Michigan Math. J. 57 (2008)

Hilbert Functions over Toric Rings

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1. Introduction

Throughout this paper, S stands for the polynomial ring $k[x_1, \dots, x_n]$ over a field k. The ring S is graded by $\deg(x_i) = 1$ for each i. The vector space of all polynomials of degree i is denoted by S_i. If J is a graded ideal, then J_i is the vector space of all polynomials in J of degree i. The Hilbert function

$$\begin{aligned} h\colon \mathbf{N} &\to \mathbf{N}, \\ i &\mapsto \dim_k J_i, \end{aligned}$$

is an important numerical invariant that measures the size of J. Macaulay's theorem [Ma] characterizes the Hilbert functions of homogeneous ideals in S. Macaulay's key idea is that every Hilbert function is attained by a lex ideal. Lex ideals are special monomial ideals defined in a simple combinatorial way. Macaulay's theorem was generalized to Betti numbers [Bi; Hu; Pa]: every lex ideal attains maximal Betti numbers among all homogenous ideals with the same Hilbert function. Furthermore, lex ideals play a key role in Hartshorne's proof of his famous result that the Hilbert scheme is connected [Ha]. These are important results, so it is interesting to find analogues over nonpolynomial rings. A lot of attention was given to the Clements–Lindström ring, which has the form $C = S/(x_1^{c_1}, \dots, x_n^{c_n})$ with $c_1 \le \dots \le c_n \le \infty$. Macaulay's theorem is known to hold in this case. Recently, there has been a lot of work on the lex-plus-powers conjecture. Another open conjecture [GHiP] is that every lex ideal in C attains maximal Betti numbers over C among all homogenous ideals in C with the same Hilbert function. The special case $c_1 = \dots = c_n = 2$ is well studied, and we have the following results.

Theorem 1.1. *Let $E = S/(x_1^2, \dots, x_n^2)$ (or one can assume that E is an exterior algebra). Then the following statements hold.*

(1) *For every graded ideal J in E, there exists a lex ideal with the same Hilbert function* [K; Kr].
(2) *The Hilbert scheme that parameterizes all graded ideals in E with a fixed Hilbert function h is connected. More precisely, every graded ideal in E with Hilbert function h is connected to the lex ideal with Hilbert function h* [PS1].

Received January 24, 2007. Revision received November 30, 2007.

(3) *If J is a graded ideal in E and if L is the lex ideal with the same Hilbert function, then the graded Betti numbers over E of J are smaller than those of L* [AHHi].

(4) *Let J be a graded ideal in E, and let L be the lex ideal with the same Hilbert function. Let $\tilde{J}$ and $\tilde{L}$ be the preimages in S of J and L, respectively. Then the graded Betti numbers over S of $\tilde{J}$ are smaller than those of $\tilde{L}$* [MPS].

(5) *There exists an explicit formula* (*that does not use homology*) *that gives the graded Betti numbers of a lex ideal over E* [AAvH].

Note that (3) and (5) are about infinite free resolutions whereas (4) is about finite ones.

We are interested in building analogues over toric rings. One can try to explore analogues of 1.1(1) over general toric rings, but we are interested in analogues of all the properties in Theorem 1.1 and therefore focus on projective toric rings. Throughout the paper, R stands for a projective toric ring.

In Section 2 we introduce monomial ideals in R. In Theorem 2.5 we show that, for every homogeneous ideal in R, there exists a monomial ideal in R with the same Hilbert function. We do not know whether this property holds over a quotient by a homogeneous binomial ideal if it is neither monomial nor toric. Our proof uses the structure of toric ideals.

In Section 3 we introduce lex ideals in R. In Theorem 3.4 we prove that an initial lex-segment generates an initial lex-segment in the next degree. This property is crucial if we are to have a useful notion of a lex ideal.

In Section 4 we raise several open problems, which are inspired by Macaulay's theorem over S and other results and conjectures in the spirit of Theorem 1.1. The starting question is, of course, to identify projective toric rings for which Macaulay's theorem holds.

It is well known that one must order the variables by $x_1 > \cdots > x_n$ in the Clements–Lindström ring C in order to make Macaulay's theorem hold; other orders might not work if the exponents $c_1, \dots, c_n$ are different. In the same way, choosing the order of the variables in the toric ring R is very important; see Remark 4.3. Establishing Macaulay's theorem for a fixed toric ring consists of two steps: (i) finding and fixing a suitable order of the variables; and (ii) proving the theorem.

In Section 5 we prove that Macaulay's theorem holds for rational normal curves. Furthermore, we describe the structure of the minimal free resolutions of monomial ideals over a rational normal curve. Theorem 5.5 states that the truncation after two steps of such a resolution is a direct sum of linear resolutions, possibly shifted in different degrees. It also provides a formula for the Betti numbers. Infinite minimal free resolutions usually have a complicated structure; it is rare that one can obtain nice structural results as in Theorem 5.5. In a project in progress [GP], we prove analogues of 1.1(2) and 1.1(3) over rational normal curves.

ACKNOWLEDGMENTS. We thank the referee for helpful suggestions. The third author is partially supported by NSF.

2. Hilbert Functions over a Projective Toric Ring

The notation introduced here will be used throughout the paper.

Let $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ be a subset of $\mathbf{N}^c \setminus \{\mathbf{0}\}$ and A the matrix with columns $\mathbf{a}_i$, and suppose that $\operatorname{rank}(A) = c$. Consider the polynomial ring $S = k[x_1, \dots, x_n]$ over a field k. The kernel of the homomorphism

$$\begin{aligned}\varphi\colon k[x_1, \dots, x_n] &\to k[t_1, \dots, t_c],\\ x_i &\mapsto \mathbf{t}^{\mathbf{a}_i} = t_1^{a_{i1}} \cdots t_c^{a_{ic}},\end{aligned}$$

is a prime ideal, which we denote by I and call the *toric ideal* associated to $\mathcal{A}$. For an integer vector $\mathbf{v} = (v_1, \dots, v_n)$, set $\mathbf{x}^{\mathbf{v}} = x_1^{v_1} \cdots x_n^{v_n}$. In this notation we have that $\varphi(\mathbf{x}^{\mathbf{v}}) = \mathbf{t}^{A\mathbf{v}}$. The *toric ring* associated to $\mathcal{A}$ is

$$S/I \cong k[\mathbf{t}^{\mathbf{a}_1}, \dots, \mathbf{t}^{\mathbf{a}_n}], \tag{2.1}$$

where the isomorphism is given by $\mathbf{x}^{\mathbf{v}} \mapsto \mathbf{t}^{A\mathbf{v}}$. Denote $\phi\colon S \to R = S/I$. If m is a monomial in S, by abuse of notation we write m for the monomial $\phi(m)$ in R.

The polynomial ring S is graded by $\deg(x_i) = 1$ for each $1 \le i \le n$. We say that I is *projective* (or that S/I is a *projective* toric ring) if I is homogeneous with respect to the standard grading of S with $\deg(x_i) = 1$ for $1 \le i \le n$. The following proposition is well known.

PROPOSITION 2.2. *The following statements are equivalent.*

(1) *I is projective.*
(2) *The points $\mathbf{a}_1, \dots, \mathbf{a}_n$ lie on a hyperplane in $\mathbf{R}^c$ not containing the origin.*
(3) *A change of coordinates (not necessarily defined over the integers) in $\mathbf{R}^c$ can change $\mathcal{A}$ to $\mathcal{A}'$ so that $I_{\mathcal{A}} = I_{\mathcal{A}'}$ and $\mathbf{a}'_1, \dots, \mathbf{a}'_n$ lie on the hyperplane $v_1 = 1$, where $(v_1, \dots, v_c)$ are the coordinates in $\mathbf{R}^c$.*

Throughout the paper, R stands for a projective toric ring S/I. This ring inherits the grading from S with $\deg(x_i) = 1$ for $1 \le i \le n$. If J is a homogeneous ideal in R, then we have that the Hilbert function of R/J is

$$\begin{aligned}\operatorname{Hilb}_R(R/J)\colon \mathbf{N} &\to \mathbf{N},\\ i &\mapsto \dim_k(R/J)_i.\end{aligned}$$

Similarly, we have that the Hilbert function of J is

$$\begin{aligned}\operatorname{Hilb}_R(J)\colon \mathbf{N} &\to \mathbf{N},\\ i &\mapsto \dim_k(J)_i.\end{aligned}$$

The following observation is useful.

LEMMA 2.3. *Let P and O be homogeneous ideals in R, and let $\tilde{P}$ and $\tilde{O}$ (respectively) be the preimages of these ideals in S. The ideals P and O have the same Hilbert function over R if and only if the ideals $\tilde{P}$ and $\tilde{O}$ have the same Hilbert function over S.*

Proof. This follows from $S/\tilde{P} \cong R/P$, $S/\tilde{O} \cong R/O$, and the additivity of the Hilbert function. □

We are interested in studying the Hilbert functions of homogeneous ideals in R. By the preceding lemma, it follows that this is equivalent to studying the Hilbert functions of the homogeneous ideals in S that contain I.

Next, we would like to reduce to the monomial case.

DEFINITION 2.4. We say that $m \in R$ is a *monomial* if there exists a preimage of m in S that is a monomial. An ideal in R is called *monomial* if it can be generated by monomials.

For the formulation of Theorem 2.5, we recall the definition of consecutive cancellations in a set of graded Betti numbers. Given a set of numbers $\{c_{i,j}\}$, we obtain a new set by a *cancellation* as follows: Fix a j, and choose i and i' such that one of the numbers is odd and the other is even; then replace $c_{i,j}$ by $c_{i,j} - 1$ and replace $c_{i',j}$ by $c_{i',j} - 1$. We have a *consecutive cancellation* when $i' = i + 1$. If we need to be specific, we call it a consecutive $\{i, j\}$-cancellation. The term "consecutive" is justified because we consider cancellations in Betti numbers of consecutive homological degrees. Over the polynomial ring S, the set of graded Betti numbers of a homogeneous ideal can be obtained by a sequence of consecutive cancellations from the graded Betti numbers of the lex ideal with the same Hilbert function. In Theorem 2.5, we consider this property over R.

THEOREM 2.5. *Let P be a homogeneous ideal in a projective toric ring $R = S/I$. Then there exists a monomial ideal M in R such that the following statements hold.*

(1) *M has the same Hilbert function as P.*
(2) *The Betti numbers of M over R are greater than or equal to those of P. Furthermore, the Betti numbers of P can be obtained from those of M by a sequence of consecutive cancellations.*
(3) *Let K and O be the preimages of M and P (respectively) in S. The Betti numbers of K over S are greater than or equal to those of O. Furthermore, the Betti numbers of O can be obtained from those of K by a sequence of consecutive cancellations.*

In order to prove this theorem we will need the next result, which introduces tools from Gröbner basis theory over quotient rings. We include its proof because we are not aware of a reference that can be cited. A reference for monomial orders, initial ideals, and results from Gröbner basis theory is [E, Chap. 15].

PROPOSITION 2.6. *Let A be a homogeneous ideal in S, and let $B \supseteq A$ be another homogeneous ideal in S.*

(1) *Let $\mathbf{w}$ be a weight vector with integer coordinates, and let $\prec_{\mathbf{w}}$ be the weight order in S induced by $\mathbf{w}$. The graded Betti numbers of the initial ideal $\mathrm{in}_{\prec_{\mathbf{w}}}(B)$ over the quotient ring $S/\mathrm{in}_{\prec_{\mathbf{w}}}(A)$ are greater than or equal to those of B over*

the ring S/A. Furthermore, the graded Betti numbers of S/B can be obtained from those of $S/\mathrm{in}_{\prec_{\mathbf{w}}}(B)$ by a sequence of consecutive cancellations.

(2) *Let $\prec$ be a monomial order in S. The graded Betti numbers of the initial ideal $\mathrm{in}_{\prec}(B)$ over the quotient ring $S/\mathrm{in}_{\prec}(A)$ are greater than or equal to those of B over the ring S/A. Furthermore, the graded Betti numbers of S/B can be obtained from those of $S/\mathrm{in}_{\prec}(B)$ by a sequence of consecutive cancellations.*

Proof. (1) Let $\mathbf{w} = (w_1, \ldots, w_n)$. Consider the polynomial ring $\tilde{S} = S[t]$ graded by $\deg(x_i) = w_i$ for $1 \le i \le n$ and $\deg(t) = 1$. Let $\tilde{B}$ be the homogenization of B in the polynomial ring $\tilde{S}$. Then t and $t-1$ are regular elements on $\tilde{S}/\tilde{B}$ (cf. [E, Thm. 15.17]). Similarly, let $\tilde{A}$ be the homogenization of A in the polynomial ring $\tilde{S}$. Then t and $t-1$ are regular elements on $\tilde{S}/\tilde{A}$.

Clearly, $\tilde{B} \supseteq \tilde{A}$. Denote by $\tilde{\mathbf{F}}$ a graded minimal free resolution of $\tilde{S}/\tilde{B}$ over $\tilde{S}/\tilde{A}$. Then $\tilde{\mathbf{F}} \otimes \tilde{S}/t$ is a minimal free resolution of $S/\mathrm{in}_{\prec_{\mathbf{w}}}(B) = \tilde{S}/\tilde{B} \otimes \tilde{S}/(t)$ over the ring $S/\mathrm{in}_{\prec_{\mathbf{w}}}(A) = \tilde{S}/\tilde{A} \otimes \tilde{S}/(t)$. Thus, the graded Betti numbers of $S/\mathrm{in}_{\prec_{\mathbf{w}}}(B)$ over the ring $S/\mathrm{in}_{\prec_{\mathbf{w}}}(A)$ coincide with the graded Betti numbers of $\tilde{S}/\tilde{B}$ over the ring $\tilde{S}/\tilde{A}$. On the other hand, $\tilde{\mathbf{F}} \otimes \tilde{S}/(t-1)$ is a nonminimal graded free resolution of $S/B = \tilde{S}/\tilde{B} \otimes \tilde{S}/(t-1)$ over the ring $S/A = \tilde{S}/\tilde{A} \otimes \tilde{S}/(t-1)$. Therefore,

$$\tilde{\mathbf{F}} \otimes \tilde{S}/(t-1) \cong \mathbf{F} \oplus \mathbf{G},$$

where $\mathbf{F}$ is a minimal graded free resolution of S/B over S/A and $\mathbf{G}$ is a trivial complex (cf. [E, Thm. 20.2]). The triviality of the complex $\mathbf{G}$ implies that the graded Betti numbers of S/B are obtained from those of $\tilde{S}/\tilde{B}$ by consecutive cancellations.

(2) Since $B \supseteq A$, by [Ba] we can choose a vector $\mathbf{w}$ with strictly positive integer coordinates and such that, with respect to the weight order induced by the weight vector $\mathbf{w}$, the initial ideal of B is $\mathrm{in}_{\prec}(B)$ and the initial ideal of A is $\mathrm{in}_{\prec}(A)$ (cf. [E, Thm. 15.16]). Therefore, (2) follows from (1). □

Proof of Theorem 2.5. Let O be the preimage of P in S. We will construct a special partial monomial order $\prec$ on S and then take M to be the image in $S/\mathrm{in}_{\prec}(I)$ of the initial ideal $\mathrm{in}_{\prec}(O)$ with respect to $\prec$. We will construct a partial monomial order $\prec$ on S such that the following two properties hold:

(a) $\mathrm{in}_{\prec}(I) = I$;

(b) if m and m' are incomparable monomials, then $m - m' \in I$.

Property (a) is useful because it implies that $S/\mathrm{in}_{\prec}(I) = R$ and hence that M is an ideal in R. Property (b) is useful because it implies that M is a monomial ideal.

We will define a partial monomial order $\prec$ using the weight orders with respect to the rows in the matrix $\mathbf{A}$ with columns $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. For $1 \le i \le c$, denote by $\mathbf{w}_i$ the weight order of the monomials in S with respect to the vector $((\mathbf{a}_1)_i, \ldots, (\mathbf{a}_n)_i)$. Observe that this is a partial order. Let m and q be two monomials in S. We say that $m \succ q$ if there exists a $1 \le j \le c$ such that

$$\mathbf{w}_j(m) > \mathbf{w}_j(q) \quad \text{and} \quad \mathbf{w}_i(m) = \mathbf{w}_i(q) \quad \text{for } 1 \le i < j.$$

This is a partial order.

Two monomials m and q are noncomparable by $\succ$ if and only if $\mathbf{w}_i(m) = \mathbf{w}_i(q)$ for all $1 \leq i \leq c$; this happens if and only if $m - q \in I$. Therefore, properties (a) and (b) both hold.

(1) The ideals O and $\text{in}_{\prec}(O)$ have the same Hilbert function because $\text{in}_{\prec}(O)$ is an initial ideal of O. By Lemma 2.3, it follows that the ideals P and M have the same Hilbert function.

(3) This follows because $\text{in}_{\prec}(O)$ is an initial ideal of O (since Proposition 2.6 can be applied with $A = 0$).

It remains to prove (2). We will apply Proposition 2.6 repeatedly. For each $0 \leq i \leq c-1$, set $N_{i+1} = \text{in}_{\mathbf{w}_{i+1}}(N_i)$ and $N_0 = O$. Note that $I = \text{in}_{\mathbf{w}_i}(I)$ for each i. By Proposition 2.6(1) it follows that the graded Betti numbers of S/N_i over S/I can be obtained from those of S/N_{i+1} over S/I by a sequence of consecutive cancellations. Finally, note that $N_c = \text{in}_{\prec}(O)$. □

We remark that this proof does not work over a quotient by a homogeneous binomial ideal if it is neither toric nor monomial. In the proof, we used the assumption that I is a toric ideal in order to have that two monomials m and q in S are noncomparable by $\succ$ if and only if $m - q \in I$.

It is worth discussing briefly the structure of monomial ideals in the projective toric ring R. Consider the *multigrading* of the polynomial ring $S = k[x_1, \dots, x_n]$ such that the variables $x_1, \dots, x_n$ have $\mathbf{N}^c$-degrees $a_1, \dots, a_n$, respectively (we say *multidegrees* instead of $\mathbf{N}^c$-degrees). For $\alpha \in \mathbf{N}^c$, the set of all monomials in S of degree α is called the *fiber* of α. We say that an ideal J in S is *multigraded* if it is homogeneous with respect to this $\mathbf{N}^c$-grading. The following result is easy and well known.

PROPOSITION 2.7. *Let M be a monomial ideal in a projective toric ring $R = S/I$. The Hilbert function of M (over R) in degree i counts the number of fibers of degree i in M.*

Proof. The toric ideal I is multigraded. By (2.1), it follows that we have the multigraded Hilbert function

$$\dim_k(R_\alpha) = \begin{cases} 1 & \text{if } \alpha \in \mathbf{N}\mathcal{A}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

All monomials in a fiber are equal in R. As a result,

$$\dim_k((R/M)_\alpha) \leq 1 \quad \text{for every } \alpha \in \mathbf{N}\mathcal{A}.$$

Hence, for a fixed $\alpha \in \mathbf{N}\mathcal{A}$, we have that M contains either the entire fiber of α or none of the monomials in it. Therefore, the Hilbert function of M (over R) in degree i counts the number of fibers of degree i in M. □

3. Lex Ideals in a Projective Toric Ring

In this section, we introduce lex ideals in a projective toric ring.

A monomial ideal M in the polynomial ring S is *lex* if the following property holds: If $m \in M$ is a monomial and if $q >_{\text{lex}} m$ is a monomial of the same degree, then $q \in M$; that is, for each $i \geq 0$, the vector space M_i is spanned by an initial lex-segment (i.e., M_i is spanned by lex-consecutive monomials of degree i starting with x_1^i). This definition does not generalize straightforwardly to the toric ring R. For example, one would like the monomial ideal $M = (a^2, ab, ac)$ to be lex in the toric ring $k[a,b,c,d]/(b^2 - ac,\ ad - bc,\ c^2 - bd)$ of the twisted cubic curve. However, it is not lex according to the definition here, because $b^2 = ac \in M$ and $ad >_{\text{lex}} b^2$ but $ad \notin M$. We must therefore introduce a new definition that will cover natural examples (as the one already given does) and that will make Theorem 3.4 work. The property in Theorem 3.4 is crucial in order to have a meaningful concept of a lex ideal.

Definitions 3.1. We introduce some definitions and illustrate them by simple examples. A *d-monomial space W* is a vector subspace of R_d spanned by monomials of degree d.

Throughout this section we fix the order of the variables to be $x_1 > \cdots > x_n$ and consider the induced lex order $\succ_{\text{lex}}$ on S; see [E, Sec. 15.2] for the definition of lex order.

The lex-greatest monomial in a fiber will be called the *top representative* of the fiber. For example, in the toric ring $k[a,b,c,d]/(b^2 - ac,\ ad - bc,\ c^2 - bd)$ of the twisted cubic curve, the fiber of the monomial b^3 is b^3, a^2d, abc and the top representative is a^2d.

We say that a d-monomial space W is a *lex space* if the following property is satisfied: If $m \in W$ is a monomial, $p \in S$ is the top representative of the fiber of m, and $q \in S_d$ is a monomial such that $q >_{\text{lex}} p$, then $q \in W$. (Recall that by abuse of notation $q \in W$ means that the image of q in R is a monomial in W.) Furthermore, if the monomials $w_1, \dots, w_r \in S$ are the top representatives of the fibers of the monomials in W, then we call $w_1, \dots, w_r$ the *S-generators of W*.

Now consider again the 2-monomial space spanned by a^2, ab, ac in the toric ring $k[a,b,c,d]/(b^2 - ac,\ ad - bc,\ c^2 - bd)$ of the twisted cubic curve. Each of the monomials a^2, ab, ac is the top representative of its fiber; thus, they are the S-generators of the space. This is a lex space.

For a monomial $m \in S$, let

$$\max(m) = \max\{i \mid x_i \text{ divides } m\}.$$

We will need the following two lemmas.

Lemma 3.2. *Let the monomial $m \in S$ be the top representative of its fiber. Set $f = \max(m)$ and $t = m/x_f$. Then*:

(1) $m = tx_f$;
(2) *t is the top representative of its fiber*; *and*
(3) $f \geq \max(t)$.

We say that $m = x_f t$ is the *distinguished factorization of m*.

Proof of Lemma 3.2. (1) and (3) are obvious. Suppose that $s \neq t$ is the top representative of the fiber of t. Then $m = tx_f$ and sx_f are in the same fiber. Since

$s >_{\text{lex}} t$ it follows that $sx_f >_{\text{lex}} m$, which contradicts the assumption that m is the top representative of its fiber. Therefore, (2) holds. □

LEMMA 3.3. *Let W be a lex d-monomial space in R_d. Let the monomial $q \in S$ be the top representative of its fiber and let $q \in W$. Fix an i such that $1 \le i \le n$. Then there exist a monomial $s \in S$ and an integer $1 \le g \le n$ such that*:

(1) *the monomials $x_i q$ and sx_g are equal in R*;
(2) *s is the top representative of its fiber*;
(3) $g \ge \max(s)$; *and*
(4) *$s \in W$.*

In this case, we say that sx_g is a *distinguished representative of $x_i q \in R$.*

Proof of Lemma 3.3. Set $i' = \max(x_i q)$ and let q' be the top representative of the fiber of the monomial $x_i q/x_{i'}$. Then $x_i q = x_{i'}(x_i q/x_{i'})$ is equal to $x_{i'}q'$ in R. Since $q' \ge_{\text{lex}} x_i q/x_{i'} \ge_{\text{lex}} q$ and since W is a lex space, it follows that $q' \in W$. We call i' the *distinguished variable index*. If $i' \ge \max(q')$ then we are done, as we can choose $g = i'$ and $s = q'$.

Otherwise, we apply the same procedure to $x_{i'}q'$. Set $i'' = \max(x_{i'}q')$ and let q'' be the top representative of the fiber of the monomial $x_{i'}q'/x_{i''}$. Then $x_{i'}q' = x_{i''}(x_{i'}q'/x_{i''})$ is equal to $x_{i''}q''$ in R. Therefore, $x_i q$ is equal to $x_{i''}q''$ in R. Since $q'' \ge_{\text{lex}} x_{i'}q'/x_{i''} \ge_{\text{lex}} q' \in W$ and since W is a lex space, it follows that $q'' \in W$. Again we call i'' the distinguished variable index. If $i'' \ge \max(q'')$ then we are done, as we can choose $g = i''$ and $s = q''$. Otherwise, we proceed as before.

This process terminates because the distinguished variable index strictly increases at each step. □

The following theorem is crucial for a useful concept of a lex ideal. It shows that a lex space generates a lex space in the next degree.

THEOREM 3.4. *Let W be a lex d-monomial space in R_d. Denote by U the $(d+1)$-monomial space in R_{d+1} spanned by the monomials*

$$\{x_i r \mid r \in S \text{ is a monomial and } r \in W,\ 1 \le i \le n\}.$$

Then U is a lex space in R_{d+1}.

Proof. The main work for proving the theorem was done in Lemmas 3.2 and 3.3.

First, note that U is spanned by the smaller set of monomials

$$\mathcal{T} = \{x_i r \mid r \text{ is the top representative of the fiber of a monomial in } W \text{ and } 1 \le i \le n\}.$$

According to Definition 3.1, we must show that (a) if $x_i r \in \mathcal{T}$ then the monomial $p \in S$ is the top representative of the fiber of $x_i r$ and (b) if $m \in S_{d+1}$ is a monomial such that $m >_{\text{lex}} p$ then $m \in U$.

Let $m = tx_f$ be the distinguished factorization of m from Lemma 3.2, and let sx_g be a distinguished representative of $x_i r$ from Lemma 3.3. Then

$$tx_f = m >_{\text{lex}} p >_{\text{lex}} sx_g.$$

The former inequality holds by assumption, and the latter inequality holds because sx_g and $x_i r$ have the same fiber and p is the top representative of this fiber. By Lemmas 3.2 and 3.3, we have

$$f \geq \max(t),$$
$$g \geq \max(s).$$

Therefore, the inequality $tx_f >_{\text{lex}} sx_g$ implies that $t \geq_{\text{lex}} s$. By Lemma 3.3, s is the top representative of its fiber, and $s \in W$. Since W is a lex space, it follows that $t \in W$. Therefore, $m = tx_f \in U$. $\square$

EXAMPLE 3.5. Consider again the lex 2-monomial space spanned by a^2, ab, ac in the toric ring $k[a,b,c,d]/(b^2 - ac,\ ad - bc,\ c^2 - bd)$ of the twisted cubic curve. Its S-generators are a^2, ab, ac. In degree 3 it generates the 3-monomial space spanned by

$$a^3,\ a^2b,\ a^2c,\ a^2d,\ ab^2,\ abc,\ abd,\ ac^2,\ acd.$$

Its S-generators are

$$a^3,\ a^2b,\ a^2c,\ a^2d,\ abd,\ acd,$$

and it is a lex space.

COROLLARY 3.6. *Let L be a monomial ideal in R. The following two conditions are equivalent.*

(1) *For every $i \geq 0$, we have that L_i is spanned by a lex space in R_i.*
(2) *Let $m_1, \ldots, m_h$ be a minimal system of monomial generators of L. Then, for every $i \in \{\deg(m_1), \ldots, \deg(m_r)\}$, L_i is spanned by a lex space in R_i.*

DEFINITION 3.7. A monomial ideal L in R is called *lex* if it satisfies the equivalent conditions in Corollary 3.6.

EXAMPLE 3.8. The ideal (a^2, ab, ac) is lex in the toric ring $A = k[a,b,c,d]/(b^2 - ac,\ ad - bc,\ c^2 - bd)$ of the twisted cubic curve. It obviously satisfies condition 3.6(2).

4. Open Problems

In this section we discuss several open problems on Hilbert functions and syzygies. These problems are probably quite challenging in general, but it is interesting and reasonable to explore them in special cases. Throughout we fix the order of the variables to be $x_1 > \cdots > x_n$ and consider the induced lex order.

Macaulay's theorem states that every Hilbert function of a homogeneous ideal in the polynomial ring S is attained by a lex ideal in S. The same property holds over the quotient ring $S/(x_1^{c_1}, \ldots, x_n^{c_n})$ for $c_1 \leq \cdots \leq c_n \leq \infty$, which is called the Clements–Lindström ring. Macaulay's key idea is that an ideal generated by an initial lex-segment in degree i has the minimal possible growth of the Hilbert function among all ideals generated by the same number of i-forms. One can explore when (i.e., over what quotient rings) this property holds. The following problem is a special case of a problem in [MP].

PROBLEM 4.1. *Find projective toric rings such that every Hilbert function of a monomial ideal is attained by a lex ideal.*

By Theorem 2.5, it follows that if every Hilbert function of a monomial ideal in R is attained by a lex ideal then every Hilbert function of a homogeneous ideal is attained by a lex ideal. In other words, Macaulay's theorem holds over R.

Furthermore, in order to establish that every Hilbert function of a monomial ideal in a fixed projective toric ring R is attained by a lex ideal, it suffices to prove that the condition in Lemma 4.2 holds.

LEMMA 4.2. *Let R be a projective toric ring. Suppose that, for every integer i and for every i-monomial space W,*

$$\dim_k R_1 L_W \leq \dim_k R_1 W.$$

Here L_W is the lex i-monomial space in R such that $\dim_k L_W = \dim_k W$. *Then, for every homogeneous ideal J in R, there exists a lex ideal K with the same Hilbert function.*

Proof. We can assume that J is a monomial ideal by Theorem 2.5. For each $i \geq 0$, let L_{J_i} be the lex i-monomial space in R such that $\dim_k L_{J_i} = \dim_k J_i$. Set $K = \bigoplus_{i \geq 0} L_{J_i}$ as vector spaces. It follows that

$$\dim_k L_{J_{i+1}} = \dim_k J_{i+1} \geq \dim_k R_1 J_i \geq \dim_k R_1 L_{J_i}.$$

Because $L_{J_{i+1}}$ and $R_1 L_i$ are both lex spaces by Theorem 3.4, we conclude that $L_{J_{i+1}} \supseteq R_1 L_i$. Therefore, K is an ideal in R.

By construction, K is a lex ideal and has the same Hilbert function as J. □

The following remark illustrates that we have freedom in the choice of the order of the variables.

REMARK 4.3. Problem 4.1 asks us to identify projective toric rings in which Lemma 4.2 holds. Recall that the order $x_1 > \cdots > x_n$ of the variables is fixed throughout Sections 3 and 4. However, for a given ring R one can choose different sets $\mathcal{A}$ that define it; in particular, we can choose different orders of the generators of the semigroup ring. We remark that a first step toward establishing Macaulay's theorem over a fixed projective toric ring is to choose a suitable order of the variables. The structure of lex ideals depends heavily on the order of the variables, as the following example shows.

Let $I = (b^2 - ac, ad - bc, c^2 - bd)$ be the defining ideal of the twisted cubic curve in the polynomial ring $k[a,b,c,d]$, and consider the monomial ideal $M = (ac, bd)$. Order the variables as $a > b > c > d$ (in this case we set $x_1 = a$, $x_2 = b$, $x_3 = c$, and $x_4 = d$). It is easy to check (by proof or by computer) that the lex ideal $L = (a^2, ab, b^2 d)$ has the same Hilbert function as M; also, it is easy to see that L is indeed lex. On the other hand, if we order the variables as $d > a > b > c$ (in this case we set $x_1 = d$, $x_2 = a$, $x_3 = b$, and $x_4 = c$), then there exists no lex ideal with the same Hilbert function as M. Any such lex ideal would have to contain $N = (d^2, da)$ because $\dim_k(M_2) = 2 = \dim_k(N_2)$, which is impossible since $7 = \dim_k(N_3) > \dim_k(M_3) = 6$.

Over the polynomial ring S, it is known [Bi; Hu; Pa] that every lex ideal has greatest Betti numbers among the homogeneous ideals with the same Hilbert function. A similar property is conjectured in a different situation related to the Clements–Lindström ring $S/(x_1^{c_1},\dots,x_n^{c_n})$ for $c_1 \le \cdots \le c_n \le \infty$ by a conjecture of Gasharov-Hibi-Peeva [GHiP] and the lex-plus-powers conjecture of Evans (see [FRi]). The analogues of these two conjectures for projective toric rings are given in the following problem, which was raised in more generality in [MP].

PROBLEM 4.4. *Let k be an infinite field (perhaps one should also assume that k has characteristic 0). Suppose that L is a lex ideal in the projective toric ring $R = S/I$. Let M be a monomial ideal with the same Hilbert function.*

(1) *Is it true that the Betti numbers of M over R are less than or equal to those of L?*
(2) *Let $\tilde{M}$ and $\tilde{L}$ be the preimages of M and L (respectively) in S. Is it true that the Betti numbers of the ideal $\tilde{M}$ over S are less than or equal to those of $\tilde{L}$?*

By Theorem 2.5, a positive answer to (1) or (2) implies that the property holds for all homogeneous ideals with the same Hilbert function as L.

EXAMPLE 4.5. Let $I = (b^2 - ac,\ ad - bc,\ c^2 - bd)$ be the defining ideal of the twisted cubic curve in the polynomial ring $k[a,b,c,d]$. Consider the monomial ideal $M = (ac, bd, d^2)$. Order the variables by $a > b > c > d$, as in Theorem 5.1. It is easy to check that the lex ideal $L = (a^2, ab, ac, ad^2, bd^2)$ has the same Hilbert function as M; it is also easy to see that the ideal L is indeed lex. Computation with the computer algebra system Macaulay 2 [GrS] shows the following Betti numbers of $k[a,b,c,d]/\tilde{M}$

total:	1	6	10	7	2
0:	1	–	–	–	–
1:	–	6	6	1	–
2:	–	–	4	6	2

and the following Betti numbers of $k[a,b,c,d]/\tilde{L}$

total:	1	8	14	9	2
0:	1	–	–	–	–
1:	–	6	8	3	–
2:	–	2	6	6	2

We see that (2) holds in this case.

However, we do not know how to verify (1) by computer because we need to work with infinite resolutions. Computation via Macaulay 2 yields only the first few Betti numbers, and one can see that the desired inequalities hold at the beginning of the resolutions.

It seems that, in order to attack Problem 4.4(1), one must first make some progress on the next problem, which is of interest in its own right.

PROBLEM 4.6. *In special cases, study the structure (or at least the Betti numbers, or the regularity) of the infinite minimal free resolution of a monomial ideal over* *the projective toric ring $R = S/I$. In particular, study the case when the monomial ideal is lex.*

Problem 4.6 is hard in general. Observe that the maximal ideal $(x_1, \dots, x_n)$ is a monomial lex ideal and that resolving $(x_1, \dots, x_n)$ is equivalent to resolving the residue field k. The minimal free resolution of k over R is known to be complicated. The Poincaré series of k (which is the generating function of the resolution) can be irrational [RSt]. Therefore, a starting condition to require when considering Problem 4.6 is that the minimal free resolution of k over R be nice. This holds in the following cases:

- when R is Koszul (e.g., for Veronese and Segre rings);
- when R is Golod (e.g., for generic toric rings [GPW]).

Since Problem 4.6 is expected to be difficult in general, it is interesting to make some headway on it in special cases—for example, with respect to Veronese rings.

Problem 4.4(2) concerns finite resolutions over the polynomial ring S. It seems that, in order to attack it, one must first obtain some results on the next problem, which is of interest on its own.

PROBLEM 4.7. *In special cases, study the structure* (*or at least the Betti numbers, or the regularity*) *of the minimal free resolution over S of an ideal of the form $I + N$, where N is a monomial ideal. In particular, study the case when the monomial ideal is lex.*

The problem is open even when I is the defining ideal of the twisted cubic curve. We should like to make some progress in the special cases of rational normal curves, Veronese rings, or Segre rings.

There is a (by now) standard technique that can be used to express the Betti numbers over S of a monomial or toric ideal in terms of homology of some simplicial complexes (see [BH; MiSt]). This technique can be applied to $I + N$. Unfortunately, the formula obtained in this way may not be particularly useful because it contains relative homology. We need the following notation: the *radical* $\mathrm{rad}(m)$ of a monomial m in S is the maximal square-free monomial dividing m.

PROPOSITION 4.8. *Multigrade S by* $\deg(x_i) = \mathbf{a}_i$ *for each* $1 \le i \le n$. *For $\alpha \in \mathbf{N}^c$, let C_α be the fiber of α and let $\Gamma(\alpha)$ be the simplicial complex on vertices $x_1, \dots, x_n$ that has faces the radicals of the monomials in C_α and all their factors. Denote by $\Gamma_N(\alpha)$ the subcomplex with faces*

$$\left\{ l \in \Gamma(\alpha) \mid \textit{there exists an } m \in C_\alpha \textit{ such that } l \textit{ divides } m \textit{ and } \tfrac{m}{l} \in I + N \right\}.$$

For each $i \ge 0$ we have

$$b^S_{i,\alpha}(I + N) = \dim \tilde{\mathrm{H}}_i(\Gamma(\alpha), \Gamma_N(\alpha); k).$$

Proof. Note that $b^S_{i,\alpha}(I + N) = b^S_{i+1,\alpha}(S/(I + N))$. We compute the Betti numbers of $S/(I + N)$ using the Koszul complex $\mathbf{K}$ that is the minimal free resolution of k over S. Let E be the exterior algebra over k on basis elements $e_1, \dots, e_n$. The complex $\mathbf{K}$ equals $S \otimes E$ as an S-module and has differential

$$d(e_{j_1} \wedge \cdots \wedge e_{j_s}) = \sum_{1 \le i \le s} (-1)^{i+1} \cdot x_{j_i} \cdot e_{j_1} \wedge \cdots \wedge \hat{e}_{j_i} \wedge \cdots \wedge e_{j_s},$$

where $\hat{e}_{j_i}$ means that e_{j_i} is omitted in the product. Then

$$\begin{aligned} b^S_{i+1,\alpha}(S/(I+N)) &= \dim_k(\mathrm{Tor}_{i+1}(S/(I+N),k)_\alpha) \\ &= \dim_k(\mathrm{H}_{i+1}(S/(I+N) \otimes \mathbf{K})_\alpha). \end{aligned}$$

Recall that $\deg(x_i) = \deg(e_i) = \mathbf{a}_i$. The component of $S/(I+N) \otimes \mathbf{K}$ in multidegree α either is zero or has basis

$$\begin{aligned} &\left\{ \frac{m}{x_{j_1}\cdots x_{j_i}} e_{j_1} \wedge \cdots \wedge e_{j_i} \;\middle|\; m \in C_\alpha,\ \frac{m}{x_{j_1}\cdots x_{j_i}} \notin I+N,\ 1 \le j_1 < \cdots < j_i \le n \right\} \\ &\quad = \left\{ \frac{m}{x_{j_1}\cdots x_{j_i}} e_{j_1} \wedge \cdots \wedge e_{j_i} \;\middle|\; m \in C_\alpha,\ \frac{\mathrm{rad}(m)}{x_{j_1}\cdots x_{j_i}} \in \Gamma(\alpha) \setminus \Gamma_N(\alpha), \right. \\ &\qquad\qquad \left. 1 \le j_1 < \cdots < j_i \le n \right\}. \end{aligned}$$

Note that the component of $S/(I+N) \otimes \mathbf{K}$ in multidegree α is 0 if $\alpha \notin \mathbf{N}\mathcal{A}$. Denote by $\mathbf{T}$ the (topological) complex computing the relative homology of the simplicial complex $\Gamma(\alpha)$ relative to its subcomplex $\Gamma_N(\alpha)$. The map

$$\begin{aligned} (S/(I+N) \otimes \mathbf{K}_{i+1})_\alpha &\to \mathbf{T}_i, \\ \frac{m}{x_{j_1}\cdots x_{j_{i+1}}} e_{j_1} \wedge \cdots \wedge e_{j_{i+1}} &\mapsto \text{the face with vertices } j_1, \ldots, j_{i+1}, \end{aligned}$$

is an isomorphism of complexes. □

In the spirit of Harthorne's theorem [Ha] and the results of Peeva and Stillman [PS1; PS2] that the Hilbert scheme over an exterior algebra is more structured than the one over S, it will be interesting to explore in the following direction.

PROBLEM 4.9. *What can be said about the structure of the Hilbert scheme parameterizing all homogeneous ideals in R with a fixed Hilbert function* (*or, equivalently, homogeneous ideals in S containing I and with a fixed Hilbert function*)? *Construct deformations* (*note that change of basis does not work in this case*).

Again, as far as we know the problem is open even in such special cases as rational normal curves, Veronese rings, Segre rings, and generic toric rings.

5. Rational Normal Curves

In this section we prove that Macaulay's theorem holds for the rational normal curves. We also study the structure of minimal free resolutions over the rational normal curves.

THEOREM 5.1. *Take the set $\mathcal{A}$ to be*

$$\mathcal{A} = \{(0,1), (1,1), (2,1), \ldots, (n-1,1)\}.$$

The toric ring of the rational normal curve is $R = S/I$, where I is the kernel of the following map:

$$\varphi\colon k[x_1,\dots,x_n] \to k[s,t],$$
$$x_i \mapsto s^{i-1}t.$$

We order the variables by $x_1 > \cdots > x_n$. If J is a homogeneous ideal in R, then there exists a lex ideal K with the same Hilbert function.

Proof. Observe that the ring R is isomorphic to $k[t, st, s^2t, \dots, s^{n-1}t]$ and that the monomials in a fixed degree i are

$$t^i >_{\text{lex}} st^i >_{\text{lex}} s^2t^i >_{\text{lex}} \cdots >_{\text{lex}} s^{i(n-1)}t^i.$$

Set $\mathbf{n} = R_1$. For an i-monomial space V in R, we say that V generates $\mathbf{n}V$ in degree $i+1$. Also, we denote by $V(j)$ the i-monomial space spanned by the lex-greatest (lex-first) j monomials in V.

Next, we will prove that Lemma 4.2 holds. Let W be an i-monomial space and set $r = \dim_k W$. Let $m_1 >_{\text{lex}} \cdots >_{\text{lex}} m_r$ be the r monomials that span W in R. Furthermore, let L be the lex i-monomial space such that $\dim_k L = r$. Then L is spanned by $t^i >_{\text{lex}} st^i >_{\text{lex}} s^2t^i >_{\text{lex}} \cdots >_{\text{lex}} s^{(r-1)}t^i$. We must show that

$$\dim_k \mathbf{n}L \le \dim_k \mathbf{n}W.$$

The proof is by induction on the dimension r of the i-monomial spaces.

Let $j = 1$. The lex space $L(1)$ is spanned by t^i, and the space $W(1)$ is spanned by m_1. Note that every monomial m in R generates n monomials in the next degree; that is, $\dim_k \mathbf{n}(km) = n$. Therefore,

$$\dim_k \mathbf{n}(L(1)) = \dim_k \mathbf{n}(W(1)) = n.$$

Suppose by induction that

$$\dim_k \mathbf{n}(L(j)) \le \dim_k \mathbf{n}(W(j)). \tag{5.2}$$

Then there exists a unique monomial in $L(j+1)$ that is not in $L(j)$—namely, $q = s^jt^i$. Furthermore, there is only one monomial in $\mathbf{n}(L(j+1))$ that is not in $\mathbf{n}(L(j))$—namely, the monomial $x_nq = s^{n-1}ts^jt^i$. Therefore,

$$\dim_k \mathbf{n}(L(j+1)) = \dim_k \mathbf{n}(L(j)) + 1. \tag{5.3}$$

Furthermore, since $m_{j+1} \in W(j+1)$ and $m_{j+1} \in W(j)$, it follows that the monomial $x_nm_{j+1} = s^{n-1}tm_{j+1}$ is in $\mathbf{n}(W(j+1))$ but not in $\mathbf{n}(W(j))$. Hence

$$\dim_k \mathbf{n}(W(j+1)) \ge \dim_k \mathbf{n}(W(j)) + 1. \tag{5.4}$$

Combining (5.2), (5.3), and (5.4) yields the desired inequality

$$\begin{aligned}\dim_k \mathbf{n}(L(j+1)) &= \dim_k \mathbf{n}(L(j)) + 1\\ &\le \dim_k \mathbf{n}(W(j)) + 1 \le \dim_k \mathbf{n}(W(j+1)).\end{aligned}$$

We have thus proved that $\dim_k \mathbf{n}L \le \dim_k \mathbf{n}W$.

By Lemma 4.2, it follows that the theorem holds. $\square$

In the rest of this section we focus on Problem 4.6. We study minimal free resolutions over rational normal curves.

THEOREM 5.5. *Let R be the toric ring of a rational normal curve, and let M be a monomial ideal in R. Denote by* $\mathbf{G}$ *the minimal free resolution of R/M over R. Then there exist bases of the free modules in* $\mathbf{G}$ *such that the truncation* $\mathbf{G}_{\geq 2}$ *is a direct sum of linear free resolutions, possibly shifted in different degrees*; *in particular, starting from the third differential map d_3, the entries in the differential maps of* $\mathbf{G}$ *are linear. The Betti numbers of R/M satisfy*

$$b^R_{j+1}(R/M) = (n-2)b^R_j(R/M) \quad \textit{for } \ j \geq 3.$$

The proof of this theorem is given later in the section. The proof uses Theorem 5.7, which will provide a precise description of the minimal free resolution.

Let $Q = k[s,t]$ be the polynomial ring in two variables over the field k. Set $p = n-1$. We consider the rational normal curve toric ring

$$\begin{aligned} T &= k[\text{all monomials of degree } p \text{ in the two variables } s \text{ and } t] \\ &= k[s^p, s^{p-1}t, \ldots, t^p]. \end{aligned} \tag{5.6}$$

By (2.1) we have that T is isomorphic to the quotient ring $R = k[x_1,\ldots,x_n]/I$ associated to the set

$$\mathcal{A} = \{(p,0),(p-1,1),\ldots,(1,p-1),(0,p)\}.$$

Thus, given a monomial ideal M in R, we can think of it as a monomial ideal in T using the isomorphism (2.1). Note that we are using a different set of points $\mathcal{A}$ than the one in the proof of Theorem 5.1.

Let $M \subset R$ be a monomial ideal generated by r minimal monomial generators. By reordering the minimal generators (if necessary), we can write M as

$$M = (s^{\mu_1}t^{\nu_1},\ldots,s^{\mu_r}t^{\nu_r}), \quad \mu_1 > \cdots > \mu_r, \ \nu_1 < \cdots < \nu_r.$$

Furthermore, we know that

$$\mu_i + \nu_i \equiv 0 \ (\mathrm{mod}\, p) \quad \text{for all } \ 1 \leq i \leq r.$$

Therefore,

$$\begin{aligned} \alpha_i &:= (\mu_{i+1} - \mu_i) \bmod p \\ &= (\nu_i - \nu_{i+1}) \bmod p, \end{aligned}$$

THEOREM 5.7. *Let T be the toric ring of the rational normal curve in* (5.6), *and let $M \subset T$ be a monomial ideal. Let* $\mathbf{F}$ *be a minimal graded free resolution of T/M over T and let d be its differential.*

(1) *The second syzygy module* $\mathrm{Syz}^T_2(T/M)$ *is minimally generated by the set* $\mathcal{B} = \bigcup_{i=1}^{r-1} \mathcal{K}_i$, *where*

$$\mathcal{K}_i = \{s^h t^{\alpha_i - h}(t^{\nu_{i+1}-\nu_i}\cdot \varepsilon_i - s^{\mu_i-\mu_{i+1}}\cdot \varepsilon_{i+1}) \mid 0 \leq h \leq \alpha_i\}$$

and $\varepsilon_1,\ldots,\varepsilon_r$ are the basis elements of F_1.

(2) *Let $q \geq 2$ and denote the q*th *Betti number of* $\mathbf{F}$ *by b^T_q. The* $(q+1)$th *syzygy module* $\mathrm{Syz}^T_{q+1}(T/M)$ *is minimally generated by the set*

$$\{s^h t^{p-1-h}(t \cdot \omega_{i,j} - s \cdot \omega_{i,j+1}) \mid 0 \le h \le p-1,\ 1 \le j \le N_i,\ 1 \le i \le N\},$$

where the $\omega_{i,j}$ are the minimal generators of F_q and

$$N_i = \begin{cases} \alpha_i & \text{if } q = 2, \\ p-1 & \text{if } q \ge 3; \end{cases} \qquad N = \begin{cases} r-1 & \text{if } q = 2, \\ b_q^T/p & \text{if } q \ge 3. \end{cases}$$

Before proving this theorem we review the simple structure of the minimal free resolution of an arbitrary monomial ideal over Q.

LEMMA 5.8. *Let $\tilde{M} = (s^{\mu_1}t^{\nu_1}, \dots, s^{\mu_r}t^{\nu_r})$ be a monomial ideal in Q, where $\mu_1 > \cdots > \mu_r$ and $\nu_1 < \cdots < \nu_r$. The minimal free resolution of $Q/\tilde{M}$ over Q is*

$$0 \longrightarrow Q^{r-1} \xrightarrow{d_2} Q^r \longrightarrow Q \longrightarrow Q/\tilde{M} \longrightarrow 0.$$

The differential map d_2 is given by

$$d_2(e_i) = t^{\nu_{i+1}-\nu_i} \cdot \varepsilon_i - s^{\mu_i - \mu_{i+1}} \cdot \varepsilon_{i+1}, \quad 1 \le i \le r-1,$$

where e_i and ε_i are the standard basis vectors in Q^{r-1} and Q^r, respectively.

Proof of Theorem 5.7. Let $\tilde{M}$ be the ideal in Q generated by the minimal monomial generators of M.

(1) Obviously, $\mathcal{B} \subset \operatorname{Syz}_2^T(T/M)$ and all the elements in $\mathcal{B}$ are linearly independent over k. In order to prove that $\mathcal{B}$ spans the syzygy submodule, we arrange the monomials in F_1 with the order $\varepsilon_1 \succ \cdots \succ \varepsilon_r$ refined by the lexicographic order on the monomials of T. Now assume there are elements in $\operatorname{Syz}_2^T(T/M)$ that are not spanned by $\mathcal{B}$, and choose such an element τ with a minimal initial term. Since $\operatorname{Syz}_2^T(T/M) \subseteq \operatorname{Syz}_2^S(Q/\tilde{M})$, we can use Lemma 5.8 to write τ as

$$\tau = \sum_{i=1}^{r-1} f_i(t^{\nu_{i+1}-\nu_i} \cdot \varepsilon_i - s^{\mu_i-\mu_{i+1}} \cdot \varepsilon_{i+1}),$$

where the f_i are polynomials in Q. On the other hand, since $\tau \in F_1$ and since F_1 is a T-module, it follows that $f_i t^{\nu_{i+1}-\nu_i}, f_i s^{\mu_i-\mu_{i+1}} \in T$. For simplicity we may assume that $f_1 \ne 0$. Then $\operatorname{in}(\tau) = mt^{\nu_2-\nu_1}\varepsilon_1$ for some monomial m of f_1. For the same reason as before we know that $mt^{\nu_2-\nu_1} \in T$. In other words, $\deg_T(m) + \nu_2 - \nu_1 \equiv 0 \pmod p$, which implies that $m = gs^h t^{\alpha_1-h}$ for some monomial $g \in T$ and some $h \in \{0, \dots, \alpha_1\}$. Subtracting $gs^h t^{\alpha_1-h}(t^{\nu_2-\nu_1} \cdot \varepsilon_1 - s^{\mu_1-\mu_2} \cdot \varepsilon_2)$ from τ yields an element, not in the span of $\mathcal{B}$, that has a smaller initial term. This contradicts the minimality property of τ.

(2) We use induction on q. For the $q = 2$ case we look at the syzygies in F_1 as described in part (1). First we note that syzygies from different $\mathcal{K}_i$ sets are T-linear independent. Hence, any generator of $\operatorname{Syz}_3^T(T/M)$ can involve only syzygies from the same $\mathcal{K}_i$ in F_1. Because all the syzygies in $\mathcal{K}_i$ are multiples of the same vector by the monomials $\{s^{\alpha_i}, s^{\alpha_i-1}t, \dots, t^{\alpha_i}\}$, finding the syzygies is equivalent to resolving the monomial ideal $J = (s^{\alpha_i}, s^{\alpha_i-1}t, \dots, t^{\alpha_i})$. Lemma 5.8 implies that the syzygies over Q originating from $\mathcal{K}_i$ are

$$\sigma_{i,j} = t \cdot \omega_{i,j} - s \cdot \omega_{i,j+1}, \quad 1 \le j \le \alpha_i,$$

where the $\omega_{i,j}$ are the basis elements in F_2 corresponding to the syzygies in $\mathcal{K}_i$. Observe that in this case $N_i = \alpha_i$ and $N = r - 1$. Using part (1), we conclude that the syzygies over T originating from $\mathcal{K}_i$ are

$$s^h t^{p-1-h} \sigma_{i,j}, \quad 0 \le h \le p-1, \ 1 \le j \le \alpha_i.$$

The inductive step for $q \ge 3$ is the same, where now $N_i = p - 1$ and $N = b_q^T/p$. □

Theorem 5.7 gives an explicit description of the minimal free resolution $\mathbf{F}$ of T/M over T. In particular, it shows that the differential maps can be written in the form of block matrices.

Counting the syzygies in the resolution yields explicit expressions for the Betti numbers.

COROLLARY 5.9. *The Betti numbers of R/M over R are given by*

$$b_j^R(R/M) = \begin{cases} r & \text{if } j = 1, \\ \sum_{i=1}^{r-1} (\alpha_i + 1) & \text{if } j = 2, \\ (n-2)^{j-3}(n-1) \sum_{i=1}^{r-1} \alpha_i & \text{if } j \ge 3. \end{cases}$$

In particular, $b_{j+1}^R(R/M) = (n-2) b_j^R(R/M)$ for $j \ge 3$.

COROLLARY 5.10. *The regularity of M is given by*

$$\operatorname{reg}_R(M) = \max_{1 \le i \le r-1} \left\{ \frac{\mu_i + \nu_{i+1} + \alpha_i}{n-1} - 2 \right\}.$$

Proof. Theorem 5.7 implies that the regularity is determined by the degrees of the generators $\omega_{i,j}$ in homological degree 2. Using Theorem 5.7, we have

$$\deg(\omega_{i,j}) = \frac{\alpha_i + (\nu_{i+1} - \nu_i)}{p} + \deg(\varepsilon_i) = \frac{\alpha_i + (\nu_{i+1} - \nu_i)}{p} + \frac{\mu_i + \nu_i}{p};$$

this yields the desired result. □

COROLLARY 5.11. *The minimal free resolution of R/M over R is finite exactly when $\alpha_i = 0$ for all i.*

COROLLARY 5.12. *If M is generated by an initial lex-segment in degree s (i.e., if M is generated by lex-consecutive monomials of degree s starting with x_1^s), then the minimal free resolution of R/M over R is s-linear.*

Proof. An initial lex-segment in R is isomorphic to an initial lex-segment in T. It follows that, for all i, $\mu_i - \mu_{i+1} = \nu_{i+1} - \nu_i = 1$ and $\alpha_i = p - 1$. The map d_2 therefore involves only elements of the form $s^h t^{p-h}$, which have degree p and thus are mapped to variables in R under the inverse map φ^{-1}. □

Proof of Theorem 5.5. Apply Theorem 5.7 and its proof, showing that the differential maps can be written in the form of block matrices. For $j \ge 3$, the differential maps d_j involve only elements of the form $s^h t^{p-h}$, which have degree p and thus are mapped to variables in R using the inverse map φ^{-1}. The relation for the Betti numbers follows from Corollary 5.9. □

EXAMPLE 5.13. Consider the ideal $M = (ab, ac)$ in the quotient ring $R = k[a,b,c,d]/(ac - b^2,\ ad - bc,\ bd - c^2)$, which is the toric ring of the twisted cubic curve. The ring R is isomorphic to $T = k[s^3, s^2t, st^2, t^3]$, and M becomes $M = (s^5t, s^4t^2)$ in the ring T. If we are to resolve (s^5t, s^4t^2) over $Q = k[s,t]$, then the minimal resolution has projective dimension 2 and the only minimal syzygy generator in homological degree 1 is $\sigma = \left(\begin{smallmatrix} t \\ -s \end{smallmatrix}\right)$. Resolving M over T via Theorem 5.7 shows that the generators of $\operatorname{Syz}_2^T(T/M)$ are $s^2\sigma, st\sigma, t^2\sigma$. To find the generators of Syz_3^T we need only resolve the monomial ideal (s^2, st, t^2), obtaining the elements

$$\eta_1 = \begin{pmatrix} t \\ -s \\ 0 \end{pmatrix}, \qquad \eta_2 = \begin{pmatrix} 0 \\ t \\ -s \end{pmatrix}.$$

The generators are given by $s^2\eta_1, st\eta_1, t^2\eta_1, s^2\eta_2, st\eta_2, t^2\eta_2$. After changing back to the ring R, we obtain the following infinite resolution of R/M over R:

$$\cdots \longrightarrow R^{12} \xrightarrow{d_4} R^6 \xrightarrow{d_3} R^3 \xrightarrow{d_2} R^2 \xrightarrow{d_1 = (ab, ac)} R \longrightarrow R/M \longrightarrow 0$$

with maps

$$d_2 = \begin{pmatrix} b & c & d \\ -a & -b & -c \end{pmatrix} \quad \text{and} \quad d_3 = \begin{pmatrix} b & c & d & 0 & 0 & 0 \\ -a & -b & -c & b & c & d \\ 0 & 0 & 0 & -a & -b & -c \end{pmatrix}.$$

Similarly, d_i consists of 2^{i-2} blocks $\begin{smallmatrix} b & c & d \\ -a & -b & -c \end{smallmatrix}$.

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