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Rational Singularities Associated to Pairs

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Dedicated to Professor Mel Hochster
on the occasion of his sixty-fifth birthday

1. Introduction and Background

Rational singularities are a class of singularities that have been heavily studied since their introduction in the 1960s. Roughly speaking, an algebraic variety has rational singularities if its structure sheaf has the same cohomology as the structure sheaf of a resolution of singularities. Rational singularities enjoy many useful properties; in particular, they are both normal and Cohen–Macaulay. Furthermore, many common varieties have rational singularities, including toric varieties and quotient varieties. Rational singularities are also known to be closely related to the singularities of the minimal model program. In particular, it is known that log terminal singularities are rational and that Gorenstein rational singularities are canonical.

There is, however, an important distinction between rational singularities and singularities of the minimal model program. In the minimal model program it is natural to consider pairs (X, D), where X is a variety and D is a $\mathbb{Q}$-divisor. In recent years the study of pairs $(X, \mathfrak{a}^c)$, where $\mathfrak{a}$ is an ideal sheaf and c is a positive real number, has also become quite common. Thus it is natural to try to extend the notion of rational singularities to pairs. We define two notions of rational pairs: a *rational pair,* which is analogous to a Kawamata log terminal (klt) pair; and a *purely rational triple,* which is analogous to a purely log terminal (plt) triple (we will discuss the characteristic-p analogues in Section 5). It is hoped that these definitions and their study will help further the understanding both of rational singularities and of log terminal pairs.

In characteristic 0, defining rational singularities for pairs has one distinct advantage over the corresponding variants of log terminal singularities. In order for (X, D) to be log terminal, one necessarily must have $K_X + D$ a $\mathbb{Q}$-Cartier divisor. Likewise, for the pair $(X, \mathfrak{a}^c)$ to be log terminal, X must necessarily be $\mathbb{Q}$-Gorenstein. One can define rational singularities for a pair $(X, \mathfrak{a}^c)$ without any such conditions on X.

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Virtually all standard properties of rational singularities transfer to pairs, as we show. In particular, summands and deformations behave well (see Corollary 4.11 and Theorem 4.13), as do various implications between log terminal and rational pairs (see Proposition 4.1 and Proposition 4.2). For the most part, the proofs are generalizations of proofs of the analogous properties of rational singularities. Because singularities of pairs come up naturally in theorems related to adjunction and inversion of adjunction, we prove that several of these results extend to rational pairs as well. In particular, we are able to prove a "rational" analogue of inversion of adjunction for log terminal pairs (see Theorem 4.14). Using a similar technique, we are able to give a remarkably short proof of an analogue of inversion of adjunction on log canonicity that uses the notion of Du Bois singularities (see Theorem 4.16).

Since the early 1980s, it has been known that rational singularities are closely related to singularities defined by the action of a Frobenius map in positive characteristic; see [F]. After the introduction of tight closure by Hochster and Hunkeke [HoHu2], a true characteristic-p analogue of rational singularities, F-rationality, was defined; see [FW]. In the next decade it was shown that a variety has rational singularities if and only if a generic positive characteristic model has F-rational singularities [H1; MeSr; Sm2]. Thus, we also define F-rationality for pairs. Directly in positive characteristic, we are able to show that F-rational pairs satisfy many of the same basic properties that rational pairs do in characteristic 0; see Propositions 6.5, 6.15, 7.3, and 7.1 as well as Theorem 7.7. Furthermore, building on the techniques of Hara and Yoshida [HY], we are able to show a direct correspondence between F-rational and rational pairs (see Theorem 6.11).

We also relate this to a notion that has existed for many years and was defined and studied for pairs in the toric setting by Blickle: the *multiplier submodule* (see [B; H2; HySm; Sm1]). Multiplier ideals and generalized test ideals (their positive-characteristic analogue) have been studied extensively in recent years as powerful invariants that measure singularities of pairs. For example, a pair is Kawamata log terminal (resp. F-regular) if and only if the corresponding multiplier ideal (resp. generalized test ideal) is the entire ring. When formulating rational singularities associated to pairs, instead of a (multiplier) ideal it is natural to consider a submodule of the canonical module, an object called the *multiplier submodule* (the characteristic-p analogue has been studied under the name "parameter test submodule"); see Definitions 3.6 and 6.4. Many questions asked about multiplier ideals can also be asked about multiplier submodules; in particular, we look at an analogue of the log canonical threshold in characteristic 0 and also in positive characteristic; see Definitions 4.7 and 7.5. We also define jumping exponents for generalized parameter test submodules and show that these numbers form a discrete set of rational numbers under certain conditions; see Definition 7.9 and Corollary 7.13.

Most of the techniques in this paper are not new. They are either techniques related to rational and F-rational singularities or techniques related to log terminal and F-regular singularities. Nonetheless, one might view these techniques extending so easily to the cases we consider as further evidence that this generalization of rational singularities to pairs is a natural one.

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2. Preliminaries in Characteristic 0

All schemes in this paper will be assumed to be separated, Noetherian, and of essentially finite type over a field. For Y a scheme, we will often work in the derived category of $\mathcal{O}_Y$-modules, denoted by $D(Y)$. The symbol $D^b(Y)$ (resp. $D^+(Y)$, $D^-(Y)$) will denote the derived category of bounded (resp. bounded below, bounded above) complexes of $\mathcal{O}_Y$-modules. Let $D_{\mathrm{coh}}(Y)$ (resp. $D_{\mathrm{qcoh}}(Y)$) denote the category of complexes of $\mathcal{O}_Y$-modules with coherent (resp. quasi-coherent) cohomology; see [Ha]. In the setting of the derived category, we will write $F^\bullet \simeq_{\mathrm{qis}} G^\bullet$ if $F^\bullet$ and $G^\bullet$ are quasi-isomorphic, and we will use $h^i(F^\bullet)$ to denote the ith cohomology of $F^\bullet$. The symbol $\omega_Y^\bullet$ will be used to denote a normalized dualizing complex on Y (see [Ha]), and ω_Y will be used to denote $h^{-\dim Y}(\omega_Y^\bullet)$.

We now state *Grothendieck duality* for proper morphisms.

THEOREM 2.1 [Ha, III.11.1, VII.3.4]. *Let $f\colon X \to Y$ be a proper morphism of Noetherian schemes of finite dimension. Suppose $\mathscr{F}^\bullet \in D^-_{\mathrm{qcoh}}(X)$ and $\mathscr{G}^\bullet \in D^+_{\mathrm{coh}}(Y)$. Then the duality morphism*

$$\mathbf{R}f_*\mathbf{R}\,\mathscr{H}\mathrm{om}_X^\bullet(\mathscr{F}^\bullet, f^!\mathscr{G}^\bullet) \to \mathbf{R}\,\mathscr{H}\mathrm{om}_Y^\bullet(\mathbf{R}f_*\mathscr{F}^\bullet, \mathscr{G}^\bullet)$$

is an isomorphism.

REMARK 2.2. The case we will consider is when $\mathscr{G}^\bullet$ is a dualizing complex for Y and the map f is a morphism of schemes of finite type over a field k such that $f^!(\omega_Y^\bullet) = \omega_X^\bullet$, giving us the following form of duality:

$$\mathbf{R}f_*\mathbf{R}\,\mathscr{H}\mathrm{om}_X^\bullet(\mathscr{F}^\bullet, \omega_X^\bullet) \cong \mathbf{R}\,\mathscr{H}\mathrm{om}_Y^\bullet(\mathbf{R}f_*\mathscr{F}^\bullet, \omega_Y^\bullet).$$

Now we define pairs, log resolutions, and some of the types of characteristic-0 singularities we will be considering; see [Kol2] or [KolM] for a more detailed introduction to these definitions. We fix X to be a Noetherian scheme of finite type over a field k of characteristic 0.

DEFINITION 2.3. A *pair* $(X, \mathfrak{a}^c)$ is the combined data of a reduced scheme X, an ideal sheaf $\mathfrak{a}$ on X, and a nonnegative rational (or even real) number c. If Z is a closed subscheme of X defined by an ideal sheaf I_Z, then we will often write (X, cZ) instead of the pair (X, I_Z^c).

DEFINITION 2.4. Suppose that X is as just described. A *resolution* of X is a proper birational map $\pi : \tilde{X} \to X$ such that $\tilde{X}$ is smooth over k. We let $\mathrm{exc}(\pi)$ denote the exceptional set of π. If $\mathfrak{a}$ is an ideal sheaf on X, then a *log resolution of* $\mathfrak{a}$

in X (or simply a *log resolution of* $(X, \mathfrak{a})$ or even a *log resolution of* $\mathfrak{a}$) is a resolution of X such that $\mathfrak{a}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-G)$ is an invertible sheaf and $\mathrm{exc}(\pi) \cup \mathrm{Supp}(G)$ is a simple normal crossings divisor.

DEFINITION 2.5. A reduced scheme X is said to have *rational singularities* if, for one resolution of X with $\pi\colon \tilde{X} \to X$, the natural map $\mathcal{O}_X \to \mathbf{R}\pi_*\mathcal{O}_{\tilde{X}}$ is a quasi-isomorphism.

REMARK 2.6. If X has rational singularities, then $\mathcal{O}_X \simeq_{\text{qis}} \mathbf{R}\pi_*\mathcal{O}_{\tilde{X}}$ for every resolution (see e.g. [KolM, 5.10] or [Kol2, 11.11]).

REMARK 2.7. It is clear from the definition that rational singularities are necessarily normal. It also follows immediately from Grauert–Riemenschneider vanishing [GR] and Grothendieck duality (see Theorem 2.1) that rational singularities are Cohen–Macaulay.

Suppose that X is a normal equidimensional $\mathbb{Q}$-Gorenstein scheme. Let $\mathfrak{a}$ be an ideal sheaf on X and suppose that $\pi\colon \tilde{X} \to X$ is a log resolution of $(X, \mathfrak{a}^c)$ with $\mathfrak{a}\mathcal{O}_{\tilde{X}} \cong \mathcal{O}_{\tilde{X}}(-G)$. Suppose that nK_X is Cartier; we then define $\pi^*(K_X)$ to be $\frac{1}{n}(\pi^*(nK_X))$, which is a $\mathbb{Q}$-divisor on $\tilde{X}$. We use $K_{\tilde{X}/X}$ to denote the unique $\mathbb{Q}$-divisor on $\tilde{X}$ that is numerically equivalent to $K_{\tilde{X}} - \pi^*(K_X)$ and supported on the exceptional set of π.

We can now write

$$K_{\tilde{X}/X} - cG = \sum_{i=1}^{n} a(X, E_i, \mathfrak{a}^c)E_i,$$

where the $a(X, E_i, \mathfrak{a}^c)$ are rational numbers and the E_i are divisors.

DEFINITION 2.8. The number $a(X, E_i, \mathfrak{a}^c)$ is called the *discrepancy of* $(X, \mathfrak{a}^c)$ *along the divisor* E_i. We say that $(X, \mathfrak{a}^c)$ has *Kawamata log terminal singularities* (or simply is *klt*) if, for a fixed log resolution π as before, all the $a(X, E_i, \mathfrak{a}^c)$ are strictly larger than -1.

REMARK 2.9. The definition of klt singularities is independent of the choice of log resolution; see [KolM]. In fact, if we view each E_i in $\tilde{X}$ as corresponding to a discrete valuation of the fraction field of X, then the numbers $a(X, E_i, \mathfrak{a}^c)$ are also independent of the choice of resolution.

DEFINITION 2.10. With notation as before, the *multiplier ideal of the pair* $(X, \mathfrak{a}^c)$, denoted by $\mathcal{J}(X, \mathfrak{a}^c)$, is defined to be $\pi_*\mathcal{O}_{\tilde{X}}(\lceil K_{\tilde{X}/X} - cG\rceil) \subseteq \mathcal{O}_X$.

REMARK 2.11. Observe that $(X, \mathfrak{a}^c)$ is klt if and only if $\mathcal{O}_{\tilde{X}}$ is naturally a subsheaf of $\mathcal{O}_{\tilde{X}}(\lceil K_{\tilde{X}/X} - cG\rceil)$. Thus we see that that $(X, \mathfrak{a}^c)$ is klt if and only if $\mathcal{J}(X, \mathfrak{a}^c) = \mathcal{O}_X$.

Remark 2.12. In a context similar to multiplier ideals, we will also often deal with restricting simple normal crossing divisors to a smooth component. In particular, we will often use (without comment) that round-down commutes with such restriction; see [La, Sec. 9.1].

Remark 2.13. One can also define log terminal singularities and multiplier ideals for a triple $(X, \Delta, \mathfrak{a}^c)$, where Δ is a $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We will not consider such definitions here because this notion does not seem as natural for rational singularities.

A key property of multiplier ideals that we will rely on is local vanishing (see [Ei]), which is essentially a corollary of Kawamata–Viehweg vanishing (see [Ka; V]). In Section 3 we state a formulation of local vanishing for multiplier ideals.

Theorem 2.14 [La, 9.4]. *Using the notation from Definition 2.10, we have*

$$R^j \pi_* \mathcal{O}_{\tilde{X}}(\lceil K_{\tilde{X}/X} - cG \rceil) = 0 \quad \textit{for } j > 0.$$

Another variation on log terminal singularities are purely log terminal singularities. We consider the situation of a triple $(X, H; \mathfrak{a}^c)$ where X is a normal $\mathbb{Q}$-Gorenstein scheme, H a reduced integral Cartier divisor with ideal sheaf I_H, $\mathfrak{a}$ another ideal sheaf, and c a nonnegative real number. A log resolution of such a triple is a simultaneous log resolution of I_H and $\mathfrak{a}$ that is also an embedded resolution of H (which is to say, the strict transform of H is smooth).

Definition 2.15. A triple $(X, H; \mathfrak{a}^c)$ has *purely log terminal singularities,* or is simply *plt,* if all the discrepancies of the triple $(X, I_H \mathfrak{a}^c)$ are greater than -1 except for those corresponding to the strict transform of H (which are necessarily equal to -1).

Definition 2.16. Let X be a normal $\mathbb{Q}$-Gorenstein scheme, H a reduced integral Cartier divisor with ideal sheaf I_H, $\mathfrak{a}$ another ideal sheaf, and c a nonnegative real number. We define the *adjoint ideal of* $(X, H; \mathfrak{a}^c)$, denoted $\operatorname{adj}(X, H; \mathfrak{a}^c)$, as follows. Let $\pi : \tilde{X} \to X$ be a log resolution of I_H and $\mathfrak{a}$ such that the strict transform $\tilde{H}$ of H is smooth (i.e., a log resolution of $(X, H; \mathfrak{a}^c)$). Let G denote the divisor on $\tilde{X}$ such that $\mathfrak{a}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-G)$. Then $\operatorname{adj}(X, H; \mathfrak{a}^c)$ is defined to be $\pi_* \mathcal{O}_{\tilde{X}}(\lceil K_{\tilde{X}/X} - cG - \pi^* H + \tilde{H} \rceil) \subseteq \mathcal{O}_X$.

Remark 2.17. We note that $(X, H; \mathfrak{a}^c)$ is plt if and only if $\operatorname{adj}(X, H; \mathfrak{a}^c) = \mathcal{O}_X$.

When H is a Weil divisor but not a Cartier divisor, one can often still define plt singularities and adjoint ideals for the triple $(X, H; \mathfrak{a}^c)$ (in fact, even further generalizations can be made). We restrict ourselves to the Cartier case because rational singularities seem best behaved in this context; see Remark 3.22 for additional discussion.

We conclude with a definition of Du Bois singularities (cf. [D; S]).

DEFINITION 2.18. Suppose that X is a reduced scheme embedded as a closed subscheme of a scheme Y with rational singularities. Let $\pi : \tilde{Y} \to Y$ be a log resolution of (Y, X) that is an isomorphism outside of X (such log resolutions exist if and only if $Y \backslash X$ is smooth). Let E denote $(\pi^{-1}(X))_{\mathrm{red}}$. Then X is said to have *Du Bois singularities* if the natural map $\mathcal{O}_X \to \mathbf{R}\pi_* \mathcal{O}_E$ is a quasi-isomorphism.

REMARK 2.19. This definition is independent of the choice of embedding or resolution. The object $\mathbf{R}\pi_*\mathcal{O}_E$ is also often denoted by $\underline{\Omega}^0_X$.

The condition that π be an isomorphism outside of X is unnecessary, as the following proposition shows (cf. [S, 4.9]).

PROPOSITION 2.20. *Suppose that X is a reduced closed subscheme of a scheme Y with rational singularities and that $Y \backslash X$ is smooth. Let $\pi : \tilde{Y} \to Y$ be a log resolution of the pair (Y, I_X), and let F denote $(\pi^{-1}(X))_{\mathrm{red}}$. Then X has Du Bois singularities if and only if the natural map $\mathcal{O}_X \to \mathbf{R}\pi_*\mathcal{O}_F$ is a quasi-isomorphism.*

Proof. It is sufficient to show that $\mathbf{R}\pi_*\mathcal{O}_F$ (or, equivalently, that $\mathbf{R}\pi_*\mathcal{O}_{\tilde{Y}}(-F)$) is independent of the choice of resolution. Since any two log resolutions can be dominated by a third, it is sufficient to consider two log resolutions $\pi_1 : Y_1 \to Y$ and $\pi_2 : Y_2 \to Y$ with a map between them $\rho : Y_2 \to Y_1$ over Y. Let $F_1 = (\pi_1^{-1}(X))_{\mathrm{red}}$ and $F_2 = (\pi_2^{-1}(X))_{\mathrm{red}} = (\rho^{-1}(F_1))_{\mathrm{red}}$. As mentioned, it is sufficient to prove that $\mathcal{O}_{Y_1}(-F_1) \to \mathbf{R}\rho_*\mathcal{O}_{Y_2}(-F_2)$ is a quasi-isomorphism. Dualizing the map and applying Grothendieck duality implies that it is sufficient to prove that $\omega_{Y_1}(F_1) \leftarrow \mathbf{R}\rho_*(\omega_{Y_2}(F_2))$ is a quasi-isomorphism.

We now apply the projection formula while twisting by $\omega_{Y_1}^{-1}(-F_1)$ (which is invertible since Y_1 is smooth). Hence we need only show that

$$\mathbf{R}\rho_*(\omega_{Y_2/Y_1}(F_2 - \rho^* F_1)) \to \mathcal{O}_{Y_1}$$

is a quasi-isomorphism. But note that $F_2 - \rho^* F_1 = -\lfloor \rho^*(1-\varepsilon)F_1 \rfloor$ for sufficiently small $\varepsilon > 0$. Thus it is sufficient to prove that the pair $(Y_1, (1-\varepsilon)F_1)$ has klt singularities by local vanishing for multiplier ideals; see [La, 9.4]. But this is true because Y_1 is smooth and F_1 is a reduced integral divisor with simple normal crossings. (Compare this proof with the proof of Theorem 4.16.) □

REMARK 2.21. Although it is hoped that the smoothness condition on $Y \backslash X$ can be removed (see [S]), it follows from [Kol] that if $\mathcal{O}_X \to \mathbf{R}\pi_*\mathcal{O}_F$ is a quasi-isomorphism (for any Y, even without rational singularities) then X has Du Bois singularities.

3. Basic Definitions and Fundamental Properties in Characteristic 0

DEFINITION 3.1. Let $(X, \mathfrak{a}^c)$ be a pair and let $\pi : \tilde{X} \to X$ with $\mathfrak{a}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-G)$ be a log resolution of $\mathfrak{a}$. We say that the pair $(X, \mathfrak{a}^c)$ has *rational singularities* (or

Kawamata rational singularities) if the natural map $\mathcal{O}_X \to \mathbf{R}\pi_* \mathcal{O}_{\tilde{X}}(\lfloor cG \rfloor)$ is a quasi-isomorphism.

REMARK 3.2. Explicitly, the pair $(X, \mathfrak{a}^c)$ has rational singularities if and only if $\mathcal{O}_X \to \pi_* \mathcal{O}_{\tilde{X}}(\lfloor cG \rfloor)$ is an isomorphism and $R^i \pi_* \mathcal{O}_{\tilde{X}}(\lfloor cG \rfloor) = 0$ for $i > 0$. Also note that the natural map of $\mathcal{O}_X$ to its normalization can be composed with the map $\pi_* \mathcal{O}_{\tilde{X}} \to \pi_* \mathcal{O}_{\tilde{X}}(\lfloor cG \rfloor)$ to obtain $\mathcal{O}_X \to \pi_* \mathcal{O}_{\tilde{X}}(\lfloor cG \rfloor)$, proving that $\mathcal{O}_X$ is a summand of its normalization and is thus normal.

REMARK 3.3. By Grothendieck duality, the pair $(X, \mathfrak{a}^c)$ has rational singularities if and only if the natural map $\mathbf{R}\pi_* \omega^{\bullet}_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(\lceil -cG \rceil) \to \omega^{\bullet}_X$ is an isomorphism (cf. [Ke+, p. 50]).

Our first goal is to prove that this definition is independent of the choice of resolution, as follows.

PROPOSITION 3.4. *Definition 3.1 is independent of the choice of resolution.*

Proof. Let $(X, \mathfrak{a}^c)$ be a pair as in Definition 3.1. Since any two log resolutions can be dominated by a third, it is enough to consider two log resolutions of $\mathfrak{a}$, X_1 and X_2 with a map between them:

$$\begin{tikzcd} X_2 \arrow[rr, "\rho"] \arrow[dr, "\pi_2"'] & & X_1 \arrow[dl, "\pi_1"] \\ & X. & \end{tikzcd}$$

We use G_1 and G_2 to denote divisors (on X_1 and X_2, respectively) such that $\mathfrak{a}\mathcal{O}_{X_1} = \mathcal{O}_{X_1}(-G_1)$ and $\mathfrak{a}\mathcal{O}_{X_2} = \mathcal{O}_{X_2}(-G_2)$. It is enough to prove that the map $\mathcal{O}_{X_1}(\lfloor cG_1 \rfloor) \to \mathbf{R}\rho_* \mathcal{O}_{X_2}(\lfloor cG_2 \rfloor)$ is a quasi-isomorphism (such a map exists because $\rho^* \lfloor cG_1 \rfloor \leq \lfloor cG_2 \rfloor$). By Grothendieck duality (since X_1 and X_2 are smooth), this is equivalent to proving the existence of a quasi-isomorphism

$$\mathbf{R}\rho_* \omega_{X_2}(-\lfloor cG_2 \rfloor) \to \omega_{X_1}(-\lfloor cG_1 \rfloor).$$

Tensoring this map with $\otimes \omega_{X_1}^{-1}$ (which is an invertible sheaf, since X_1 is smooth) then reduces our question to independence of the definition of multiplier ideals (after an application of local vanishing for multiplier ideals [La, 9.4]), because $\rho^* G_1 = G_2$ and G_1 is a simple normal crossings divisor. (See also [GR, Sec. 2].)

□

Our next main goal is to explore how varying the constant c or varying the ideal $\mathfrak{a}$ affects whether the pair in question has rational singularities. In the process of doing this, we will introduce a notion analogous to the multiplier ideal and will also prove a technical result (Theorem 3.11), related to [Ko2, Thm. 1] and [KolM, 5.13], which will be used to give a simple proof that log terminal pairs are rational and that summands of (appropriate) rational pairs are rational. The essential ingredient in all of this is the following (vanishing) lemma. This lemma, which will be obvious to experts, can be thought of as either a generalization of Grauert–Riemenschneider vanishing [GR] or a slight modification of the usual formulation of local vanishing for multiplier ideals [La, 9.4.1].

LEMMA 3.5. *Suppose that X is a reduced equidimensional scheme and that $\mathfrak{a}$ is an ideal sheaf on X. Further suppose that $\pi : \tilde{X} \to X$ is a log resolution of $\mathfrak{a}$ with $\mathfrak{a}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-G)$. Then, for any nonnegative real number c and for all $i > 0$, we have $h^i(\mathbf{R}(\pi_* \omega_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(\lceil -cG \rceil))) = 0$.*

Proof. First note that we may assume X is normal since the map π factors through the normalization of X and since finite maps have no higher cohomology. Thus, we may also assume that X is irreducible. We then reduce to the case when $\mathfrak{a}$ is a (locally) principal ideal sheaf by choosing general elements of $\mathfrak{a}$; see [La, 9.2.22–9.2.28]. The proof is then the same as the proof of [La, 9.4.1, 9.4.17] except that we do not need to pull back K_X. The essential ingredient is the Kawamata–Viehweg vanishing theorem [Ka; V]. □

Smith [Sm1] has noted that, when dealing with rational singularities and related notions, instead of working with (analogues of) multiplier ideals one should rather work with submodules of the canonical module. This idea was further studied in [H2]. Hence the following definition is natural (cf. Remark 6.4).

DEFINITION 3.6 [B]. The *multiplier submodule* of a pair $(X, \mathfrak{a}^c)$ is defined to be the image of $\pi_*(\omega_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(\lceil -cG \rceil))$ inside ω_X. We will denote it by $\mathcal{J}(\omega_X, \mathfrak{a}^c)$.

It is easy to see that this submodule is independent of the choice of resolution. From this point of view, Lemma 3.5 can be thought of as *local vanishing for multiplier submodules.*

LEMMA 3.7. *If X is a reduced equidimensional scheme as before, then the natural map $\pi_*(\omega_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(\lceil -cG \rceil)) \to \omega_X$ is injective.*

Proof. Consider the exact triangle

$$\mathcal{O}_X \longrightarrow \mathbf{R}\pi_* \mathcal{O}_{\tilde{X}} \longrightarrow C^\bullet \xrightarrow{+1}$$

and note that $\dim(\operatorname{Supp}(h^i(C^\bullet))) < \dim X - i$. By an easy analysis of a spectral sequence, one has $\mathbb{H}^d_{\mathfrak{m}}(C^\bullet) = 0$, which implies that there is a surjection $H^d_{\mathfrak{m}}(\mathcal{O}_X) \to \mathbb{H}^d_{\mathfrak{m}}(\mathbf{R}\pi_* \mathcal{O}_{\tilde{X}})$ (for any maximal ideal $\mathfrak{m}$). By local duality [Ha, V, Thm. 6.2], the natural map $\pi_* \omega_{\tilde{X}} \to \omega_X$ is an injection. But then we are done because $\omega_{\tilde{X}}(\lceil -cG \rceil) \subset \omega_{\tilde{X}}$ and π_* is left exact. Compare with [LiTe, Sec. 2, Rem. (b)] and [Ke, Sec. 1]. □

COROLLARY 3.8. *Suppose that X is a reduced equidimensional Cohen–Macaulay scheme and that $\mathfrak{a}$ is an ideal sheaf on X. Then $(X, \mathfrak{a}^c)$ has rational singularities if and only if the multiplier submodule of X, $\pi_*(\omega_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(\lceil -cG \rceil))$, is equal to ω_X.*

At this point it is natural to mention a (characteristic-free) definition for rational singularities of pairs that makes sense even when X is not known to have a resolution. This slight generalization of a definition of Lipman and Teissier will appear later in the paper when we compare rational and F-rational pairs (see Theorem 6.11).

DEFINITION 3.9 (cf. [LiTe, Sec. 2]). Let $(R, \mathfrak{m})$ be a d-dimensional reduced local ring, let $\mathfrak{a} \subseteq R$ be an ideal such that $\mathfrak{a}$ contains elements not contained in any minimal prime of R, and let $t \geq 0$ be a real number. Then $(R, \mathfrak{a}^t)$ is *pseudo-rational* if (a) R is normal, Cohen–Macaulay, and analytically unramified and (b) for any proper birational morphism $\pi : Y \to X := \operatorname{Spec} R$ from a normal scheme Y such that $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-G)$ is invertible, the map

$$\delta_\pi : H^d_{\mathfrak{m}}(R) \to H^d_E(\mathcal{O}_Y(\lfloor tG \rfloor))$$

is injective. Here $E = \pi^{-1}(\mathfrak{m})$ denotes the closed fiber of π and δ_π is the map induced by $\mathcal{O}_{\operatorname{Spec} R} \to \mathbf{R}\pi_*\mathcal{O}_Y(\lfloor tG \rfloor)$.

REMARK 3.10. In addition, when R is essentially of finite type over a field of characteristic 0, a straightforward application of local duality (see [Ha, V, Thm. 6.2]) implies that $(R, \mathfrak{a}^t)$ is pseudo-rational if and only if $(\operatorname{Spec} R, \mathfrak{a}^t)$ has rational singularities.

Now we come to the promised generalization of a result of Kovács [Ko2].

THEOREM 3.11. *Suppose that $(X, \mathfrak{a}^c)$ is a pair such that $\pi : \tilde{X} \to X$ is a log resolution of $\mathfrak{a}$. If the natural map*

$$\mathcal{O}_X \to \mathbf{R}\pi_*\mathcal{O}_{\tilde{X}}(\lfloor cG \rfloor)$$

has a left inverse (meaning that there exists a map $\mathcal{O}_{\tilde{X}}(\lfloor cG \rfloor) \to \mathcal{O}_X$ such that the composition $\mathcal{O}_X \to \mathbf{R}\pi_*\mathcal{O}_{\tilde{X}}(\lfloor cG \rfloor) \to \mathcal{O}_X$ is a quasi-isomorphism), then $(X, \mathfrak{a}^c)$ has rational singularities.

The proof is virtually the same as the one found in [Ko2]; we simply use Lemma 3.5 instead of Grauert–Riemenschneider vanishing.

Proof of Theorem 3.11. Since π factors through the normalization of X, we immediately see that $\mathcal{O}_X$ is a summand of its own normalization and thus is itself normal. Therefore, we may assume without loss of generality that X is irreducible (and, in particular, equidimensional). Now, apply Grothendieck duality to derive the composition

$$\omega^{\bullet}_X \to \mathbf{R}\pi_*\omega^{\bullet}_{\tilde{X}}(-\lfloor cG \rfloor) \to \omega^{\bullet}_X.$$

By Lemma 3.5 and since the composition is an isomorphism, we have $h^i(\omega^{\bullet}_X) = 0$ for $i \neq -\dim X$. This implies that X is Cohen–Macaulay. It is now enough to show that

$$\pi_*(\omega_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(-\lfloor cG \rfloor)) \to \omega_X$$

is an isomorphism. However, the map is injective by Lemma 3.7, and it is surjective because it is a split surjection (by assumption). □

One could have given an indirect argument that X is Cohen–Macaulay by first showing that X is rational, but Kovács's argument generalizes quite well to pairs and is really no longer than an indirect argument.

COROLLARY 3.12. *Suppose that X is a reduced scheme, $\mathfrak{a}$ is an ideal sheaf, and $c_1 < c_2$ are nonnegative real numbers. If $(X, \mathfrak{a}^{c_2})$ has rational singularities then so does $(X, \mathfrak{a}^{c_1})$. Furthermore, for $\mathfrak{b}$ another ideal sheaf with $\mathfrak{a} \subseteq \mathfrak{b}$, if $(X, \mathfrak{a}^{c_1})$ is rational then so is $(X, \mathfrak{b}^{c_1})$.*

Proof. Let $\pi : \tilde{X} \to X$ be a log resolution of $\mathfrak{a}$. Then we have the following composition:

$$\mathcal{O}_X \to \mathbf{R}\pi_* \mathcal{O}_{\tilde{X}}(\lfloor c_1 G \rfloor) \to \mathbf{R}\pi_* \mathcal{O}_{\tilde{X}}(\lfloor c_2 G \rfloor).$$

This is a quasi-isomorphism by assumption, proving that $(X, \mathfrak{a}^{c_1})$ has rational singularities by Theorem 3.11. The proof of the second statement is similar. □

COROLLARY 3.13. *Suppose that the pair $(X, \mathfrak{a}^c)$ has rational singularities. Then X has rational singularities and, in particular, is Cohen–Macaulay.*

REMARK 3.14. In the previous two corollaries, one can avoid working with the derived category by first dualizing and then considering containments of multiplier submodules.

We conclude this section with a definition of purely rational singularities (cf. Definition 6.14).

DEFINITION 3.15. Let X be a normal scheme, H an integral reduced Cartier divisor with ideal sheaf I_H, $\mathfrak{a}$ another ideal sheaf with no minimal prime among the components of H, and c a nonnegative real number. Suppose that $\pi : \tilde{X} \to X$ is a log resolution of H and $\mathfrak{a}$, where $\tilde{H}$, the strict transform of H, is smooth (i.e., π is a log resolution of the triple $(X, H; \mathfrak{a}^c)$). We use G to denote the divisor on $\tilde{X}$ such that $\mathfrak{a}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-G)$. Then $(X, H; \mathfrak{a}^c)$ has *purely rational singularities* if the natural map

$$\mathcal{O}_X \to \mathbf{R}\pi_* \mathcal{O}_{\tilde{X}}(\lfloor cG + \pi^* H - \tilde{H} \rfloor)$$

is a quasi-isomorphism.

REMARK 3.16. By Grothendieck duality, $(X, H; \mathfrak{a}^c)$ has purely rational singularities if and only if

$$\mathbf{R}\pi_*(\omega^{\bullet}_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(\lceil -cG - \pi^* H + \tilde{H} \rceil)) \to \omega^{\bullet}_X$$

is a quasi-isomorphism.

We also define the adjoint submodule.

DEFINITION 3.17. Suppose X is a reduced scheme, H a Cartier divisor, and $\mathfrak{a}$ an ideal sheaf with no minimal primes in common with any of the components of H. The *adjoint submodule* of a triple $(X, H; \mathfrak{a}^c)$ is defined to be the image of $\pi_*(\mathcal{O}_{\tilde{X}}(\lceil K_X - cG - \pi^*H + \tilde{H}\rceil)$ inside ω_X, where π is defined as before. We denote the adjoint submodule by $\mathrm{adj}(\omega_X, H; \mathfrak{a}^c)$.

We now show that the notions of purely rational singularities and the adjoint submodule are well-defined.

PROPOSITION 3.18. *With notation as in Definition 3.15, the definition of purely rational singularities is independent of the choice of resolution* (*more generally, the adjoint submodule* $\pi_*(\mathcal{O}_{\tilde{X}}(\lceil K_X - cG - \pi^*H + \tilde{H}\rceil) \subset \omega_X$ *is well-defined*). *Furthermore,*

$$h^i(\mathbf{R}\pi_*(\mathcal{O}_{\tilde{X}}(\lceil K_{\tilde{X}} - cG - \pi^*H + \tilde{H}\rceil))) = 0$$

for $i > 0$, *so that* $(X, H; \mathfrak{a}^c)$ *has purely rational singularities if and only if* X *is Cohen–Macaulay and*

$$\pi_*(\mathcal{O}_{\tilde{X}}(\lceil K_X - cG - \pi^*H + \tilde{H}\rceil)) \to \mathcal{O}_{\tilde{X}}(K_X)$$

is surjective (*in other words, if and only if the adjoint submodule is equal to* ω_X).

Proof. To show that $h^i(\mathbf{R}\pi_*\omega_{\tilde{X}}(-\lfloor cG + \pi^*H - \tilde{H}\rfloor)) = 0$ for $i > 0$, it is enough to show that $h^i(\mathbf{R}\pi_*\omega_{\tilde{X}}(-\lfloor cG - \tilde{H}\rfloor)) = 0$ by the projection formula, since H is Cartier. Thus, consider the short exact sequence

$$0 \to \omega_{\tilde{X}}(-\lfloor cG\rfloor) \to \omega_{\tilde{X}}(-\lfloor cG - \tilde{H}\rfloor) \to \omega_{\tilde{H}}(-\lfloor cG\rfloor) \to 0.$$

If we apply $\mathbf{R}\pi_*$ then the higher cohomology of $\mathbf{R}\pi_*\omega_{\tilde{X}}(-\lfloor cG\rfloor)$ is zero by Lemma 3.5 and likewise the higher cohomology of $\mathbf{R}\pi_*\omega_{\tilde{H}}(-\lfloor cG\rfloor)$ is also zero. This proves the vanishing we desired.

It is now sufficient to prove that the adjoint submodule is well-defined. Since any two log resolutions can be dominated by a third, we consider the case of two log resolutions of $(X, H; \mathfrak{a}^c)$, X_1 and X_2, with a map between them:

$$\begin{tikzcd} X_2 \arrow[rr, "\rho"] \arrow[dr, "\pi_2"'] & & X_1 \arrow[dl, "\pi_1"] \\ & X. & \end{tikzcd}$$

As before, we assume that the strict transforms $\tilde{H}_1$ and $\tilde{H}_2$ of H (in X_1 and X_2, respectively) are smooth. We define G_i to be the divisor on X_i such that $\mathfrak{a}\mathcal{O}_{X_i} = \mathcal{O}_{X_i}(-G_i)$; observe that $\rho^*G_1 = G_2$ and $\rho^*\pi_1^*H = \pi_2^*H$.

It is now sufficient to show that the map

$$\rho_*\omega_{X_2}(-\lfloor cG_2 + \pi_2^*H - \tilde{H}_2\rfloor) \to \omega_{X_1}$$

has image $\omega_{X_1}(-\lfloor cG_1 + \pi_1^*H - \tilde{H}_1\rfloor)$. By the projection formula (twisting by $\otimes(\omega_{X_1}^{-1}(\pi_1^*H - \tilde{H}_1)))$, it is sufficient to show that the image of the map

$$\rho_*\mathcal{O}_{X_2}(\lceil K_{X_2/X_1} - cG_2 + \tilde{H}_2 - \rho^*\tilde{H}_1\rceil) \to \mathcal{O}_{X_1}(\pi_1^*H - \tilde{H}_1)$$

is equal to $\mathcal{O}_{X_1}(-\lfloor cG_1\rfloor) = \mathcal{O}_{X_1}(-\lfloor cG_1\rfloor - \tilde{H}_1 + \tilde{H}_1)$. But this follows because the definition of the usual adjoint ideal is independent of choice of resolution. $\square$

REMARK 3.19. With notation as before, the previous proof implies we have a short exact sequence

$$0 \to \mathcal{J}(\omega_X, \mathfrak{a}^c) \to \operatorname{adj}(\omega_X, H; \mathfrak{a}^c) \otimes \mathcal{O}_X(H) \to \mathcal{J}(\omega_H, (\mathfrak{a}|_H)^c) \to 0.$$

Note that this is essentially the same as [La, 9.3.44]. Also see Theorem 4.14.

Using the vanishing in Proposition 3.18, one can prove the following analogue of Theorem 3.11.

THEOREM 3.20. *If the natural map* $\mathcal{O}_X \to \mathbf{R}\pi_*\mathcal{O}_{\tilde{X}}(\lfloor cG + \pi^*H - \tilde{H}\rfloor)$ *has a left inverse—in other words, if there exists a map* $\mathbf{R}\pi_*\mathcal{O}_{\tilde{X}}(\lfloor cG + \pi^*H - \tilde{H}\rfloor) \to \mathcal{O}_X$ *such that the composition*

$$\mathcal{O}_X \to \mathbf{R}\pi_*\mathcal{O}_{\tilde{X}}(\lfloor cG + \pi^*H - \tilde{H}\rfloor) \to \mathcal{O}_X$$

is a quasi-isomorphism—then $(X, H; \mathfrak{a}^c)$ *has purely rational singularities.*

The proof is the same as in Theorem 3.11.

REMARK 3.21. We note that if $(X, H; \mathfrak{a}^c)$ has purely rational singularities then, by Theorem 3.11, $(X, \mathfrak{a}^c I_H^{(1-\varepsilon)})$ has (Kawamata) rational singularities for every ε satisfying $1 \geq \varepsilon \geq 0$. In particular, $(X, \mathfrak{a}^c)$ has rational singularities.

REMARK 3.22. Let us briefly discuss the case where H is not Cartier. In this case, one can consider $\pi_*(\mathcal{O}_{\tilde{X}}(\lceil K_X - cG + \tilde{H}\rceil) \subset \omega_X(H)$ instead of the adjoint submodule. One still has a vanishing for the higher cohomology, and many results still work. However, this object seems somewhat contrived and doesn't seem as closely related to the adjoint ideals as defined, for example, in [La]. For this reason, we restrict ourselves to the Cartier case.

4. Log Terminal Singularities, Deformations, Summands, and Adjunction

In this section we show how rational pairs relate to log terminal pairs, prove that pairs with rational singularities behave well with respect to deformation and summands, and conclude by showing that rational pairs satisfy several "inversion of adjunction" results often observed for log terminal pairs. We also give a simple proof of a result related to inversion of adjunction on log canonicity that uses the notion of Du Bois singularities.

First we relate log terminal and rational singularities associated to pairs. In particular, we show that Kawamata log terminal pairs are rational and that rational pairs $(X, \mathfrak{a}^c)$ with X Gorenstein are Kawamata log terminal; see also [E2]. We then compute an example to show that these notions are distinct even when X is $\mathbb{Q}$-Gorenstein. Compare the following two results with Propositions 6.5 and 6.15.

PROPOSITION 4.1. *If* $(X, \mathfrak{a}^c)$ *is rational* (*resp.* $(X, H; \mathfrak{a}^c)$ *is purely rational*) *and if* X *is Gorenstein, then* $(X, \mathfrak{a}^c)$ *is klt* (*resp.* $(X, H; \mathfrak{a}^c)$ *is plt*).

Proof. Let $\pi : \tilde{X} \to X$ be a log resolution of $\mathfrak{a}$. By Remark 3.3 we have a quasi-isomorphism $\mathbf{R}(\pi_* \omega^{\bullet}_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(\lceil -cG \rceil)) \simeq_{\text{qis}} \omega^{\bullet}_X$. But then, since ω_X is a line bundle, we have $\pi_* \mathcal{O}_{\tilde{X}}(\lceil K_{\tilde{X}/X} - cG \rceil) \cong \mathcal{O}_X$ by the projection formula. Thus the pair is klt. The proof of the plt case is the same. □

PROPOSITION 4.2. *Suppose that X is $\mathbb{Q}$-Gorenstein. If $(X, \mathfrak{a}^c)$ is klt (respectively, $(X, H; \mathfrak{a}^c)$ is plt), then $(X, \mathfrak{a}^c)$ is also rational (respectively, $(X, H; \mathfrak{a}^c)$ is purely rational).*

Proof. This statement is local, so we may assume that X is affine. Let $\pi : \tilde{X} \to X$ be a log resolution of $\mathfrak{a}$. Now, since $(X, \mathfrak{a}^c)$ is klt, we have a natural inclusion

$$\mathcal{O}_{\tilde{X}} \subseteq \mathcal{O}_{\tilde{X}}(\lceil K_{\tilde{X}/X} - cG \rceil).$$

This implies an inclusion

$$\mathcal{O}_{\tilde{X}}(\lfloor cG \rfloor) \subseteq \mathcal{O}_{\tilde{X}}(\lceil K_{\tilde{X}/X} - cG \rceil + \lfloor cG \rfloor) \subseteq \mathcal{O}_{\tilde{X}}(\lceil K_{\tilde{X}/X} \rceil).$$

Applying $\mathbf{R}\pi_*$ gives us a composition

$$\mathcal{O}_X \to \mathbf{R}\pi_* \mathcal{O}_X(\lfloor cG \rfloor) \to \mathbf{R}\pi_* \mathcal{O}_{\tilde{X}}(\lceil K_{\tilde{X}/X} \rceil).$$

But $\mathbf{R}\pi_* \mathcal{O}_{\tilde{X}}(\lceil K_{\tilde{X}/X} \rceil)$ is quasi-isomorphic to $\mathcal{O}_X$ because X is log terminal (using local vanishing for multiplier ideals [La, 9.4]), which completes the proof of the klt case by Theorem 3.11.

In the plt case, the proof is analogous. We begin with the inclusion $\mathcal{O}_{\tilde{X}} \subseteq \mathcal{O}_{\tilde{X}}(\lceil K_{\tilde{X}/X} - cG - \pi^* H + \tilde{H} \rceil)$ and observe that $\pi^* H - \tilde{H}$ is a integral divisor. This gives us an inclusion $\mathcal{O}_{\tilde{X}}(\lfloor cG + \pi^* H - \tilde{H} \rfloor) \subseteq \mathcal{O}_{\tilde{X}}(\lceil K_{\tilde{X}/X} \rceil)$, where $\tilde{H}$ is the strict transform of H. We then apply $\mathbf{R}\pi_*$ and use Theorem 3.20, which completes the proof. □

REMARK 4.3. Compare the preceding proof with [Ko2, Thm. 4].

REMARK 4.4. If X is not $\mathbb{Q}$-Gorenstein but $((X, \Delta), \mathfrak{a}^c)$ is klt (in particular, $K_X + \Delta$ is $\mathbb{Q}$-Cartier), then the same proof implies that $(X, \mathfrak{a}^c)$ is rational. In this case it seems natural to try to show that (X, Δ) is a rational pair. However, there is no clear way to pull back Δ as a divisor because it is not $\mathbb{Q}$-Cartier (by assumption).

REMARK 4.5. One could also give a more indirect and less homological proof of Theorem 4.2 by comparing multiplier ideals and multiplier submodules.

We now present an example of a pair with a log terminal underlying scheme that has rational singularities and not log terminal singularities.

EXAMPLE 4.6. Consider the surface singularity $X = \operatorname{Spec} \mathbb{C}[x^3, x^2y, xy^2, y^3]$. This is a surface with cyclic quotient singularities and so it is, in particular, log terminal. First we consider the scheme's resolution and how this affects its canonical divisor. This singularity can be resolved by a single blow-up $\pi : \tilde{X} \to X$ at the ideal $\mathfrak{m} = (x^3, x^2y, xy^2, y^3)$.

The canonical module ω_X can be identified with the ideal (x^2y, xy^2). Note that this identification yields $\omega_X^{(3)} \cong (x^3y^3)$. It is now easy to see that the pair $(X, \mathfrak{m}^t)$ is rational for any $0 < t < 1$ but is not klt for t sufficiently close to 1, since X's discrepancy along the exceptional divisor is equal to $-\frac{1}{3}$.

This example suggests our next definition. As an analogue of the log canonical threshold, one can define the following rational number (cf. Definition 7.5).

DEFINITION 4.7. Let X be a scheme with rational singularities and let $\mathfrak{a}$ be an ideal sheaf. We define the *rational threshold* of the pair $(X, \mathfrak{a})$, denoted by $\operatorname{rt}(X, \mathfrak{a})$, to be equal the following number:

$$\operatorname{rt}(X, \mathfrak{a}) = \sup\{t > 0 \mid (X, \mathfrak{a}^t) \text{ has rational singularities}\}.$$

In Example 4.6, the log canonical threshold of the pair $(X, \mathfrak{a})$ is equal to $\frac{2}{3}$ whereas the rational threshold is equal to 1. More generally, suppose that X is a variety with a log resolution $\pi : \tilde{X} \to X$ that has only a single reduced exceptional divisor E, which dominates P and was obtained by blowing up the same ideal P, where $P\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-E)$; then the rational threshold of (X, P) is always an integer. On the other hand, there are many examples of varieties with noninteger rational thresholds, since the rational threshold and the log canonical threshold of a Gorenstein scheme clearly coincide.

Let us consider now a broader set of examples: the Veronese subrings. We will use a slightly different approach from the preceding example. The following generalization of a lemma by Kovács will be useful in this computation.

LEMMA 4.8 [Ko1, Lemma 3.3]. *Suppose that X is a Cohen–Macaulay scheme of essentially finite type over a field of characteristic* 0. *Suppose that $\mathfrak{a}$ is an ideal sheaf and t is a positive rational number. Let Σ be the subset of X, where $(X, \mathfrak{a}^t)$ does not have rational singularities. Let $\pi : \tilde{X} \to X$ be a log resolution of $(X, \mathfrak{a}^t)$ with $\mathfrak{a}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-G)$. Then $R^i(\mathcal{O}_{\tilde{X}}(\lfloor tG \rfloor)) = 0$ for all $0 < i < \dim X - \dim \Sigma - 1$.*

Proof. The proof is virtually the same as in [Ko1]; one simply uses Theorem 3.5 instead of Grauert–Riemenschneider vanishing. □

EXAMPLE 4.9. Suppose $S = k[x_1, \dots, x_d]$. Let R be the rth Vernonese subring, $R = k[x_1^r, x_1^{r-1}x_2, x_1^{r-1}x_2^2, \dots, x_{d-1}x_d^{r-1}, x_d^r]$. We shall study the rational threshold of the pair (Spec $R, \mathfrak{m}^t$), where $\mathfrak{m}$ is the maximal ideal of the origin. It is clear that the pair can be resolved with a single blow-up, and to study that blow-up we can use a set of d charts that correspond to placing each x_i^r in the denominator. (Note: this implies that the rational threshold must be an integer.) Fix $X = \operatorname{Spec} R$. Let $\pi : \tilde{X} \to X$ be the aforementioned resolution, and let E be the exceptional divisor (note that $\mathfrak{m}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-E)$). Since R is a Cohen–Macaulay isolated singularity, by Lemma 4.8 it is sufficient to understand the cohomology $R^{d-1}\pi_*\mathcal{O}_{\tilde{X}}(\lfloor tE \rfloor)$.

We use Čech cohomology to interpret this object. Using the charts corresponding to $x_1^r, \dots, x_d^r$, we see that an arbitrary element of $R^{d-1}\pi_*\mathcal{O}_{\tilde{X}}(\lfloor tE \rfloor)$ looks like $f/(x_1^r x_2^r \dots x_d^r)^c$. Observe that the order of vanishing of f along E must be greater than or equal to $cd - \lfloor t \rfloor$. A natural first nonzero element of the cohomology group would seem to be $(x_1^{r-1} x_2^{r-1} \dots x_d^{r-1})/(x_1^r x_2^r \dots x_d^r)$; unfortunately, that numerator doesn't always exist in this context because in some sense the numerator's order of vanishing on E is $d(r-1)/r$, which is not always an integer. To see that the pair is nonrational, it is natural to seek a cohomology element that vanishes on E to degree $\lfloor d(r-1)/r \rfloor - d = \lfloor -d/r \rfloor$, which by assumption must be greater than or equal to $-\lfloor t \rfloor$. It is not hard to see that such a nonzero element exists—assuming the arithmetic is satisfied by modifying the original "nonexistent" element. In other words, if $t \geq \lceil d/r \rceil$ then $(X, \mathfrak{m}^t)$ cannot be a rational pair, which means that $\operatorname{rt}(X, \mathfrak{m}) \leq \lceil d/r \rceil$. On the other hand, the log canonical threshold $\operatorname{lt}(X, \mathfrak{m})$ is equal to d/r and clearly $\operatorname{lt}(X, \mathfrak{m}) < \operatorname{rt}(X, \mathfrak{m})$. Thus we have $d/r \leq \operatorname{rt}(X, \mathfrak{m}) \leq \lceil d/r \rceil$. Therefore, since $\operatorname{rt}(X, \mathfrak{m})$ is an integer, it must be equal to $\lceil d/r \rceil$.

See Example 7.8 for a study of the same class of singularities using positive-characteristic techniques (an explicit proof of the fact that $\operatorname{rt}(X, \mathfrak{m}) \geq \lceil d/r \rceil$ is given in positive characteristic).

We now prove that summands of pairs with rational singularities are rational (cf. [Bo]). In fact, we prove a more general result that is analogous to the full generality of [Ko2, Thm. 1]. The proof is relatively short (the key ingredient is Theorem 3.11) and was inspired by a similar result in [Ko2].

THEOREM 4.10. *Suppose that $\rho \colon Y \to X$ is a dominant morphism of reduced schemes such that every component of Y dominates a component of X. Let $\mathfrak{a}$ be an ideal sheaf on X and suppose that $(Y, (\mathfrak{a}\mathcal{O}_Y)^c)$ is rational. Further suppose that the natural map $\mathcal{O}_X \to \mathbf{R}\rho_* \mathcal{O}_Y$ has a left inverse (i.e., there exists a map $\delta \colon \mathbf{R}\rho_* \mathcal{O}_Y \to \mathcal{O}_X$ such that $\mathcal{O}_X \to \mathbf{R}\rho_* \mathcal{O}_Y \to \mathcal{O}_X$ is a quasi-isomorphism). Then $(X, \mathfrak{a}^c)$ has rational singularities as well.*

Proof. Let the maps $\pi : \tilde{X} \to X$ and $\pi' : \tilde{Y} \to Y$ be log resolutions of $(X, \mathfrak{a}^c)$ and $(Y, (\mathfrak{a}\mathcal{O}_{\tilde{Y}})^c)$, respectively. Let G be the divisor on $\tilde{X}$ such that $\mathfrak{a}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-G)$, and let F be the divisor on $\tilde{Y}$ such that $\mathfrak{a}\mathcal{O}_{\tilde{Y}} = \mathcal{O}_{\tilde{Y}}(-F)$. We can choose these resolutions so that there is map $\gamma : \tilde{Y} \to \tilde{X}$ such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\gamma} & \tilde{X} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi'} \\ Y & \xrightarrow[\rho]{} & X. \end{array}$$

Note that $\gamma^* G = F$. We will show that there is a natural map

$$\mathcal{O}_{\tilde{X}}(\lfloor cG \rfloor) \to \gamma_* \mathcal{O}_{\tilde{Y}}(\lfloor cF \rfloor) = \gamma_* \mathcal{O}_{\tilde{Y}}(\lfloor c(\gamma^* G) \rfloor).$$

By composition with the map

$$\mathcal{O}_{\tilde{X}}(\lfloor cG \rfloor) \to \gamma_* \gamma^* \mathcal{O}_{\tilde{X}}(\lfloor cG \rfloor) = \gamma_* \mathcal{O}_{\tilde{Y}}(\gamma^* \lfloor cG \rfloor),$$

we see that it suffices to show the existence of a natural inclusion $\mathcal{O}_{\tilde{Y}}(\gamma^* \lfloor cG \rfloor) \subseteq \mathcal{O}_{\tilde{Y}}(\lfloor c(\gamma^* G) \rfloor)$. But this is true because—even though round-down does not commute with pull-backs—there is always an inequality $\gamma^* \lfloor cG \rfloor \leq \lfloor c(\gamma^* G) \rfloor$.

Now consider the diagram

$$\begin{tikzcd} \mathbf{R}\rho_*\mathbf{R}\pi'_*\mathcal{O}_{\tilde{Y}}(\lfloor cF\rfloor) & \mathbf{R}\pi_*\mathcal{O}_{\tilde{X}}(\lfloor cG\rfloor) \arrow[l] \\ \mathbf{R}\rho_*\mathcal{O}_Y \arrow[u, "\mathbf{R}\rho_*(p')"'] & \mathcal{O}_X. \arrow[l] \arrow[u, "p"] \end{tikzcd}$$

Since $(Y, (\mathfrak{a}\mathcal{O}_Y)^c)$ is rational, it follows that $p'\colon \mathcal{O}_Y \to \mathbf{R}\pi'_*\mathcal{O}_{\tilde{Y}}(\lfloor cF\rfloor)$ is a quasi-isomorphism. We thus consider the composition

$$\mathcal{O}_X \to \mathbf{R}\pi_*\mathcal{O}_{\tilde{X}}(\lfloor cG\rfloor) \to \mathbf{R}\rho_*\mathbf{R}\pi'_*\mathcal{O}_{\tilde{Y}}(\lfloor cF\rfloor) \simeq_{\text{qis}} \mathbf{R}\rho_*\mathcal{O}_Y \to \mathcal{O}_X,$$

where the final map in the composition exists by hypothesis. This composition must be a quasi-isomorphism by construction, creating a left inverse of p. By Theorem 3.11, the proof is now complete. □

COROLLARY 4.11. *Suppose R and S are domains, $\mathfrak{a}$ is an ideal of R, and R is a summand of S (e.g., suppose that R is normal and that $R \to S$ is a finite map). If $(S, (\mathfrak{a}S)^c)$ is rational, then so is $(R, \mathfrak{a}^c)$.*

Compare this corollary with Proposition 7.1.

REMARK 4.12. Observe that the converse of Corollary 4.11 is not true. Of course, even when $\mathfrak{a} = R$, by [Si] the converse can fail for a canonical cover. We can demonstrate another type of failure by using Example 4.6. Let

$$X = \operatorname{Spec} \mathbb{C}[x^3, x^2y, xy^2, y^3]$$

and let $Y = \mathbb{C}[x, y]$ be its canonical cover. Let $\mathfrak{a} = (x^3, x^2y, xy^2, y^3)$ and note that $\mathfrak{a}\mathcal{O}_Y = (x, y)^3$. Then $(X, \mathfrak{a}^{0.9})$ has rational singularities but $(Y, ((x, y)^3)^{0.9})$ clearly does not.

We now explore how rational pairs deform; see [E1, Thm. 2] or [Kol2, 11.15].

THEOREM 4.13. *Suppose that $(X, \mathfrak{a}^c)$ is a pair, that H is a Cartier divisor on X, and that H has no common components with $V(\mathfrak{a})$. If the pair $(H, (\mathfrak{a}|_H)^c)$ has rational singularities, then so does $(X, \mathfrak{a}^c)$ near H.*

Proof [E1]. Let x be a point of X also contained in H. Since it is enough to prove the problem at the stalk associated to x, we assume $X = \operatorname{Spec} R$ with (R, m) local, $H = \operatorname{Spec} R/f$ for some regular element $f \in R$, and $\mathfrak{a}$ is an ideal of R that has no common minimal primes with $(f) = I_H$. Since $(H, (\mathfrak{a}|_H)^c)$ is rational, it follows that H and thus X is Cohen–Macaulay. Let $\pi : \tilde{X} \to X$ be a resolution of $(X, \mathfrak{a}^c)$ that is also simultaneously a resolution of H and let G denote the divisor such that $\mathfrak{a}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-G)$. Let $\bar{H}$ be the total transform of H (i.e., $\bar{H}$ is the scheme defined by $f\mathcal{O}_{\tilde{X}}$), and let $\tilde{H}$ denote the strict transform of H. Note that there is a natural inclusion of schemes $\tilde{H} \to \bar{H}$. Consider now the following diagram:

$$
\begin{tikzcd}
 & & & \pi_*(\omega_{\tilde{H}} \otimes \mathcal{O}_{\tilde{X}}(-\lfloor cG \rfloor)) \arrow[d] & \\
0 \arrow[r] & \pi_*(\omega_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(-\lfloor cG \rfloor)) \arrow[r, "\times f"] \arrow[d, hook, "\psi"] & \pi_*(\omega_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(-\lfloor cG \rfloor)) \arrow[r] \arrow[d, hook, "\psi"] & \pi_*(\omega_{\tilde{H}} \otimes \mathcal{O}_{\tilde{X}}(-\lfloor cG \rfloor)) \arrow[r] \arrow[d, two heads, "\phi"] & 0 \\
0 \arrow[r] & \omega_X \arrow[r, "\times f"] & \omega_X \arrow[r] & \omega_H \arrow[r] & 0.
\end{tikzcd}
$$

The bottom row is exact because H is Cohen–Macaulay, and the top row is exact by Lemma 3.5. The map labeled ϕ is surjective because the vertical composition from $\pi_*(\omega_{\tilde{H}} \otimes \mathcal{O}_{\tilde{X}}(-\lfloor cG \rfloor))$ is an isomorphism. Hence, by Lemma 3.5, it is enough to show that ψ is surjective.

Let C be the cokernel of ψ. That ϕ is surjective means that $C \xrightarrow{\times f} C$ is surjective (by the snake lemma). But this contradicts Nakayama's lemma, completing the proof. □

We conclude this section with several results related to adjunction (cf. [KolM, 5.6; La, 9.5.11, 9.5.17; Sh]). The first result could be thought of as an analogue of adjunction and inversion of adjunction for log terminal singularities, and in a sense it is the easy case because we work only with Cartier divisors (cf. [Kol+, Chaps. 16, 17]). We also obtain a positive-characteristic analogue later in Theorem 7.3.

THEOREM 4.14. *Suppose that X is a normal scheme and that H is a Cartier divisor on X. Further suppose that $\mathfrak{a}$ is an ideal sheaf whose support does not contain any component of H and that c is a nonnegative real number. Then $(H, (\mathfrak{a}|_H)^c)$ has rational singularities if and only if $(X, H; \mathfrak{a}^c)$ has purely rational singularities near H.*

Proof. By Remark 3.19, we have a short exact sequence that maps to another short exact sequence:

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{J}(\omega_X, \mathfrak{a}^c) \arrow[r] \arrow[d, "\alpha"] & \operatorname{adj}(\omega_X, H; \mathfrak{a}^c) \otimes \mathcal{O}_X(H) \arrow[r] \arrow[d, "\beta"] & \mathcal{J}(\omega_H, (\mathfrak{a}|_H)^c) \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \omega_X \arrow[r] & \omega_X(H) \arrow[r] & \omega_H \arrow[r] & 0.
\end{tikzcd}
$$

The bottom row is exact on the right because X is Cohen–Macaulay near H under any assumption.

Suppose first that $(H, (\mathfrak{a}|_H)^c)$ has rational singularities; then so does $(X, \mathfrak{a}^c)$ near H. After localizing, we assume that $(X, \mathfrak{a}^c)$ is rational. These observations imply that the maps α and γ are isomorphisms, which proves that β is an isomorphism as well. Untwisting by $\mathcal{O}_X(H)$ implies that $(X, H; \mathfrak{a}^c)$ has purely rational singularities.

Conversely, if $(X, H; \mathfrak{a}^c)$ has purely rational singularities then $(X, \mathfrak{a}^c)$ has rational singularities by Remark 3.21. Hence α and β are isomorphisms, which implies that γ is an isomorphism as well, completing the proof. □

REMARK 4.15. One could, of course, dualize the proof of Theorem 4.14 and use the same argument in the derived category. If H is not Cartier then one could prove the same result using the suggested definition from Remark 3.22—assuming that $(X, \mathfrak{a}^c)$ already had rational singularities for the "only if" (rational implies purely rational) implication.

We also have the following result, which can be viewed as an analogue to the "adjunction direction" for log canonical singularities.

THEOREM 4.16. *Suppose that X is a reduced scheme and that H is a Cartier divisor. If $(X, (1-\varepsilon)H)$ is rational for all sufficiently small $\varepsilon > 0$, then H has Du Bois singularities.*

Proof. Let $\pi : \tilde{X} \to X$ be a log resolution of (X, H). Let $\bar{H}$ be the total transform of H, and let E be the reduced pre-image of H under π; in particular $\bar{H}_{\mathrm{red}} = E$. Because $\mathcal{O}_X \simeq_{\mathrm{qis}} \mathbf{R}\pi_*\mathcal{O}_{\tilde{X}}(\lfloor(1-\varepsilon)\bar{H}\rfloor)$ for all ε sufficiently close to zero, we have

$$\begin{aligned}\mathcal{O}_X(-H) &\simeq_{\mathrm{qis}} \mathbf{R}\pi_*\mathcal{O}_{\tilde{X}}(\lfloor(1-\varepsilon)\bar{H} - \bar{H}\rfloor) \\ &\simeq_{\mathrm{qis}} \mathbf{R}\pi_*\mathcal{O}_{\tilde{X}}(\lfloor -\varepsilon\bar{H}\rfloor) \simeq_{\mathrm{qis}} \mathbf{R}\pi_*\mathcal{O}_{\tilde{X}}(-E)\end{aligned}$$

for ε sufficiently small. Hence

$$\begin{array}{ccccccc}\mathcal{O}_X(-H) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_H & \xrightarrow{+1} & \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathbf{R}\pi_*\mathcal{O}_{\tilde{X}}(-E) & \longrightarrow & \mathbf{R}\pi_*\mathcal{O}_{\tilde{X}} & \longrightarrow & \mathbf{R}\pi_*\mathcal{O}_E & \xrightarrow{+1} & \end{array}$$

and the first two vertical arrows are quasi-isomorphisms. But then the third arrow is also a quasi-isomorphism, which proves that H has Du Bois singularities by [Kol, 2.4] (see also [Koll, 12.8]).

There is a partial converse to Theorem 4.16, which can be thought of as an analogue to inversion of adjunction for log canonicity (cf. [K]).

THEOREM 4.17. *Suppose that X is a reduced scheme and that H is a Cartier divisor on X. Further suppose that $X \setminus H$ is smooth. Then H has Du Bois singularities if and only if $(X, (1-\varepsilon)H)$ is rational near H for all sufficiently small $\varepsilon > 0$.*

Proof. We set up the proof in the same way as for Theorem 4.16, but now we observe that $\mathcal{O}_H \simeq_{\mathrm{qis}} \mathbf{R}\pi_*\mathcal{O}_E$ if and only if H has Du Bois singularities, since $X - H$ is smooth. Note that if H is Du Bois then X automatically has rational singularities (and thus is also Cohen–Macaulay) by [S, 5.1]. □

REMARK 4.18. When working with any ambient X that has rational singularities (see [S]), we expect that $\mathcal{O}_H \simeq_{\mathrm{qis}} \mathbf{R}\pi_*\mathcal{O}_E$ if and only if H has Du Bois singularities. Therefore, the condition that $X - H$ is smooth could possibly be replaced by the condition that $X - H$ is rational without otherwise altering the proof.

Remark 4.19. We also have a positive-characteristic analogue of the previous two theorems using F-injective instead of Du Bois singularities; see Proposition 7.7.

It is a conjecture of Kollár that log canonical singularities are Du Bois, and the foregoing proof shows that this conjecture is closely related to inversion of adjunction on log canonicity. Recent work by Kovács, Schwede, and Smith [SKoSm] has shown that (semi)log canonical singularities are Du Bois in the case of Cohen–Macaulay schemes. That result and the previous argument also give a short homological proof of inversion of adjunction on log canonicity (at least for the case where X has Gorenstein rational singularities and is smooth outside H).

Theorem 4.17 suggests that it might be natural to consider Du Bois singularities for pairs, and it perhaps even suggests a definition. However, there are certain technicalities associated with such a definition when the ambient space is not "nice". In positive characteristic we do propose an analogous definition, at least in the Cohen–Macaulay case; see Definition 7.5.

5. Positive-Characteristic Preliminaries

In this section we recall the definitions of generalizations of tight closure and F-singularities of pairs. The reader is referred to [HY; T1; T2; TW; TY] for details.

Throughout the following sections, all rings are excellent reduced Noetherian commutative rings with identity. Let R be a reduced ring of characteristic $p > 0$. We denote by R° the set of elements of R that are not in any minimal prime ideal of R. Let $F\colon R \to R$ be the Frobenius map that sends x to x^p, and let R viewed as an R-module via the e-times iterated Frobenius map $F^e\colon R \to R$ be denoted by ${}^eR$. Since R is reduced, we can identify $F^e\colon R \to {}^eR$ with the natural inclusion map $R \hookrightarrow R^{1/p^e}$. Also, for any ideal I of R and for any power q of p, we denote by $I^{[q]}$ the ideal of R generated by the qth powers of elements of I. We say that R is *F-finite* if ${}^1R$ (or $R^{1/p}$) is a finitely generated R-module. For example, any algebra essentially of finite type over a perfect field is F-finite.

Let M be an R-module. For each integer $e \geq 1$, we denote $\mathbb{F}^e(M) = \mathbb{F}^e_R(M) := {}^eR \otimes_R M$ and regard it as an R-module by the action of R from the left. Then we have the induced e-times Frobenius map $F^e\colon M \to \mathbb{F}^e(M)$. The image of $z \in M$ via this map is denoted by $z^q := F^e(z) \in \mathbb{F}^e(M)$. For an R-submodule N of M, we denote by $N_M^{[q]}$ the image of the induced map $\mathbb{F}^e(N) \to \mathbb{F}^e(M)$. If $M = R$ and N is an ideal I of R, then $I_R^{[q]} = I^{[q]}$.

Definition 5.1 [HY, Def. 6.1; T2, Def. 3.1]. Let R be a reduced ring of characteristic $p > 0$, let $\mathfrak{a} \subseteq R$ be an ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, and let $t \geq 0$ be a real number. Let $N \subseteq M$ be (not necessarily finitely generated) R-modules.

(i) The *$\mathfrak{a}^t$-tight closure* $N_M^{*\mathfrak{a}^t}$ of N in M is defined to be the submodule of M consisting of all elements $z \in M$ for which there exists a $c \in R^\circ$ such that

$$c\mathfrak{a}^{\lceil tq \rceil} z^q \subseteq N_M^{[q]}$$

for all large $q = p^e$. The $\mathfrak{a}^t$-tight closure of an ideal I of R is simply defined by $I^{*\mathfrak{a}^t} := I_R^{*\mathfrak{a}^t}$.

(ii) Let $x \in R^\circ$ such that $\mathfrak{a}$ is not contained in any minimal prime of xR. Then the *divisorial* $(x; \mathfrak{a}^t)$*-tight closure* $N_M^{\mathrm{div}*(x;\mathfrak{a}^t)}$ of N in M is defined to be the submodule of M satisfying the following condition: an element $z \in M$ belongs to $N_M^{\mathrm{div}*(x;\mathfrak{a}^t)}$ if there exists a $c \in R^\circ$ that is not in any minimal prime of xR and such that

$$cx^{q-1}\mathfrak{a}^{\lceil tq\rceil}z^q \subseteq N_M^{[q]}$$

for all large $q = p^e$. The divisorial $(x; \mathfrak{a}^t)$-tight closure of an ideal I of R is simply defined by $I^{\mathrm{div}*(x;\mathfrak{a}^t)} := I_R^{\mathrm{div}*(x;\mathfrak{a}^t)}$.

REMARK 5.2. If $\mathfrak{a} = R$ then $\mathfrak{a}^t$-tight closure is nothing but classical tight closure; that is, the classical tight closure I^* of an ideal $I \subseteq R$ is equal to I^{*R^t} for any $t \geq 0$. We refer the reader to [Hu] for the classical tight closure theory.

DEFINITION 5.3 [HY, Def. 6.3]. Let $R, \mathfrak{a}, t$ be as in Definition 5.1. An element $c \in R^\circ$ is called an $\mathfrak{a}^t$*-test element* if, for every ideal $I \subseteq R$, we have $cz^q\mathfrak{a}^{\lceil tq\rceil} \subseteq I^{[q]}$ for all $q = p^e$ whenever $z \in I^{*\mathfrak{a}^t}$.

A local ring R of characteristic $p > 0$ is said to be *F-rational* if $I^* = I$ for all ideals $I \subseteq R$ generated by a system of parameters for R (see [FW] for details).

LEMMA 5.4 [TY]. *Let* $(R, \mathfrak{m})$ *be an excellent reduced local ring of characteristic* $p > 0$. *Let* $\mathfrak{a} \subseteq R$ *be an ideal such that* $\mathfrak{a} \cap R^\circ \neq \emptyset$ *and let* $t \geq 0$ *be a real number.*

(1) *Let* $\hat{R}$ *denote the* $\mathfrak{m}$*-adic completion of* R. *Then* $I^{*\mathfrak{a}^t}\hat{R} = (I\hat{R})^{*(\mathfrak{a}\hat{R})^t}$ *for all* $\mathfrak{m}$*-primary ideals* I *of* R.

(2) *If* R *is equidimensional and if* S *is a multiplicatively closed set in* R, *then* $I^{*\mathfrak{a}^t}R_S = (IR_S)^{*(\mathfrak{a}R_S)^t}$ *for all ideals* I *generated by a subsystem of parameters for* R.

(3) *Let* $c \in R^\circ$ *such that* R_c *is Gorenstein F-rational. Then some power* c^n *of* c *is an* $\mathfrak{a}^t$*-test element for all ideals* $\mathfrak{a} \subseteq R$ *such that* $\mathfrak{a} \cap R^\circ \neq \emptyset$ *and for all real numbers* $t \geq 0$.

DEFINITION 5.5 [T1, Def. 3.1]. Let $\mathfrak{a}$ be an ideal of an F-finite reduced ring R of characteristic $p > 0$ such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, and let $t \geq 0$ be a real number.

(i) The pair $(R, \mathfrak{a}^t)$ is said to be *F-pure* if, for all large $q = p^e$, there exists an element $d \in \mathfrak{a}^{\lfloor t(q-1)\rfloor}$ such that the natural inclusion $d^{1/q}R \hookrightarrow R^{1/q}$ splits as an R-module homomorphism.

(ii) The pair $(R, \mathfrak{a}^t)$ is said to be *strongly F-regular* if, for every $c \in R^\circ$, there exist $q = p^e$ and $d \in \mathfrak{a}^{\lceil tq\rceil}$ such that the natural inclusion $(cd)^{1/q}R \hookrightarrow R^{1/q}$ splits as an R-module homomorphism.

(iii) Let $x \in R^\circ$ such that $\mathfrak{a}$ is not contained in any minimal prime of xR. The triple $(R, x; \mathfrak{a}^t)$ is said to be *divisorially F-regular* if, for every $c \in R^\circ$ that is not in any minimal prime of xR, there exist $q = p^e$ and $d \in \mathfrak{a}^{\lceil tq \rceil}$ such that the natural inclusion $(cdx^{q-1})^{1/q}R \hookrightarrow R^{1/q}$ splits as an R-module homomorphism.

Definition 5.6 [TW, Def. 2.1]. Let R and $\mathfrak{a}$ be as in Definition 5.5. Assume in addition that R is a strongly F-regular ring; that is, assume the pair (R, R^1) is strongly F-regular. Then the F-pure threshold $\operatorname{fpt}(\mathfrak{a})$ of $\mathfrak{a}$ is defined to be

$$\begin{aligned}\operatorname{fpt}(\mathfrak{a}) &= \{t \in \mathbb{R}_{\geq 0} \mid (R, \mathfrak{a}^t) \text{ is strongly } F\text{-regular}\}\\ &= \{t \in \mathbb{R}_{\geq 0} \mid (R, \mathfrak{a}^t) \text{ is } F\text{-pure}\}.\end{aligned}$$

Remark 5.7. (1) When $\mathfrak{a} = R$, the strong F-regularity (resp. F-purity) of $(R, \mathfrak{a}^t)$ is equivalent to that of R. See [HoHu1; HoHu2; HoRo] for more on F-pure rings and strongly F-regular rings.

(2) If $(R, \mathfrak{a}^t)$ is strongly F-regular then it is F-pure. If $(R, x; \mathfrak{a}^t)$ is divisorially F-regular, then $(R, x\mathfrak{a}^t)$ is F-pure and $(R, x^{1-\varepsilon}\mathfrak{a}^t)$ is strongly F-regular for any $1 \geq \varepsilon > 0$ (see [HW]).

(3) If $(R, \mathfrak{a}^t)$ is strongly F-regular (resp. $(R, x; \mathfrak{a}^t)$ is divisorially F-regular), then $I^{*\mathfrak{a}^t} = I$ (resp. $I^{\operatorname{div}*(x, \mathfrak{a}^t)} = I$) for all ideals $I \subseteq R$. If R is F-finite $\mathbb{Q}$-Gorenstein, then the converse also holds. The reader is referred to [T1, Cor. 3.5] (resp. [T2, Rem. 3.2]).

6. Basic Definitions and Fundamental Properties in Positive Characteristic

In [FW], Fedder and Watanabe defined the notion of F-rational rings. In this section, we introduce the notion of F-rationality for a pair $(R, \mathfrak{a}^t)$ of a ring R of characteristic $p > 0$ and an ideal $\mathfrak{a} \subseteq R$ with real exponent $t \geq 0$.

Definition 6.1 (cf. [FW]). Let $\mathfrak{a}$ be an ideal of a reduced ring R of characteristic $p > 0$ such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, and let $t \geq 0$ be a real number. When R is local, $(R, \mathfrak{a}^t)$ is said to be *F-rational* if $I^{\mathfrak{a}^t *} = I$ for every ideal I generated by a system of parameters for R. When R is not local, we say that $(R, \mathfrak{a}^t)$ is F-rational if the localization $(R_\mathfrak{m}, \mathfrak{a}^t_\mathfrak{m})$ is F-rational for every maximal ideal $\mathfrak{m}$ of R.

Proposition 6.2. *Let $\mathfrak{a} \subseteq \mathfrak{b}$ be ideals of a reduced ring R of characteristic $p > 0$ such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, and let $t \geq 0$ be a real number.*

(1) *If $(R, \mathfrak{a}^t)$ is F-rational, then so is $(R, \mathfrak{a}^s)$ for all $0 \leq s \leq t$.*
(2) *If $(R, \mathfrak{a}^t)$ is F-rational, then so is $(R, \mathfrak{b}^t)$. For $\mathfrak{a}$ a reduction of $\mathfrak{b}$, $(R, \mathfrak{a}^t)$ is F-rational if and only if $(R, \mathfrak{b}^t)$ is F-rational.*
(3) *If $(R, \mathfrak{a}^t)$ is F-rational, then R is F-rational and, in particular, is normal. Moreover, if R is locally excellent then R is Cohen–Macaulay.*

Proof. This follows immediately from [HY, Prop. 1.3] and [Hu, Thm. 4.2].

Lemma 6.3. *Let $(R, \mathfrak{m})$ be a d-dimensional excellent reduced local ring of characteristic $p > 0$. Let $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, and let $t \geq 0$ be a real number. Then the following three conditions are equivalent.*

(1) *$(R, \mathfrak{a}^t)$ is F-rational.*
(2) *R is equidimensional and $I^{*\mathfrak{a}^t} = I$ for some ideal I generated by a system of parameters for R.*
(3) *R is Cohen–Macaulay and $0^{*\mathfrak{a}^t}_{H^d_{\mathfrak{m}}(R)} = 0$ in $H^d_{\mathfrak{m}}(R)$. The latter condition is equivalent to saying that, for each $c \in R^\circ$, there exist $q = p^e$ and $c' \in \mathfrak{a}^{\lceil tq \rceil}$ such that $cc'F^e \colon H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R)$ is injective, where $F^e \colon H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R)$ denotes the induced e-times iterated Frobenius map on $H^d_{\mathfrak{m}}(R)$.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious, so we will prove the implication (2) $\Rightarrow$ (3). First note that R is Cohen–Macaulay because $I^{*\mathfrak{a}^t} = I^* = I$ and R is equidimensional (see [Hu, Thm. 4.2]). We choose a system of parameters $x_1, \dots, x_d$ in R such that $I = (x_1, \dots, x_d)$ and let x denote the product of $x_1, \dots, x_d$.

Claim. $(x_1^m, \dots, x_d^m)^{*\mathfrak{a}^t} = (x_1^m, \dots, x_d^m)$ for each integer $m \geq 1$.

Proof of Claim. Let $y \in (x_1^m, \dots, x_d^m)^{*\mathfrak{a}^t}$. Without loss of generality we may assume that $y(x_1, \dots, x_d) \subseteq (x_1^m, \dots, x_d^m)$. Since R is Cohen–Macaulay, one has $y \in (x_1^m, \dots, x_d^m, x^{m-1})$. We write y as $y = \sum_{i=1}^d a_i x_i^m + b x^{m-1}$, where $a_i \in R$ for all $i = 1, \dots, d$ and $b \in R$. By definition, there exists a $c \in R^\circ$ such that $c\mathfrak{a}^{\lceil tq \rceil} y^q \in (x_1^{mq}, \dots, x_d^{mq})$ for all large $q = p^e$. Then $c\mathfrak{a}^{\lceil tq \rceil} b^q \in (x_1^q, \dots, x_d^q)$. Hence $b \in (x_1, \dots, x_d)^{*\mathfrak{a}^t} = (x_1, \dots, x_d)$, which implies that $y \in (x_1^m, \dots, x_d^m)$. □

Fix an arbitrary element $\eta = [z/x^m] \in 0^{*\mathfrak{a}^t}_{H^d_{\mathfrak{m}}(R)}$. By the definition of $\mathfrak{a}^t$-tight closure, there exists a $c \in R^\circ$ such that $0 = c\mathfrak{a}^{\lceil tq \rceil}\eta^q = c\mathfrak{a}^{\lceil tq \rceil}[z^q/x^{mq}]$ for all large $q = p^e$. This implies that, for large n, $c\mathfrak{a}^{\lceil tq \rceil} z^q x^n \in (x_1^{n+mq}, \dots, x_d^{n+mq})$. Since R is Cohen–Macaulay, we then obtain that $c\mathfrak{a}^{\lceil tq \rceil} z^q \in (x_1^{mq}, \dots, x_d^{mq})$ for all large $q = p^e$; this yields $z \in (x_1^m, \dots, x_d^m)^{*\mathfrak{a}^t} = (x_1^m, \dots, x_d^m)$, where the last equality follows from the preceding claim. Then $\eta = 0$; that is, $0^{*\mathfrak{a}^t}_{H^d_{\mathfrak{m}}(R)} = 0$.

Next we will show the implication (3) $\Rightarrow$ (1). Take any system of parameters $x_1, \dots, x_d$ in R and let x represent the product of $x_1, \dots, x_d$. Fix any element $z \in (x_1, \dots, x_d)^{*\mathfrak{a}^t}$, and consider the element $\xi = [z/x] \in H^d_{\mathfrak{m}}(R)$. By definition, we can choose an element $d \in R^\circ$ such that $d\mathfrak{a}^{\lceil tq \rceil} z^q \in (x_1^q, \dots, x_d^q)$ for all large $q = p^e$. This implies that $d\mathfrak{a}^{\lceil tq \rceil}\xi^q = 0$ for all large $q = p^e$; that is, $\xi \in 0^{*\mathfrak{a}^t}_{H^d_{\mathfrak{m}}(R)}$. Since $\xi = 0$ by assumption, we obtain $z \in (x_1, \dots, x_d)$. □

REMARK 6.4. Let the notation be as in Lemma 6.3 and assume in addition that R is a homomorphic image of a Gorenstein local ring. Then one could define the *generalized parameter test submodule* $\tau(\omega_R, \mathfrak{a}^t)$ associated to $(R, \mathfrak{a}^t)$ as

$$\tau(\omega_R, \mathfrak{a}^t) = \mathrm{Ann}_{\omega_R}(0^{*\mathfrak{a}^t}_{H^d_{\mathfrak{m}}(R)}) \subseteq \omega_R.$$

This is a characteristic-p analogue of the multiplier submodule (see Definition 3.6). By Lemma 6.3, $(R, \mathfrak{a}^t)$ is F-rational if and only if R is Cohen–Macaulay and $\tau(\omega_R, \mathfrak{a}^t) = \omega_R$. Employing the same strategy as in [HyVi], we can use the generalized parameter test submodule $\tau(\omega_R, \mathfrak{a}^t)$ to recover the Briançon–Skoda theorem for F-rational rings [AHu, Thm. 3.6]: If $(R, \mathfrak{m})$ is an excellent F-rational local ring of dimension d that is a homomorphic image of a Gorenstein local ring, then $\overline{I^{n+d-1}} \subseteq I^n$ for all ideals $I \subseteq R$ and integers $n \geq 0$.

PROPOSITION 6.5. *Let $\mathfrak{a}$ be an ideal of a locally excellent reduced ring R of characteristic $p > 0$ such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, and let $t \geq 0$ be a real number.*

(1) *If $(R, \mathfrak{a}^t)$ is strongly F-regular then it is F-rational. If R is F-finite Gorenstein, then the converse also holds.*
(2) *Let S be a multiplicatively closed set in R. If $(R, \mathfrak{a}^t)$ is F-rational, then the localization $(R_S, \mathfrak{a}_S^t)$ is also F-rational.*
(3) *Assume, in addition, that R is local. Then $(R, \mathfrak{a}^t)$ is F-rational if and only if $(\hat{R}, (\mathfrak{a}\hat{R})^t)$ is F-rational.*

Proof. (1) By Remark 5.7, strongly F-regular pairs are F-rational. We thus consider the converse implication. Since strong F-regularity commutes with localization, we may assume that $(R, \mathfrak{m})$ is an F-finite reduced local ring. By [T1, Lemma 3.4], $(R, \mathfrak{a}^t)$ is strongly F-regular if and only if, for each $c \in R^\circ$, there exist $q = p^e$ and $c' \in \mathfrak{a}^{\lceil tq \rceil}$ such that $cc'F^e \colon E \to \mathbb{F}(E)$ is injective, where $F^e \colon E \to \mathbb{F}(E)$ is the e-times Frobenius map induced on the injective hull $E = E_R(R/\mathfrak{m})$ of the residue field $R/\mathfrak{m}$. Hence, if $(R, \mathfrak{a}^t)$ is Gorenstein F-rational, then by Lemma 6.3 it is strongly F-regular because in this case $H_{\mathfrak{m}}^{\dim R}(R) \cong E$.

(2) We may assume that R is a Cohen–Macaulay local ring, and it suffices to show that $(R_P, \mathfrak{a}_P^t)$ is F-rational for every prime ideal P of R. Let $x_1, \ldots, x_i$ be any elements of P whose images in R_P form a system of parameters for R_P. We can choose elements $x_{i+1}, \ldots, x_d$ of R such that $x_1, \ldots, x_d$ form a system of parameters for R. Set $I = (x_1, \ldots, x_i)$ and $I_n = (x_1, \ldots, x_i, x_{i+1}^n, \ldots, x_d^n)$ for each integer $n \geq 1$. By assumption, $I_n^{*\mathfrak{a}^t} = I_n$ for all $n \geq 1$. This implies that

$$I = \bigcap_n I_n = \bigcap_n I_n^{*\mathfrak{a}^t} = I^{*\mathfrak{a}^t}.$$

Because I is generated by a subsystem of parameters for R, by Lemma 5.4(2) it follows that $(IR_P)^{*\mathfrak{a}_P^t} = I^{*\mathfrak{a}^t}R_P = IR_P$. In other words, $(R_P, \mathfrak{a}_P^t)$ is F-rational.

(3) Let I be an ideal of R generated by a system of parameters for R. By Lemma 5.4(1), $I^{*\mathfrak{a}^t} = I$ if and only if $(I\hat{R})^{*(\mathfrak{a}\hat{R})^t} = I\hat{R}$. Thus, the assertion is obvious. □

DEFINITION 6.6. Let $\mathfrak{a}$ be an ideal of a reduced local ring R of characteristic $p > 0$ such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, and let $t \geq 0$ be a real number. Then an element $c \in R^\circ$ is called a *parameter* $\mathfrak{a}^t$*-test element* if, for every ideal I generated by a system of parameters for R, we have $cz^q\mathfrak{a}^{\lceil tq \rceil} \subseteq I^{[q]}$ for all $q = p^e$ whenever $z \in I^{*\mathfrak{a}^t}$.

REMARK 6.7. Let R be a Cohen–Macaulay reduced local ring of characteristic $p > 0$, and let $c \in R^\circ$ be a parameter $\mathfrak{a}^t$-test element. Then, by the same argument as the proof of Lemma 6.3, we can easily check that $(R, \mathfrak{a}^t)$ is F-rational if and only if there exist $q = p^e$ and $c' \in \mathfrak{a}^{\lceil tq \rceil}$ such that $cc'F^e \colon H_{\mathfrak{m}}^d(R) \to H_{\mathfrak{m}}^d(R)$ is injective, where $F^e \colon H_{\mathfrak{m}}^d(R) \to H_{\mathfrak{m}}^d(R)$ denotes the induced e-times iterated Frobenius map on $H_{\mathfrak{m}}^d(R)$.

LEMMA 6.8. *Let* $(R, \mathfrak{m})$ *be a* d*-dimensional excellent reduced equidimensional local ring of characteristic* $p > 0$. *Let* $c \in R^\circ$ *such that* R_c *is* F*-rational. Then some power* c^n *of* c *is a parameter* $\mathfrak{a}^t$*-test element for all ideals* $\mathfrak{a} \subseteq R$ *such that* $\mathfrak{a} \cap R^\circ \neq \emptyset$ *and, for all real numbers,* $t \geq 0$.

Proof. Making use of gamma construction, by an argument analogous to the proof of [V, Thm. 3.9] we can reduce to the case where R is an F-finite reduced local ring that is a homomorphic image of a Gorenstein local ring. Let $c' \in R^\circ$ be an R- and $\mathfrak{a}^t$-test element (we can take such an element by Lemma 5.4(3)), and let $F^e\colon H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R)$ denote the induced e-times iterated Frobenius map on $H^d_{\mathfrak{m}}(R)$. We now claim (cf. [V, Thm. 1.13]) that there exist $q_0 = p^{e_0}$ and $n \in \mathbb{N}$ such that the nth power c^n of c kills $\operatorname{Ker}(c'F^{e_0})$.

Take any system of parameters $x_1, \dots, x_d$ in R and let x denote the product of $x_1 \dots x_d$. Fix any $z \in (x_1, \dots, x_d)^{*\mathfrak{a}^t}$ and consider the element $\xi = [z/x] \in H^d_{\mathfrak{m}}(R)$. Since c' is an $\mathfrak{a}^t$-test element, one has $c'\mathfrak{a}^{\lceil tq_0q \rceil} z^{q_0q} \in (x_1^{q_0q}, \dots, x_d^{q_0q})$ for all $q = p^e$, which implies that $c'\mathfrak{a}^{\lceil tq_0q \rceil}\xi^{q_0q} = 0$ in $H^d_{\mathfrak{m}}(R)$. In particular, $\mathfrak{a}^{\lceil tq \rceil}\xi^q$ is contained in $\operatorname{Ker}(c'F^{e_0})$ and thus, by our claim, $c^n\mathfrak{a}^{\lceil tq \rceil}\xi^q = 0$. Then there exists an integer $k \geq 0$ such that $c^n\mathfrak{a}^{\lceil tq \rceil}z^q x^k \in (x_1^{q+k}, \dots, x_d^{q+k})$. Applying the colon-capturing property of classical tight closure and [HoHu, Lemma 12.9], one has some power c^m of c such that $c^m c^n \mathfrak{a}^{\lceil tq \rceil} z^q \in (x_1^q, \dots, x_d^q)$ for all $q = p^e$. Since m is independent of the choice of $x_1, \dots, x_d$, z, $\mathfrak{a}$, and t, it follows that c^{m+n} is an $\mathfrak{a}^t$-test element for all ideals $\mathfrak{a} \subseteq R$ such that $\mathfrak{a} \cap R^\circ \neq \emptyset$ and for all real numbers $t \geq 0$. □

THEOREM 6.9 (cf. [Sm2, Thm. 3.1]). *Let* R *be an excellent reduced local ring of characteristic* $p > 0$, *let* $\mathfrak{a}$ *be an ideal of* R *such that* $\mathfrak{a} \cap R^\circ \neq \emptyset$, *and let* $t \geq 0$ *be a real number. If* $(R, \mathfrak{a}^t)$ *is* F*-rational, then it is pseudo-rational.*

Proof. Because R is excellent and F-rational, it is also Cohen–Macaulay, normal, and analytically unramified. Let π and δ_π be as in Definition 3.9. Then, by [HY, Prop. 3.8], $\operatorname{Ker}(\delta_\pi) \subseteq 0^{*\mathfrak{a}^t}_{H^d_{\mathfrak{m}}(R)}$. Since $(R, \mathfrak{a}^t)$ is F-rational by Lemma 6.3, this implies that $\operatorname{Ker}(\delta_\pi) = 0$. □

Let R be an algebra of essentially finite type over a field k of characteristic 0. Let $\mathfrak{a} \subseteq R$ be an ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, and let $t \geq 0$ be a real number. One can choose a finitely generated $\mathbb{Z}$-subalgebra A of k and a subalgebra R_A of R of essentially finite type over A such that the natural map $R_A \otimes_A k \to R$ is an isomorphism and $\mathfrak{a}_A R = \mathfrak{a}$, where $\mathfrak{a}_A := \mathfrak{a} \cap R_A \subseteq R_A$. Given a closed point $s \in \operatorname{Spec} A$ with residue field $\kappa = \kappa(s)$, we denote the corresponding fibers over s by R_κ and $\mathfrak{a}_\kappa$. Then we refer to a triple $(\kappa, R_\kappa, \mathfrak{a}_\kappa)$, for a general closed point $s \in \operatorname{Spec} A$ with residue field $\kappa = \kappa(s)$ of sufficiently large characteristic $p \gg 0$, as *reduction to characteristic* $p \gg 0$ of $(k, R, \mathfrak{a})$. The pair $(R_\kappa, \mathfrak{a}_\kappa^t)$ inherits the properties possessed by the original pair $(R, \mathfrak{a}^t)$ (the size of p depends on t). Furthermore, given a log resolution $f\colon \tilde{X} \to X = \operatorname{Spec} R$ of $(X, \mathfrak{a})$, we can reduce this entire setup to characteristic $p \gg 0$.

Definition 6.10. In the situation just described, $(R, \mathfrak{a}^t)$ is said to be of *strongly F-regular* (resp. *F-pure, F-rational*) *type* if the reduction to characteristic $p \gg 0$ of $(R, \mathfrak{a}^t)$ is strongly F-regular (resp. F-pure, F-rational).

Theorem 6.11 (cf. [H1; MeSr]). *Let R be a finitely generated algebra over a field of characteristic* 0. *Let $\mathfrak{a} \subseteq R$ be an ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, and let $t \geq 0$ be a real number. Then* $(\operatorname{Spec} R, \mathfrak{a}^t)$ *has rational singularities if and only if* $(R, \mathfrak{a}^t)$ *is of F-rational type.*

Proof. Since the assertion is local, we may assume that $(R, \mathfrak{m})$ is a d-dimensional normal Cohen–Macaulay local ring of essentially finite type over a field of characteristic 0. Fix a log resolution $\pi : Y \to X := \operatorname{Spec} R$ of $\mathfrak{a}$ such that $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-G)$, let $E := \pi^{-1}(\mathfrak{m})$ be the closed fiber of π, and let $\delta_\pi : H^d_{\mathfrak{m}}(R) \to H^d_E(\mathcal{O}_Y(\lfloor tG \rfloor))$ be as in Definition 3.9. Then, by Remark 3.10, $(\operatorname{Spec} R, \mathfrak{a}^t)$ has rational singularities if and only if the map δ_π is injective. After reduction to characteristic $p \gg 0$, we can assume that R is a normal Cohen–Macaulay local ring of essentially finite type over a perfect field of characteristic p together with a log resolution $\pi : Y \to X := \operatorname{Spec} R$ of $(X, \mathfrak{a})$ such that $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-G)$. Then it suffices to show that $(R, \mathfrak{a}^t)$ is F-rational if and only if the map δ_π is injective—but this follows immediately from the combination of [HY, Thm. 6.9] and Lemma 6.3. $\square$

Remark 6.12. In fact, one can use the same techniques to show an equivalence between the multiplier submodule and the parameter test submodule (cf. [HY, Thm. 6.8]).

Remark 6.13. In [Sm2], Smith gave a characterization of F-rational rings in terms of the stability of submodules of $H^d_{\mathfrak{m}}(R)$ under the action of Frobenius. Using the technique of Hara and Yoshida (see [HY, Prop. 1.15]), one can prove an analogous generalization to F-rational pairs.

We now consider another variant of F-rational pairs corresponding to the pure rationality defined in Definition 3.15.

Definition 6.14. Let x be a nonzero divisor of a reduced ring R of characteristic $p > 0$, and let $\mathfrak{a} \subseteq R$ be an ideal that is not contained in any minimal prime of xR. Let $t \geq 0$ be a real number. When R is local, the triple $(R, x; \mathfrak{a}^t)$ is said to be *divisorially F-rational* if $I^{\operatorname{div}*(x;\,\mathfrak{a}^t)} = I$ for every ideal I generated by a system of parameters for R. When R is not local, we say that $(R, x; \mathfrak{a}^t)$ is divisorially F-rational if the localization $(R_{\mathfrak{m}}, \mathfrak{a}^t_{\mathfrak{m}})$ is divisorially F-rational for every maximal ideal $\mathfrak{m}$ of R.

We can prove analogues of Proposition 6.2, Lemma 6.3, and Proposition 6.5 for divisorial F-rationality.

Proposition 6.15. *Let x be a nonzero divisor of a reduced ring R of characteristic $p > 0$, and let $\mathfrak{a} \subseteq R$ be an ideal that is not contained in any minimal prime of xR. Let $t \geq 0$ be a real number.*

(1) *If* $(R, x; \mathfrak{a}^t)$ *is divisorially F-rational, then* $(R, x^{1-\varepsilon}\mathfrak{a}^t)$ *is F-rational for all* $1 \geq \varepsilon > 0$; *in particular,* $(R, \mathfrak{a}^t)$ *is F-rational.*
(2) *Assume in addition that R is locally excellent. If* $(R, x; \mathfrak{a}^t)$ *is divisorially F-regular, then it is divisorially F-rational*; *if R is F-finite Gorenstein, then the converse also holds.*

Proof. Part (2) follows from the combination of Lemma 6.16 and an argument similar to the proof of Proposition 6.5(1). So we will prove only part (1). Without loss of generality, we may assume that R is local. Let $I \subseteq R$ be an ideal generated by a system of parameters for R, and let $z \in I^{*x^{1-\varepsilon}\mathfrak{a}^t}$. By definition, there exists a $c \in R^\circ$ such that $cx^{\lceil(1-\varepsilon)q\rceil}\mathfrak{a}^{\lceil tq\rceil}z^q \subseteq I^{[q]}$ for all large $q = p^e$. Then one can choose an element $d \in R^\circ$ that is not in any minimal prime of xR such that dx^n lies in the ideal cR for some $n \in \mathbb{N}$. Taking sufficiently large $q = p^e$ so that $n + \lceil(1-\varepsilon)q\rceil \leq q-1$, one has $dx^{q-1}\mathfrak{a}^{\lceil tq\rceil}z^q \subseteq I^{[q]}$. This implies that $z \in I^{\operatorname{div}*(x;\mathfrak{a}^t)} = I$ because $(R, x; \mathfrak{a}^t)$ is divisorially F-rational. Thus, $(R, x^{1-\varepsilon}\mathfrak{a}^t)$ is F-rational. □

LEMMA 6.16. *Let* $(R, \mathfrak{m})$ *be a d-dimensional excellent reduced local ring of characteristic* $p > 0$, *and let* $t \geq 0$ *be a real number. Fix* $x \in R^\circ$ *and let* $\mathfrak{a} \subseteq R$ *be an ideal that is not contained in any minimal prime of xR. Then the following three conditions are equivalent.*

(1) $(R, \mathfrak{a}^t)$ *is divisorially F-rational.*
(2) *R is equidimensional and* $I^{\operatorname{div}*(x;\mathfrak{a}^t)} = I$ *for some ideal I generated by a system of parameters for R.*
(3) *R is Cohen–Macaulay and* $0^{\operatorname{div}*(x;\mathfrak{a}^t)}_{H^d_\mathfrak{m}(R)} = 0$ *in* $H^d_\mathfrak{m}(R)$. *The latter condition is equivalent to saying that, for each* $c \in R^\circ$ *that is not in any minimal prime of xR, there exist* $q = p^e$ *and* $c' \in \mathfrak{a}^{\lceil tq\rceil}$ *such that* $cc'x^{q-1}F^e \colon H^d_\mathfrak{m}(R) \to H^d_\mathfrak{m}(R)$ *is injective, where* $F^e \colon H^d_\mathfrak{m}(R) \to H^d_\mathfrak{m}(R)$ *denotes the induced e-times iterated Frobenius map on* $H^d_\mathfrak{m}(R)$.

Proof. The proof is essentially the same as that of Lemma 6.3. □

REMARK 6.17. Let the notation be as in Lemma 6.16 and assume in addition that R is a homomorphic image of a Gorenstein local ring. Then one could define the *divisorial test submodule* $\tau^{\operatorname{div}}(\omega_R, x; \mathfrak{a}^t)$ associated to $(R, x; \mathfrak{a}^t)$ as

$$\tau^{\operatorname{div}}(\omega_R, x; \mathfrak{a}^t) = \operatorname{Ann}_{\omega_R}(0^{\operatorname{div}*(x;\mathfrak{a}^t)}_{H^d_\mathfrak{m}(R)}) \subseteq \omega_R.$$

This is a characteristic-p analogue of the adjoint submodule (see Definition 3.17). By Lemma 6.16, $(R, x; \mathfrak{a}^t)$ is divisorially F-rational if and only if $\tau^{\operatorname{div}}(\omega_R, x; \mathfrak{a}^t) = \omega_R$ and R is Cohen–Macaulay.

7. Geometric Properties

In fixed prime characteristic, F-rational pairs satisfy several nice properties that are analogous to those of rational pairs.

PROPOSITION 7.1. *Let $R \hookrightarrow S$ be a pure finite local homomorphism of local domains of characteristic $p > 0$. Let $\mathfrak{a}$ be a nonzero ideal of R and let $t \geq 0$ be a real number. If $(S, (\mathfrak{a}S)^t)$ is F-rational, then so is $(R, \mathfrak{a}^t)$.*

Proof. Let $I \subseteq R$ be an ideal generated by a system of parameters for R. Then it is easy to check that $I^{*\mathfrak{a}^t}S \subseteq (IS)^{*(\mathfrak{a}S)^t}$. Since IS is generated by a system of parameters for S, it follows by assumption that $(IS)^{*(\mathfrak{a}S)^t} = IS$. Thus

$$I^{*\mathfrak{a}^t} = I^{*\mathfrak{a}^t}S \cap R \subseteq (IS)^{*(\mathfrak{a}S)^t} \cap R = I;$$

that is, $(R, \mathfrak{a}^t)$ is F-rational. □

REMARK 7.2. Suppose R and S are domains, $\mathfrak{a}$ is an ideal of R, and R is a direct summand of S. If $(S, (\mathfrak{a}S)^t)$ is strongly F-regular, then $(R, \mathfrak{a}^t)$ is also strongly F-regular and, in particular, F-rational. However, even if $(S, (\mathfrak{a}S)^t)$ is F-rational, $(R, \mathfrak{a}^t)$ is not necessarily F-rational in general (see [W; HWY] for counterexamples). The reader should compare Proposition 7.1 with Corollary 4.11.

PROPOSITION 7.3. *Let $(R, \mathfrak{m})$ be an excellent reduced local ring of characteristic $p > 0$, and let $x \in \mathfrak{m}$ be a nonzero divisor of R. Denote $S := R/xR$. Let $\mathfrak{a} \subseteq R$ be an ideal that is not contained in any minimal prime of xR, and let $t \geq 0$ be a real number. Then $(S, (\mathfrak{a}S)^t)$ is F-rational if and only if $(R, x; \mathfrak{a}^t)$ is divisorially F-rational.*

Proof. First assume that $(S, (\mathfrak{a}S)^t)$ is F-rational. Note that both S and R are normal and Cohen–Macaulay by Proposition 6.2. We choose elements $y_1, \dots, y_{d-1}$ in R such that $x, y_1, \dots, y_{d-1}$ forms a system of parameters for R. Let $z \in (x, y_1, \dots, y_{d-1})^{\mathrm{div}*(x;\, \mathfrak{a}^t)}$. Then there exists a $c \in R \setminus xR$ such that $c\mathfrak{a}^{\lceil tq \rceil}x^{q-1}z^q \subseteq (x^q, y_1^q, \dots, y_{d-1}^q)$ for all large $q = p^e$. Since $x, y_1, \dots, y_{d-1}$ is an R-regular sequence, one has $c\mathfrak{a}^{\lceil tq \rceil}z^q \in (x, y_1^q, \dots, y_{d-1}^q)$. This implies that $\bar{c}(\mathfrak{a}S)^{\lceil tq \rceil}\bar{z}^q \in (\overline{y_1}^q, \dots, \overline{y_{d-1}}^q)$ where $\bar{c}, \bar{z}, \overline{y_1}, \dots, \overline{y_{d-1}}$ are the images of $c, z, y_1, \dots, y_{d-1}$ in S, respectively. Since $\bar{c} \in S \setminus \{0\} = S^\circ$, we have

$$\bar{z} \in (\overline{y_1}, \dots, \overline{y_{d-1}})^{*(\mathfrak{a}S)^t} = (\overline{y_1}, \dots, \overline{y_{d-1}}).$$

Thus, z lies in $(x, y_1, \dots, y_{d-1})$ and $(R, x; \mathfrak{a}^t)$ is divisorially F-rational by Lemma 6.16. The converse argument just reverses this. □

As a corollary of Proposition 7.3 we obtain the correspondence between pure rationality and divisorial F-rationality. Let R be an algebra of essentially finite type over a field of characteristic 0, and let $t \geq 0$ be a real number. Let x be a nonzero divisor of R and let $\mathfrak{a} \subseteq R$ be an ideal that is not contained in any minimal prime of xR. Then $(R, x; \mathfrak{a}^t)$ is said to be of *divisorially F-rational* if reduction to characteristic $p \gg 0$ of $(R, x; \mathfrak{a}^t)$ is divisorially F-rational (see the paragraph before Definition 6.10 for the meaning of “reduction to characteristic $p \gg 0$”).

COROLLARY 7.4. *Let notation be as before and assume in addition that R is a normal local ring of essentially finite type over a field of characteristic 0. Then*

$(\operatorname{Spec} R, \operatorname{div}(x), \mathfrak{a}^t)$ has purely rational singularities if and only if $(R, x, \mathfrak{a}^t)$ is of divisorially F-rational type.

Proof. This follows from the combination of Theorem 6.11, Proposition 7.3, and Theorem 4.14. □

DEFINITION 7.5. Let $\mathfrak{a}$ be an ideal of a Cohen–Macaulay reduced ring R of characteristic $p > 0$ such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, and let $t \geq 0$ be a real number.

(i) When $(R, \mathfrak{m})$ is a d-dimensional local ring and $F^e \colon H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R)$ denotes the induced e-times iterated Frobenius map on $H^d_{\mathfrak{m}}(R)$, the pair $(R, \mathfrak{a}^t)$ is said to be *F-injective* if, for all large $q = p^e$, there exists a $c \in \mathfrak{a}^{\lfloor t(q-1) \rfloor}$ such that $cF^e \colon H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R)$ is injective. When R is not local, we say that $(R, \mathfrak{a}^t)$ is F-injective if the localization $(R_{\mathfrak{m}}, \mathfrak{a}^t_{\mathfrak{m}})$ is F-injective for every maximal ideal $\mathfrak{m}$ of R.

(ii) Suppose that R is F-rational. Then we define the F-injective threshold $\operatorname{fit}(\mathfrak{a})$ of $\mathfrak{a}$ to be

$$\operatorname{fit}(\mathfrak{a}) = \sup\{t \in \mathbb{R}_{\geq 0} \mid (R, \mathfrak{a}^t) \text{ is } F\text{-rational}\}.$$

We briefly study the properties of F-injective pairs, which are needed in a subsequent proposition. Let $\mathfrak{a}$ be an ideal of a reduced ring R of characteristic $p > 0$ such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, and let $t \geq 0$ be a real number. Then the $\mathfrak{a}^t$*-Frobenius closure* of an ideal $I \subseteq R$ is defined to be the ideal of R consisting of all the elements $x \in R$ for which $\mathfrak{a}^{\lfloor t(q-1) \rfloor} x^q \subseteq I^{[q]}$ for all large $q = p^e$. This ideal is denoted by $I^{F\mathfrak{a}^t}$.

LEMMA 7.6. *Let notation be as before and assume in addition that $(R, \mathfrak{m})$ is a d-dimensional Cohen–Macaulay local ring.*

(1) *If $(R, \mathfrak{a}^t)$ is F-rational then it is F-injective.*

(2) *$(R, \mathfrak{a}^t)$ is F-injective if and only if, for every (resp. some) ideal I generated by a system of parameters for R, one has $I^{F\mathfrak{a}^t} = I$.*

(3) *Suppose that R is an excellent F-rational local ring and that $\mathfrak{a}$ is a principal ideal. Then $(R, \mathfrak{a}^t)$ is F-injective if and only if $t \leq \operatorname{fit}(\mathfrak{a})$.*

Proof. Part (1) is obvious by Lemma 6.3, and part (2) follows from an argument similar to the proof of Lemma 6.3. So we will prove only part (3).

First we show that

$$\operatorname{fit}(\mathfrak{a}) = \sup\{t \in \mathbb{R}_{\geq 0} \mid (R, \mathfrak{a}^t) \text{ is } F\text{-injective}\}.$$

To check this, it is enough to show that if $(R, \mathfrak{a}^t)$ is F-injective for some $t > 0$ then $(R, \mathfrak{a}^{t-\varepsilon})$ is F-rational for all $t \geq \varepsilon > 0$. By Lemma 6.8, the unit 1 is a parameter $\mathfrak{a}^{t-\varepsilon}$-test element. Choose sufficiently large $q = p^e$ so that $\lceil (t-\varepsilon)q \rceil \leq \lfloor t(q-1) \rfloor$. Then the F-injectivity of $(R, \mathfrak{a}^t)$ implies that there exists a $c \in \mathfrak{a}^{\lceil (t-\varepsilon)q \rceil}$ such that $cF^e \colon H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R)$ is injective, where $F^e \colon H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R)$ denotes the induced e-times iterated Frobenius map on $H^d_{\mathfrak{m}}(R)$. By Remark 6.7, this is equivalent to the F-rationality of $(R, \mathfrak{a}^{t-\varepsilon})$.

To complete the proof of part (3), it only remains to show that $(R, \mathfrak{a}^{\operatorname{fit}(\mathfrak{a})})$ is F-injective. Let $I \subseteq R$ be an ideal generated by a system of parameters for R. Let

$$\nu(p^e) := \max\{r \in \mathbb{N} \mid \mathfrak{a}^r z^{p^e} \not\subseteq I^{[q]} \text{ for all } z \in R \setminus I\}.$$

Because R is F-injective, the invariant $\nu(p^e)$ is well-defined.

Claim (cf. [MuTW]).

$$\mathrm{fit}(\mathfrak{a}) = \lim_{e\to\infty} \frac{\nu(p^e)}{p^e} = \inf_e \frac{\nu(p^e)+1}{p^e}.$$

Proof of Claim. Since $\mathfrak{a}$ is a principal ideal it follows that, if $\mathfrak{a}^{\nu(q)+1}z^q$ lies in $I^{[q]}$, then $\mathfrak{a}^{p(\nu(q)+1)}z^{pq}$ lies in $I^{[pq]}$. Thus, $(\nu(q)+1)/q \geq (\nu(pq)+1)/pq$; that is,

$$\lim_{e\to\infty} \frac{\nu(p^e)}{p^e} = \inf_e \frac{\nu(p^e)+1}{p^e}.$$

Because $(R, \mathfrak{a}^{\mathrm{fit}(\mathfrak{a})-\varepsilon})$ is F-injective and $(R, \mathfrak{a}^{\mathrm{fit}(\mathfrak{a})+\varepsilon})$ is never F-injective for all $1 \geq \varepsilon > 0$, by part (2) one has $\lfloor(\mathrm{fit}(\mathfrak{a})-\varepsilon)(q-1)\rfloor \leq \nu(q) < \lfloor(\mathrm{fit}(\mathfrak{a})+\varepsilon)(q-1)\rfloor$ for infinitely many $q = p^e$. This implies that

$$\mathrm{fit}(\mathfrak{a}) - \varepsilon \leq \lim_{e\to\infty} \frac{\nu(p^e)}{p^e} \leq \mathrm{fit}(\mathfrak{a}) + \varepsilon.$$

Since ε can take arbitrarily small values, we obtain the assertion. □

By the claim just proved, $\lfloor \mathrm{fit}(\mathfrak{a})(q-1)\rfloor \leq \nu(q)$ for every $q = p^e$, which means that $I^{F\mathfrak{a}^{\mathrm{fit}(\mathfrak{a})}} = I$. □

THEOREM 7.7. *Let $(R, \mathfrak{m})$ be an excellent reduced local ring of characteristic $p > 0$, and let $x \in \mathfrak{m}$ be a nonzero divisor of R. If $(R, x^{1-\varepsilon})$ is F-rational for all sufficiently small $1 \gg \varepsilon > 0$, then R/xR is Cohen–Macaulay and F-injective (i.e., the pair $(R/xR, (R/xR)^1)$ is F-injective). When the localized ring R_x is F-rational, the converse implication also holds.*

Proof. Without loss of generality, we may assume that R is Cohen–Macaulay.

Claim. (R, x) is F-injective if and only if R/xR is F-injective.

Proof of Claim. We choose elements $y_1, \dots, y_{d-1}$ in R such that $x, y_1, \dots, y_{d-1}$ is a system of parameters for R. An element $z \in R$ lies in $(x, y_1, \dots, y_{d-1})^{Fx}$ if and only if $z^q \in (x, y_1^q, \dots, y_{d-1}^q)$ for all large $q = p^e$, because $x, y_1, \dots, y_{d-1}$ is an R-regular sequence. This is equivalent to saying that $\bar{z}^q \in (\overline{y_1}^q, \dots, \overline{y_{d-1}}^q)$ for all large $q = p^e$; that is, $\bar{z} \in (\overline{y_1}, \dots, \overline{y_{d-1}})^F$, where $\bar{z}, \overline{y_1}, \dots, \overline{y_{d-1}}$ are the images of $z, y_1, \dots, y_{d-1}$ (respectively) in S. Thus, by Lemma 7.6(2), we obtain the assertion. □

If $(R, x^{1-\varepsilon})$ is F-rational for all sufficiently small $1 \gg \varepsilon > 0$, then by Lemma 7.6(3) it follows that (R, x) is F-injective. To complete the proof of this theorem, by the preceding claim it remains only to show that if (R, x) is F-injective and R_x is F-rational then $(R, x^{1-\varepsilon})$ is F-rational for all $1 \geq \varepsilon > 0$. Since R_x is F-rational, it follows from Lemma 6.8 that some power x^n of x is a parameter x-test element. Choose sufficiently large $q = p^e$ so that $\lceil(1-\varepsilon)q\rceil + n \leq q-1$. Then the F-injectivity of (R, x) implies that $x^n x^{\lceil(1-\varepsilon)q\rceil} F^e \colon H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R)$ is injective, where $F^e \colon H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R)$ denotes the induced e-times iterated

Frobenius map on $H^d_{\mathfrak{m}}(R)$. By Remark 6.7, this is equivalent to the F-rationality of $(R, x^{1-\varepsilon})$. $\square$

EXAMPLE 7.8. Consider the rth Veronese subring $R = S^{(r)}$ of the d-dimensional formal power series ring $S = k[[x_1, \dots, x_d]]$ over a perfect field k of characteristic $p > 0$. It is well known that R is strongly F-regular. By [TW, Ex. 2.4(ii)], the F-pure threshold $\mathrm{fpt}(\mathfrak{m})$ of the maximal ideal $\mathfrak{m}$ of R is equal to d/r; that is, $(R, \mathfrak{m}^t)$ is strongly F-regular if and only if $t < d/r$. We will show that the F-injective threshold $\mathrm{fit}(\mathfrak{m})$ of $\mathfrak{m}$ is equal to $\lceil d/r \rceil$.

Let $I = (x_1^r, x_2^r, \dots, x_d^r)$. Then $(R, \mathfrak{m}^t)$ is F-rational if and only if $I^{*\mathfrak{m}^t}$ contains none of the monomials $x_1^{i_1} \dots x_d^{i_d}$ in R with $r - 2 \le i_j \le r - 1$ for all $j = 1, \dots, d$. Put $n = \lceil d/r \rceil$ and $z = x_1^{r-2} \cdots x_{rn-d}^{r-2} x_{rn-d+1}^{r-1} \cdots x_d^{r-1} \in R$. Since $x_1^{2q-2} \cdots x_{rn-d}^{2q-2} x_{rn-d+1}^{q-1} \cdots x_d^{q-1}$ is in $\mathfrak{m}^{n(q-1)} \subseteq \mathfrak{m}^{\lceil (n-\varepsilon)q \rceil}$ for all large $q = p^e$, it follows that $z^q \mathfrak{m}^{\lceil (n-\varepsilon)q \rceil}$ is not contained in $I^{[q]}$. Thus, z does not belong to $I^{*\mathfrak{m}^{n-\varepsilon}}$ (here, 1 is an $\mathfrak{m}^{n-\varepsilon}$-test element by Lemma 5.4). Similarly, we can show that $I^{*\mathfrak{m}^{n-\varepsilon}}$ contains none of the monomials $x_1^{i_1} \dots x_d^{i_d}$ in R with $r-2 \le i_j \le r-1$ for all $j = 1, \dots, d$. This means $\mathrm{fit}(\mathfrak{m}) \ge n = \lceil d/r \rceil$. We leave it for the reader to check that $\mathrm{fit}(\mathfrak{m}) \le \lceil d/r \rceil$.

We conclude this section with a proof of a special case of the discreteness and rationality of F-injective thresholds. More generally, we introduce a new invariant that is a generalization of F-injective thresholds and then study its properties.

DEFINITION 7.9. Let R be a reduced local ring of characteristic $p > 0$ that is a homomorphic image of a Gorenstein local ring, and let $\mathfrak{a} \subseteq R$ be an ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$. We say that a real number $t > 0$ is a *jumping exponent for generalized parameter test submodules* $\tau(\omega_R, \mathfrak{a}^*)$ if $\tau(\omega_R, \mathfrak{a}^t) \subsetneq \tau(\omega_R, \mathfrak{a}^{t-\varepsilon})$ for all $\varepsilon > 0$.

If R is excellent and F-rational, then it follows from Remark 6.4 that the smallest jumping exponent for the generalized parameter test submodules $\tau(\omega_R, \mathfrak{a}^*)$ is the F-injective threshold $\mathrm{fit}(\mathfrak{a})$ of $\mathfrak{a}$.

LEMMA 7.10. *Let $(R, \mathfrak{m})$ be a complete local domain of characteristic $p > 0$, and let $\mathfrak{a} \subseteq R$ be an ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$.*

(1) *For every nonnegative real number t, there exists an $\varepsilon > 0$ such that*

$$\tau(\omega_R, \mathfrak{a}^t) = \tau(\omega_R, \mathfrak{a}^{t'})$$

for all $t' \in [t, t+\varepsilon)$.

(2) *If α is a jumping exponent for the generalized parameter test submodules $\tau(\omega_R, \mathfrak{a}^*)$, then so is $p\alpha$.*

(3) *If $\mathfrak{a}$ is generated by m elements then, for every $t \ge m$,*

$$\tau(\omega_R, \mathfrak{a}^t) = \tau(\omega_R, \mathfrak{a}^{t-1})\mathfrak{a}.$$

Proof. (1) Let $c \in R^\circ$ be a parameter $\mathfrak{a}^s$-test element for every $s \ge 0$, and fix any $d \in \mathfrak{a} \cap R^\circ$. Then cd is also a parameter $\mathfrak{a}^s$-test element for every $s \ge 0$. Denote by N_e the submodule of $H^d_{\mathfrak{m}}(R)$ consisting of all elements $\xi \in H^d_{\mathfrak{m}}(R)$

such that $cd\mathfrak{a}^{\lceil tp^e\rceil}\xi^{p^e} = 0$ in $H^d_{\mathfrak{m}}(R)$. By definition, one can see that $0^{*\mathfrak{a}^t}_{H^d_{\mathfrak{m}}(R)} = \bigcap_{e\in\mathbb{N}} N_e$. Since $H^d_{\mathfrak{m}}(R)$ is an Artinian R-module, there exists an integer m such that $0^{*\mathfrak{a}^t}_{H^d_{\mathfrak{m}}(R)} = \bigcap_{e=0}^m N_e$. Put $\varepsilon = 1/p^m$; we will prove that $\tau(\mathfrak{a}^t) = \tau(\mathfrak{a}^{t+\varepsilon})$. Let $\xi \in 0^{*\mathfrak{a}^{t+\varepsilon}}_{H^d_{\mathfrak{m}}(R)}$. Because c is a parameter $\mathfrak{a}^{t+\varepsilon}$-test element, $c\mathfrak{a}^{\lceil (t+\varepsilon)q\rceil}\xi^q = 0$ for all $q = p^e$; in particular, $c\mathfrak{a}^{\lceil tp^e\rceil+1}\xi^{p^e} = 0$ for all $e = 0,\dots,m$. Since d is in $\mathfrak{a}$, we know that ξ lies in $\bigcap_{e=0}^m N_e = 0^{*\mathfrak{a}^t}_{H^d_{\mathfrak{m}}(R)}$.

(2) Let $c \in R^\circ$ be a parameter $\mathfrak{a}^t$-test element for every $t \ge 0$, and fix any $\varepsilon > 0$. Since α is a jumping exponent for generalized parameter test submodules $\tau(\omega_R, \mathfrak{a}^*)$, there exists a $\xi \in 0^{*\mathfrak{a}^\alpha}_{H^d_{\mathfrak{m}}(R)}$ that is not contained in $0^{*\mathfrak{a}^{\alpha-\varepsilon}}_{H^d_{\mathfrak{m}}(R)}$. This means that $c\mathfrak{a}^{\lceil \alpha q\rceil}\xi^q = 0$ in $H^d_{\mathfrak{m}}(R)$ for all $q = p^e$, but $c\mathfrak{a}^{\lceil (\alpha-\varepsilon)q\rceil}\xi^q \ne 0$ in $H^d_{\mathfrak{m}}(R)$ for infinitely many $q = p^e$. Put $\eta = \xi^p \in H^d_{\mathfrak{m}}(R)$. Then $c\mathfrak{a}^{\lceil p\alpha q\rceil}\eta^q = 0$ in $H^d_{\mathfrak{m}}(R)$ for all $q = p^e$, but $c\mathfrak{a}^{\lceil p(\alpha-\varepsilon)q\rceil}\eta^q \ne 0$ for infinitely many $q = p^e$. This implies that η belongs to $0^{*\mathfrak{a}^{p\alpha}}_{H^d_{\mathfrak{m}}(R)}$ but not to $0^{*\mathfrak{a}^{p(\alpha-\varepsilon)}}_{H^d_{\mathfrak{m}}(R)}$. Thus, by Matlis duality, $\tau(\omega_R, \mathfrak{a}^{p\alpha}) \subsetneq \tau(\omega_R, \mathfrak{a}^{p(\alpha-\varepsilon)})$.

(3) By the proof of [HT, Thm. 4.1], $0^{*\mathfrak{a}^t}_{H^d_{\mathfrak{m}}(R)} = (0^{*\mathfrak{a}^{t-1}}_{H^d_{\mathfrak{m}}(R)} : \mathfrak{a})_{H^d_{\mathfrak{m}}(R)}$ for every real number $t \ge m$. Because $\mathrm{Ann}_{H^d_{\mathfrak{m}}(R)}(\tau(\omega_R, \mathfrak{a}^{t-1})\mathfrak{a})$ is equal to $(0^{*\mathfrak{a}^{t-1}}_{H^d_{\mathfrak{m}}(R)} : \mathfrak{a})_{H^d_{\mathfrak{m}}(R)} = 0^{*\mathfrak{a}^t}_{H^d_{\mathfrak{m}}(R)}$, by Matlis duality one has

$$\begin{aligned}\tau(\omega_R, \mathfrak{a}^t) &= \mathrm{Ann}_{\omega_R}(0^{*\mathfrak{a}^t}_{H^d_{\mathfrak{m}}(R)})\\ &= \mathrm{Ann}_{\omega_R}(\mathrm{Ann}_{H^d_{\mathfrak{m}}(R)}(\tau(\omega_R, \mathfrak{a}^{t-1})\mathfrak{a})) = \tau(\omega_R, \mathfrak{a}^{t-1})\mathfrak{a}. \qquad \square\end{aligned}$$

THEOREM 7.11. *Let $(R,\mathfrak{m})$ be an excellent F-rational local ring of characteristic $p > 0$ that is a homomorphic image of a Gorenstein local ring, and fix $g \in R^\circ$. Let $\alpha, \beta > 0$ be integers and write $\gamma = \alpha/(p^\beta - 1)$. Then there exists a $c \in (0,\gamma)$ for which $\tau(\omega_R, g^t) = \tau(\omega_R, g^c)$ for all $t \in [c,\gamma)$.*

Proof. For any integers $m, n > 0$, we denote by $N_{m,n}$ the submodule of $H^d_{\mathfrak{m}}(R)$ consisting of all elements $\xi \in H^d_{\mathfrak{m}}(R)$ such that $g^m\xi^{p^n} = 0$ in $H^d_{\mathfrak{m}}(R)$.

Claim. $\tau(\omega_R, g^{m/p^n}) = \mathrm{Ann}_{\omega_R}(N_{m,n})$.

Proof of Claim. First note that, by Lemma 6.8, the unit 1 is a parameter g^t-test element for every $t \ge 0$. Then one can see that $0^{*g^{m/p^n}}_{H^d_{\mathfrak{m}}(R)} = \bigcap_{e\ge n} N_{mp^{e-n},e}$. Now it suffices to show that $N_{k,e} \subseteq N_{kp,e+1}$ for all integers $k, e > 0$, but this is obvious. $\square$

Fix the $R[\theta; f^\beta]$-module structure on $H^d_{\mathfrak{m}}(R)$ given by $\theta\xi = g^\alpha\xi^{p^\beta}$ for all $\xi \in H^d_{\mathfrak{m}}(R)$. Then $N_{\alpha(1+p^\beta+\dots+p^{\beta(s-1)}),s\beta}$ coincides with the kernel of θ^s as an $R[\theta; f^\beta]$-module. Thus, $\{N_{\alpha(1+p^\beta+\dots+p^{\beta(s-1)}),s\beta}\}_{s\ge1}$ forms an ascending chain of $R[\theta; f^\beta]$-modules, and by the Hartshorne–Speiser–Lyubeznik theorem (see [L, Prop. 4.4]) it stabilizes at some $s = \nu$. For all $s \ge 1$, the preceding claim shows that

$$\tau(\omega_R, g^{\alpha(1+p^\beta+\dots+p^{\beta(s-1)})/p^{s\beta}}) = \mathrm{Ann}_{\omega_R}(N_{\alpha(1+p^\beta+\dots+p^{\beta(s-1)}),s\beta}).$$

Thus, the stabilization of $\{N_{\alpha(1+p^\beta+\dots+p^{\beta(s-1)}),s\beta}\}_{s\ge1}$ implies the stabilization of a family of generalized parameter test submodules $\{\tau(\omega_R, g^{\alpha(1+p^\beta+\dots+p^{\beta(s-1)})/p^{s\beta}})\}_{s\ge1}$ for $s \ge \nu$. Since

$$\frac{\alpha(1+p^{\beta}+\cdots+p^{\beta(s-1)})}{p^{s\beta}}=\frac{\alpha}{p^{s\beta}}\frac{p^{s\beta}-1}{p^{\beta}-1}$$

is an increasing sequence that converges to γ as s approaches infinity, we may take $c=\alpha(1+p^{\beta}+\cdots+p^{\beta(\nu-1)})/p^{\nu\beta}$. □

COROLLARY 7.12. *Let $(R,\mathfrak{m})$ be a complete F-rational local ring of characteristic $p>0$, and fix $g\in R^{\circ}$. Then the set of jumping exponents for generalized parameter test submodules $\tau(\omega_R,g^*)$ cannot have a rational accumulation point.*

Proof. Assume to the contrary that the set of jumping exponents for generalized parameter test submodules $\tau(\omega_R,g^*)$ has a rational accumulation point γ. Then there exists a sequence $\{c_n\}_{n\geq 1}$ of jumping exponents converging to γ. If we write γ in the form of $\alpha/p^d(p^{\beta}-1)$ then, by Lemma 7.10(2), $\{p^d c_n\}_{n\geq 1}$ is a sequence of jumping exponents again and is converging to $\alpha/(p^{\beta}-1)$. This contradicts Theorem 7.11. □

COROLLARY 7.13. *Let $(R,\mathfrak{m})$ be a complete F-rational local ring of characteristic $p>0$, and fix $g\in R^{\circ}$. Then jumping exponents for generalized parameter test submodules $\tau(\omega_R,g^*)$ are rational and have no accumulation points.*

Proof. Applying Lemma 7.10 and Corollary 7.12 to [KatLZ, Prop. 4.2], we obtain the assertion. □

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