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Longest Alternating Subsequences of Permutations

Richard P. Stanley

Dedicated to Mel Hochster on the occasion of his sixty-fifth birthday

1. Introduction

Let $\mathfrak{S}_n$ denote the symmetric group of permutations of $1, 2, \dots, n$, and let $w = w_1 \cdots w_n \in \mathfrak{S}_n$. An *increasing subsequence* of w of length k is a subsequence $w_{i_1} \cdots w_{i_k}$ satisfying

$$w_{i_1} < w_{i_2} < \cdots < w_{i_k}.$$

There has been much recent work on the length $\mathrm{is}_n(w)$ of the longest increasing subsequence of a permutation $w \in \mathfrak{S}_n$. A highlight is the asymptotic determination of the expectation $E(n)$ of is_n by Logan–Shepp [11] and Vershik–Kerov [18]:

$$E(n) := \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \mathrm{is}_n(w) \sim 2\sqrt{n}, \quad n \to \infty. \tag{1}$$

Baik, Deift, and Johansson [3] obtained a vast strengthening of this result—namely, the limiting distribution of $\mathrm{is}_n(w)$ as $n \to \infty$. In particular, for w chosen uniformly from $\mathfrak{S}_n$,

$$\lim_{n \to \infty} \mathrm{Prob}\left(\frac{\mathrm{is}_n(w) - 2\sqrt{n}}{n^{1/6}} \le t \right) = F(t), \tag{2}$$

where $F(t)$ is the Tracy–Widom distribution. The proof uses a result of Gessel [9] that gives a generating function for the quantity

$$u_k(n) = \#\{w \in \mathfrak{S}_n : \mathrm{is}(w) \le k\}.$$

Namely, define

$$U_k(x) = \sum_{n \ge 0} u_k(n) \frac{x^{2n}}{n!^2}, \quad k \ge 1;$$

$$I_i(2x) = \sum_{n \ge 0} \frac{x^{2n+i}}{n!\,(n+i)!}, \quad i \in \mathbb{Z}.$$

The function I_i is the *hyperbolic Bessel function* of the first kind of order i. Note that $I_i(2x) = I_{-i}(2x)$. Gessel then showed that

$$U_k(x) = \det(I_{i-j}(2x))_{i,j=1}^{k}.$$

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In this paper we will develop an analogous theory for *alternating subsequences*—that is, subsequences $w_{i_1}\cdots w_{i_k}$ of w satisfying

$$w_{i_1} > w_{i_2} < w_{i_3} > w_{i_4} < \cdots w_{i_k}.$$

According to our definition, an alternating sequence $a, b, c, \dots$ (of length ≥ 2) must begin with a descent $a > b$. Let $\mathrm{as}(w) = \mathrm{as}_n(w)$ denote the length (number of terms) of the longest alternating subsequence of $w \in \mathfrak{S}_n$, and let

$$a_k(n) = \#\{w \in \mathfrak{S}_n : \mathrm{as}(w) = k\}.$$

For instance, $a_1(w) = 1$, corresponding to the permutation $12\cdots n$, while $a_n(n)$ is the total number of alternating permutations in $\mathfrak{S}_n$. This number is customarily denoted E_n. A celebrated result of André [1] (see [16, Sec. 3.16]) states that

$$\sum_{n\geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x. \tag{3}$$

The numbers E_n were first considered by Euler (using (3) as their definition) and are known as *Euler numbers.* Because of (3), E_{2n} is also known as a *secant number* and E_{2n-1} as a *tangent number.*

Define

$$\begin{aligned} b_k(n) &= \#\{w \in \mathfrak{S}_n : \mathrm{as}(w) \leq k\} \\ &= a_1(n) + a_2(n) + \cdots + a_k(n) \end{aligned} \tag{4}$$

so that, for example, $b_k(n) = n!$ for $k \geq n$. Also define the generating functions

$$A(x,t) = \sum_{k,n\geq 0} a_k(n) t^k \frac{x^n}{n!}, \tag{5}$$

$$B(x,t) = \sum_{k,n\geq 0} b_k(n) t^k \frac{x^n}{n!}.$$

Our main result (Theorem 2.3) consists of the formulas

$$\begin{aligned} B(x,t) &= \frac{1+\rho+2te^{\rho x}+(1-\rho)e^{2\rho x}}{1+\rho-t^2+(1-\rho-t^2)e^{2\rho x}}, \\ A(x,t) &= (1-t)B(x,t), \end{aligned} \tag{6}$$

where $\rho = \sqrt{1-t^2}$.

As a consequence, we obtain explicit formulas for $a_k(n)$ and $b_k(n)$:

$$\begin{aligned} b_k(n) &= \frac{1}{2^{k-1}} \sum_{\substack{r+2s\leq k \\ r\equiv k \ (\mathrm{mod}\ 2)}} (-2)^s \binom{k-s}{(k+r)/2}\binom{n}{s} r^n; \\ a_k(n) &= b_k(n) - b_{k-1}(n). \end{aligned}$$

By equation (6) we also obtain formulas for the factorial moments

$$\nu_k(n) = \frac{1}{n!} \sum_{w\in\mathfrak{S}_n} \mathrm{as}(w)(\mathrm{as}(w)-1)\cdots(\mathrm{as}(w)-k+1).$$

For instance, the mean $\nu_1(n)$ and variance $\mathrm{var}(\mathrm{as}_n) = \nu_2(n) + \nu_1(n) - \nu_1(n)^2$ are given by

$$\begin{aligned} \nu_1(n) &= \frac{4n+1}{6}, \quad n \geq 2, \\ \mathrm{var}(\mathrm{as}_n) &= \frac{8}{45}n - \frac{13}{180}, \quad n \geq 4. \end{aligned} \tag{7}$$

The limiting distribution of as_n (the analogue of equation (2)) was obtained independently by Pemantle and Widom, as discussed at the end of Section 3. Instead of the Tracy–Widom distribution as in (2), this time we obtain a Gaussian distribution.

NOTE. We can give an alternative description of $b_k(n)$ in terms of pattern avoidance. If $v = v_1 v_2 \cdots v_k \in \mathfrak{S}_k$, then we say that a permutation $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$ *avoids* v if w has no subsequence $w_{i_1} w_{i_2} \cdots w_{i_k}$ whose terms are in the same relative order as v [6, Chap. 4.5; 17, Sec. 7]. If $X \subset \mathfrak{S}_k$, then we say that $w \in \mathfrak{S}_n$ *avoids* X if w avoids all $v \in X$. Now note that $b_{k-1}(n)$ is the number of permutations $w \in \mathfrak{S}_n$ that avoid all E_k alternating permutations in $\mathfrak{S}_k$.

After seeing the first draft of this paper, Miklós Bóna pointed out that the statistic as_n can be expressed very simply in terms of a previously considered statistic on $\mathfrak{S}_n$: the number of *alternating runs.* Hence many of our results can also be deduced from known results on alternating runs. This development is discussed further in Section 4. In particular, it follows from [20] that the polynomials $T_n(t) = \sum_k a_k(n) t^k$ have interlacing real zeros. This result can be used to give a third proof (in addition to the proofs of Pemantle and Widom) that the limiting distribution of as_n is Gaussian.

2. The Main Generating Function

The key result that allows us to obtain explicit formulas is the following lemma.

LEMMA 2.1. *Let $w \in \mathfrak{S}_n$. Then there is an alternating subsequence of w of maximum length that contains n.*

Proof. Let $a_1 > a_2 < \cdots a_k$ be an alternating subsequence of w of maximum length $k = \mathrm{as}(w)$, and suppose that n is not a term of this subsequence. If n precedes a_1 in w, then we can replace a_1 by n and obtain an alternating subsequence of length k containing n. If n appears between a_i and a_{i+1} in w, then we can similarly replace the larger of a_i and a_{i+1} by n. Finally, suppose that n appears to the right of a_k. If k is even then we can append n to the end of the subsequence to obtain a longer alternating subsequence, contradicting the definition of k. But if k is odd then we can replace a_k by n, again obtaining an alternating subsequence of length k containing n. □

We can use Lemma 1.2 to obtain a recurrence for $a_k(n)$, beginning with the initial condition $a_0(0) = 1$.

LEMMA 2.2. *Let* $1 \leq k \leq n+1$. *Then*

$$a_k(n+1) = \sum_{j=0}^{n} \binom{n}{j} \sum_{\substack{2r+s=k-1 \\ r,s\geq 0}} (a_{2r}(j) + a_{2r+1}(j)) a_s(n-j). \tag{8}$$

Proof. We can choose a permutation $w = a_1 \cdots a_{n+1} \in \mathfrak{S}_{n+1}$ such that $\mathrm{as}(w) = k$ as follows. First choose $0 \leq j \leq n$ such that $a_{j+1} = n+1$. Then choose in $\binom{n}{j}$ ways the set $\{a_1, \ldots, a_j\}$. For $s \geq 0$ we can choose in $a_s(n-j)$ ways a permutation $w' = a_{j+2} \cdots a_{n+1}$ satisfying $\mathrm{as}(w') = s$. Next we choose a permutation $w'' = a_1 \cdots a_j$ such that the longest *even* length of an alternating subsequence of w'' is $2r = k-1-s$. We can choose w'' to satisfy either $\mathrm{as}(w'') = 2r$ or $\mathrm{as}(w'') = 2r+1$. The concatenation $w = w''(n+1)w' \in \mathfrak{S}_{n+1}$ will then satisfy $\mathrm{as}(w) = k$, and conversely all such w arise in this way. Hence equation (8) follows. □

Now write

$$F_k(x) = \sum_{n\geq 0} a_k(n) \frac{x^n}{n!}.$$

For example, $F_0(x) = 1$ and $F_1(x) = e^x - 1$. Multiplying (8) by $x^n/n!$ and summing on $n \geq 0$ gives

$$F_k'(x) = \sum_{2r+s=k-1} (F_{2r}(x) + F_{2r+1}(x)) F_s(x). \tag{9}$$

Observe that

$$A(x,t) = \sum_{k\geq 0} F_k(x) t^k,$$

where $A(x,t)$ is defined by (5). Since $k-1-s$ is even in (9), we need to work with the even part $A_e(x,t)$ and odd part $A_o(x,t)$ of $A(x,t)$, which are defined by

$$\begin{aligned} A_e(x,t) &= \sum_{k\geq 0} F_{2k}(x) t^{2k} \\ &= \frac{1}{2}(A(x,t) + A(x,-t)), \\ A_o(x,t) &= \sum_{k\geq 0} F_{2k+1}(x) t^{2k+1} \\ &= \frac{1}{2}(A(x,t) - A(x,-t)). \end{aligned} \tag{10}$$

Multiply equation (9) by t^k and sum on $k \geq 0$. We obtain

$$\frac{\partial A(x,t)}{\partial x} = tA_e(x,t)A(x,t) + A_o(x,t)A(x,t). \tag{11}$$

Substituting $-t$ for t yields

$$\frac{\partial A(x,-t)}{\partial x} = -tA_e(x,t)A(x,-t) - A_o(x,t)A(x,-t). \tag{12}$$

Adding and subtracting equations (11) and (12) gives the following system of differential equations for $A_e = A_e(x,t)$ and $A_o = A_o(x,t)$:

$$\frac{\partial A_e}{\partial x} = tA_eA_o + A_o^2, \tag{13}$$

$$\frac{\partial A_o}{\partial x} = tA_e^2 + A_eA_o. \tag{14}$$

Hence we must solve this system of equations in order to find

$$A(x,t) = A_e(x,t) + A_o(x,t).$$

THEOREM 2.3. *We have*

$$B(x,t) = \frac{1+\rho+2te^{\rho x}+(1-\rho)e^{2\rho x}}{1+\rho-t^2+(1-\rho-t^2)e^{2\rho x}}, \tag{15}$$

$$A(x,t) = (1-t)B(x,t) \tag{16}$$

$$= (1-t)\frac{1+\rho+2te^{\rho x}+(1-\rho)e^{2\rho x}}{1+\rho-t^2+(1-\rho-t^2)e^{2\rho x}}, \tag{17}$$

where $\rho = \sqrt{1-t^2}$.

Proof. We can simply verify that the stated expression (17) for $A(x,t)$ satisfies (13) and (14) with the initial condition $A(0,t) = 1$, a routine computation (especially with the use of a computer). The relationship (16) between $A(x,t)$ and $B(x,t)$ is then an immediate consequence of (4), which is equivalent to $a_k(n) = b_k(n) - b_k(n-1)$.

It might be of interest, though, to explain how the formula (17) for $A(x,t)$ can be derived if the answer is not known in advance. If we divide equation (13) by (14), the result is

$$\frac{\partial A_e/\partial x}{\partial A_o/\partial x} = \frac{A_o}{A_e}.$$

Therefore, $\frac{\partial}{\partial x}(A_e^2 - A_o^2) = 0$ and so $A_e^2 - A_o^2$ is independent of x. This observation suggests computing the generating function in t for $A_e^2 - A_o^2$, which a computer shows is equal to $1 + O(t^N)$ for a large value of N. Assuming then that $A_e^2 - A_o^2 = 1$ (or even proving it combinatorially), we can substitute $\sqrt{1-A_e^2}$ for A_o in (13) to obtain

$$\frac{\partial A_e}{\partial x} = tA_e\sqrt{A_e^2-1} + A_e^2 - 1,$$

a single differential equation for A_e. This equation can routinely be solved by separation of variables (though some care must be taken to choose the correct branch of the resulting integral, including the correct sign of $\sqrt{A_e^2-1}$); we will spare the reader the details. A similar argument yields A_o, so we obtain $A = A_e + A_o$. □

NOTE. Ira Gessel has pointed out the following simplified expression for $B(x,t)$:

$$B(x,t) = \frac{2/\rho}{1-\frac{1-\rho}{t}e^{\rho x}} - \frac{1}{\sqrt{1-t^2}}. \tag{18}$$

3. Consequences

A number of corollaries follow from Theorem 2.3. The first gives explicit expressions for $a_k(n)$ and $b_k(n)$, as stated in the Introduction. I am grateful to Ira Gessel for providing the proof given here.

COROLLARY 3.1. *For all* $k,n \geq 1$,

$$b_k(n) = \frac{1}{2^{k-1}} \sum_{\substack{r+2s\leq k \\ r\equiv k \ (\mathrm{mod}\, 2)}} (-2)^s \binom{k-s}{(k+r)/2}\binom{n}{s} r^n, \tag{19}$$

$$a_k(n) = b_k(n) - b_{k-1}(n). \tag{20}$$

Proof. Define $b'_k(n)$ to be the right-hand side of (19), and set

$$B'(x,t) = \sum_{k,n\geq 0} b'_k(n) t^k \frac{x^n}{n!}.$$

Set $n = s+m$ and $k = r+2s+2l$, so that

$$\begin{aligned} B'(x,t) &= \sum_{r,s,l,m} (-1)^s 2^{1-r-s-2l}\binom{r+s+2l}{r+s+l}\binom{s+m}{s} r^{s+m} t^{r+2s+2l}\frac{x^{s+m}}{(s+m)!} \\ &= 2\sum_{r,s\geq 0}\left(\frac{t}{2}\right)^r \frac{(-rt^2x/2)^s}{s!}\left[\sum_l \binom{r+s+2l}{l}\left(\frac{t^2}{4}\right)^l\right]\left[\sum_m \frac{(rx)^m}{m!}\right]. \end{aligned} \tag{21}$$

The sum on m is e^{rx}. Now let

$$C(u) = \sum_{n\geq 0} C_n u^n = \frac{1-\sqrt{1-4x}}{2x},$$

the generating function for the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$. If $G(x) = xC(x)$ then $G(x) = (x-x^2)^{\langle -1\rangle}$, where ${}^{\langle -1\rangle}$ denotes compositional inverse. It is then an immediate consequence of the Lagrange inversion formula [16, Thm. 5.4.2] that

$$C(u)^a = \sum_{k\geq 0}\frac{a}{k+a}\binom{2k-1+a}{k}u^k.$$

Differentiating both sides of $(uC(u))^a$ with respect to u, we obtain the formula

$$\sum_k \binom{2k+a}{k}u^k = \frac{C(u)^a}{\sqrt{1-4u}}.$$

Hence the sum on l in equation (21) is

$$\frac{C(t^2/4)^{r+s}}{\sqrt{1-t^2}} = \frac{1}{\rho}\left(\frac{2-2\rho}{t^2}\right)^{r+s}.$$

Thus

$$\begin{aligned} B'(x,t) &= \frac{2}{\rho}\sum_{r,s\geq 0}\left(\frac{t}{2}\right)^r \frac{(-rt^2x/2)^s}{s!}e^{rx}\left(\frac{2-2\rho}{t^2}\right)^{r+s} \\ &= \frac{2}{\rho}\sum_r \left(\frac{1-\rho}{t}e^x\right)^r \sum_s \frac{(-r(1-\rho)x)^s}{s!} \\ &= \frac{2}{\rho}\sum_r \left(\frac{1-\rho}{t}e^x\right)^r e^{-r(1-\rho)x} \\ &= \frac{2}{\rho}\left(1-\frac{1-\rho}{t}e^{\rho x}\right)^{-1}, \end{aligned}$$

and the proof of (19) follows from (18). Equation (20) is then an immediate consequence of (4). □

By Corollary 3.1, when k is fixed $b_k(n)$ is a linear combination of $k^n, (k-2)^n, (k-4)^n, \ldots$ with coefficients that are polynomials in n. For $k \leq 6$ we have

$$\begin{aligned} b_2(n) &= 2^{n-1}, \\ b_3(n) &= \frac{1}{4}(3^n - 2n + 3), \\ b_4(n) &= \frac{1}{8}(4^n - 2(n-2)2^n), \\ b_5(n) &= \frac{1}{16}(5^n - (2n-5)3^n + 2(n^2 - 5n + 5)), \\ b_6(n) &= \frac{1}{32}(6^n - 2(n-3)4^n + (2n^2 - 12n + 15)2^n). \end{aligned}$$

As a further application of Theorem 2.3 we can obtain the factorial moment-generating function

$$F(x,t) = \sum_{j,n\geq 0} \nu_j(n) x^n \frac{t^j}{j!},$$

where

$$\nu_j(n) = \frac{1}{n!}\sum_{w\in\mathfrak{S}_n} (\mathrm{as}(w))_j = \frac{1}{n!}\sum_k a_k(n)(k)_j$$

and

$$(h)_j = h(h-1)\cdots(h-j+1).$$

Namely, we have

$$\begin{aligned} \left.\frac{\partial^j A(x,t)}{\partial t^j}\right|_{t=1} &= \sum_{n\geq 0}\frac{1}{n!}\sum_{k\geq 0} a_k(n)(k)_j x^n \\ &= \sum_{n\geq 0} \nu_j(n) x^n. \end{aligned}$$

On the other hand, by Taylor's theorem we have

$$A(x,t) = \sum_{j\geq 0} \left.\frac{\partial^j A(x,t)}{\partial t^j}\right|_{t=1} \frac{(t-1)^j}{j!}.$$

It follows that

$$F(x,t) = A(x,t+1). \tag{22}$$

(It is far from obvious from the form of $A(x,t+1)$ obtained by substituting $t+1$ for t in (17) that it even has a Taylor series expansion at $t=0$.) From equations (17) and (22) it is easy to compute (using a computer) the generating functions

$$M_j(x) = \sum_{n\geq 0} \nu_j(n) x^n$$

for small j. For $1 \leq j \leq 4$ we have

$$\begin{aligned}
M_1(x) &= \frac{6x - 3x^2 + x^3}{6(1-x)^2},\\
M_2(x) &= \frac{90x^2 - 15x^4 + 6x^5 - x^6}{90(1-x)^3},\\
M_3(x) &= \frac{2520x^3 - 315x^4 + 189x^5 - 231x^6 + 93x^7 - 18x^8 + 2x^9}{1260(1-x)^4},\\
M_4(x) &= \frac{N_4(x)}{9450(1-x)^5},
\end{aligned}$$

where

$$\begin{aligned}
N_4(x) = {} & 47250x^4 - 3780x^6 + 2880x^7 - 2385x^8 + 1060x^9\\
& - 258x^{10} + 36x^{11} - 3x^{12}.
\end{aligned}$$

It is not difficult to see that, in general, $M_j(x)$ is a rational function of x with denominator $(1-x)^{j+1}$. It follows from standard properties of rational generating functions [15, Sec. 4.3] that, for fixed j, $\nu_j(n)$ is a polynomial in n of degree j for n sufficiently large. In particular,

$$\nu_1(n) = \frac{4n+1}{6}, \quad n \geq 2; \tag{23}$$

$$\nu_2(n) = \frac{40n^2 - 24n - 19}{90}, \quad n \geq 4;$$

$$\nu_3(n) = \frac{1120n^3 - 2856n^2 + 440n + 1581}{3780}, \quad n \geq 6.$$

Observe that $\nu_1(n)$ is just the expectation (mean) of as_n. The simple formula $(4n+1)/6$ for this quantity should be contrasted with the situation for the length $\mathrm{is}_n(w)$ of the longest increasing subsequence of $w \in \mathfrak{S}_n$, where even the asymptotic formula $E(n) \sim 2\sqrt{n}$ for the expectation is a highly nontrivial result [17, Sec. 3]. A simple proof of (23) follows from (28) and an argument of Knuth [10, Exer. 5.1.3.15].

From the formulas for $\nu_1(n)$ and $\nu_2(n)$ we easily compute the variance $\mathrm{var}(\mathrm{as}_n)$ of as_n:

$$\mathrm{var}(\mathrm{as}_n) = \nu_2(n) + \nu_1(n) - \nu_1(n)^2 = \frac{32n-13}{180}, \quad n \geq 4. \tag{24}$$

We now consider a further application of Theorem 2.3. Let

$$T_n(t) = \sum_{k=0}^{n} a_k(n) t^k. \tag{25}$$

For instance,

$$\begin{aligned}
T_1(t) &= t,\\
T_2(t) &= t + t^2,\\
T_3(t) &= t + 3t^2 + 2t^3,\\
T_4(t) &= t + 7t^2 + 11t^3 + 5t^4,\\
T_5(t) &= t + 15t^2 + 43t^3 + 45t^4 + 16t^5,\\
T_6(t) &= t + 31t^2 + 148t^3 + 268t^4 + 211t^5 + 61t^6,\\
T_7(t) &= t + 63t^2 + 480t^3 + 1344t^4 + 1767t^5 + 1113t^6 + 272t^7.
\end{aligned}$$

COROLLARY 3.2. *The polynomial $T_n(t)$ is divisible by $(1+t)^{\lfloor n/2 \rfloor}$. Moreover, if $U_n(t) = T_n(t)/(1+t)^{\lfloor n/2 \rfloor}$, then*

$$U_{2n}(-1) = -U_{2n+1}(-1) = \frac{(-1)^n E_{2n+1}}{2^n},$$

where E_{2n+1} denotes a tangent number.

Proof. Let $A_e(x,t)$ and $A_o(x,t)$ be the even and odd parts of $A(x,t)$ as in equations (10). By the definition of $A_e(x)$ we have

$$A_e\left(\frac{x}{\sqrt{1+t}}, t\right) = \sum_{n \geq 0} \frac{T_{2n}(t)}{(1+t)^n} \frac{x^{2n}}{(2n)!}.$$

With the help of a computer we establish that

$$\begin{aligned}
\lim_{t \to -1} A_e\left(\frac{x}{\sqrt{1+t}}, t\right) &= \operatorname{sech}^2 \frac{x}{\sqrt{2}}\\
&= \sum_{n \geq 0} \frac{(-1)^n E_{2n+1}}{2^n} \frac{x^{2n}}{(2n)!}.
\end{aligned}$$

Hence the desired result is true for $T_{2n}(t)$. Similarly,

$$\begin{aligned}
\lim_{t \to -1} \sqrt{1+t}\, A_o\left(\frac{x}{\sqrt{1+t}}, t\right) &= -\sqrt{2} \tanh \frac{x}{\sqrt{2}}\\
&= -\sum_{n \geq 0} \frac{(-1)^n E_{2n+1}}{2^n} \frac{x^{2n+1}}{(2n+1)!},
\end{aligned}$$

proving the result for $T_{2n+1}(t)$. □

By Corollary 3.2, $T_n(-1) = 0$ for $n \geq 2$. In other words, for $n \geq 2$ we have

$$\#\{w \in \mathfrak{S}_n : \mathrm{as}_n(w) \text{ even}\} = \#\{w \in \mathfrak{S}_n : \mathrm{as}_n(w) \text{ odd}\} = \frac{n!}{2}.$$

A simple combinatorial proof of this fact follows from switching the last two elements of w; it is easy to see that this operation either increases or decreases $\mathrm{as}_n(w)$ by 1, as first pointed out by M. Bóna and P. Pylyavskyy. More generally, a combinatorial proof of Corollary 3.2 is a consequence of equation (28) to follow and an argument of Bóna [6, Lemma 1.40].

The formulas (23) and (24) for the mean and variance of as_n suggest in analogy with (2) that as_n will have a limiting distribution $K(t)$ defined by

$$K(t) = \lim_{n\to\infty} \mathrm{Prob}\left(\frac{\mathrm{as}_n(w) - 2n/3}{\sqrt{n}} \leq t\right)$$

for all $t \in \mathbb{R}$, where w is chosen uniformly from $\mathfrak{S}_n$. Indeed, we have that $K(t)$ is a Gaussian distribution with variance $8/45$:

$$K(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{t\sqrt{45}/4} e^{-s^2}\, ds. \tag{26}$$

It was pointed out by R. Pemantle (private communication) that equation (26) is a consequence of [13, Thms. 3.1, 3.3, or 3.5] and possibly [5]. An independent proof was also given by Widom [19], and in the next section we offer an additional method.

4. Relationship to Alternating Runs

A *run* of a permutation $w = w_1 \cdots w_n \in \mathfrak{S}_n$ is a maximal factor (subsequence of consecutive elements) that is increasing. An *alternating run* is a maximal factor that is increasing or decreasing. (Perhaps "birun" would be a better term.) For instance, the permutation 64283157 has four alternating runs (642, 28, 831, and 157). Let $g_k(n)$ be the number of permutations $w \in \mathfrak{S}_n$ with k alternating runs. As pointed out by Bóna [7], it is easy to see that

$$a_k(n) = \frac{1}{2}(g_{k-1}(n) + g_k(n)), \quad n \geq 2. \tag{27}$$

If we define $G_n(t) = \sum_k g_k(n)t^k$, then (27) is equivalent to

$$T_n(t) = \frac{1}{2}(1+t)G_n(t), \tag{28}$$

where $T_n(t)$ is defined by (25).

Research on the numbers $g_k(n)$ goes back to the nineteenth century; for references see Bóna [6, Sec. 1.2] and Knuth [10, Exer. 5.1.3.15–16]. In particular, let $A_n(t)$ denote the nth *Eulerian polynomial*; that is,

$$A_n(t) = \sum_{w \in \mathfrak{S}_n} t^{1+\mathrm{des}(w)},$$

where $\operatorname{des}(w)$ denotes the number of descents of w (the size of the descent set defined in equation (29)). It was shown by David and Barton [8, pp. 157–162] and stated more concisely by Knuth [10, p. 605] that

$$G_n(t) = \left(\frac{1+t}{2}\right)^{n-1} (1+w)^{n+1} A_n\left(\frac{1-w}{1+w}\right), \quad n \geq 2,$$

where $w = \sqrt{(1-t)/(1+t)}$. Theorem 2.3 is then a straightforward consequence of the well-known generating function

$$\sum_{n \geq 0} A_n(t) \frac{x^n}{n!} = \frac{1-t}{1-te^{(1-t)x}}$$

(see e.g. [6, Thm. 1.7]).

It is also well known [6, Thm. 1.10] that the Eulerian polynomial $A_n(t)$ has only real zeros and that the zeros of $A_n(t)$ and $A_{n+1}(t)$ interlace. From this fact Wilf [20] showed that the polynomials $G_n(t)$ have (interlacing) real zeros, and hence by (28) the polynomials $T_n(t)$ also have real zeros. It is then a consequence of standard results (e.g., [4, Thm. 2]) that the numbers $a_k(n)$ for fixed n are asymptotically normal as $n \to \infty$, yielding another proof of (26).

5. Open Problems

In this section we mention three directions in which our work in this paper could be generalized.

1. Let $\operatorname{is}(m, w)$ denote the length of the longest subsequence of $w \in \mathfrak{S}_n$ that is a union of m increasing subsequences, so that $\operatorname{is}(w) = \operatorname{is}(1, w)$. The numbers $\operatorname{is}(m, w)$ have many interesting properties, as summarized in [17, Sec. 4]. Can anything be said about the analogue for alternating sequences—that is, the length $\operatorname{as}(m, w)$ of the longest subsequence of w that is a union of m alternating subsequences? This question can also be formulated in terms of the lengths of the alternating runs of w.

2. Can the results for increasing subsequences and alternating subsequences be generalized to other "patterns"? More specifically, let σ be a (finite) word in the letters U and D; for example, $\sigma = UUDUD$. Let σ^∞ denote the infinite word $\sigma\sigma\sigma\cdots$, as in

$$(UUD)^\infty = UUDUUDUUD\cdots.$$

In this example we have, for instance, that $UUDUUDU$ is a prefix of σ^∞ of length 7.

Let $\tau = a_1 a_2 \cdots a_{m-1}$ be a word of length $m-1$ in the letters U and D. A sequence $v = v_1 v_2 \cdots v_m$ of integers is said to have *descent word* τ if $v_i > v_{i+1}$ whenever $a_i = D$ but $v_i < v_{i+1}$ whenever $a_i = U$. Hence v is increasing if and only if $\tau = U^{m-1}$, and v is alternating if and only if $\tau = (DU)^{j-1}$ or $\tau = (DU)^{j-1}D$ according as $m = 2j-1$ or $m = 2j$.

Now let $w \in \mathfrak{S}_n$ and define $\operatorname{len}_\sigma(w)$ to be the length of longest subsequence of w whose descent word is a prefix of σ^∞. Thus $\operatorname{len}_U(w) = \operatorname{is}_n(w)$ and $\operatorname{len}_{DU}(w) = \operatorname{as}_n(w)$. What can be said in general about $\operatorname{len}_\sigma(w)$? In particular, let

$$E_\sigma(n) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \mathrm{len}_\sigma(w),$$

the expectation of $\mathrm{len}_\sigma(w)$ for $w \in \mathfrak{S}_n$. Note that $E_U(n) \sim 2\sqrt{n}$ by (1) and that $E_{DU}(n) \sim 2n/3$ by (7). Is it true that for any σ we have $E_\sigma(n) \sim \alpha n^c$ for some $\alpha, c > 0$? Or at least that for some $c > 0$ (depending on σ) we have

$$\lim_{n \to \infty} \frac{\log E_\sigma(n)}{\log n} = c$$

(in which case can we determine c explicitly)?

3. The *descent set* $D(w)$ of a permutation $w = w_1 \cdots w_n$ is defined by

$$D(w) = \{i : w_i > w_{i+1}\} \subseteq [n-1], \tag{29}$$

where $[n-1] = \{1, 2, \ldots, n-1\}$. Thus w is alternating if and only if $D(w) = \{1, 3, 5, \ldots\} \cap [n-1]$. Let $S \subseteq [k-1]$. What can be said about the number $b_{k,S}(n)$ of permutations $w \in \mathfrak{S}_n$ that avoid all $v \in \mathfrak{S}_k$ satisfying $D(v) = S$? In particular, what is the value $L_{k,S} = \lim_{n\to\infty} b_{k,S}(n)^{1/n}$? (It follows from [2] and [12], generalized in an obvious way, that this limit exists and is finite.) For example, if $S = \emptyset$ or $S = [k-1]$, then it follows from [14] that $L_{k,S} = (k-1)^2$. On the other hand, if $S = \{1, 3, 5, \ldots\} \cap [k-1]$ then it follows from (19) that $L_{k,S} = k-1$.

Added in proof. The statement in part 3 of Section 5 that $L_{k,S}$ exists is open, since Arratia's paper [2] deals with the avoidance of a *single* permutation.

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Department of Mathematics 2-375
Massachusetts Institute of Technology
Cambridge, MA 02139

rstan@math.mit.edu