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Gröbner Bases and Betti Numbers of Monoidal Complexes

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To Mel Hochster on his 65th birthday

1. Introduction

Combinatorial commutative algebra is a branch of combinatorics, discrete geometry, and commutative algebra. On the one hand, problems from combinatorics or discrete geometry are studied using techniques from commutative algebra; on the other hand, questions in combinatorics motivated various results in commutative algebra. Since the fundamental papers of Stanley (see [13] for the results) and Hochster [8; 9], combinatorial commutative algebra has been a growing and active field of research. See also Bruns and Herzog [7], Villarreal [16], Miller and Sturmfels [11], and Sturmfels [15] for classical and recent results and new developments in this area of mathematics.

Stanley–Reisner rings and affine monoid algebras are two of the classes of rings considered in combinatorial commutative algebra. In this paper we consider toric face rings associated to monoidal complexes. They generalize Stanley–Reisner rings by allowing a more general incidence structure than simplicial complexes and more general rings associated with their faces—namely, affine monoid algebras instead of polynomial rings.

In cooperation with M. Brun and B. Ichim, the authors have studied the local cohomology of toric face rings in previous work [1; 3; 10]. One of the main results is a general version of Hochster's formula for the local cohomology of a Stanley–Reisner ring (see [7] or [13]) even beyond toric face rings.

In this paper we want to generalize Hochster's formulas for the graded Betti numbers of a Stanley–Reisner ring [9] and affine monoid rings [11, Thm. 9.2] to toric face rings. Such a generalization is indeed possible for monoidal complexes that, roughly speaking, can be embedded into a space $\mathbb{Q}^d$ (see Theorem 4.5). As counterexamples show, full generality does not seem possible. One of the problems encountered is to construct a suitable grading. This forces us to consider grading monoids that are not necessarily cancellative.

Another topic treated in Section 3 is initial ideals (of the defining ideals) of toric face rings with respect to monomial (pre)orders defined by weights. Indeed, toric face rings come up naturally in the study of initial ideals of affine monoid algebras. In this regard we generalize results of Sturmfels [15]. We will pay special attention to the question of when the initial ideal is radical, monomial, or both

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(see Theorem 3.7 and Theorem 3.8). This gives us an opportunity to indicate a "simplicial" proof of Hochster's famous theorem on the Cohen–Macaulay property of affine normal monoid domains [8]. For unexplained terminology we refer the reader to [6] and [7].

2. Monoidal Complexes and Toric Face Rings

A *cone* is a subset of a space $\mathbb{R}^d$ of type $\mathbb{R}_+ x_1 + \cdots + \mathbb{R}_+ x_n$ with $x_1, \dots, x_n \in \mathbb{R}^d$. The *dimension* of a cone C is the vector space dimension of $\mathbb{R}C$. A *face* of C is a subset of type $C \cap H$, where H is a *support hyperplane* of C (i.e., a hyperplane H for which C is contained in one of the two closed halfspaces H^+, H^- determined by H). A *rational* cone is generated by elements $x \in \mathbb{Q}^d$. A *pointed* cone has $\{0\}$ as a face.

A *fan* in $\mathbb{R}^d$ is a finite collection $\mathcal{F}$ of cones in $\mathbb{R}^d$ satisfying the following conditions:

(i) all the faces of each cone $C \in \mathcal{F}$ belong to $\mathcal{F}$, too;
(ii) the intersection $C \cap D$ of $C, D \in \mathcal{F}$ is a face of C and of D.

We want to investigate more general configurations of cones, giving up the condition that all cones are contained in a single space but retaining the incidence structure. A *conical complex* consists of

(i) a finite set Σ of sets,
(ii) a cone $C_c \subseteq \mathbb{R}^{\delta_c}$ ($\delta_c = \dim \mathbb{R}C_c$) for each $c \in \Sigma$, and
(iii) a bijection $\pi_c \colon C_c \to c$ for each $c \in \Sigma$ such that the following conditions are satisfied:
  (a) for each face C' of C_c ($c \in \Sigma$) there exists a $c' \in \Sigma$ with $\pi_c(C') = c'$;
  (b) for all $c, d \in \Sigma$ there exist faces C' of C_c and D' of D_d such that $c \cap d = \pi_c(C') \cap \pi_d(D')$ and the restriction of $\pi_d^{-1} \circ \pi_c$ to C' is an isomorphism of the cones C' and D'.

Here an *isomorphism* of cones C, D is a bijective map $\varphi \colon C \to D$ that extends to an isomorphism of the vector spaces $\mathbb{R}C$ and $\mathbb{R}D$. Simplifying the notation, we write Σ also for the conical complex. A fan $\mathcal{F}$ is a conical complex in a natural way: fans are nothing but embedded conical complexes.

As introduced in the definition, δ_c will always denote the dimension of C_c so that $\mathbb{R}C_c$ can be identified with $\mathbb{R}^{\delta_c}$. The elements $c \in \Sigma$ are called the *faces* of Σ. Similarly, one defines *rays* and *facets* of Σ as (respectively) 1-dimensional and maximal faces of Σ. The *dimension* of Σ is the maximal dimension of a facet of Σ. We denote by $|\Sigma| = \bigcup_{c \in \Sigma} c$ the *support* of Σ. Identifying C_c with c, we may consider C_c as a subset of $|\Sigma|$. Then we can treat $|\Sigma|$ almost like an (embedded) fan. The main difference is that it makes no sense to speak of concepts like convexity globally, although locally in the cones C_c we may consider convex subsets. The complex Σ is *rational* and *pointed* (respectively) if all cones C_c, $c \in \Sigma$, are rational and pointed. We call Σ *simplicial* if all cones C_c, $c \in \Sigma$, are simplicial—that is, if they are generated by linearly independent vectors.

In order to define interesting algebraic objects associated to a conical complex, one needs a corresponding discrete structure. A *monoidal complex* $\mathcal{M}$ supported by a conical complex Σ is a set of monoids $(M_c)_{c\in\Sigma}$ such that:

(i) for each $c \in C$, the monoid M_c is an affine (i.e., finitely generated) monoid contained in $\mathbb{Z}_c^{\delta}$;
(ii) $M_c \subseteq C_c$ and $\mathbb{R}_+ M_c = C_c$ for every $c \in \Sigma$;
(iii) for all $c, d \in \Sigma$, the map $\pi_d^{-1} \circ \pi_c$ restricts to a monoid isomorphism between $M_c \cap \pi_c^{-1}(c \cap d)$ and $M_d \cap \pi_d^{-1}(c \cap d)$.

In other words, we have chosen for every $c \in \Sigma$ an affine monoid M_c that generates C_c and whose intersection with a face C_d of C_c is just M_d. The monoidal complex naturally associated to a single affine monoid M is simply denoted by M; it is supported on the conical complex formed by the faces of the cone $\mathbb{R}_+ M$.

The simplest examples of *monoidal complexes* are those associated with rational fans $\mathcal{F}$. For each cone $C \in \mathcal{F}$ we choose $M_C = C \cap \mathbb{Z}^d$. These monoids are finitely generated by Gordan's lemma. Moreover, they are normal: recall that an affine monoid M is *normal* if $M = \mathrm{gp}(M) \cap \mathbb{R}_+ M$.

REMARK 2.1. Let $\mathrm{gp}(M)$ denote the group of differences of a monoid M. The groups $\mathrm{gp}(M_C)$ of the monoids in the monoidal complex associated with a fan $\mathcal{F}$ form again a monoidal complex in a natural way, since $\mathrm{gp}(M_D) = \mathrm{gp}(M_C) \cap \mathbb{R}D$ if D is a face of C. Its underlying conical complex is given by the collection of the vector spaces $\mathbb{R}C$, $C \in \mathcal{F}$.

In general, the compatibility condition between the passage to faces and the formation of groups of differences need not be satisfied. Nevertheless, the rational structures defined by the monoids M_c (namely, the rational subspaces $\mathbb{Q}\,\mathrm{gp}(M_c)$ of $\mathbb{R}^{\delta_c}$) are compatible with the passage to faces. This follows from condition (ii): both monoids $\mathrm{gp}(M_c) \cap \mathbb{R}C_d$ and $\mathrm{gp}(M_d)$ are contained in $\mathbb{Z}^{\delta_d}$ and have the same rank δ_d.

Note that the monoids M_c form a direct system of sets with respect to the embeddings $\pi_d^{-1} \circ \pi_c \colon M_c \to M_d$, where $c, d \in \Sigma$ and $c \subseteq d$. We set

$$|\mathcal{M}| = \varinjlim M_c.$$

In general, there exists no global monoid structure on $|\mathcal{M}|$, but it carries a partial monoid structure because we can consider each monoid M_c as a subset of $|\mathcal{M}|$ in the natural way. If there exists a $c \in \Sigma$ such that $a, b \in M_c$, then $a + b$ is their sum in M_c and, as an element of $|\mathcal{M}|$, this sum is independent of the choice of c.

Next we choose a field K and define the *toric face ring* $K[\mathcal{M}]$ of $\mathcal{M}$ (over K) as follows. As a K-vector space, let

$$K[\mathcal{M}] = \bigoplus_{a\in|\mathcal{M}|} Kt^a.$$

We set

$$t^a \cdot t^b = \begin{cases} t^{a+b} & \text{if } a, b \in M_c \text{ for some } c \in \Sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Multiplication in $K[\mathcal{M}]$ is defined as the K-bilinear extension of this product; it turns $K[\mathcal{M}]$ into a K-algebra. In the following, the elements of $|\mathcal{M}|$ are called *monomials.*

There exist at least two other natural descriptions of toric face rings of a monoidal complex. The first is a realization as an inverse limit of the affine monoid rings $K[M_c]$, $c \in \Sigma$. For $c \in \Sigma$ and a face d of c there exists a natural projection map, the *face projection map* $K[M_c] \to K[M_d]$ that sends monomials t^a to zero if $a \notin M_d$. With respect to these maps we may consider the inverse limit $\varprojlim K[M_c]$ as follows.

PROPOSITION 2.2. *Let $\mathcal{M}$ be a monoidal complex supported on a conical complex Σ. Then*

$$K[\mathcal{M}] \cong \varprojlim K[M_c].$$

For the proof of the proposition we introduce some more notation. Let $c \in \Sigma$ and let $\mathfrak{p}_c$ be the ideal of $K[\mathcal{M}]$ that is generated by all monomials t^a with $a \notin M_c$. Then there is a natural isomorphism of K-algebras $K[M_c] \cong K[\mathcal{M}]/\mathfrak{p}_c$. In particular, $\mathfrak{p}_c$ is a prime ideal. Moreover, if $d \subset c$ and $c, d \in \Sigma$, then the natural epimorphism $K[\mathcal{M}]/\mathfrak{p}_c \to K[\mathcal{M}]/\mathfrak{p}_d$ coincides with the map induced by the projection map $K[M_c] \to K[M_d]$. We identify these maps in the following proof.

Proof of Proposition 2.2. Observe that each of the ideals $\mathfrak{p}_c$ has a K-basis consisting of monomials of $K[\mathcal{M}]$. Hence the following equations are satisfied for $c, d, e \in \Sigma$:

(i) $\mathfrak{p}_c + \mathfrak{p}_d = \mathfrak{p}_{c \cap d}$;
(ii) $\mathfrak{p}_c \cap (\mathfrak{p}_d + \mathfrak{p}_e) = \mathfrak{p}_c \cap \mathfrak{p}_d + \mathfrak{p}_c \cap \mathfrak{p}_e$;
(iii) $\mathfrak{p}_c + \mathfrak{p}_d \cap \mathfrak{p}_e = (\mathfrak{p}_c + \mathfrak{p}_d) \cap (\mathfrak{p}_c + \mathfrak{p}_e)$ for all c, d, e.

Now it follows easily that $\varprojlim K[\mathcal{M}]/\mathfrak{p}_c$ is isomorphic to $K[\mathcal{M}]/\bigcap_{c \in \Gamma} \mathfrak{p}_c$ (see e.g. [1, Ex. 3.3]). But $\bigcap_{c \in \Gamma} \mathfrak{p}_c = 0$, and so

$$\varprojlim K[M_c] \cong \varprojlim K[\mathcal{M}]/\mathfrak{p}_c \cong K[\mathcal{M}] \Big/ \bigcap_{c \in \Gamma} \mathfrak{p}_c \cong K[\mathcal{M}]. \qquad \square$$

As the second natural description we want to characterize a toric face ring as a quotient of a polynomial ring. It is not difficult to compute the defining ideal of such a presentation. In view of Theorem 3.4 (to follow), we must consider elements of $K[\mathcal{M}]$ that are either monomials t^a ($a \in \mathcal{M}$) or 0. For a uniform notation we augment $|\mathcal{M}|$ by an element $-\infty$ and set $t^{-\infty} = 0$.

PROPOSITION 2.3. *Let $\mathcal{M}$ be a monoidal complex supported on a conical complex Σ, and let $(a_e)_{e \in E}$ be a family of elements of $|\mathcal{M}| \cup \{-\infty\}$ generating $K[\mathcal{M}]$ as a K-algebra. (Equivalently, $\{a_e : e \in E\} \cap M_c$ generates M_c for each $c \in \Sigma$.) Then the kernel $I_{\mathcal{M}}$ of the surjection*

$$\varphi\colon K[X_e : e \in E] \to K[\mathcal{M}], \quad \varphi(X_e) = t^{a_e},$$

is generated by

(i) *all monomials* $\prod_{h\in H} X_h$, *where* H *is a subset of* E *for which* $\{a_h : h \in H\}$ *is not contained in any monoid* M_c, $c \in \Gamma$; *and*

(ii) *all binomials* $\prod_{g\in G} X_g^{i_g} - \prod_{h\in H} X_h^{j_h}$, *where* $G, H \subset E$, *all* a_g, a_h *are contained in a monoid* M_c *for some* $c \in \Sigma$, *and* $\sum_{g\in G} i_g a_g = \sum_{h\in H} j_h a_h$.

Moreover, the monomials in (i) *are all monomials contained in* $I_{\mathcal{M}}$. *A binomial is contained in* $I_{\mathcal{M}}$ *if either both its monomials are contained in the family of monomials given in* (i) *or it is in the list of the binomials in* (ii).

Proof. It is clear that $I_{\mathcal{M}}$ contains the ideal J generated by all the monomials and binomials listed in (i) and (ii).

For the converse, let f be a polynomial such that $\varphi(f) = 0$. Then we can assume that all monomials of f map to elements of $|\mathcal{M}|$, since all other monomials belong to $I_{\mathcal{M}}$. Now let $c \in \Sigma$, and define f_c to be the polynomial that arises as the sum of those terms of f whose monomials are mapped to elements of $M_c \subset |\mathcal{M}|$. Then $\varphi(f_c) = 0$ as well. It is well known and easy to show that f_c then belongs to the ideal in $R[X_e : e \in E]$ generated by all those binomials in (ii) for which $a_g, a_h \in M_c$. (Equivalently, the binomials in (ii) for M_c generate the presentation ideal of $K[M_c]$ over a polynomial ring in the variables X_e, where $e \in E$ and $a_e \in M_c$.) We may thus replace f by $f - f_c$ and then finish the proof by induction on the number of terms of f.

It is clear that a monomial belongs to $I_{\mathcal{M}}$ if and only if it is contained in the family of monomials given in (i). If a binomial is an element of $I_{\mathcal{M}}$, then either both monomials belong to this ideal or none of the monomials belong. In the latter case the binomial must be one of the family of binomials given in (ii), since no other binomials belong to the kernel of the map $K[X_e : e \in E] \to K[\mathcal{M}]$. This follows directly from the construction of the ring $K[\mathcal{M}]$. □

EXAMPLES 2.4. (i) Let $\mathcal{F}$ be a rational fan in $\mathbb{R}^d$, and let $\mathcal{M}$ be the conical complex associated with $\mathcal{F}$. Then the algebra $K[\mathcal{M}]$ is the toric face ring introduced by Stanley [12].

(ii) Let Δ be an abstract simplicial complex on the vertex set $[n] = \{1, \dots, n\}$. Then Δ has a geometric realization by considering the simplices $\operatorname{conv}(e_{i_1}, \dots, e_{i_m})$ such that $\{i_1, \dots, i_m\}$ belongs to Δ (here $e_1, \dots, e_n$ is the canonical basis of $\mathbb{R}^n$). The cones over the faces of the geometric realization form a fan $\mathcal{F}$ whose toric face ring R given by (i) is nothing but the Stanley–Reisner ring of Δ. In fact, according to Proposition 2.3, the kernel of the natural epimorphism $K[X_1, \dots, X_n] \to R$ is generated by those monomials $X_{j_1} \cdots X_{j_r}$ such that $\{j_1, \dots, j_r\} \notin \Delta$.

Algebras associated with monoidal complexes therefore generalize Stanley–Reisner rings by allowing arbitrary conical complexes as their combinatorial skeleton and, consequently, monoid algebras as their ring-theoretic flesh.

(iii) The polyhedral algebras of [5] are another special case of algebras associated with monoidal complexes. For them, the cones are generated by lattice polytopes and the monoids are the polytopal monoids considered in [5].

REMARK 2.5. Toric face rings were defined in [2] by their presentation ideals given in Proposition 2.3. Thus Proposition 2.2 is equivalent to [2, Thm. 4.7]. In

[2] the generators of the affine monoids M_c for $c \in \Sigma$ were fixed at the start, but in this paper we fix only the monoids and are free to choose generators whenever we like. This approach leads directly to a natural description of the toric face rings. Using arguments like those here (e.g., the prime ideals in the proof of Proposition 2.2), one obtains alternative and slightly more compact proofs than those in [2].

We have already used the fact that the zero ideal of $K[\mathcal{M}]$ is the intersection of the prime ideals $\mathfrak{p}_c$. This implies that $K[\mathcal{M}]$ is reduced.

Let Σ be a conical complex. A conical complex Γ is a *subdivision* of Σ if $|\Gamma| = |\Sigma|$ and if each face $c \in \Sigma$ is the union of faces $d \in \Gamma$. The subdivision is called a *triangulation* if Γ is simplicial. We call a subdivision Γ *rational* if all cones C_d $(d \in \Gamma)$ are rational.

Suppose that Γ is a subdivision of Σ, and let $\mathcal{M}$ be a monoidal complex supported by Γ and c a face of Σ. In the situation of Proposition 2.3, for the toric face ring $K[\mathcal{M}]$ we let S_c be the polynomial subring of $S = K[X_e : e \in E]$ generated by those X_e for which $a_e \in C_c$. Furthermore, let $\mathcal{M}_c$ be the monoidal subcomplex of $\mathcal{M}$ consisting of all faces D_d of Γ $(d \subset c)$ and their associated monoids.

Because $\mathcal{M}_c$ is a monoidal subcomplex, one has the natural epimorphism $K[\mathcal{M}] \to K[\mathcal{M}_c]$ generalizing the face projection. It is given by $t^a \mapsto t^a$ whenever $a \in C_c$ and by $t^a \mapsto 0$ otherwise. But we have also an embedding $K[\mathcal{M}_c] \to K[\mathcal{M}]$, since points of $|\mathcal{M}_c|$ that are contained in a face of Γ are also contained in a face of $\mathcal{M}_c$.

In order to encode the incidence structure of Σ, we let A_Σ denote the ideal in S generated by the squarefree monomials $\prod_{h \in H} X_h$ for which $\{a_h : h \in H\}$ is not contained in a face of Σ.

PROPOSITION 2.6. *Let notation be as in Proposition 2.3.*

(i) *The embedding $K[\mathcal{M}_c] \to K[\mathcal{M}]$ is a section of the projection $K[\mathcal{M}] \to K[\mathcal{M}_c]$ and thus makes $K[\mathcal{M}_c]$ a retract of $K[\mathcal{M}]$.*

(ii) *Let $c_1, \dots, c_n$ be the facets of Σ, and set $\mathcal{M}_i = \mathcal{M}_{c_i}$. Then*

$$I_{\mathcal{M}} = A_\Sigma + SI_{\mathcal{M}_1} + \dots + SI_{\mathcal{M}_n}.$$

Moreover, for each face $c \in \Sigma$ we have $I_{\mathcal{M}_c} = S_c \cap I_{\mathcal{M}}$.

Proof. Part (i) is evident. The representation of $I_{\mathcal{M}}$ in part (ii) follows immediately from Proposition 2.3: none of the binomial relations is lost on the right-hand side, which contains also all the monomial relations because these are contained either in one of the $I_{\mathcal{M}_i}$ or in A_Σ. The equation $I_{M_c} = S_c \cap I_{\mathcal{M}}$ restates part (i) as lifted to the presentations of the algebras. $\square$

In particular, we can apply Proposition 2.6 in the case $\Gamma = \Sigma$.

3. Toric Face Rings and Initial Ideals

Next we want to compute initial ideals of the presentation ideals of monoidal complexes considered in Proposition 2.3. Recall that a *weight vector* for a polynomial ring $S = K[X_1, \dots, X_n]$ is an element $w \in \mathbb{N}^n$, where $\mathbb{N}$ denotes the set of nonnegative integers. Given this vector, we assign X_i the *weight* w_i. It is easy to see that this is equivalent to endowing S with a positive $\mathbb{Z}$-grading under which the monomials are homogeneous. Thus, the whole terminology of graded rings (with the prefix w) can be applied. In particular, we can speak of the *w-degree* of a monomial; it is defined by

$$\deg_w X^a = \sum_{i=1}^{n} a_i w_i = a \cdot w.$$

A weight vector w determines a *weight (pre-)order* if one sets

$$X^a \leq_w X^b \iff a \cdot w \leq b \cdot w.$$

The only axiom of a monomial order (as considered below) not satisfied is antisymmetry: for $n > 1$ there always exist distinct monomials X^a and X^b such that simultaneously $X^a \leq_w X^b$ and $X^b \leq_w X^a$.

The *w-initial component* $\mathrm{in}_w(f)$ of a polynomial f is just its w-homogeneous component of highest degree. Let $V \subseteq S$ be a subspace. Then the *w-initial subspace* $\mathrm{in}_w(V)$ is the subspace generated by the polynomials $\mathrm{in}_w(f)$, $f \in V$. Observe that, for an ideal $I \subseteq S$, the w-initial subspace $\mathrm{in}_w(I)$ is again an ideal of S. Now well-known results for monomial orders hold also for weight orders. As an example, for subspaces $V_1 \subseteq V_2 \subseteq S$ we have $\mathrm{in}_w(V_1) = \mathrm{in}_w(V_2)$ if and only if $V_1 = V_2$.

A *monomial order* $<$ on S is a total order of the monomials of S such that (i) $1 < X^a$ for all monomials X^a and (ii) $X^a < X^b$ implies $X^{a+c} < X^{b+c}$ for all monomials X^a, X^b, X^c. Now we can speak similarly of initial terms $\mathrm{in}_<(f)$ and initial subspaces $\mathrm{in}_<(V)$ with respect to $<$. Recall that a *Gröbner basis* of I is a set of elements of I whose initial monomials generate $\mathrm{in}_<(I)$. Such a set always exists and then also generates I.

It is an important fact that a monomial order can always be approximated by a weight order if only finitely many monomials are concerned: for an ideal I of S there exists a weight vector $w \in \mathbb{N}^n$ such that $\mathrm{in}_<(I) = \mathrm{in}_w(I)$. Conversely, given a weight vector $w \in \mathbb{N}^n$ and a monomial order $<'$, we can refine the weight order $<_w$ to a monomial order $<$ by setting $X^a < X^b$ if $a \cdot w < b \cdot w$ or if $a \cdot w = b \cdot w$ and $X^a <' X^b$. Observe also that the w-initial terms of a Gröbner basis of I with respect to $<$ generate $\mathrm{in}_w(I)$. For more details and general results on weight orders and monomial orders, we refer the reader to [4] or [15].

The ideal given in Proposition 2.3 has a special structure. It is generated by monomials and binomials, and this property persists in the passage to initial ideals.

LEMMA 3.1. *Let $I \subset K[X_1, \dots, X_n]$ be an ideal generated by monomials and binomials and let $w \in \mathbb{N}^n$ be a weight vector. Then $\mathrm{in}_w(I)$ is generated by the monomials and the initial components of the binomials in I.*

Proof. We refine the weight order to a monomial order $<$. Using the Buchberger algorithm to compute a Gröbner basis for I, one enlarges the given set of generators of I consisting of monomials and binomials only by adding more monomials and binomials. The corresponding initial components with respect to the weight order $<_w$ then generate $\mathrm{in}_w(I)$. □

It is a useful consequence of Lemma 3.1 that the decomposition of the ideal $I_{\mathcal{M}}$ in Proposition 2.6 is passed onto their initial ideals. We need this only for the trivial subdivision of Σ by itself, but it can easily be generalized to the setting of Proposition 2.6. (Also see [3, Thm. 5.9] for a related result.)

PROPOSITION 3.2. *Consider the presentation of $K[\mathcal{M}]$ as a residue class ring of $S = K[X_e : e \in E]$ as in Proposition 2.3, a weight vector w on S, and the induced weight vectors for the subalgebras $S_c = K[X_e : a_e \in M_c]$, $c \in \Sigma$. We have*

$$\mathrm{in}_w(I_\Sigma) = A_{\mathcal{M}} + S \cdot \mathrm{in}_w(I_{M_1}) + \cdots + S \cdot \mathrm{in}_w(I_{M_n}),$$

where again $c_1, \dots, c_n$ are the facets of Σ and $M_i = M_{c_i}$. Moreover, $\mathrm{in}_w(I_{M_c}) = S_c \cap \mathrm{in}_w(I_{\mathcal{M}})$ for all $c \in \Sigma$.

Proof. It is clear that the right-hand side is contained in $\mathrm{in}_w(I_\Sigma)$. For the converse inclusion it is enough to consider the system of generators of $I_{\mathcal{M}}$ described in Proposition 3.1, and there is nothing to say about the monomials in $I_{\mathcal{M}}$. Let f be the initial component of a binomial g in $I_{\mathcal{M}}$. According to Proposition 2.3 there are two cases: (1) g belongs to $A_{\mathcal{M}}$; then so does f. (2) $g \in I_{M_i}$ for some i; then $f \in \mathrm{in}_w(I_{M_i})$, and we are done with the decomposition of $\mathrm{in}_w(I_\Sigma)$.

The equality $\mathrm{in}_w(I_{M_c}) = S_c \cap \mathrm{in}_w(I_{\mathcal{M}})$ is left to the reader. It is easily derived from Proposition 2.3 and Proposition 3.1. □

Recall that a function $f\colon X \to \mathbb{R}$ on a convex set X is called *convex* if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

for all $x, y \in X$ and $t \in [0,1]$. A function $f\colon |\Sigma| \to \mathbb{R}$ on a conical complex Σ is called convex if it is convex on all the cones C_c for $c \in \Sigma$. For a function $f\colon |\Sigma| \to \mathbb{R}$, a connected subset W of a facet C_c of Σ is a *domain of linearity* if it is maximal with respect to the following property: $f|_W$ can be extended to an affine function on $\mathbb{R}C_c$. Now a subdivision Γ of a conical complex Σ is said to be *regular* if there exists a convex function $f\colon |\Sigma| \to \mathbb{R}$ whose domains of linearity are facets of Γ. Such a function is called a *support function* for the subdivision Γ.

Let $(a_e)_{e\in E}$ be a family of elements of $|\mathcal{M}|$ such that $\{a_e : e \in E\} \cap M_c$ generates M_c for each $c \in \Sigma$. Now we choose a polynomial ring $S = K[X_e : e \in E]$ and define the surjective homomorphism $\varphi\colon K[X_e : e \in E] \to K[\mathcal{M}]$ that maps X_e to t^{a_e} as considered in Proposition 2.3. Let $w = (w_e)_{e\in E}$ be a weight vector for S.

On the one hand, the weight vector w determines initial ideals, especially the initial ideal $\mathrm{in}_w(I_{\mathcal{M}})$. On the other hand, w determines also a conical subdivision Γ_w of the conical complex Σ as follows. First, every cone $C_c \subseteq \mathbb{R}^{\delta_c}$ and the weight vector w define a cone

$$C_c' = \mathbb{R}_+((a_e, w_e) : e \in E \text{ such that } a_e \in C_c) \subseteq \mathbb{R}^{\delta_c+1}.$$

The projection on the first δ_c coordinates maps C_c' onto C_c. The *bottom* of C_c' with respect to C_c consists of all points $(a, h_a) \in C_c'$ such that the line segment $[(a,0),(a,h_a)]$ intersects C_c' only in (a,h_a). In other words, $h_a = \min\{h' : (a,h') \in C_c'\}$. Clearly $h_a > 0$ for all $a \in C_c$, $a \neq 0$. The bottom is a subcomplex of the boundary of C_c' (or C_c' itself). Note that its projection onto C_c defines a conical subdivision of the cone C_c. Second, the collection of these conical subdivisions of the cones C_c constitutes a conical subdivision of Σ.

Now we show that this subdivision is regular (as defined previously). Toward this end, we define the function $\mathrm{ht}_w : |\Sigma| \to \mathbb{R}$ as follows. For $a \in |\Sigma|$ there exists a minimal face $c \in \Sigma$ such that $a \in C_c$. Construct C_c' as before, using the weight vector w. Then we define

$$\mathrm{ht}_w(a) = \min\{h' \in \mathbb{R} : (a,h') \in C_c'\};$$

that is, $\mathrm{ht}_w(a)$ is the unique vector in the bottom of C_c' that is projected on C_c via the projection map on the first δ_c coordinates.

PROPOSITION 3.3. *Let $\mathcal{M}$ be a monoidal complex supported on a conical complex Σ, let $(a_e)_{e\in E}$ be a family of elements of $|\mathcal{M}|$ such that $\{a_e : e \in E\} \cap M_c$ generates M_c for each $c \in \Sigma$, and let $w = (w_e)_{e\in E}$ be a weight vector.*

(i) *For $c \in \Sigma$, $b_1, \dots, b_m \in C_c$, and $\alpha_i > 0$ $(i = 1, \dots, m)$, we have*

$$\mathrm{ht}_w\left(\sum_{i=1}^{m} \alpha_i b_i\right) \le \sum_{i=1}^{m} \alpha_i \,\mathrm{ht}_w(b_i). \tag{1}$$

In particular, ht_w is a convex function on $|\Sigma|$.

(ii) *Its domains of linearity are the cones D_d for facets d of Γ_w; that is, equality holds in (1) if and only if there exists a facet of Γ_w containing $b_1, \dots, b_m$.*

Therefore, Γ_w is a regular subdivision of Σ.

Part (i) uses only the definition of ht and that the cones C_c' are closed under $\mathbb{R}_+$-linear combinations. Part (ii) reflects the fact that an $\mathbb{R}_+$-linear combination of points in the boundary of a cone C lies in the boundary if and only if all points (with nonzero coefficients) belong to a facet of C (cf. [6, Lemma 7.16]).

Since the weights w_e are positive, it follows that the cones C_c' are pointed even if C_c is not. Thus all faces of Γ_w are pointed, too.

For each D_d with $d \in \Gamma_w$, we let $N_{d,w}$ be the monoid generated by all $a_e \in D_d$ for which $\mathrm{ht}_w(a_e) = w_e$. The cones D_d and the monoids $N_{d,w}$ form a monoidal complex $\mathcal{M}_w = \mathcal{M}_{\Gamma_w}$ supported by the conical complex Γ_w, the *monoidal complex defined by* w. Observe that each extreme ray of a cone D_d of Γ_w is the image of an extreme ray of C_c' for some $c \in \Sigma$. The latter contains a point (a_e, w_e) and therefore $w_e = \mathrm{ht}_w(a_e)$, which implies $D_d = \mathbb{R}_+ N_{d,w}$. The remaining conditions for a monoidal complex are fulfilled as well. It is important to note that the monoidal complex $\mathcal{M}_w$ is dependent not only on Γ_w or the pair (Γ_w, E) but also on the chosen weight w.

The algebra $K[\mathcal{M}_w]$ is again a residue class ring of the polynomial ring $K[X_e : e \in E]$ under the assignment

$$X_e \mapsto \begin{cases} t^{a_e} & \text{if } a_e \in |\mathcal{M}_w|, \\ 0 & \text{otherwise.} \end{cases}$$

The kernel of this epimorphism is denoted by $J_{\mathcal{M}_w}$. It is, of course, just the presentation ideal of the toric face ring $K[\mathcal{M}_w]$ supported by the conical complex Γ_w (and here we must allow that indeterminates X^e go to 0).

One cannot expect that $\operatorname{in}_w(I_{\mathcal{M}}) = J_{\mathcal{M}_w}$, since $J_{\mathcal{M}_w}$ is always a radical ideal but $\operatorname{in}_w(I_{\mathcal{M}})$ need not be radical. However, this is the only obstruction. The next theorem generalizes a result of Sturmfels [14; 15], who proved it for the case where the conical complex is induced by a single monoid and the subdivision Γ_w is a triangulation. So Theorem 3.4 is essentially equivalent to [2, Thm. 5.11]. See Remark 2.5 for the difference between the two approaches.

THEOREM 3.4. *Let $\mathcal{M}$ be a monoidal complex supported on a conical complex Σ, let $(a_e)_{e\in E}$ be a family of elements of $|\mathcal{M}|$ such that $\{a_e : e \in E\} \cap M_c$ generates M_c for each $c \in \Sigma$, and let $w = (w_e)_{e\in E}$ be a weight vector. Moreover, let $\mathcal{M}_w$ be the monoidal complex defined by w. Then the ideal $J_{\mathcal{M}_w}$ is the radical of the initial ideal* $\operatorname{in}_w(I_{\mathcal{M}})$.

Proof. For a single monoid this is just [6, Thm. 7.18], and we reduce the general case to it.

We remarked previously that the ideal $J_{\mathcal{M}_w}$ is, by construction, the presentation ideal of a toric face ring. The underlying complex is Γ_w, a subdivision of Σ. We apply Proposition 2.6 to this subdivision of Σ and to the facets of Σ, which correspond to single monoids $M_1, \dots, M_n$. Thus

$$J_{\mathcal{M}_w} = A_\Sigma + J_{(M_1)_w} + \cdots + J_{(M_n)_w}. \tag{2}$$

By [6, Thm. 7.18] we have $J_{(M_1)_w} = \operatorname{Rad} \operatorname{in}_w(I_{M_i})$, and therefore

$$J_{\mathcal{M}_w} = A_\Sigma + \operatorname{Rad} S \cdot \operatorname{in}_w(I_{M_1}) + \cdots + \operatorname{Rad} S \cdot \operatorname{in}_w(I_{M_n}).$$

The right-hand side is certainly contained in $\operatorname{Rad} \operatorname{in}_w(I_{\mathcal{M}})$, and it contains $\operatorname{in}_w(I_{\mathcal{M}})$ by Proposition 3.2. Since $J_{\mathcal{M}_w}$ is a radical ideal, we are done. □

Because $\operatorname{Rad} \operatorname{in}_w(I_{\mathcal{M}}) = J_{\mathcal{M}_w}$, we always have the inclusion $\operatorname{in}_w(I_{\mathcal{M}}) \subseteq J_{\mathcal{M}_w}$. It is a natural question to characterize the cases where we have equality, which holds exactly when the monoids $N_{d,w}$ are determined by their cones.

COROLLARY 3.5. *Given the hypotheses of Theorem 3.4, the following statements are equivalent*:

(i) $\operatorname{in}_w(I_{\mathcal{M}})$ *is a radical ideal*;
(ii) *for all facets $d \in \Gamma_w$ one has $N_{d,w} = M_c \cap D_d$, where $c \in \Sigma$ is the smallest face such that $d \subseteq c$.*

Proof. Condition (ii) clearly depends only on the facets of Σ, but this holds likewise for condition (i). The equality $J_{\mathcal{M}_w} = \operatorname{in}_w(I_{\mathcal{M}})$ is passed to the facets: we obtain the corresponding ideals for the facets c_i by intersection with S_{c_i}, and in the converse direction we use equation (2) and Proposition 3.2. Therefore, it is enough to consider the case of a single cone, for which the corollary is part of [6, Cor. 7.20]. □

Before presenting another corollary we characterize the cases in which $\operatorname{in}_w(I_{\mathcal{M}})$ is a monomial ideal. We say that a monoidal complex is *free* if all its monoids are free commutative monoids. Evidently this implies that the associated conical complex is simplicial, but the converse does not hold. The free monoidal complexes are exactly those derived from abstract simplicial complexes (cf. Example 2.4(ii)). We note the following obvious consequence of Theorem 3.4.

LEMMA 3.6. *Given the hypotheses of Theorem 3.4, the following statements are equivalent*:

(i) $\operatorname{Rad}\operatorname{in}_w(I_{\mathcal{M}})$ *is a (squarefree) monomial ideal*;
(ii) $\mathcal{M}_w$ *is a free monoidal complex.*

In particular, if these equivalent conditions hold then Γ_w *is a regular triangulation of* Σ.

For the next result we recall the definition of unimodular cones. Let $L \subseteq \mathbb{R}^d$ be a lattice (i.e., L is a subgroup of $\mathbb{R}^d$ generated by $\mathbb{R}$-linearly independent elements), and assume that $L \subseteq \mathbb{Q}^d$. Let $C \subseteq \mathbb{R}^d$ be a rational pointed cone. Since for each extreme ray R of C the monoid $R \cap L$ is normal and of rank 1, there exists a unique generator e of this monoid. We call these generators the *extreme generators of C* with respect to L. When C is simplicial we call C *unimodular* with respect to L if the sublattice of L generated by the extreme generators of C with respect to L generates a direct summand of L.

THEOREM 3.7. *With the same assumptions as in Theorem 3.4, the following statements are equivalent*:

(i) *the ideal* $\operatorname{in}_w(I_{\mathcal{M}})$ *is a monomial radical ideal*;
(ii) *the conical complex* Γ_w *is a triangulation of* Σ, *the extreme generators of a cone* D_d *for* $d \in \Gamma_w$ *with respect to* $\operatorname{gp}(M_c)$ *generate the monoid* $N_{d,w}$, *and* D_d *is unimodular with respect to* $\operatorname{gp}(M_c)$.

Proof. It follows from Corollary 3.5 and Lemma 3.6 that $\operatorname{in}_w(I_{\mathcal{M}})$ is a monomial radical ideal if and only if the following statements hold.

(a) $\mathcal{M}_w$ is free.
(b) For a facet $d \in \Gamma_w$, let $c \in \Sigma$ be the smallest face such that $d \subseteq c$; then $N_{d,w} = M_c \cap D_d$.

It remains to show the equivalence of (a) and (b) to (ii). Yet both sides of this equivalence depend only on the single monoids M_c and the restrictions of $\mathcal{M}_w$ to them. In the case of a single monoid, the theorem is part of [6, Cor. 7.20]. □

Now we can give a nice criterion for the normality of the monoids in a monoidal complex in terms of an initial ideal with respect to a weight vector.

THEOREM 3.8. *The following statements are equivalent*:

(i) *all monoids M_c of the monoidal complex $\mathcal{M}$ are normal*;

(ii) *there exists a family of elements $(a_e)_{e\in E}$ of $|\mathcal{M}|$ such that $\{a_e : a_e \in M_c\}$ generates M_c for each monoid M_c of $\mathcal{M}$ and a weight vector $w = (w_e)_{e\in E}$ such that $\mathrm{in}_w(I_{\mathcal{M}})$ is a monomial radical ideal.*

Proof. Proving (ii) $\Rightarrow$ (i) again reduces to the case of a single monoid M by Proposition 3.2. In this case, (ii) implies that M is the union of free monoids with the same group as M. Then the normality of M follows immediately.

For (i) $\Rightarrow$ (ii) we must construct a regular unimodular triangulation Γ of $\mathbb{R}_+ M$ by elements of M, which we choose as a system of generators. The weight of X_e is then chosen as the value of the support function of the triangulation at a_e.

The existence of such a triangulation is a standard result (see e.g. [6, Thm. 2.70], where it is stated for a single monoid M). The construction goes through for monoidal complexes as well (and the proof implicitly makes use of this fact). However, there is one subtle point to be taken into account: If M is normal and F is a face of the cone $\mathbb{R}_+ M$, then $\mathrm{gp}(M \cap F) = \mathrm{gp}(M) \cap \mathbb{R}F$. This condition ensures that the groups $\mathrm{gp}(M_c)$ again form a monoidal complex and that unimodularity of a free submonoid does not depend on the monoid M_c in which it is considered. $\square$

In the investigation of a normal monoid M, one is usually not interested in an arbitrary system of generators of M but rather in $\mathrm{Hilb}(M)$. It is well known that one cannot always find a (regular) unimodular triangulation by elements of $\mathrm{Hilb}(M)$, and this limits considerably the value of results like Theorem 3.8. Nevertheless, the theorem is powerful when the unimodularity of certain triangulations is given automatically.

Theorem 3.8 can be used to prove that monoid algebras of normal affine monoids are Cohen–Macaulay. This result is due to Hochster [8].

COROLLARY 3.9. *Let M be a normal affine monoid. Then the monoid algebra $K[M]$ is Cohen–Macaulay for every field K.*

Proof. We may assume that M is positive. In fact, $M = U(M) \oplus M'$, where $U(M)$ is the group of units of M and M' is a normal affine monoid that is positive. Moreover, $K[M]$ is a Laurent polynomial extension of $K[M']$ and so we may replace M by M'.

It follows from Theorem 3.8 that there exists a system of generators $(a_e)_{e\in E}$ of M and a weight vector $w = (w_e)_{e\in E}$ such that $K[M] = S/I_M$, where $S = K[X_e : e \in E]$ and $\mathrm{in}_w(I_M)$ is a monomial radical ideal. Thus, $\mathrm{in}_w(I_M) = I_\Delta$ for an abstract simplicial complex Δ on the vertex set E. Now standard results from Gröbner basis theory yield that $K[M]$ is Cohen–Macaulay if the Stanley–Reisner ring $K[\Delta] = S/I_\Delta$ is Cohen–Macaulay.

Observe that Δ is a triangulation of a cross-section of $\mathbb{R}_+M$. Now one can use, for example, a theorem of Munkres [7, 5.4.6], which states that the Cohen–Macaulay property of $K[\Delta]$ depends only on the topological type of $|\Delta|$. A cross-section of a pointed cone is homeomorphic to a simplex, whose Stanley–Reisner ring is certainly Cohen–Macaulay. $\square$

4. Betti Numbers of Toric Face Rings

A consequence of Proposition 2.3 is a presentation of a toric face ring $K[\mathcal{M}]$ over a polynomial ring S. It is a natural question to determine not only the Betti numbers of $K[\mathcal{M}]$ over S but also the graded Betti numbers (if there exists a natural grading). The first question is, of course, which grading is a natural one to consider: $\mathbb{Z}^d$ may not be the best choice to start with even if Σ is a fan in $\mathbb{R}^d$ and the monoids in $\mathcal{M}$ are embedded in $\mathbb{Z}^d$.

At first we recall a few facts from graded homological algebra. Let H be an (additive) commutative monoid that is positive; in other words, H has no invertible elements except 0. Usually one defines graded structures on rings and modules via groups. If H is *cancellative* (i.e., if $a+b=a+c$ implies $b=c$ for $a,b,c \in H$), then H can be naturally embedded into the abelian (Grothendieck) group G of H. One can thus define terms like H-graded by considering G-graded objects whose homogeneous components with degrees not in H are zero. But we will have to consider noncancellative monoids, so it may be impossible to embed H into a group.

Hence we introduce H-graded objects directly. Let R be a commutative ring and M an R-module (where we as always assume that R is commutative and not trivial). An *H-grading* of R is a decomposition $R = \bigoplus_{h \in H} R_h$ of R as abelian groups such that $R_h \cdot R_g \subseteq R_{h+g}$ for all $h, g \in H$. A graded ring together with an H-grading is called an *H-graded ring*. Now assume that R is an H-graded ring. A *grading* of M is a decomposition $M = \bigoplus_{h \in H} M_h$ of M as abelian groups such that $R_h \cdot M_g \subseteq M_{h+g}$ for all $h, g \in H$. An H-graded R-module M together with an H-grading is called an *H-graded module*; M_h is called the *h-homogeneous component* of M, and an element $x \in M_h$ is said to be *homogeneous* of degree $\deg x = h$.

From now on we assume that R is a Noetherian H-graded ring. The finitely generated (hereafter, f.g.) H-graded R-modules build a category. The morphisms are the homogeneous R-module homomorphisms $\varphi\colon M \to M'$ (i.e., $\varphi(M_h) \subseteq M'_h$ for all $h \in H$). For $h \in H$ we let $M(-h)$ be the H-graded R-module with homogeneous components $M(-h)_g = \bigoplus_{h' \in H,\, g = h'+h} M_{h'}$ for $g \in H$. In particular, $R(-h)$ is a free R-module of rank 1 with generator sitting in degree h. Since kernels of homogeneous maps of f.g. H-graded R-modules are again f.g. H-graded and since there exist f.g. free H-graded R-modules, it follows that every f.g. H-graded R-module has a free (hence projective) resolution

$$F_\bullet\colon\ \cdots \to F_n \to \cdots \to F_0 \to 0,$$

where F_n is a finite direct sum of free modules of the form $R(-h)$ for some $h \in H$ and where all maps are homogeneous and R-linear.

Next we want to pose a condition on H and specialize the considered class of rings. We say that H is *cancellative with respect to* 0 if $a+b=a$ implies $b=0$ for $a,b \in H$. Let K be a field. An H-graded K-algebra R is a Noetherian K-algebra $R=\bigoplus_{h\in H} R_h$ with $R_0=K$. Because H is positive, all homogeneous units of R must belong to R_0, and R has the unique H-graded maximal ideal $\mathfrak{m}=\bigoplus_{h\in H\setminus\{0\}} R_h$. We see that R is an H-graded local ring, a notion defined in the obvious way. Observe that $\mathfrak{m}$ is also maximal in R. The ring R behaves like a local ring by the following lemma.

LEMMA 4.1. *Assume that H is cancellative with respect to* 0 *and that R is an H-graded K-algebra. Then Nakayama's lemma holds*: *If M is a f.g. H-graded module and if $N \subseteq M$ is a f.g. H-graded submodule such that $M=N+\mathfrak{m}M$, then $M=N$. In particular, homogeneous elements $x_1,\dots,x_n$ are a minimal system of generators of M if and only if their residue classes constitute a K-vector space basis of $M/\mathfrak{m}M$, in which case we write $n=\mu(M_{\mathfrak{m}})$.*

Proof. We may assume without loss of generality that $N=0$. Now let $x_1,\dots,x_n$ be a minimal system of generators of homogeneous elements of M. Since $M=\mathfrak{m}M$ we have

$$x_n=\sum_{i=1}^{n} a_i x_i,$$

where $a_i \in \mathfrak{m}$. For all homogeneous components a_{ij} of some a_i, we may without loss of generality assume that $\deg x_n=\deg a_{ij}+\deg x_i$. Fix a homogeneous component a_{in} of a_n. Then $\deg x_n=\deg a_{in}+\deg x_n$ implies $\deg a_{in}=0$ because H is cancellative with respect to 0. Therefore, $a_{in}\in K\cap\mathfrak{m}$ and so $a_{in}=0$. Hence $a_n=0$ and x_n is a linear combination of $x_1,\dots,x_{n-1}$, in contradiction to the minimality of the system of generators. □

EXAMPLE 4.2. Let $F=\mathbb{N}^n$ for some $n\ge 0$, and set $\deg a=\sum a_i$ for $a\in F$. Let M be a quotient of F by a homogeneous congruence—that is, a congruence in which $x\sim y$ implies $\deg x=\deg y$. Then M is cancellative at 0, but in general it is not cancellative.

Now we can re-prove many well-known results from local and $\mathbb{Z}$-graded ring theory (see e.g. [7, Sec. 1.5]).

For example, let $x_1,\dots,x_n$ be a minimal system of generators of M, where $\deg x_i=h_i\in H$, and let $\varphi\colon F=\bigoplus_{i=1}^n R(-h_i)\to M$ be the homogeneous map sending the generator e_i of $R(-h_i)$ to x_i. Then we claim that $\operatorname{Ker}\varphi\subseteq\mathfrak{m}F$. Indeed, otherwise it follows that the residue classes of $x_1,\dots,x_n$ are not a K-vector space basis of $M/\mathfrak{m}M$ and thus, by Nakayama's lemma, $x_1,\dots,x_n$ is not a minimal system of generators. This is a contradiction. Consequently, there exist minimal H-graded free resolutions; that is, given a f.g. H-graded module M, there exists an H-graded free resolution $F_\bullet$ of M such that $\operatorname{Ker}\partial_n\subseteq\mathfrak{m}F_n$ for all n. Writing $F_n=\bigoplus_{h\in H} R(-h)^{\beta^R_{n,h}(M)}$, we call the $\beta^R_{n,h}(M)$ the *H-graded Betti numbers* of M. Up to homogeneous isomorphism of complexes, $F_\bullet$ is uniquely determined by the requirement that $\operatorname{Ker}\partial_n\subseteq\mathfrak{m}F_n$ for all n. The numbers $\beta^R_{n,h}(M)$ are also uniquely determined. Indeed, $\operatorname{Tor}_n^R(M, K)$ is an H-graded module considered as an R- or K-module (where $K = R/\mathfrak{m}$) and we have $\dim_K \operatorname{Tor}_n^R(M, K)_h = \beta_{n,h}^R(M)$, which is easily verified. More results in this direction can be easily verified: a f.g. H-graded R-module is projective if and only if it is free; $\operatorname{proj\,dim} M = \operatorname{proj\,dim} M_{\mathfrak{m}}$; and so forth.

Next we want to apply the theory discussed so far to the situation of toric face rings. Let $\mathcal{M}$ be a monoidal complex supported on a pointed conical complex Σ, and let $(a_e)_{e \in E}$ be a family of elements of $|\mathcal{M}|$ such that $\{a_e : e \in E\} \cap M_c$ generates M_c for each $c \in \Sigma$.

According to Proposition 2.3, the defining ideal $I_{\mathcal{M}}$ of the toric face ring $K[\mathcal{M}]$ of a monoidal complex is a sum

$$I_{\mathcal{M}} = A_{\mathcal{M}} + B_{\mathcal{M}},$$

where $A_{\mathcal{M}}$ is an ideal generated by squarefree monomials and $B_{\mathcal{M}}$ is a binomial ideal containing no monomials. This follows because every binomial generator vanishes on the vector $(1)_{e \in E}$ but a monomial here has value 1.

Recall that a congruence relation on a commutative monoid M is an equivalence relation $\sim$ such that, for $a, b, c \in M$ with $a \sim b$, we have $a + c \sim b + c$. Now $M/{\sim}$ is again a commutative monoid in a natural way.

Consider the free monoid $\mathbb{N}^E$ with generators f_e for $e \in E$. Note that $S = K[X_e : e \in E]$ is the monoid algebra of $\mathbb{N}^E$; the monomials in S are denoted by $X^a = \prod_{e \in E} X_e^{a_e}$. On $\mathbb{N}^E$ we define the congruence relation $a \sim b$ for $a, b \in \mathbb{N}^E$ if and only if $X^a - X^b \in B_{\mathcal{M}}$ is a binomial. We let $H_{\mathcal{M}}$ denote the monoid $\mathbb{N}^E/{\sim}$. It is well known and not hard to see that $S/B_{\mathcal{M}}$ is exactly the monoid algebra of the monoid $H_{\mathcal{M}}$.

Lemma 4.3. *$H_{\mathcal{M}}$ is a commutative positive monoid with monoid algebra $S/B_{\mathcal{M}}$.*

Proof. We need only show that $H_{\mathcal{M}}$ is positive. Let $\bar{g}, \bar{h} \in H_{\mathcal{M}}$ for $g, h \in \mathbb{N}^E$ such that $\overline{g+h} = \bar{0}$, and assume that $\bar{g}, \bar{h} \neq \bar{0}$. It follows from the definition of $H_{\mathcal{M}}$ that $X^{g+h} - 1 \in B_{\mathcal{M}}$. But $B_{\mathcal{M}}$ is generated by binomials that vanish on the zero vector $(0)_{e \in E}$ because all monoids M_c for $c \in \Sigma$ are positive. The binomial $X^{g+h} - 1$ does not vanish on (0), and this yields a contradiction. □

We saw that from the algebraic point of view it is useful for $H_{\mathcal{M}}$ to be cancellative with respect to 0. But this property is not strong enough for a combinatorial description of the Betti numbers, as a counterexample will show. The next lemma describes a stronger cancellation property for monoidal complexes associated with fans.

Lemma 4.4. *Assume that Σ is a rational pointed fan in $\mathbb{R}^n$ and that $\mathcal{M}$ is a monoidal complex supported on Σ such that $M_c \subseteq \mathbb{Z}^n$ for $c \in \Sigma$. (We do not require that $M_c = C_c \cap \mathbb{Z}^n$.)*

(i) *If $\bar{i} + \bar{j} = \bar{i} + \bar{k}$ for $\bar{i}, \bar{j}, \bar{k} \in H_{\mathcal{M}}$, where $i, j, k \in \mathbb{N}^E$, then $X^j - X^k \in I_{\mathcal{M}}$.*
(ii) *The monoid $H_{\mathcal{M}}$ is cancellative with respect to* 0.
(iii) *If $X^j - X^k \in I_{\mathcal{M}}$ and $X^j, X^k \notin I_{\mathcal{M}}$, then $\bar{i} = \bar{j}$ in $H_{\mathcal{M}}$.*

Proof. (i) Note that the toric face ring $K[\mathcal{M}]$ has a natural $\mathbb{Z}^n$-grading induced by the embeddings $M_c \subseteq \mathbb{Z}^n$ for $c \in \Sigma$. Then also the polynomial ring $K[X_e : e \in E]$ is $\mathbb{Z}^n$-graded if we give X_e the degree $a_e \in \mathbb{Z}^n$. Observe that the ideal $B_{\mathcal{M}}$ is then $\mathbb{Z}^n$-graded because the generators are homogeneous with respect to this grading. Then $K[X_e : e \in E]/B_{\mathcal{M}}$ is $\mathbb{Z}^n$-graded. Equivalently, we obtain the monoid homomorphism $\varphi\colon H_{\mathcal{M}} \to \mathbb{Z}^n$, $\bar{i} \mapsto \sum_{e\in E} i_e a_e$.

Now $\bar{i} + \bar{j} = \bar{i} + \bar{k}$ implies that $\sum_{e\in E}(i_e + j_e)a_e = \sum_{e\in E}(i_e + k_e)a_e$ in $\mathbb{Z}^n$, and thus $\sum_{e\in E} j_e a_e = \sum_{e\in E} k_e a_e$. It follows that $X^j - X^k \in \operatorname{Ker}(K[X_e : e \in E] \to K[\mathcal{M}]) = I_{\mathcal{M}}$.

(ii) It follows from (i) that $H_{\mathcal{M}}$ is cancellative with respect to 0, because $X^j - 1 \notin I_{\mathcal{M}}$.

(iii) If $X^j - X^k \in I_{\mathcal{M}}$ and $X^j, X^k \notin I_{\mathcal{M}}$, then $X^j - X^k \in B_{\mathcal{M}}$ by the last observation of Proposition 2.3. Hence $\bar{i} = \bar{j}$ in $H_{\mathcal{M}}$. □

Let $S = K[X_e : e \in E]$. Observe that all rings S, $S/B_{\mathcal{M}}$, and $K[\mathcal{M}]$ are naturally $H_{\mathcal{M}}$-graded. For S we set $\deg X^i = \bar{i} \in H_{\mathcal{M}}$. Since $X^a - 1 \notin I_{\mathcal{M}}$ we have that $H_{\mathcal{M}}$ is positive, and S is an H-graded local ring. If $H_{\mathcal{M}}$ is cancellative with respect to 0, then one can apply Lemma 4.1 to f.g. $H_{\mathcal{M}}$-graded S-modules like $K[\mathcal{M}]$. In particular, we can speak about minimal $H_{\mathcal{M}}$-graded resolutions. Our next goal is to determine the corresponding $H_{\mathcal{M}}$-graded Betti numbers.

For $\bar{h} \in H_{\mathcal{M}}$ we define

$$\Delta_{\bar{h}} = \left\{ F \subseteq E : \bar{h} = \bar{g} + \sum_{e\in F} \bar{f}_e \text{ for some } \bar{g} \in H_{\mathcal{M}} \right\}.$$

We see immediately that $\Delta_{\bar{h}}$ is a simplicial complex whose vertex set is a subset of E; we call $\Delta_{\bar{h}}$ the *squarefree divisor complex* of $\bar{h}$. Moreover, we need a special subcomplex of $\Delta_{\bar{h}}$ defined as follows:

$$\Delta_{\bar{h},\mathcal{M}} = \left\{ F \subseteq E : \bar{h} = \bar{g} + \sum_{e\in F} \bar{f}_e \text{ for some } \bar{g} \in H_{\mathcal{M}} \text{ such that } X^g \in I_{\mathcal{M}} \right\}.$$

For an arbitrary simplicial complex Δ on some ordered vertex set E (with order $<$), let $\tilde{\mathcal{C}}(\Delta)$ denote the augmented oriented chain complex of Δ with coefficients in K—that is, the complex

$$\tilde{\mathcal{C}}_{\bullet}(\Delta)\colon 0 \to \mathcal{C}_{\dim\Delta} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{C}_0 \xrightarrow{\partial} \mathcal{C}_{-1} \to 0,$$

where $\mathcal{C}_i = \bigoplus_{F\in\Delta,\,\dim F = i} KF$ and $\partial(F) = \sum_{F'\in\Delta,\,\dim F' = i-1} \varepsilon(F,F')F'$. Here $\varepsilon(F,F')$ is 0 if $F' \not\subseteq F$. Otherwise it is $(-1)^k$ if $F = \{e_0,\dots,e_i\}$ for elements $e_0 < \cdots < e_i$ in E, and $F' = \{e_0,\dots,e_{k-1},e_{k+1},\dots,e_i\}$. Further we let $\tilde{H}(\Delta)_i = H_i(\tilde{\mathcal{C}}(\Delta)_{\bullet})$ be the ith reduced simplicial homology group of Δ. If Δ' is a subcomplex of Δ, then we denote by $\tilde{\mathcal{C}}(\Delta,\Delta')_{\bullet} = \tilde{\mathcal{C}}(\Delta)_{\bullet}/\tilde{\mathcal{C}}(\Delta')_{\bullet}$ the relative augmented oriented chain complex of Δ and Δ' and by $H_i(\Delta,\Delta')$ the ith homology of this complex.

THEOREM 4.5. *Let $\mathcal{M}$ be a monoidal complex supported on a rational pointed conical complex Σ. Let $(a_e)_{e\in E}$ be a family of elements of $|\mathcal{M}|$ such that $\{a_e : e \in E\} \cap M_c$ generates M_c for each $c \in \Sigma$.*

(i) *For $\bar{h} \in H_{\mathcal{M}}$ such that $K[\mathcal{M}]_{\bar{h}} \neq 0$ (equivalently, for $\sum_e h_e a_e \in |\mathcal{M}_\Sigma|$) and $i \in \mathbb{N}$, we have*

$$\beta^S_{i\bar{h}}(K[\mathcal{M}]) = \dim_K \tilde{H}_{i-1}(\Delta_{\bar{h}}).$$

Moreover, if $\sum_e h_e a_e \in M_c$ for some $c \in \Sigma$ then $\beta^S_{i\bar{h}}(K[\mathcal{M}]) = \beta^S_{i\bar{h}}(K[M_c])$.

(ii) *Suppose that Σ is a rational pointed fan in $\mathbb{R}^n$ and that $M_c \subseteq \mathbb{Z}^n$ for $c \in \Sigma$. For $\bar{h} \in H_{\mathcal{M}}$ such that $K[\mathcal{M}]_{\bar{h}} = 0$ and $i \in \mathbb{N}$, we have*

$$\beta^S_{i\bar{h}}(K[\mathcal{M}]) = \dim_K \tilde{H}_{i-1}(\Delta_{\bar{h}}, \Delta_{\bar{h},\mathcal{M}}).$$

Proof. Let $S = K[X_e : e \in E]$. We fix an arbitrary total order $<$ on E. Let $\mathcal{K}_\bullet(K[\mathcal{M}])$ denote the Koszul complex of X_e ($e \in E$) tensored with $K[\mathcal{M}]$. This complex is naturally $H_{\mathcal{M}}$-graded, and its $H_{\mathcal{M}}$-graded homology is exactly $\operatorname{Tor}_\bullet(K, K[\mathcal{M}])$ (see e.g. [7] for details). Hence we can use this complex to determine the numbers $\beta^S_{i\bar{h}}(K[\mathcal{M}])$ in (i) and (ii). We have

$$\mathcal{K}_i(K[\mathcal{M}]) = \bigoplus_{F\subseteq E,\,|F|=i} K[\mathcal{M}]\Big(-\sum_{e\in F} \bar{f}_e\Big),$$

and the differential $\partial_i : \mathcal{K}_i(K[\mathcal{M}]) \to \mathcal{K}_{i-1}(K[\mathcal{M}])$ is given on the component

$$K[\mathcal{M}]\Big(-\sum_{e\in F} \bar{f}_e\Big) \to K[\mathcal{M}]\Big(-\sum_{e\in F} \bar{f}_e\Big)$$

for $F', F \subseteq E$ as the zero map for $F' \not\subseteq F$ or otherwise as the multiplication $\varepsilon(F, F')X_{e_k}$, where

$$\varepsilon(F, F') = \begin{cases} 0 & \text{if } F' \not\subseteq F, \\ (-1)^{k-1} & \text{if } F = \{e_1 < \cdots < e_i\},\ F' = F \setminus \{e_k\}. \end{cases}$$

For $\beta^S_{i\bar{h}}(K[\mathcal{M}])$ we must first determine $\mathcal{K}_i(K[\mathcal{M}])_{\bar{h}}$. Thus we compute

$$K[\mathcal{M}]\Big(-\sum_{e\in F} \bar{f}_e\Big)_{\bar{h}} = \bigoplus_{\bar{h}'\in H_{\mathcal{M}},\, \bar{h}'+\sum_{e\in F}\bar{f}_e=\bar{h}} K[\mathcal{M}]_{\bar{h}'}. \tag{3}$$

We remark that such an $\bar{h}'$ exists if and only if $F \in \Delta_{\bar{h}}$.

In case (i) we assume that $K[\mathcal{M}]_{\bar{h}} \neq 0$ (i.e., $X^h \notin I_{\mathcal{M}}$). It follows from $\bar{h}' + \sum_{e\in F} \bar{f}_e = \bar{h}$ that $X^h - X^{h'} \cdot \prod_{e\in F} X_e \in I_{\mathcal{M}}$. Applying the last statement of Proposition 2.3, we see that there exists a $c \in \Sigma$ such that $a_e \in M_c$ if at least one of the numbers h_e, h'_e is not zero or if $e \in F$. But this situation coincides with the computation of $\beta^S_{i\bar{h}}(K[M_c])$, so we obtain $\beta^S_{i\bar{h}}(K[\mathcal{M}]) = \beta^S_{i\bar{h}}(K[M_c])$. We can replace Σ by the fan associated to C and $\mathcal{M}_\Sigma$ by its restriction to this fan. Note that we could replace M_c by any M_d such that $\sum_e h_e a_e \in M_d$, which takes care of the last assertion of (i). In particular, we may assume that Σ is a rational pointed fan in $\mathbb{R}^n$.

Since $X^h - X^{h'} \cdot \prod_{e\in F} X_e \in I_{\mathcal{M}}$ and since $X^h \notin I_{\mathcal{M}}$, it follows from Proposition 2.3 that $X^{h'} \notin I_{\mathcal{M}}$. Assume that there exists another $\bar{h}'' \in H_{\mathcal{M}}$ such that $\bar{h}''+\sum_{e\in F} \bar{f}_e = \bar{h}$ and $X^{h''} \notin I_{\mathcal{M}}$. We obtain from Lemma 4.4 and $\bar{h}'+\sum_{e\in F} \bar{f}_e = \bar{h}'' + \sum_{e\in F} \bar{f}_e$ that $X^{h'} - X^{h''} \in B_{\mathcal{M}}$. Hence $\bar{h}' = \bar{h}''$ in $H_{\mathcal{M}}$; in other words, $\bar{h}'$ is uniquely determined if it exists. Hence we have

$$\mathcal{K}_i(K[\mathcal{M}]) \cong \bigoplus_{F\in\Delta_{\bar{h}},\,|F|=i} KF$$

in case (i).

We now consider $F' \in \Delta_{\bar{h}}$ such that $|F'| = i-1$. Begin by choosing $\bar{h}'$ such that $\bar{h}' + \sum_{e\in F} \bar{f}_e = \bar{h}$ and choosing $\bar{h}''$ such that $\bar{h}'' + \sum_{e\in F'} \bar{f}_e = \bar{h}$. The differential $\mathcal{K}_i(K[\mathcal{M}]) \to \mathcal{K}_{i-1}(K[\mathcal{M}])$ on the component $KF \to KF'$ (which corresponds to $K[\mathcal{M}]_{\bar{h}'} \to K[\mathcal{M}]_{\bar{h}''}$) is given by

$$\partial_i(F) = \begin{cases} 0 & \text{if } F' \not\subseteq F, \\ \varepsilon(F,F')F' & \text{if } F = \{e_1 < \cdots < e_i\},\ F' = F \setminus \{e_k\}. \end{cases}$$

Hence we see that the complex $\mathcal{K}_\bullet(K[\mathcal{M}])_{\bar{h}}$ coincides with $\tilde{\mathcal{C}}_{\bullet-1}(\Delta_{\bar{h}})$, and this yields

$$\beta^S_{i\bar{h}}(K[\mathcal{M}]) = \dim_K \tilde{H}_{i-1}(\Delta_{\bar{h}}).$$

It remains to prove (ii). By hypothesis, Σ is a rational pointed fan in $\mathbb{R}^n$ at the outset, and we take $\bar{h} \in H_{\mathcal{M}}$ such that $K[\mathcal{M}]_{\bar{h}} = 0$ (i.e., $X^h \in I_{\mathcal{M}}$). We again consider equation (3) to compute $\beta^S_{i\bar{h}}(K[\mathcal{M}])$. We still have that $\bar{h}'$ with $\bar{h}' + \sum_{e\in F} \bar{f}_e = \bar{h}$ exists if and only if $F \in \Delta_{\bar{h}}$. Such an $\bar{h}'$ is again uniquely determined by Lemma 4.4, as can be seen analogously to case (i). (Note that here we need the assumption that Σ is a fan.)

Now $K[\mathcal{M}]_{\bar{h}'} = 0$ if and only if $F \in \Delta_{\bar{h},\mathcal{M}}$. Hence

$$\mathcal{K}_i(K[\mathcal{M}]) \cong \bigoplus_{F\in\Delta_{\bar{h}}\setminus\Delta_{\bar{h},\mathcal{M}},\,|F|=i} KF.$$

Similarly to the proof of (i), we see that the complex $\mathcal{K}_\bullet(K[\mathcal{M}])_{\bar{h}}$ coincides with the complex $\tilde{\mathcal{C}}_{\bullet-1}(\Delta_{\bar{h}}, \Delta_{\bar{h},\mathcal{M}})$. It follows that

$$\beta^S_{i\bar{h}}(K[\mathcal{M}]) = \dim_K \tilde{H}_{i-1}(\Delta_{\bar{h}}, \Delta_{\bar{h},\mathcal{M}}),$$

and this concludes the proof. □

One can easily generalize Theorem 4.5(ii) in the following way. If $\mathcal{M}$ satisfies the properties of Lemma 4.4, then the proof of Theorem 4.5(ii) works for $\mathcal{M}$. However, in general one cannot expect the compact combinatorial formula to be true for all monoidal complexes without any further assumptions. Indeed, we have the following counterexample.

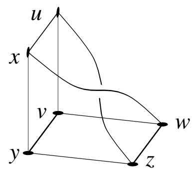

Figure 1 Möbius strip as a monoidal complex

EXAMPLE 4.6. Consider the Möbius strip as a monoidal complex $\mathcal{M}$ (see Figure 1) by viewing each quadrangle as a unit square and choosing the monoid over it as the corresponding monoid. Together with the compatibility conditions, this determines $\mathcal{M}$ completely.

The ideal $I_{\mathcal{M}}$ is generated by the binomials resulting from the unit squares and monomials

$$X_xX_z - X_uX_w,\ X_yX_w - X_vX_z,\ X_xX_v - X_uX_y,\ X_uX_vX_w,\ \text{and}\ X_uX_vX_z.$$

The other monomials are redundant (e.g., $X_uX_yX_z = X_z(X_uX_y - X_xX_v) + X_v(X_xX_z - X_uX_w) + X_vX_uX_w$). Because the binomial relations are homogeneous, $H_{\mathcal{M}}$ is cancellative with respect to 0.

Let x_a stand for the residue class of X_a, and choose the degree

$$\bar{h} = x_ux_vx_z = x_ux_wx_y = x_xx_yx_z = x_vx_wx_x \in H_{\mathcal{M}}.$$

This equation shows that Lemma 4.4 does not hold for $\mathcal{M}$.

Since $X^h \in I_{\mathcal{M}}$, it follows that $K[\mathcal{M}]_{\bar{h}} = 0$. The degree-$\bar{h}$ component of the Koszul complex is

$$\mathcal{K}_\bullet(K[\mathcal{M}])_{\bar{h}} \colon 0 \to K^4 \to K^{12} \to K^9 \to 0 \to 0,$$

where K^9 is in homological degree 1, and the Betti numbers are

$$\beta^S_{i\bar{h}}(K[\mathcal{M}]) = \begin{cases} 1 & \text{for } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now we consider the complex $\tilde{\mathcal{C}}_\bullet(\Delta_{\bar{h}}, \Delta_{\bar{h},\mathcal{M}})$, which is given by

$$0 \to K^4 \to K^{12} \to K^6 \to 0 \to 0$$

with K^6 in homological degree 0. Hence $\tilde{H}_1(\Delta_{\bar{h}}, \Delta_{\bar{h},\mathcal{M}}) = K^d$ for some $d \geq 2$, and the formula of Theorem 4.5(ii) does not hold in this case.

The results of this section imply, in particular, the well-known Tor formula of Hochster for Stanley–Reisner rings (see [9]). Recall that if Δ is a simplicial complex on the vertex set $[n]$ and if $S = K[X_1, \dots, X_n]$, then $K[\Delta] = S/I_\Delta$ is the Stanley–Reisner ring of Δ, where $I_\Delta = \big(\prod_{i \in F} X_i : F \notin \Delta\big)$ is the Stanley–Reisner ideal of Δ. Now all considered rings have a natural $\mathbb{Z}^n$-grading. It is well known that $\beta^S_{ia}(K[\Delta]) = 0$ if a is not a squarefree vector—that is, if a is not a 0–1 vector. (One can either show this via the results of this section or prove it directly.) For a squarefree vector a with support $W = \{i \in [n] : a_i = 1\}$, we write $\beta^S_{iW}(K[\Delta]) = \beta^S_{ia}(K[\Delta])$ for the corresponding Betti number.

COROLLARY 4.7 (Hochster). *Let Δ be a simplicial complex on the vertex set $[n]$. Then, for $W \subseteq [n]$,*

$$\beta^S_{iW}(K[\Delta]) = \dim_K \tilde{H}_{|W|-i-1}(\Delta_W),$$

where $\Delta_W = \{F \in \Delta : F \subseteq W\}$.

Proof. In Example 2.4 it was observed that there exists a rational pointed fan Σ and an embedded monoidal complex $\mathcal{M}$ such that $K[\mathcal{M}] = K[\Delta]$. Hence the binomial ideal $B_{\mathcal{M}}$ is 0, and $I_{\mathcal{M}} = I_\Delta$ is generated by squarefree monomials. The monoid $H_{\mathcal{M}}$ is nothing but the free monoid $\mathbb{N}^n$ in this case; thus, the induced grading is just the natural $\mathbb{N}^n$-grading on $K[\Delta]$. It remains to observe that the complex $\tilde{\mathcal{C}}_{\bullet-1}(\Delta_{\bar{h}}, \Delta_{\bar{h},\mathcal{M}})$ coincides with the complex $\tilde{\mathcal{C}}_{|W|-\bullet-1}(\Delta_W)$, which determines the homology $\tilde{H}_{|W|-i-1}(\Delta_W)$. This concludes the proof. □

REMARK 4.8. Let Δ be a simplicial complex on $[n]$. Hochster computed also the local cohomology of the Stanley–Reisner ring as a $\mathbb{Z}^n$-graded K-vector space in terms of combinatorial data of the given complex (see e.g. [7; 13]). Whereas the Tor formula for all cases is restricted to embedded monoidal complexes (or to complexes that behave like these), one can prove a Hochster formula for the local cohomology in great generality. In fact, the goal of showing such a formula for toric face rings was one of the starting points for the systematic study of such rings. See [10] for the case of embedded monoidal complexes and [1] for classes of rings that include toric face rings as a special case.

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