mathcal{A}}(R) = \sum_{n\geq 0} I_n/I_{n+1}.$$

In this paper we study certain strata of these algebras. For that we will focus on the role of the Hilbert polynomial of the Hilbert function $\lambda(R/I_{n+1})$, $n \gg 0$,

$$H^1_{\mathcal{A}}(n) = P^1_{\mathcal{A}}(n) = \sum_{i=0}^{d} (-1)^i e_i(\mathcal{A}) \binom{n+d-i}{d-i},$$

particularly of its coefficients $e_0(\mathcal{A})$ and $e_1(\mathcal{A})$. Two of our principal aims are to establish relationships between the coefficients $e_i(\mathcal{A})$ for $i = 0, 1$ and (marginally) $e_2(\mathcal{A})$. If R is a Cohen–Macaulay ring then there are numerous related developments; see especially those given and discussed in [8] and [28]. The situation is quite different in the non–Cohen–Macaulay case. Just to illustrate the issue, suppose $d > 2$ and consider a comparison between $e_0(I)$ and $e_1(I)$, a subject that has received considerable attention. It is often possible to pass to a reduction $R \to S$, with $\dim S = 2$ or even $\dim S = 1$, so that $e_0(I) = e_0(IS)$ and $e_1(I) = e_1(IS)$. If R is Cohen–Macaulay, then this is straightforward. However, in general the relationship between $e_0(IS)$ and $e_1(IS)$ may involve other invariants of S, some of which may not be easily traceable all the way to R.

Our perspective is partly influenced by the interpretation of the coefficient e_1 as a *tracking number* (see [6] for the original terminology and also [26])—that is, as a numerical positional tag of the algebra $\mathcal{R}(\mathcal{A})$ in the set of all such algebras with the same multiplicity. The coefficient e_1 under various circumstances is also called the *Chern number* or *Chern coefficient* of the algebra.

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This paper is organized around the following list of questions and conjectural statements about the values of e_1 for very general filtrations associated to the $\mathfrak{m}$-primary ideals of a local Noetherian ring $(R, \mathfrak{m})$.

1. *Conjecture 1: the negativity conjecture.* For every ideal J that is generated by a system of parameters, $e_1(J) < 0$ if and only if R is not Cohen–Macaulay.
2. *Conjecture 2: the positivity conjecture.* For every $\mathfrak{m}$-primary ideal I, for its integral closure filtration $\mathcal{A}$ we have

$$e_1(\mathcal{A}) \geq 0.$$

2. *Conjecture 3: the uniformity conjecture.* For each Noetherian local ring R, there exist two functions $\mathbf{f}_l(\cdot), \mathbf{f}_u(\cdot)$ defined with some extended multiplicity degree over R and such that, for each $\mathfrak{m}$-primary ideal I and any I-good filtration $\mathcal{A}$,

$$\mathbf{f}_l(I) \leq e_1(\mathcal{A}) \leq \mathbf{f}_u(I).$$

4. For any two minimal reductions J_1, J_2 of an $\mathfrak{m}$-primary ideal I,

$$e_1(J_1) = e_1(J_2).$$

For general local rings the conjectures may fail for reasons that will be illustrated by examples. We will settle Conjecture 1 for domains that are essentially of finite type over fields by making use of the existence of special maximal Cohen–Macaulay modules (Theorem 3.2). The lower bound in Conjecture 3 is also settled for general rings through the use of extended degree functions (Corollary 7.7). The upper bound uses the technique of the Briançon–Skoda theorem on perfect fields (Theorem 6.1). We bring no real understanding to the last question.

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2. e_1 as a Tracking Number

Let $(R, \mathfrak{m})$ be a Noetherian local domain of dimension d that is a quotient of a Gorenstein ring. For an $\mathfrak{m}$-primary ideal I, we shall consider the set of all graded subalgebras $\mathbf{A}$ of the integral closure of $R[It]$,

$$R[It] \subset \mathbf{A} \subset \bar{\mathbf{A}} = \overline{R[It]}.$$

We will assume that $\bar{\mathbf{A}}$ is a finite $R[It]$-algebra and denote the set of these algebras by $\mathfrak{S}(I)$. If the algebra

$$\mathbf{A} = \sum I_n t^n$$

comes with a filtration that is decreasing ($I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$), then it has an associated graded ring

$$\operatorname{gr}(\mathbf{A}) = \sum_{n=0}^{\infty} I_n/I_{n+1}.$$

PROPOSITION 2.1. *If* $\mathbf{A}$ *satisfies condition* S_2 *of Serre, then* $\{I_n, n \geq 0\}$ *is a decreasing filtration.*

Proof. See [34, Prop. 4.6]. $\square$

We shall describe the role of the Hilbert coefficient $e_1(\cdot)$ in our study of the normalization of $R[It]$, following [26] and [25]. For each $\mathbf{A} = \sum A_n t^n \in \mathfrak{S}(I)$, we consider the Hilbert polynomial (for $n \gg 0$)

$$\lambda(R/A_{n+1}) = \sum_{i=0}^{d} (-1)^i e_i(\mathbf{A}) \binom{n+d-i}{d-i}.$$

The multiplicity $e_0(\mathbf{A})$ is constant across the whole set $\mathfrak{S}(I)$: $e_0(\mathbf{A}) = e_0(I)$. The next proposition shows the role of $e_1(\mathbf{A})$ in tracking $\mathbf{A}$ in the set $\mathfrak{S}(I)$.

PROPOSITION 2.2. *Let* $(R, \mathfrak{m})$ *be a normal, Noetherian local domain that is a quotient of a Gorenstein ring, and let* I *be an* $\mathfrak{m}$*-primary ideal. For algebras* $\mathbf{A}$ *and* $\mathbf{B}$ *of* $\mathfrak{S}(I)$:

(i) *If the algebras* $\mathbf{A}$ *and* $\mathbf{B}$ *satisfy* $\mathbf{A} \subset \mathbf{B}$, *then* $e_1(\mathbf{A}) \leq e_1(\mathbf{B})$.
(ii) *If* $\mathbf{B}$ *is the* S_2*-ification of* $\mathbf{A}$, *then* $e_1(\mathbf{A}) = e_1(\mathbf{B})$.
(iii) *If the algebras* $\mathbf{A}$ *and* $\mathbf{B}$ *satisfy the condition* S_2 *of Serre and* $\mathbf{A} \subset \mathbf{B}$, *then* $e_1(\mathbf{A}) = e_1(\mathbf{B})$ *if and only if* $\mathbf{A} = \mathbf{B}$.

Proof. Assertions (i) and (ii) follow directly from the relationship between Krull dimension and the degree of Hilbert polynomials. The exact sequence of graded $R[It]$-modules

$$0 \longrightarrow \mathbf{A} \longrightarrow \mathbf{B} \longrightarrow \mathbf{B}/\mathbf{A} \longrightarrow 0$$

gives that the dimension of $\mathbf{B}/\mathbf{A}$ is at most $d-1$. Moreover, $\dim \mathbf{B}/\mathbf{A} = d-1$ if and only if its multiplicity is

$$\deg(\mathbf{B}/\mathbf{A}) = e_1(\mathbf{B}) - e_1(\mathbf{A}) > 0.$$

Assertion (iii) follows because with $\mathbf{A}$ and $\mathbf{B}$ satisfying S_2, the quotient $\mathbf{B}/\mathbf{A}$ is nonzero, will satisfy S_1, and therefore has Krull dimension $d-1$. $\square$

COROLLARY 2.3. *Suppose there is a sequence of distinct algebras*

$$\mathbf{A}_0 \subset \mathbf{A}_1 \subset \cdots \subset \mathbf{A}_n = \overline{R[It]},$$

in $\mathfrak{S}(I)$ *that satisfy the condition* S_2 *of Serre. Then*

$$n \leq e_1(\overline{R[It]}) - e_1(\mathbf{A}_0) \leq e_1(\overline{R[It]}) - e_1(I).$$

This result highlights the importance of having lower bounds for $e_1(I)$ and upper bounds for $e_1(\overline{R[It]})$. For simplicity we denote the last coefficient as $\bar{e}_1(I)$. In the Cohen–Macaulay case, $e_1(J) = 0$ for any parameter ideal J. Upper bounds for $\bar{e}_1(I)$ were given in [26]. For instance, [26, Thm. 3.2(a),(b)] shows that if R is a Cohen–Macaulay algebra of type t that is essentially of finite type over a perfect field k and if δ is a nonzero divisor in the Jacobian ideal $\mathrm{Jac}_k(R)$, then

$$\bar{e}_1(I) \leq \frac{t}{t+1}[(d-1)e_0(I) + e_0(I + \delta R/\delta R)]$$

and

$$\bar{e}_1(I) \leq (d-1)[e_0(I) - \lambda(R/\bar{I})] + e_0(I + \delta R/\delta R).$$

3. Cohen–Macaulayness and the Negativity of e_1

Given the role of the Hilbert coefficient e_1 as a tracking number in the normalization of blowup algebras, it is of interest to know its signature.

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. If R is Cohen–Macaulay, then $e_1(J) = 0$ for an ideal J generated by a system of parameters $x_1, \dots, x_d$. As a consequence, for any $\mathfrak{m}$-primary ideal I we have $e_1(I) \geq 0$. If $d = 1$ then the property $e_1(J) = 0$ is characteristic of Cohen–Macaulayness. For $d \geq 2$, the situation is somewhat different. Given the ring $R = k[x, y, z]/(z(x, y, z))$, for $T = H^0_{\mathfrak{m}}(R)$ and $S = k[x, y] = R/T$ it follows that $e_1(R) = e_1(S) = 0$.

We shall argue that the negativity of $e_1(J)$ is an expression of the lack of Cohen–Macaulayness of R in numerous classes of rings. To provide a framework, we state the following.

CONJECTURE 3.1. *Let R be a Noetherian local ring that admits an embedding into a big Cohen–Macaulay module. Then, for a parameter ideal J, $e_1(J) < 0$ if and only if R is not Cohen–Macaulay.*

This places restrictions on R; in particular, R must be unmixed and equidimensional. As a matter of fact, we will be concerned almost exclusively with integral domains that are essentially of finite type over a field.

We next establish the *small* version of the conjecture.

THEOREM 3.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$. Suppose there is an embedding*

$$0 \to R \to E \to C \to 0,$$

where E is a finitely generated maximal Cohen–Macaulay R-module. If R is not Cohen–Macaulay, then $e_1(J) < 0$ for any parameter ideal J.

Proof. We may assume that the residue field of R is infinite. We argue by induction on d. For $d = 2$, let J be a parameter ideal. If R is not Cohen–Macaulay, then $\operatorname{depth} C = 0$.

Let $J = (x, y)$; we may assume that x is a superficial element for the purpose of computing $e_1(J)$ and that x is also superficial relative to C. In other words, x is not contained in any associated prime of C distinct from $\mathfrak{m}$.

Tensoring the theorem's exact sequence by $R/(x)$ yields the exact complex

$$0 \to T = \operatorname{Tor}_1^R(R/(x), C) \to R/(x) \to E/xE \to C/xC \to 0,$$

where T is a nonzero module of finite support. Denote by S the image of $R' = R/(x)$ in E/xE, and note that S is a Cohen–Macaulay ring of dimension 1. By the Artin–Rees theorem, $T \cap (y^n)R' = 0$ for $n \gg 0$ and thus, from the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T \cap (y^n)R' & \longrightarrow & (y^n)R' & \longrightarrow & (y^n)S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T & \longrightarrow & R' & \longrightarrow & S & \longrightarrow & 0,
\end{array}$$

it follows that the Hilbert polynomial of the ideal yR' is

$$e_0 n - e_1 = e_0(yS)n + \lambda(T).$$

Thus

$$e_1(J) = -\lambda(T) < 0, \tag{1}$$

as claimed.

Assume now that $d \geq 3$, and let x be a superficial element for J and for the modules E and C. In the exact sequence

$$0 \to T = \mathrm{Tor}_1^R(R/(x), C) \to R' = R/(x) \to E/xE \to C/xC \to 0, \tag{2}$$

T is either zero—in which case we continue with the induction procedure—or T is a nonzero module of finite support.

First we recall an elementary rule for the calculation of Hilbert coefficients. Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let $\mathcal{A} = \{I_n, n \geq 0\}$ be a filtration as before. Given a finitely generated R-module M, denote by $e_i(M)$ the Hilbert coefficients of M for the filtration $\mathcal{A}M = \{I_n M, n \geq 0\}$.

Proposition 3.2. *Let*

$$0 \to A \to B \to C \to 0$$

be an exact sequence of finitely generated R-modules. If $r = \dim A < s = \dim B$, then $e_i(B) = e_i(C)$ for $i < s - r$.

To continue with the proof of Theorem 3.1, if $T \neq 0$ in the exact sequence (2) then, by Proposition 3.2, $e_1(JR') = e_1(J(R'/T))$ and we thus have the embedding $R'/T \hookrightarrow E/xE$. By the induction hypothesis, it suffices to prove that if R'/T is Cohen–Macaulay then R', and hence R, will be Cohen–Macaulay. This is the content of [18, Prop. 2.1]. For convenience, we give the details.

We may assume that R is a complete local ring. Because R is embedded in a maximal Cohen–Macaulay module, any associated prime of R is an associated prime of E and so is equidimensional. Consider the following exact sequences:

$$0 \to T \to R' \to S = R'/T \to 0;$$

$$0 \to R \xrightarrow{x} R \to R' \to 0.$$

From the first sequence, taking local cohomology yields

$$0 \to H^0_{\mathfrak{m}}(T) = T \to H^0_{\mathfrak{m}}(R') \to H^0_{\mathfrak{m}}(S) = 0,$$

since $H^i_{\mathfrak{m}}(T) = 0$ for $i > 0$ and S is Cohen–Macaulay of dimension ≥ 2; one also has $H^1_{\mathfrak{m}}(S) = H^1_{\mathfrak{m}}(R') = 0$. From the second sequence, since the associated primes of R have dimension d, it follows that $H^1_{\mathfrak{m}}(R)$ is a finitely generated R-module. Finally, $H^1_{\mathfrak{m}}(R) = 0$ by Nakayama's lemma and so $T = H^0_{\mathfrak{m}}(R') = 0$. □

We now analyze what is required to extend the proof to big Cohen–Macaulay cases. We will assume that R is an integral domain and that E is a big balanced Cohen–Macaulay module (see [Chap. 8; 30]). Embed R into E,

$$0 \longrightarrow R \longrightarrow E \longrightarrow C \longrightarrow 0.$$

The preceding argument for $d \geq 3$ will work if in the induction argument we can pick $x \in J$ to be superficial for the Hilbert polynomial of J, avoiding the finite set of associated primes of E and all associated primes of C that are different from $\mathfrak{m}$. It is this last condition that is the most troublesome.

There is one case where this can be overcome—namely, when R is a complete local ring and E is countably generated. Indeed, C will be countably generated and $\operatorname{Ass}(C)$ will be a countable set. The prime avoidance result of [4, Lemma 3] allows for the choice of x. Let us apply these ideas in an important case.

THEOREM 3.2. *Let $(R, \mathfrak{m})$ be a Noetherian local integral domain essentially of finite type over a field. If R is not Cohen–Macaulay, then $e_1(J) < 0$ for any parameter ideal J.*

Proof. Let A be the integral closure of R and let $\hat{R}$ be its completion. Tensor the embedding $R \subset A$ to obtain

$$0 \longrightarrow \hat{R} \longrightarrow \hat{R} \otimes_R A = \hat{A}.$$

From the properties of pseudo-geometric local rings (see [21, Sec. 37]), $\hat{A}$ is a reduced semilocal ring with a decomposition

$$\hat{A} = A_1 \times \cdots \times A_r,$$

where each A_i is a complete local domain of dimension $\dim R$ that is finite over $\hat{R}$.

For each A_i, we make use of [11, Thm. 3.1] and [12, Prop. 1.4] to pick a countably generated big balanced Cohen–Macaulay A_i-module and thereby an $\hat{R}$-module. Collecting the E_i yields the embedding

$$\hat{R} \longrightarrow A_1 \times \cdots \times A_r \longrightarrow E = E_1 \oplus \cdots \oplus E_r.$$

Since E is a countably generated big balanced Cohen–Macaulay $\hat{R}$-module, the foregoing argument shows that if $\hat{R}$ is not Cohen–Macaulay then $e_1(J\hat{R}) < 0$. This suffices to prove the assertion about R. □

REMARK 3.3. There are other classes of local rings that admit big balanced Cohen–Macaulay modules. A crude way to handle this would be as follows. Let E be such a module and assume it has a set of generators of cardinality s. Let X be a set of indeterminates of cardinality $> s$. The local ring $S = R[X]_{\mathfrak{m}[X]}$ is R-flat and has the same depth as R. If $E' = S \otimes_R E$ is a big balanced Cohen–Macaulay

S-module whose residue field of cardinality is greater than that of the corresponding module C, then prime avoidance would again work. Experts have cautioned that E' may not be balanced.

EXAMPLE 3.4. We will consider some classes of examples.

1. Let $(R, \mathfrak{m})$ be a regular local ring, let F be a nonzero (finitely generated) free R-module, and let N be a nonfree submodule of F. Then the idealization (trivial extension) of R by N, $S = R \oplus N$, is a non–Cohen–Macaulay local ring. If we pick $E = R \oplus F$ then Theorem 3.2 implies that $e_1(J) < 0$ for any parameter ideal $J \subset S$. It is not difficult to give an explicit formula for $e_1(J)$ in this case.

2. Let $R = \mathbb{R} + (x, y)\mathbb{C}[x, y] \subset \mathbb{C}[x, y]$ for x, y distinct indeterminates. Although R is not Cohen–Macaulay, its localization S at the maximal irrelevant ideal is a Buchsbaum ring. It is easy to verify that $e_1(x, y) = -1$ and that $e_1(S) = 0$ for the $\mathfrak{m}$-adic filtration of S.

Observe that R has an isolated singularity. For these rings, [25, Thm. 5] can be extended (it does not require the Cohen–Macaulay condition) and therefore describes bounds for $e_1(\bar{\mathbf{A}})$ of integral closures. Thus, if $\mathbf{A}$ is the Rees algebra of the parameter ideal J then

$$e_1(\bar{\mathbf{A}}) - e_1(J) \leq (d - 1 + \lambda(R/L))e_0(J),$$

where L is the Jacobian of R. In this example, $d = 2$, $\lambda(R/L) = 1$, and

$$e_1(\bar{\mathbf{A}}) - e_1(J) \leq 2e_0(J).$$

3. Let k be a field of characteristic 0 and let $f = x^3 + y^3 + z^3$ be a polynomial of $k[x, y, z]$. Set $A = k[x, y, z]/(f)$ and let R be the Rees algebra of the maximal irrelevant ideal $\mathfrak{m}$ of A. By the Jacobian criterion, R is normal. Because the reduction number of $\mathfrak{m}$ is 2, R is not Cohen–Macaulay. Furthermore, it is easy to verify that R is not contained in any Cohen–Macaulay domain that is finite over R.

Let $S = R_{\mathcal{M}}$, where $\mathcal{M}$ is the irrelevant maximal ideal of R. The first superficial element (in the reduction to dimension 2) can be chosen to be prime. Now one takes the integral closure of S, which will be a maximal Cohen–Macaulay module.

The argument extends to geometric domains in any characteristic if depth $R = d - 1$.

REMARK 3.5. Uniform lower bounds for e_1 are rare but still exist in special cases. For example, if R is a generalized Cohen–Macaulay ring then, according to [10, Thm. 5.4],

$$e_1(J) \geq -\sum_{i=1}^{d-1} \binom{d-2}{i-1} \lambda(H^i_{\mathfrak{m}}(R)),$$

with equality if R is Buchsbaum.

It should be observed that uniform lower bounds may not always exist. For instance, if $A = k[x, y, z]$ and if R is the idealization of (x, y), then $e_1(J) = -n$ for the ideal $J = (x, y, z^n)$.

The Koszul homology modules $H_i(J)$ of J are the first place to look for bounds for $e_1(J)$. We recall [3, Thm. 4.6.10] that the multiplicity of J is given by the formula

$$e_0(J) = \lambda(R/J) - \sum_{i=1}^{d} (-1)^{i-1} h_i(J),$$

where $h_i(J)$ is the length of $H_i(J)$. The summation term is nonnegative and vanishes only if R is Cohen–Macaulay. Unfortunately, it does not give us bounds for $e_1(J)$. There is a formula involving these terms in the special case when J is generated by a d-sequence. Then the corresponding *approximation complex* is acyclic, and the Hilbert–Poincaré series of J is

$$\frac{\sum_{i=0}^{d} (-1)^i h_i(J) t^i}{(1-t)^d}$$

[14, Cor. 4.6]; hence

$$e_1(J) = \sum_{i=1}^{d} (-1)^i i h_i(J).$$

Later we shall prove the existence of lower bounds more generally by making use of extended degree functions.

4. Bounds on $\bar{e}_1(I)$ via the Briançon–Skoda Number

In this section we discuss the role of Briançon–Skoda type theorems (see [1; 20]) in determining some relationships between the coefficients $e_0(I)$ and $\bar{e}_1(I)$. We follow the treatment given in [26, Thm. 3.1] and [25, Thm. 5] but formulated for the non–Cohen–Macaulay case. We will provide a short proof along the lines of [20] for the special case we need: $\mathfrak{m}$-primary ideals in a local ring. The general case is treated by Hochster and Huneke in [16, 1.5.5 and 4.1.5]. Let k be a perfect field, let R be a reduced and equidimensional k-algebra essentially of finite type, and assume either that R is affine with $d = \dim R$ or that $(R, \mathfrak{m})$ is local with $d = \dim R + \operatorname{trdeg}_k(R/\mathfrak{m})$. Recall that the *Jacobian ideal* $\operatorname{Jac}_k(R)$ of R is defined as the dth Fitting ideal of the module of differentials $\Omega_k(R)$; it can be computed explicitly from a presentation of the algebra. By varying Noether normalizations one deduces from [20, Thm. 2] that the Jacobian ideal $\operatorname{Jac}_k(R)$ is contained in the conductor $R\colon \bar{R}$ of R (see also [23], [2, 3.1], and [15, 2.1]). Here $\bar{R}$ denotes the integral closure of R in its total ring of fractions.

THEOREM 4.1. *Let k be a perfect field, let R be a reduced local k-algebra essentially of finite type with the property S_2 of Serre, and let I be an ideal with a minimal reduction generated by g elements. Denote by $\mathbf{D} = \sum_{n\geq 0} D_n t^n$ the S_2-ification of $R[It]$. Then, for every integer n,*

$$\operatorname{Jac}_k(R)\overline{I^{n+g-1}} \subset D_n.$$

Proof. The proof is lifted from [26, 3.1] with the modification required by the use of $\mathbf{D}$ at the end.

We may assume that k is infinite. Then, passing to a minimal reduction, we may suppose that I is generated by g generators. Let S be a finitely generated k-subalgebra of R so that $R = S_{\mathfrak{p}}$ for some $\mathfrak{p} \in \operatorname{Spec}(S)$, and write $S = k[x_1, \ldots, x_e] = k[X_1, \ldots, X_e]/\mathfrak{a}$ with $\mathfrak{a} = (h_1, \ldots, h_t)$ an ideal of height c. Notice that S is reduced and equidimensional. Let $K = (f_1, \dots, f_g)$ be an S-ideal with $K_{\mathfrak{p}} = I$, and consider the extended Rees ring $B = S[Kt, t^{-1}]$. Now B is a reduced and equidimensional affine k-algebra of dimension $e - c + 1$.

Let $\varphi\colon k[X_1, \dots, X_e, T_1, \dots, T_g, U] \twoheadrightarrow B$ be the k-epimorphism mapping X_i to x_i, T_i to $f_i t$, and U to t^{-1}. Its kernel has height $c + g$ and contains the ideal $\mathfrak{b}$ generated by $\{h_i, T_j U - f_j \mid 1 \le i \le t, 1 \le j \le g\}$. Consider the Jacobian matrix of these generators,

$$\Theta = \left(\begin{array}{c|cccc} \frac{\partial h_i}{\partial X_j} & & 0 & & \\ \hline & U & & & T_1 \\ * & & \ddots & & \vdots \\ & & & U & T_g \end{array} \right).$$

Notice that $I_{c+g}(\Theta) \supset I_c\big(\big(\frac{\partial h_i}{\partial X_j}\big)\big) U^{g-1}(T_1, \dots, T_g)$. Applying φ, we obtain

$$\mathrm{Jac}_k(B) \supset I_{c+g}(\Theta) B \supset \mathrm{Jac}_k(S) K t^{-g+2}.$$

Thus $\mathrm{Jac}_k(S) K t^{-g+2}$ is contained in the conductor of B. Localizing at $\mathfrak{p}$, we see that $\mathrm{Jac}_k(R) I t^{-g+2}$ is in the conductor of the extended Rees ring $R[It, t^{-1}]$. Hence $\mathrm{Jac}_k(R) I \overline{I^{n+g-1}} \subset I^{n+1}$ for every n, which yields

$$\mathrm{Jac}_k(R) \overline{I^{n+g-1}} \subset I^{n+1} : I \subset D_{n+1} : I = D_n$$

because $\mathrm{gr}(\mathbf{D})_+$ has positive grade. □

This result, together with an application of [25, Thm. 5], gives the following estimation.

Corollary 4.1. *Let k be a perfect field, let $(R, \mathfrak{m})$ be a normal and reduced local k-algebra essentially of finite type of dimension d, and let I be an $\mathfrak{m}$-primary ideal. If the Jacobian ideal L of R is $\mathfrak{m}$-primary then, for any minimal reduction J of I,*

$$\bar{e}_1(I) - e_1(J) \le (d + \lambda(R/L) - 1) e_0(I).$$

Moreover, if $L \ne R$ and $\bar{I} \ne \mathfrak{m}$, then one can replace -1 by -2.

5. Lower Bounds for $\bar{e}_1$

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \ge 1$, and let I be an $\mathfrak{m}$-primary ideal. If R is Cohen–Macaulay, then the original lower bound for $e_1(I)$ was provided by Narita [22] and Northcott [24]:

$$e_1(I) \ge e_0(I) - \lambda(R/I).$$

This bound has been improved in several ways (see [29] for a detailed discussion). For non–Cohen–Macaulay rings, estimates for $e_1(I)$ are in a state of flux.

In this section we experiment with a special class of non–Cohen–Macaulay rings and methods in seeking lower bounds for $\bar{e}_1(I)$. We assume that R is a normal domain and that the minimal reduction J of I is generated by a d-sequence.

This is the case of normal Buchsbaum rings, examples of which can be constructed by a machinery developed in [9] (see also [31]).

Let $\mathbf{A} = R[Jt]$ and $\mathbf{B} = \overline{R[Jt]}$. The corresponding Sally module $S_{\mathbf{B}/\mathbf{A}}$ is defined by the exact sequence

$$0 \longrightarrow B_1 t\mathbf{A} \longrightarrow \mathbf{B}_+ \longrightarrow S_{\mathbf{A}}(\mathbf{B}) = \bigoplus_{n\geq 2} B_n/B_1 J^{n-1} \longrightarrow 0. \qquad (3)$$

LEMMA 5.1. *Let $(R, \mathfrak{m})$ be an analytically unramified local ring, and let J be an ideal generated by a system of parameters $x_1, \dots, x_d$. Then*

$$R/\bar{J} \otimes_R \operatorname{gr}_J(R) \simeq R/\bar{J}[T_1, \dots, T_d].$$

Proof. Let $R[T_1, \dots, T_d] \to \mathbf{A}$ be a minimal presentation of $\mathbf{A} = R[Jt]$. According to [27, Thms. 3.1 and 3.6], the presentation ideal $\mathcal{L}$ has all coefficients in $\bar{J}$; that is, $\mathcal{L} \subset \bar{J}R[T_1, \dots, T_d]$. □

PROPOSITION 5.2. *Let $(R, \mathfrak{m})$ be a normal, analytically unramified local domain, and let J be an ideal generated by a system of parameters $J = (x_1, \dots, x_d)$ of linear type. The Sally module $S_J(\mathbf{B})$ defined previously is either 0 or a module of dimension d and multiplicity*

$$e_0(S_J(\mathbf{B})) = \deg(S_J(\mathbf{B})) = \bar{e}_1(I) - e_0(I) - e_1(J) + \lambda(R/\bar{I}).$$

Proof. By [14, Cor. 4.6], since R is normal of dimension $d \geq 2$ it follows that depth $\operatorname{gr}_J(R) \geq 2$. Now we make use of [17, Lemma 1.1] (see also [34, Prop. 3.11]) to establish that depth $R[Jt] \geq 2$ as well. Then, using [34, Thm. 3.53], we have that $R[Jt]$ satisfies condition S_2 of Serre.

Consider the exact sequence

$$0 \longrightarrow B_1 R[Jt] \longrightarrow R[Jt] \longrightarrow R/\bar{J} \otimes_R \operatorname{gr}_J(R) \longrightarrow 0.$$

By Lemma 5.1, the last algebra is a polynomial ring in d variables. Therefore, $B_1 R[Jt]$ satisfies condition S_2 of Serre. Thus, in the defining sequence (3) of $S_J(\mathbf{B})$, either S vanishes (and $B_n = B_1 J^n$ for $n \geq 2$) or $\dim S_J(\mathbf{B}) = d$. In the latter case, the calculation of the Hilbert function (see [34, Rem. 2.17]) of $S_J(\mathbf{B})$ yields the asserted expression for its multiplicity. □

6. Existence of General Bounds

We shall now treat bounds for $\bar{e}_1(I)$ for several classes of geometric local rings. Suppose $\dim R = d > 0$. Let I be an $\mathfrak{m}$-primary ideal, and let $\mathcal{A} = \{A_n, n \geq 0\}$ be a filtration integral over the I-adic filtration. We may assume that $I = J = (x_1, \dots, x_d)$ is a parameter ideal. We will consider some reductions on $R \to R'$ such that $e_i(\mathcal{A}) = e_i(\mathcal{A}')$, $i = 0, 1$, for $\mathcal{A}' = \{A_n R', n \geq 0\}$.

Using superficial sequences $\mathbf{x} = \{x_1, \dots, x_{d-1}\}$ of length $d - 1$ is the technique of choice for Cohen–Macaulay rings. In general, as in equation (1), one needs more control over $H^0_{\mathfrak{m}}(R/(\mathbf{x}))$. Let us first examine the case of generalized Cohen–Macaulay local rings by examining the effect of certain reductions.

1. If $d \geq 2$ and $R' = R/H^0_{\mathfrak{m}}(R)$, then $e_i(\mathcal{A}) = e_i(\mathcal{A}')$ for $i = 0, 1$ by Proposition 3.2. In addition, $H^i_{\mathfrak{m}}(R) = H^i_{\mathfrak{m}}(R')$ for $i \geq 1$.

2. Another property is that if $d \geq 2$ and x_1 is a superficial element for $\mathcal{A}$ (i.e., R-regular if $d = 2$), then preservation will hold when passing to $R_1 = R/(x_1)$. As for the lengths of the local cohomology modules, if x_1 is R-regular then, by

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow R_1 \longrightarrow 0,$$

we have the exact sequence

$$0 \longrightarrow H^0_{\mathfrak{m}}(R_1) \longrightarrow H^1_{\mathfrak{m}}(R) \longrightarrow H^1_{\mathfrak{m}}(R) \longrightarrow H^1_{\mathfrak{m}}(R_1) \longrightarrow H^2_{\mathfrak{m}}(R) \longrightarrow \cdots,$$

which gives

$$\begin{aligned} \lambda(H^0_{\mathfrak{m}}(R_1)) &\leq \lambda(H^1_{\mathfrak{m}}(R)), \\ \lambda(H^1_{\mathfrak{m}}(R_1)) &\leq \lambda(H^1_{\mathfrak{m}}(R)) + \lambda(H^2_{\mathfrak{m}}(R)), \\ &\vdots \\ \lambda(H^{d-2}_{\mathfrak{m}}(R_1)) &\leq \lambda(H^{d-2}_{\mathfrak{m}}(R)) + \lambda(H^{d-1}_{\mathfrak{m}}(R)). \end{aligned}$$

3. Now we combine the two transformations. Let $T = H^0_{\mathfrak{m}}(R)$, set $R' = R/T$, let $x \in I$ be a superficial element for $\mathcal{A}R'$, and set $R_1 = R'/(x)$. Because x is regular on R', we have the exact sequence

$$0 \longrightarrow T/xT \longrightarrow R/(x) \longrightarrow R'/xR' \longrightarrow 0$$

and the associated exact sequence

$$0 \longrightarrow T/xT \longrightarrow H^0_{\mathfrak{m}}(R/(x)) \longrightarrow H^0_{\mathfrak{m}}(R'/xR') \longrightarrow 0,$$

since $H^1(T/xT) = 0$. Observe that this gives

$$\frac{R/(x)}{H^0_{\mathfrak{m}}(R/(x))} \simeq \frac{R'/xR'}{H^0_{\mathfrak{m}}(R'/xR')}.$$

There are two consequences to this calculation:

$$\begin{aligned} \lambda(H^0_{\mathfrak{m}}(R/(x))) &\leq \lambda(H^0_{\mathfrak{m}}(R)) + \lambda(H^0_{\mathfrak{m}}(R'/xR')) \\ &\leq \lambda(H^0_{\mathfrak{m}}(R)) + \lambda(H^1_{\mathfrak{m}}(R)); \\ \lambda(H^i_{\mathfrak{m}}(R/(x))) &\leq \lambda(H^i_{\mathfrak{m}}(R)) + \lambda(H^{i+1}_{\mathfrak{m}}(R)), \quad 1 \leq i \leq d-2. \end{aligned}$$

PROPOSITION 6.1. *Let $(R, \mathfrak{m})$ be a generalized Cohen–Macaulay local ring of positive depth, with I and $\mathcal{A}$ as before. If $d \geq 2$, consider a sequence of $d-1$ reductions of the type $R \to R/(x)$ and denote by S the ring $R/(x_1, \ldots, x_{d-1})$. Then* $\dim S = 1$ *and*

$$\lambda(H^0_{\mathfrak{m}}(S)) \leq T(R) = \sum_{i=1}^{d-1} \binom{d-2}{i-1} \lambda(H^i_{\mathfrak{m}}(R)).$$

Moreover, if R is a Buchsbaum ring then equality holds.

Next we illustrate another elementary but useful kind of reduction. Let $(R, \mathfrak{m})$ be a Noetherian local domain of dimension $d \geq 2$ and let I be an $\mathfrak{m}$-primary ideal. Suppose R has a finite extension S that satisfies the condition S_2 of Serre,

$$0 \to R \to S \to C \to 0.$$

Consider the polynomial ring $R[x, y]$ and tensor the sequence by

$$R' = R[x, y]_{\mathfrak{m}[x,y]}.$$

Then there are $a, b \in I$ such that the ideal generated by polynomial $\mathbf{f} = ax + by$ has the following type of primary decomposition:

$$(\mathbf{f}) = P \cap Q,$$

where P is a minimal prime of $\mathbf{f}$ and Q is $\mathfrak{m}R'$-primary. In the sequence

$$0 \to R' \to S' = R' \otimes_R S \to C' = R' \otimes_R C \to 0,$$

we can find an $a \in I$ that is superficial for C and also pick a $b \in I$ such that (a, b) has codimension 2. Now reduce the second sequence modulo $\mathbf{f} = ax + by$. Noting that $\mathbf{f}$ will be superficial for C', we will have an exact sequence

$$0 \to T \to R'/(\mathbf{f}) \to S'/\mathbf{f}S' \to C'/\mathbf{f}C' \to 0$$

in which T has finite length and $S'/\mathbf{f}S'$ is an integral domain, since a, b is a regular sequence in S. This suffices to establish the assertion.

Finally, we consider generic reductions on R. Let $\mathbf{X}$ be a $d \times (d-1)$ matrix $\mathbf{X} = (x_{ij})$ in $d(d-1)$ indeterminates, and let C be the local ring $R[\mathbf{X}]_{\mathfrak{m}[\mathbf{X}]}$. The filtration $\mathcal{A}C$ has the same Hilbert polynomial as $\mathcal{A}$. If $I = (x_1, \ldots, x_d)$ then we can define the ideal

$$(f_1, \ldots, f_{d-1}) = (x_1, \ldots, x_d) \cdot \mathbf{X}.$$

PROPOSITION 6.2. *Let R be an analytically unramified and generalized Cohen–Macaulay integral domain, and let I and $\mathcal{A}$ be defined as before. Then $S = C/(f_1, \ldots, f_{d-1})$ is a local ring of dimension* 1 *such that $\lambda(H^0_{\mathfrak{m}}(S)) \leq T(R)$ and $S/H^0_{\mathfrak{m}}(S)$ is an integral domain.*

The next result shows the existence of bounds for $\bar{e}_1(I)$ as in [26].

We outline the strategy to find bounds for $e_1(\mathcal{A})$. Suppose that $(R, \mathfrak{m})$ is a local domain essentially of finite type over a field and that I is an $\mathfrak{m}$-primary ideal, and denote by $\mathcal{A}$ a filtration as before. We first achieve a reduction to a one-dimensional ring $R \to R'$, where $e_0(I) = e_0(IR')$, $e_1(\mathcal{A}) = e_1(\mathcal{A}R')$, and T is a prime ideal in the sequence

$$0 \to T = H^0_{\mathfrak{m}}(R') \to R' \to S \to 0.$$

Since S is a one-dimensional integral domain, we have the following result.

PROPOSITION 6.3. *In these conditions,*

$$e_1(\mathcal{A}) = e_1(\mathcal{A}R') = e_1(\mathcal{A}S) - \lambda(T)$$

and

$$e_1(\mathcal{A}S) \leq \bar{e}_1(I) = \lambda(\bar{S}/S),$$

where $\bar{S}$ is the integral closure of S.

This shows that, in order to find bounds for $e_1(\mathcal{A})$, one needs to trace back to the original ring R the properties of T and S. As for T, this is realized if R is a generalized Cohen–Macaulay ring for the reductions described in Proposition 6.1.

THEOREM 6.1 (Existence of bounds). *Let $(R, \mathfrak{m})$ be a local integral domain of dimension d that is essentially of finite type over a perfect field, and let I be an $\mathfrak{m}$-primary ideal. If R is a generalized Cohen–Macaulay ring and if δ is a nonzero element of the Jacobian ideal of R then, for any I-good filtration $\mathcal{A}$ as before,*

$$e_1(\mathcal{A}) < (d-1)e_0(I) + e_0((I, \delta)/(\delta)) - T.$$

Proof. This is a consequence of the proof of [26, Thm. 3.2(a)]. The assertion there is that

$$\bar{e}_1(I) \leq \frac{t}{t+1}[(d-1)e_0(I) + e_0((I+\delta R)/\delta R)],$$

where t is the Cohen–Macaulay type of R. Here we apply it to the reduction in Proposition 6.3,

$$\bar{e}_1(IS) < (d-1)e_0(I) + e_0((I+\delta R)/\delta R),$$

while dropping the term involving t (over which we lose control in the reduction process). Key to the conclusion is that the element δ survives the reduction. □

7. Extended Degrees and Lower Bounds for e_1

The derivation of upper bounds for $e_1(\mathcal{A})$ above required that R be a generalized Cohen–Macaulay ring. Let us eliminate this requirement by working with the variation of the extended degree function hdeg [7; 33] labeled hdeg_I (see [19; 34; p. 142]). The same method will provide lower bounds for $e_1(I)$.

COHOMOLOGICAL DEGREES. Let $(R, \mathfrak{m})$ be a Noetherian local ring (or a standard graded algebra over an Artinian local ring) of infinite residue field. We denote by $\mathcal{M}(R)$ the category of finitely generated R-modules (or the corresponding category of graded R-modules).

A general class of these functions was introduced in [7], and a prototype was defined earlier in [33]. In [13], Gunston carried out a more formal examination of such functions in order to introduce his own construction of a new cohomological degree. One of the points that must be taken care of is that of an appropriate *generic hyperplane* section. Let us recall the setting.

DEFINITION 7.1. For $(R, \mathfrak{m})$ a local ring, a *notion of genericity* on $\mathcal{M}(R)$ is a function

$$U\colon \{\text{isomorphism classes of } \mathcal{M}(R)\} \longrightarrow \{\text{nonempty subsets of } \mathfrak{m}\setminus\mathfrak{m}^2\}$$

that is subject to the following conditions for each $A \in \mathcal{M}(R)$:

(i) if $f - g \in \mathfrak{m}^2$, then $f \in U(A)$ if and only if $g \in U(A)$;
(ii) the set $\overline{U(A)} \subset \mathfrak{m}/\mathfrak{m}^2$ contains a nonempty Zariski-open subset;
(iii) if depth $A > 0$ and $f \in U(A)$, then f is regular on A.

There is a similar definition for graded modules. We shall usually switch notation, denoting the algebra by S.

Another extension is that associated to an $\mathfrak{m}$-primary ideal I [19]: A notion of genericity on $\mathcal{R}$ with respect to I is a function

$$U\colon \{\text{isomorphism classes of } \mathcal{M}(R)\} \longrightarrow \{\text{nonempty subsets of } I\setminus\mathfrak{m}I\}$$

that is subject to the following conditions for each $A \in \mathcal{M}(R)$:

(i) if $f - g \in \mathfrak{m}I$, then $f \in U(A)$ if and only if $g \in U(A)$;
(ii) the set $\overline{U(A)} \subset I/\mathfrak{m}I$ contains a nonempty Zariski-open subset;
(iii) if depth $A > 0$ and $f \in U(A)$, then f is regular on A.

Fixing a notion of genericity $U(\cdot)$, one has the following extension of the classical multiplicity.

DEFINITION 7.2. A *cohomological degree* or *extended multiplicity function* is a function

$$\operatorname{Deg}(\cdot)\colon \mathcal{M}(R) \mapsto \mathbb{N}$$

that satisfies the following conditions.

(i) If $L = \Gamma_{\mathfrak{m}}(M)$ is the submodule of elements of M that are annihilated by a power of the maximal ideal and if $\bar{M} = M/L$, then
$$\operatorname{Deg}(M) = \operatorname{Deg}(\bar{M}) + \lambda(L), \tag{4}$$
where $\lambda(\cdot)$ is the ordinary length function.
(ii) *Bertini's rule.* If M has positive depth, then there exists an $h \in \mathfrak{m}\setminus\mathfrak{m}^2$ such that
$$\operatorname{Deg}(M) \geq \operatorname{Deg}(M/hM). \tag{5}$$
(iii) *Calibration rule.* If M is a Cohen–Macaulay module, then
$$\operatorname{Deg}(M) = \deg(M), \tag{6}$$
where $\deg(M)$ is the ordinary multiplicity of M.

For the case of a notion of genericity relative to an $\mathfrak{m}$-primary ideal I, we have $\deg(M) = e(I; M)$, the Samuel multiplicity of M relative to I.

The existence of cohomological degrees in arbitrary dimensions was established in [33], which allows us to state the following definition.

DEFINITION 7.3. Let M be a finitely generated graded module over the graded algebra A, and let S be a Gorenstein graded algebra mapping onto A with maximal graded ideal $\mathfrak{m}$. Set $\dim S = r$ and $\dim M = d$. The *homological degree* of M is the integer

$$\operatorname{hdeg}(M) = \deg(M) + \sum_{i=r-d+1}^{r} \binom{d-1}{i-r+d-1} \cdot \operatorname{hdeg}(\operatorname{Ext}^i_S(M,S)). \tag{7}$$

This expression becomes more compact when $\dim M = \dim S = d > 0$:

$$\operatorname{hdeg}(M) = \deg(M) + \sum_{i=1}^{d} \binom{d-1}{i-1} \cdot \operatorname{hdeg}(\operatorname{Ext}^i_S(M,S)). \tag{8}$$

REMARK 7.4. Note that this definition morphs easily into an extended degree (denoted hdeg_I), where Samuel multiplicities relative to I are used. The definition of hdeg can be extended to any Noetherian local ring S by setting $\operatorname{hdeg}(M) = \operatorname{hdeg}(\hat{S} \otimes_S M)$. On other occasions, we may also assume that the residue field of S is infinite, an assumption that can be realized by replacing $(S, \mathfrak{m})$ with the local ring $S[X]_{\mathfrak{m}S[X]}$. In fact, if X is any set of indeterminates then the localization is still a Noetherian ring; hence the residue field can be assumed to have any cardinality, as we shall assume in the proofs.

SPECIALIZATION AND TORSION. One of the uses of extended degrees is the following. Let M be a module and let $\mathbf{x} = \{x_1, \dots, x_r\}$ be a superficial sequence for the module M relative to an extended degree Deg. How can we estimate the length of $H^0_{\mathfrak{m}}(M)$ in terms of M?

First consider the case of $r = 1$. Let $H = H^0_{\mathfrak{m}}(M)$ and write

$$0 \longrightarrow H \longrightarrow M \longrightarrow M' \longrightarrow 0. \tag{9}$$

Reduction modulo x_1 gives the exact sequence

$$0 \longrightarrow H/x_1H \longrightarrow M/x_1M \longrightarrow M'/x_1M' \longrightarrow 0. \tag{10}$$

From the first sequence we have $\operatorname{Deg}(M) = \operatorname{Deg}(H) + \operatorname{Deg}(M')$, and from the second we have

$$\operatorname{Deg}(M/x_1M) - \operatorname{Deg}(H/x_1H) = \operatorname{Deg}(M'/x_1M') \le \operatorname{Deg}(M').$$

Taking local cohomology of the second exact sequence yields the short exact sequence

$$0 \longrightarrow H/x_1H \longrightarrow H^0_{\mathfrak{m}}(M/x_1M) \longrightarrow H^0_{\mathfrak{m}}(M'/x_1M') \longrightarrow 0,$$

from which follows the estimation

$$\begin{aligned}\operatorname{Deg}(H^0_{\mathfrak{m}}(M/x_1M)) &= \operatorname{Deg}(H/x_1H) + \operatorname{Deg}(H^0_{\mathfrak{m}}(M'/x_1M')) \\ &\le \operatorname{Deg}(H/x_1H) + \operatorname{Deg}(M'/x_1M') \\ &\le \operatorname{Deg}(H) + \operatorname{Deg}(M') = \operatorname{Deg}(M).\end{aligned}$$

We continue these observations in our next result.

PROPOSITION 7.5. *Let M be a module and let $\{x_1, \dots, x_r\}$ be a superficial sequence relative to M and* Deg. *Then*

$$\lambda(H^0_{\mathfrak{m}}(M/(x_1, \dots, x_r)M)) \le \operatorname{Deg}(M).$$

Now we derive a more precise formula using hdeg. It will be of use later.

THEOREM 7.1. *Let M be a module of dimension $d \geq 2$ and let $\mathbf{x} = \{x_1, \ldots, x_{d-1}\}$ be a superficial sequence for M and* hdeg. *Then*

$$\lambda(H^0_{\mathfrak{m}}(M/(\mathbf{x})M)) \leq \lambda(H^0_{\mathfrak{m}}(M)) + T(M).$$

Proof. Consider the exact sequence

$$0 \longrightarrow H = H^0_{\mathfrak{m}}(M) \longrightarrow M \longrightarrow M' \longrightarrow 0.$$

We have $\operatorname{Ext}^i_S(M, S) = \operatorname{Ext}^i_S(M', S)$ for $d > i \geq 0$, and therefore $T(M) = T(M')$. On the other hand, reduction modulo $\mathbf{x}$ gives

$$\begin{aligned}\lambda(H^0_{\mathfrak{m}}(M/(\mathbf{x})M)) &\leq \lambda(H^0_{\mathfrak{m}}(M'/(\mathbf{x})M')) + \lambda(H/(\mathbf{x})H) \\ &\leq \lambda(H^0_{\mathfrak{m}}(M'/(\mathbf{x})M')) + \lambda(H),\end{aligned}$$

which shows that it is enough to prove the assertion for M'.

If $d > 2$, then the argument in the main theorem of [33] can be used to pass to M'/x_1M'. This reduces all the way to the case $d = 2$. Write $h = \mathbf{x}$. The assertion requires that $\lambda(H^0_{\mathfrak{m}}(M/hM)) \leq \operatorname{hdeg}(\operatorname{Ext}^1_S(M, S))$. We have the cohomology exact sequence

$$\operatorname{Ext}^1_S(M, S) \xrightarrow{h} \operatorname{Ext}^1_S(M, S) \longrightarrow \operatorname{Ext}^2_S(M/hM, S) \longrightarrow \operatorname{Ext}^0_S(M, S) = 0,$$

where

$$\lambda(H^0_{\mathfrak{m}}(M/hM)) = \operatorname{hdeg}(\operatorname{Ext}^2_S(M/hM, S)).$$

If $\operatorname{Ext}^1_S(M, S)$ has finite length then the assertion is clear. Otherwise, $L = \operatorname{Ext}^1_S(M, S)$ is a module of dimension 1 over a discrete valuation domain V with h for its parameter. By the fundamental theorem for such modules,

$$V = V^r \oplus \left(\bigoplus_{j=1}^{s} V/h^{e_j}V\right),$$

so that multiplication by h yields

$$\lambda(L/hL) = r + s \leq r + \sum_{j=1}^{s} e_j = \operatorname{hdeg}(L).$$

An alternative argument at this point is to consider the exact sequence (we may assume $\dim S = 1$)

$$0 \longrightarrow L_0 \longrightarrow L \xrightarrow{h} L \longrightarrow L/hL \longrightarrow 0,$$

where both L_0 and L/hL have finite length. If H denotes the image of the multiplication by h on L, then by dualizing we obtain the short exact sequence

$$0 \longrightarrow \operatorname{Hom}_S(L, S) \xrightarrow{h} \operatorname{Hom}_S(L, S) \longrightarrow \operatorname{Ext}^1_S(L/hL, S) \longrightarrow \operatorname{Ext}^1_S(L, S),$$

which shows that

$$\lambda(L/hL) \leq \deg(L) + \lambda(L_0) = \mathrm{hdeg}(L),$$

as desired. □

We now employ the extended degree hdeg_I to derive lower bounds for $e_1(I)$. We begin by making a crude comparison between $\mathrm{hdeg}(M)$ and $\mathrm{hdeg}_I(M)$.

PROPOSITION 7.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be an $\mathfrak{m}$-primary ideal. Suppose $\mathfrak{m}^r \subset I$. If M is an R-module of dimension d, then*

$$\mathrm{hdeg}_I(M) \leq r^d \cdot \deg(M) + r^{d-1} \cdot (\mathrm{hdeg}(M) - \deg(M)).$$

Proof. If r is the index of nilpotency of R/I then, for any R-module L of dimension s,

$$\lambda(L/(\mathfrak{m}^r)^n L) \geq \lambda(L/I^n L).$$

The Hilbert polynomial of L gives

$$\lambda(L/(\mathfrak{m}^r)^n L) = \deg(M)\frac{r^s}{s!}n^s + \text{lower terms}.$$

We now apply this estimate to the definition of $\mathrm{hdeg}(M)$, taking into account that its terms are evaluated at modules of decreasing dimension. □

THEOREM 7.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, let I be an $\mathfrak{m}$-primary ideal, and let M be a finitely generated R-module of dimension $d \geq 1$. Let $\mathbf{x} = \{x_1, \ldots, x_r\}$ be a superficial sequence in I relative to M and* hdeg_I. *Then*

$$\mathrm{hdeg}_I(M/(\mathbf{x})M) \leq \mathrm{hdeg}_I(M).$$

Moreover, if $r < d$ then

$$H^0(M/(\mathbf{x})M) \leq \mathrm{hdeg}_I(M) - e(I; M).$$

If we apply this to R, passing to $R' = R/(x_1, \ldots, x_{d-1})$, then we have the estimate for $H^0_{\mathfrak{m}}(R')$ so that equation (1) can be used as follows.

COROLLARY 7.7 (Lower bound for $e_1(I)$). *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$. If I is an $\mathfrak{m}$-primary ideal, then*

$$e_1(I) \geq -\mathrm{hdeg}_I(R) + e_0(I).$$

(*Note that* $\mathrm{hdeg}_I(R) - e_0(I)$ *is the Cohen–Macaulay deficiency of R relative to the degree function* hdeg_I.)

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