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Lower Bounds for Hilbert–Kunz Multiplicities in Local Rings of Fixed Dimension

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Dedicated to Mel Hochster on the occasion of his 65th birthday

1. Introduction

Let $(R, \mathfrak{m}, k)$ be a local ring of positive characteristic p—that is, quasi-local (only one maximal ideal) and Noetherian. Let $q = p^e$, where e is a nonnegative integer. For any ideal I of R we denote $I^{[q]} = (i^q : i \in I)$.

For an $\mathfrak{m}$-primary ideal I, one can consider the Hilbert–Samuel multiplicity and the Hilbert–Kunz multiplicity of I with respect to R.

Definition 1.1. Let I be an $\mathfrak{m}$-primary ideal in $(R, \mathfrak{m})$. Let $\lambda(\cdot)$ denote the usual length function.

1. *The Hilbert–Samuel multiplicity of R at I* is defined by

$$\mathrm{e}(I) = \mathrm{e}(I; R) := \lim_{n\to\infty} d!\,\frac{\lambda(R/I^n)}{n^d}.$$

   The limit exists and is a positive integer.
2. *The Hilbert–Kunz multiplicity of R at I* is defined by

$$\mathrm{e}_{\mathrm{HK}}(I) = \mathrm{e}_{\mathrm{HK}}(I; R) := \lim_{q\to\infty} \frac{\lambda(R/I^{[q]})}{q^d}.$$

   Monsky has shown that the latter limit exists and is positive.

The Hilbert–Samuel multiplicty of R, denoted $\mathrm{e}(R)$, is by definition $\mathrm{e}(\mathfrak{m})$. Similarly, the Hilbert–Kunz multiplicity of R, denoted $\mathrm{e}_{\mathrm{HK}}(R)$, is $\mathrm{e}_{\mathrm{HK}}(\mathfrak{m})$.

It is known that, for parameter ideals I, one has $\mathrm{e}(I) = \mathrm{e}_{\mathrm{HK}}(I)$. The following sequence of inequalities is also known to hold whenever I is $\mathfrak{m}$-primary:

$$\max\left\{1, \frac{\mathrm{e}(I)}{d!}\right\} \le \mathrm{e}_{\mathrm{HK}}(I) \le \mathrm{e}(I).$$

We call a local ring R *formally unmixed* if $\hat{R}$ is equidimensional and $\mathrm{Min}(\hat{R}) = \mathrm{Ass}(\hat{R})$—in other words, $\dim(\hat{R}/P) = \dim(\hat{R})$ for all its minimal primes P and if all associated primes of $\hat{R}$ are minimal. Nagata calls such rings *unmixed*. However, throughout our paper, a local unmixed ring is a local ring R that is equidimensional and for which $\mathrm{Min}(R) = \mathrm{Ass}(R)$.

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In this paper we investigate rings that have small Hilbert–Kunz multiplicity. It is known that a formally unmixed local ring of characteristic p is regular if and only if $\mathrm{e_{HK}}(R) = 1$. In fact, similar statements that hold true for the Hilbert–Samuel multiplicity are considered classical. (The unmixedness assumption is essential because there are examples of nonregular rings that are not formally unmixed with $\mathrm{e_{HK}}(R) = 1$; the reason is that neither Hilbert–Samuel multiplicity nor Hilbert–Kunz multiplicity can pick up lower-dimensional components of $\hat{R}$.) Since $\mathrm{e}(R)$ is always a positive integer, it follows that $\mathrm{e}(R) \geq 2$ if R is formally unmixed but not regular. The situation is much more subtle with the Hilbert–Kunz multiplicity because it often takes on noninteger values. So the question becomes: If one fixes the dimension d, how close to 1 can $\mathrm{e_{HK}}(R)$ be (when R is formally unmixed but not regular)? What can be said about the structure of rings of small Hilbert–Kunz multiplicity? This problem has been intensively studied in recent years (with success mostly for rings of small dimension) by Blickle and Enescu [3], Enescu and Shimomoto [5], and Watanabe and Yoshida [15; 16; 17]. In this paper we develop techniques that shed light on this problem independent of dimension. We show that if R is not regular then there exists a lower bound—strictly greater than 1 and depending only on d—for its Hilbert–Kunz multiplicity.

The goal is at least twofold: find the constants

$$\varepsilon_{\mathrm{HK}}(d,p) = \inf\{\mathrm{e_{HK}}(R) - 1 : R \text{ nonregular, formally unmixed, } \dim R = d, \operatorname{char} R = p\}$$

and

$$\varepsilon_{\mathrm{HK}}(d) = \inf\{\varepsilon_{\mathrm{HK}}(d,p) : p > 0\}$$

(as introduced in [3]) and then describe the structure of the rings with small Hilbert–Kunz multiplicity from both an algebraic and geometric point of view.

It is known that $\varepsilon_{\mathrm{HK}}(d,p) \geq 1/(d!\,p^d)$ by results in [3]. However, as $p \to \infty$, the right-hand side clearly tends toward 0, so this does not give a positive lower bound for $\varepsilon_{\mathrm{HK}}(d)$. A by-product of our work is that it leads to a proof that $\varepsilon_{\mathrm{HK}}(d) > 0$, answering positively a problem raised in [3, Sec. 3]. We should mention that a conjecture of Watanabe and Yoshida [17] asserts that if $(R, \mathfrak{m}, k)$ has residue field equal to $\overline{\mathbf{F}_p}$, $p > 2$, then $\mathrm{e_{HK}}(R) \geq \mathrm{e_{HK}}(R_{p,d})$, where $R_{p,d} = \overline{\mathbf{F}_p}[[x_0,\dots,x_d]]/(x_0^2 + \cdots + x_d^2)$. This conjecture has been answered positively for dimensions $d = 1,2,3,4$ (the difficult cases of dimensions 3 and 4 are due to Watanabe and Yoshida) and in the case of complete intersections by Enescu and Shimomoto [5].

The starting point of our investigation is the following theorem.

THEOREM 1.2 (Blickle–Enescu). *Let R be an unmixed d-dimensional ring that is a homomorphic image of a Cohen–Macaulay local ring of characteristic $p > 0$. Let $d \geq 2$. If*

$$\mathrm{e_{HK}}(R) \leq 1 + \max\{1/d!, 1/\mathrm{e}(R)\},$$

then R is Cohen–Macaulay and F-rational.

REMARK 1.3. The proof of this result shows that, in fact, the inequality $\mathrm{e}_{\mathrm{HK}}(R) < \frac{\mathrm{e}(R)}{\mathrm{e}(R)-1}$ forces R to be Cohen–Macaulay and F-rational.

In fact, the hypotheses of Theorem 1.2 suffice to show that R must be (strongly) F-regular. This is the content of Corollary 3.6, reprinted here for convenience.

COROLLARY. *Let $(R, \mathfrak{m}, k)$ be a formally unmixed ring of characteristic p with* $\dim(R) = d \geq 2$. *If* $\mathrm{e}_{\mathrm{HK}}(R) \leq 1 + \max\{1/d!, 1/\mathrm{e}(R)\}$, *then R is F-regular and Gorenstein. If R is excellent, then R is strongly F-regular.*

Theorem 4.12 gives a positive lower bound for $\varepsilon(d)$ that does not depend on p, as follows.

THEOREM. *Let $(R, \mathfrak{m}, k)$ be a formally unmixed local ring of positive characteristic p and dimension $d \geq 2$. If R is not regular, then*

$$\mathrm{e}_{\mathrm{HK}}(R) \geq 1 + \frac{1}{d \cdot (d!(d-1)+1)^d}.$$

This result shows that $\varepsilon(d) > 0$, but our techniques can be refined to give sharper estimates. In a future paper we will give results that are considerably better but at the cost of much more technical arguments; we have opted here to give instead a more accessible proof of the fact that such an $\varepsilon(d)$ exists. Although the aforementioned conjecture of Watanabe and Yoshida is still open, we have developed techniques that—for the first time—work regardless of dimension or additional hypotheses on the rings.

In dealing with Hilbert–Kunz multiplicities it often useful to assume that the rings under study are either formally unmixed or unmixed and homomorphic images of Cohen–Macaulay rings. This will also be the case in our paper.

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2. Definitions and Known Results

First we would like to review some definitions and results that will be useful later. Throughout the paper, R will be a Noetherian ring containing a field of characteristic p, where p is prime. Also, q will denote p^e, a varying power of p.

If I is an ideal in R then $I^{[q]} = (i^q : i \in I)$, where $q = p^e$ is a power of the characteristic. Let $R^\circ = R \setminus \bigcup P$, where P runs over the set of all minimal primes of R. An element x is said to belong to the *tight closure* of the ideal I if there exists a $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all sufficiently large $q = p^e$. The tight closure of I is denoted by I^*. By a *parameter ideal* we mean an ideal generated by a full system of parameters in a local ring R. A tightly closed ideal of R is an ideal I such that $I = I^*$.

Let $F\colon R \to R$ be the Frobenius homomorphism $F(r) = r^p$. We denote by F^e the eth iteration of F; that is, $F^e(r) = r^q$ and $F^e\colon R \to R$. One can regard R as an R-algebra via the homomorphism F^e. As an abelian group this R-algebra equals R, but it has a different scalar multiplication. We will denote this new algebra by $R^{(e)}$. For an R-module M we let $F^e(M) = R^{(e)} \otimes_R M$, where we consider this an R-module via $R^{(e)}$; in other words, $a(r \otimes m) = (ar) \otimes m$, but $r \otimes (am) = a^q r \otimes m$. For an element $m \in M$, let $m^q = 1 \otimes m \in F^e(M)$. If $N \subseteq M$ then we denote the image of $F^e(N)$ in $F^e(M)$ by $N^{[q]}$, and this is the same as the submodule of $F^e(M)$ generated by the elements n^q for $n \in N$. We then say that $x \in M$ is in the tight closure of N in M, denoted N^*_M, if there exists a $c \in R^0$ such that $cx^q \in N^{[q]}$ for all $q \gg 0$.

DEFINITION 2.1. R is *F-finite* if $R^{(1)}$ is module finite over R or, equivalently (in the case that R is reduced), if $R^{1/p}$ is module finite over R. Also, R is called *F-pure* if the Frobenius homomorphism is a pure map (i.e., if $F \otimes_R M$ is injective for every R-module M).

If R is F-finite, then $R^{1/q}$ is module finite over R for every q. Moreover, any quotient and localization of an F-finite ring is F-finite. Any finitely generated algebra over a perfect field is F-finite. An F-finite ring is excellent.

DEFINITION 2.2. A reduced Noetherian F-finite ring R is *strongly F-regular* if, for every $c \in R^0$, there exists a q such that the R-linear map $R \to R^{1/q}$ sending 1 to $c^{1/q}$ splits over R or, equivalently, if $Rc^{1/q} \subset R^{1/q}$ splits over R.

The notion of strong F-regularity localizes well, and all ideals are tightly closed in strongly F-regular rings. Regular rings are strongly F-regular, and strongly F-regular rings are Cohen–Macaulay and normal.

Let $E_R(k)$ be the injective hull of the residue field of R. Then a reduced F-finite ring R is strongly F-regular if and only if $0^*_{E_R} = 0$, (see e.g. [14, 7.1.2]). More generally, when $(R, \mathfrak{m})$ is reduced and excellent (but not necessarily F-finite), we will say that R is strongly F-regular if $0^*_{E_R} = 0$.

DEFINITION 2.3. A ring R is called F-rational if all parameter ideals are tightly closed. A ring R is called weakly F-regular if all ideals are tightly closed. The ring R is F-regular if and only if $S^{-1}R$ is weakly F-regular for all multiplicative sets $S \subset R$.

Regular rings are (strongly) F-regular. For Gorenstein rings, the notions of F-rationality and F-regularity coincide (and if, in addition, the ring is excellent, then they coincide with strong F-regularity).

DEFINITION 2.4. Let $I \subseteq J$ be two $\mathfrak{m}$-primary ideals in $(R, \mathfrak{m}, k)$, and let M be a finitely generated R-module. Then the Hilbert–Kunz multiplicity of I on M is

$$\mathrm{e}_{\mathrm{HK}}(I; M) = \lim_{q \to \infty} \frac{1}{q^d} \lambda(M/I^{[q]}M),$$

and the relative Hilbert–Kunz multiplicity of I and J on M is

$$\mathrm{e}_{\mathrm{HK}}(I, J; M) = \mathrm{e}_{\mathrm{HK}}(I; M) - \mathrm{e}_{\mathrm{HK}}(J; M).$$

When $M = R$, we simply drop M from the notation.

PROPOSITION 2.5 (Associativity formula; see [17, Prop. 1.2(5)]). *Let* $(R, \mathfrak{m}, k)$ *be a local ring and* I *an* $\mathfrak{m}$*-primary ideal of* R. *Denote* $\mathrm{Assh}(R) = \{P \in \mathrm{Ass}(R) : \dim(R/P) = \dim(R)\}$. *Then*

$$\mathrm{e}_{\mathrm{HK}}(I; M) = \sum_{P \in \mathrm{Assh}(R)} \lambda_{R_P}(M_P) \cdot \mathrm{e}_{\mathrm{HK}}(I; R/P).$$

REMARK 2.6. The associativity formula immediately implies that if $\mathrm{e}_{\mathrm{HK}}(R) < 2$ then $\mathrm{Assh}(R)$ contains one element, and if this element is the prime P then the P-primary component of 0 is P. Thus, if R is unmixed and $\mathrm{e}_{\mathrm{HK}}(R) < 2$, then R is a domain.

We will also need the following technical notion.

DEFINITION 2.7. Let $(R, \mathfrak{m}, k)$ be a local ring of positive characteristic p, and let $J \subset I$ be $\mathfrak{m}$-primary ideals. Define the *star length* of J in I, denoted $\lambda^*(I/J)$, to be the minimum length n of a sequence of ideals

$$J^* = I_0 \subset I_1 \subset \cdots \subset I_n = I^*$$

such that, for each k, we have $I_{k+1} = (I_k, x_k)^*$ for some element x_k with $\mathfrak{m}x_k \subset I_k$.

The definition of star length was introduced by Hanes [6], who also noted some of the following basic properties of the star length function.

PROPOSITION 2.8. *Let* $J \subset I$ *be any* $\mathfrak{m}$*-primary ideals of a local ring* $(R, \mathfrak{m}, k)$ *of prime characteristic* $p > 0$. *Then the following statements hold.*

(a) $\lambda^*(I/J) \le \lambda(I/J)$ *and* $\lambda^*(I/J) = \lambda^*(I^*/J^*)$.
(b) $\mathrm{e}_{\mathrm{HK}}(J) \le \mathrm{e}_{\mathrm{HK}}(I) + \lambda^*(I/J)\,\mathrm{e}_{\mathrm{HK}}(R)$; *moreover,* $\mathrm{e}_{\mathrm{HK}}(J) \le \lambda^*(R/J)\,\mathrm{e}_{\mathrm{HK}}(R)$.

Our next proposition offers a natural characterization of strong F-regularity in terms of the relative Hilbert–Kunz multiplicity.

PROPOSITION 2.9. *Let* $(R, \mathfrak{m}, k)$ *be an excellent local ring. Then the following are equivalent*:

(1) R *is strongly F-regular*;
(2) $\inf\{\mathrm{e}_{\mathrm{HK}}(I, J) \mid I \subsetneq J\} > 0$;
(3) $\inf\{\mathrm{e}_{\mathrm{HK}}(I, (I, x)) \mid I$ *is* $\mathfrak{m}$*-primary and irreducible and* x *is a socle element modulo* $I\} > 0$.

Proof. Let $E_R(k) = E$. By [2, Thm. 0.2], R is strongly F-regular if and only if $\liminf \lambda(R/0 :_{F^e(E)} u^q)/q^d > 0$ (the theorem is stated there for F-finite rings, but the proof works also in the excellent case).

We first show that (1) implies (3). Let $I \subseteq R$ be irreducible and $\mathfrak{m}$-primary, and let x be a socle element modulo I. There is then an injection $R/I \hookrightarrow E$ sending x to u. Applying Frobenius gives a map $R/I^{[q]} \to F^e(E)$ sending x^q to u^q, from which it is clear that $I^{[q]} : x^q \subseteq 0 :_{F^e(E)} u^q$. Hence $\mathrm{e}_{\mathrm{HK}}(I,(I,x)) \geq \liminf \lambda(R/0 :_{F^e(E)} u^q)/q^d > 0$.

To see that (3) implies (2), we first note that it suffices to take $J = (I, y)$ for a socle element y modulo I. In this case we can embed $R/I \hookrightarrow R/I_1 \oplus \cdots \oplus R/I_t$, where each I_n is irreducible, and $y \mapsto (x, 0, \ldots, 0)$, where x is the socle element modulo I_1. It is then clear, after applying Frobenius, that $\mathrm{e}_{\mathrm{HK}}(I, J) \geq \mathrm{e}_{\mathrm{HK}}(I_1, (I_1, x))$.

Clearly, (2) implies (3).

Suppose that (3) holds but that R (of dimension d) is not strongly F-regular. Choose $c \in R^0$ such that $cu^q = 0$ in $F^e(E)$ for all q. Then $\dim R/cR = d - 1$. Let $\mathrm{e}_1 = \mathrm{e}_{\mathrm{HK}}(R)$ and $\mathrm{e}_2 = \mathrm{e}_{\mathrm{HK}}(R/cR)$. Fix q_0 such that

$$\lambda(R/(c, \mathfrak{m}^{[q_0]})) \leq (\mathrm{e}_2 + 1)q_0^{d-1}.$$

Because $cu^{q_0} = 0$, we can choose an irreducible ideal I with socle representative x such that $cx^{q_0} \in I^{[q_0]}$. Since $\mathfrak{m}x \subseteq I$, we see that $(\mathfrak{m}^{[q_0]}, c)^{[q]} x^{q_0 q} \subseteq I^{[q_0 q]}$ for all q. Hence, for large q we have

$$\begin{aligned}\lambda\left(\frac{R}{I^{[q_0 q]} : x^{q_0 q}}\right) &\leq \lambda\left(\frac{R}{(\mathfrak{m}^{[q_0]}, c)^{[q]}}\right)\\ &\leq \lambda\left(\frac{R}{(\mathfrak{m}^{[q_0]}, c)}\right)(\mathrm{e}_1 + 1)q^d \leq (\mathrm{e}_2 + 1)q_0^{d-1}(\mathrm{e}_1 + 1)q^d.\end{aligned}$$

Dividing by $(q_0 q)^d$ and taking limits then shows that

$$\mathrm{e}_{\mathrm{HK}}(I, (I, x)) \leq \frac{(\mathrm{e}_2 + 1)(\mathrm{e}_1 + 1)}{q_0}.$$

Since q_0 may be taken arbitrarily large (this will change the ideal I), we have contradicted the assumption (3). □

In later sections we will often want to obtain a minimal reduction of an ideal in a local ring. The standard technique is to pass to a faithfully flat extension. Our next remark merely summarizes several well-known facts that will be needed.

REMARK 2.10. Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic p.

(a) Assume that $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a flat local homomorphism with $\mathfrak{n} = \mathfrak{m}S$ (e.g., completion).
  (i) For any $\mathfrak{m}$-primary ideal $I \subseteq R$, we have $\mathrm{e}_{\mathrm{HK}}(IS) = \mathrm{e}_{\mathrm{HK}}(I)$. In particular, $\mathrm{e}_{\mathrm{HK}}(S) = \mathrm{e}_{\mathrm{HK}}(R)$.
  (ii) If R is Cohen–Macaulay (CM) with canonical module ω_R, then S is CM with canonical module $\omega_S = \omega_R \otimes S$.

(b) Let Y be an indeterminate over R and set $S = R[Y]_{\mathfrak{m}R[Y]}$. Then S is (i) faithfully flat with maximal ideal extended from R and (ii) residue field isomorphic to $k(Y)$ (hence infinite). Part (a) then applies.

(c) If R has infinite residue field then $\mathfrak{m}$ has a minimal reduction $\mathbf{x} = x_1, \dots, x_d$ with $\mathrm{e}(R) = \mathrm{e}((\mathbf{x})) = \mathrm{e}_{\mathrm{HK}}((\mathbf{x}))$, and if R is CM then the common value is also equal to $\lambda(R/(\mathbf{x}))$. If R has finite residue field, then parts (a) and (b) may be applied in order to change to the case where the residue field is infinite.

3. Hilbert–Kunz Lower Bounds via Duality

This section will present various lower bounds for the Hilbert–Kunz multplicity of a ring $(R, \mathfrak{m}, k)$ of fixed multiplicity and dimension.

LEMMA 3.1. *If* $(R, \mathfrak{m})$ *is local of dimension* d, $I \subseteq J$ *are* $\mathfrak{m}$*-primary ideals,* $c \in R^\circ$, *and* M *is finitely generated over* R, *then*

$$\lim_{q\to\infty} \frac{1}{q^d} \lambda\left(\frac{J^{[q]}M}{(cJ^{[q]} + I^{[q]})M} \right) = 0.$$

Proof. Let $n = \mu(M)$ and $k = \mu(J)$. Then one can see that there is a surjection

$$\left(\frac{R}{cR}\right)^{nk} \to \frac{J^{[q]}M}{(cJ^{[q]} + I^{[q]})M} \to 0$$

and that the kernel contains $I^{[q]}\left(\frac{R}{cR}\right)^{nk}$.

Since $\dim R/cR = d - 1$, we observe that

$$\lim_{q\to\infty} \frac{1}{q^d} \lambda\left(\left(\frac{R}{cR + I^{[q]}} \right)^{nk} \right) = 0,$$

which implies our statement. ☐

We are now ready to formulate an important technical result leading to a series of corollaries that are the main goal of this section.

THEOREM 3.2. *Let* $(R, \mathfrak{m})$ *be a Cohen–Macaulay ring with system of parameters* $\mathbf{x} = x_1, \dots, x_d$. *Let* $e = \lambda(R/(\mathbf{x}))$. *Suppose that* $I \supseteq (\mathbf{x})$, *and set* $J = (\mathbf{x})^* : I$.

Let $a = \lambda^*(R/I)$, $f = \lambda^*(R/J)$, *and* $b = \lambda(((\mathbf{x})^* : I)/(\mathbf{x}))$. *Then* $\mathrm{e}_{\mathrm{HK}}(R) \geq \frac{e}{f+a}$ *and so, in particular,*

$$\mathrm{e}_{\mathrm{HK}}(R) \geq \frac{e}{e - b + a}.$$

Proof. Completing R leaves the Hilbert–Kunz multiplicity unaffected, can only increase the star lengths (a and f), and can only decrease b. So to prove the desired formulas we may complete R. We may therefore assume that R has a q_0-weak test element c.

Let ω_R be the canonical module of R. We have $\mathrm{Assh}(R) = \mathrm{Ass}(R)$ and, for each $P \in \mathrm{Ass}(R)$, $\lambda_{R_P}(\omega_P) = \lambda_{R_P}(R_P)$. Hence, by applying the associativity formula in Remark 2.5 to compute $\mathrm{e}_{\mathrm{HK}}(I; R)$ and $\mathrm{e}_{\mathrm{HK}}(I; \omega)$, we see that they are equal. Thus $\mathrm{e}_{\mathrm{HK}}(I_1, I_2; \omega_R) = \mathrm{e}_{\mathrm{HK}}(I_1, I_2)$ whenever $I_1 \subseteq I_2$ are $\mathfrak{m}$-primary ideals.

Since $\mathbf{x}$ is a system of parameters (s.o.p.), it follows that $\mathrm{e}_{\mathrm{HK}}((\mathbf{x})) = \mathrm{e}((\mathbf{x})) = e$. Also, $\mathrm{e}_{\mathrm{HK}}((\mathbf{x})) = \mathrm{e}_{\mathrm{HK}}(J) + \mathrm{e}_{\mathrm{HK}}((\mathbf{x}), J)$. By Proposition 2.8, it follows that

$\mathrm{e}_{\mathrm{HK}}(J) \leq \lambda^*(R/J)\,\mathrm{e}_{\mathrm{HK}}(R) = f\,\mathrm{e}_{\mathrm{HK}}(R)$. The heart of the proof is seeing that $\mathrm{e}_{\mathrm{HK}}((\mathbf{x}), J; \omega_R) \leq a\,\mathrm{e}_{\mathrm{HK}}(R)$ and so $\mathrm{e}_{\mathrm{HK}}((\mathbf{x}), J) = \mathrm{e}_{\mathrm{HK}}((\mathbf{x}), J; \omega_R) \leq a\,\mathrm{e}_{\mathrm{HK}}(R)$.

Indeed, $\omega_R/(\mathbf{x})^{[q]}\omega_R$ is the canonical module of the Artinian ring $R/(\mathbf{x})^{[q]}$, so it is injective over it. By Matlis duality over complete Artinian rings, we obtain that $\lambda(R/I^{[q]}) = \lambda(\mathrm{Hom}(R/I^{[q]}, \omega_R/(\mathbf{x})^{[q]}\omega_R))$.

Note that, by the definition of J and the fact that c is a q_0-weak test element, we have $cJ^{[q]} \subseteq (\mathbf{x})^{[q]} : I^{[q]}$ for all $q \geq q_0$. Thus, for all $q \geq q_0$,

$$\frac{(cJ^{[q]} + (\mathbf{x})^{[q]})\omega_R}{(\mathbf{x})^{[q]}\omega_R} \subseteq \frac{(\mathbf{x})^{[q]}\omega_R : I^{[q]}}{(\mathbf{x})^{[q]}\omega_R} = \mathrm{Hom}\left(\frac{R}{I^{[q]}}, \frac{\omega_R}{(\mathbf{x})^{[q]}\omega_R}\right).$$

By the equality

$$\lambda\left(\frac{J^{[q]}\omega_R}{(\mathbf{x})^{[q]}\omega_R}\right) = \lambda\left(\frac{J^{[q]}\omega_R}{(cJ^{[q]} + (\mathbf{x})^{[q]})\omega_R}\right) + \lambda\left(\frac{(cJ^{[q]} + (\mathbf{x})^{[q]})\omega_R}{(\mathbf{x})^{[q]}\omega_R}\right),$$

Lemma 3.1, Matlis duality, and Proposition 2.8, we have

$$\mathrm{e}_{\mathrm{HK}}((\mathbf{x}), J; \omega_R) \leq \mathrm{e}_{\mathrm{HK}}(I; \omega_R) = \mathrm{e}_{\mathrm{HK}}(I) \leq a\,\mathrm{e}_{\mathrm{HK}}(R).$$

In conclusion,

$$\begin{aligned} e = \mathrm{e}_{\mathrm{HK}}((\mathbf{x}), R) &= \mathrm{e}_{\mathrm{HK}}(J, R) + \mathrm{e}_{\mathrm{HK}}((\mathbf{x}), J) \\ &\leq f\,\mathrm{e}_{\mathrm{HK}}(R) + a\,\mathrm{e}_{\mathrm{HK}}(R) = (f + a)\,\mathrm{e}_{\mathrm{HK}}(R), \end{aligned}$$

proving the theorem's first inequality. The last inequality follows because $f = \lambda^*(R/J) \leq \lambda(R/J) = e - b$. □

The next corollary shows how useful Theorem 3.2 can be when R is not Gorenstein. Note that the lower bound for $\mathrm{e}_{\mathrm{HK}}(R)$ does not depend on the dimension of the ring.

COROLLARY 3.3. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay ring of CM-type t and multiplicity* $\mathrm{e} = \mathrm{e}(R)$. *Then*

$$\mathrm{e}_{\mathrm{HK}}(R) \geq \frac{\mathrm{e}}{\mathrm{e} - t + 1}.$$

Proof. We may assume by Remark 2.10 that the residue field is infinite, so there exists a s.o.p. $\mathbf{x}$ with $\mathrm{e}(R) = \lambda(R/(\mathbf{x}))$. Now apply Theorem 3.2 with $I = \mathfrak{m}$ (so $a = 1$ and $b \geq t$). □

COROLLARY 3.4. *Let $(R, \mathfrak{m})$ be a nonregular Cohen–Macaulay ring of minimal multiplicity. Then* $\mathrm{e}_{\mathrm{HK}}(R) \geq \mathrm{e}(R)/2$.

Proof. By the structure theorem of Sally [13], R has type $t = \mathrm{e}(R) - 1$. Therefore, $\mathrm{e}_{\mathrm{HK}}(R) \geq \frac{\mathrm{e}(R)}{\mathrm{e}(R) - (\mathrm{e}(R) - 1) + 1} = \frac{\mathrm{e}(R)}{2}$. □

COROLLARY 3.5. *Let $(R, \mathfrak{m}, k)$ be a local Cohen–Macaulay ring of characteristic p and dimension d. If* $\mathrm{e}_{\mathrm{HK}}(R) < \frac{\mathrm{e}}{\mathrm{e}-1}$, *then R is Gorenstein and F-regular* (*and hence strongly F-regular if R is also excellent*).

Proof. We may assume that R is not regular. If R is not Gorenstein then t, the type of R, is at least 2. Theorem 3.2 then shows that $\mathrm{e}_{\mathrm{HK}} \geq \frac{\mathrm{e}}{\mathrm{e}-t+1} \geq \frac{\mathrm{e}}{\mathrm{e}-1}$. Hence R is Gorenstein, and we are done by Theorem 1.2. □

We can now state the desired generalization of Theorem 1.2. The improvement is replacing "F-rational" by an appropriate form of "F-regular" in the conclusion.

COROLLARY 3.6. *Let $(R,\mathfrak{m},k)$ be a formally unmixed ring of characteristic p with* $\dim(R) = d \geq 2$. *If* $\mathrm{e}_{\mathrm{HK}}(R) \leq 1 + \max\{1/d!, 1/\mathrm{e}(R)\}$, *then R is F-regular and Gorenstein. If R is excellent, then R is strongly F-regular.*

Proof. Let $\mathrm{e} = \mathrm{e}(R)$. We can pass to the completion and assume that R is complete and unmixed. We remark that, for an excellent Gorenstein ring, strong F-regularity and F-regularity are equivalent. Moreover, if the completion of a ring R is F-regular then R is F-regular. Hence, by Theorem 1.2 we may assume that R is Cohen–Macaulay.

If R is not strongly F-regular, then $\mathrm{e}_{\mathrm{HK}}(R) \geq \frac{\mathrm{e}}{\mathrm{e}-1} > 1 + \frac{1}{\mathrm{e}}$. Therefore, $1 + \frac{1}{d!} \geq \mathrm{e}_{\mathrm{HK}}(R) > 1 + \frac{1}{\mathrm{e}}$, which implies that $\mathrm{e} > d!$ and $\mathrm{e}_{\mathrm{HK}}(R) \geq \frac{\mathrm{e}}{d!} > \frac{d!+1}{d!}$. This is a contradiction.

If $\mathrm{e} \geq d! + 1$ then, since $\mathrm{e}_{\mathrm{HK}}(R) > \mathrm{e}/d!$ (this inequality is due to Hanes [7]), it follows that $\mathrm{e}_{\mathrm{HK}}(R) > 1 + \frac{1}{d!} > 1 + \frac{1}{\mathrm{e}}$—a contradiction. Thus $\mathrm{e} \leq d!$ and so $\mathrm{e}_{\mathrm{HK}}(R) \leq 1 + \frac{1}{\mathrm{e}} < \frac{\mathrm{e}}{\mathrm{e}-1}$, which implies that R is Gorenstein. □

It should be remarked that Corollaries 3.5 and 3.6 are closely related to recent unpublished results of D. Hanes, who independently proved in particular that, under the assumptions of Corollary 3.6, the ring R is Gorenstein and F-regular.

Theorem 3.2 yields some interesting results when we apply it to Gorenstein rings that are not F-regular.

COROLLARY 3.7. *Let $(R,\mathfrak{m})$ be a Gorenstein ring of dimension d and embedding dimension $v = \mu(\mathfrak{m})$. Let $\mathrm{e} = \mathrm{e}(R)$. If either R or $\hat{R}$ is not F-regular, then*

$$\mathrm{e}_{\mathrm{HK}}(R) \geq \frac{\mathrm{e}}{\mathrm{e} - v + d}.$$

Proof. Non-F-regularity passes to the completion, so we may assume that R is complete. By Remark 2.10, we may assume that the residue field is infinite—and that $\mathbf{x}$ is a s.o.p. with $\mathrm{e}(R) = \lambda(R/(\mathbf{x}))$—while preserving the nonweak-F-regularity of R. If u denotes a socle element modulo $(\mathbf{x})$ then $u \in (\mathbf{x})^*$. We can now apply Theorem 3.2 with $I = \mathfrak{m}$. Then

$$a = \lambda^*(R/\mathfrak{m}) = 1 \quad \text{and} \quad b = \lambda(((\mathbf{x})^* : \mathfrak{m})/(\mathbf{x})) \geq v - d + 1,$$

since in the 0-dimensional Gorenstein ring $S = R/(\mathbf{x})$ we have $(u)S : \mathfrak{m}S = 0 : \mathfrak{m}^2 S$ and $\lambda(0 : \mathfrak{m}^2 S) = \lambda(S/\mathfrak{m}^2 S) = v - d + 1$. The corollary now follows. □

REMARK 3.8. It is possible in "pathological" (e.g., nonexcellent) cases for a ring to be weakly F-regular while its completion is not. Loepp and Rotthaus construct such an example, which is Gorenstein, in [12]. Corollary 3.7 applies in this case.

Corollary 3.7 can be improved, and this improvement—although interesting on its own—will also prove useful in Section 4. We first establish some notation. For a graded ring $G = \bigoplus_{i \geq 0} G_i$ that is finitely generated over an Artinian ring G_0, let $k_i = \lambda(G_i)$. If $\lambda(G) < \infty$, let $r = \max\{i \mid G_i \neq 0\}$. We note that if $(S,\mathfrak{n})$ is a Gorenstein ring of dimension 0 and if G is the associated graded ring of S at $\mathfrak{n}$, then G_r is generated by the image of the socle element and so $k_r = 1$.

COROLLARY 3.9. *Let* $(R,\mathfrak{m})$ *be a non-F-regular Gorenstein local ring of dimension d and multiplicity* $\mathrm{e}=\mathrm{e}(R)$, *and let* $\mathbf{x}=x_1,\dots,x_d$ *be a minimal reduction of* $\mathfrak{m}$. *Let G be the associated graded ring of* $R/(\mathbf{x})$ (*at its maximal ideal*), *and let r and* k_i $(0\le i\le r)$ *be as before. Then*

$$\mathrm{e}_{\mathrm{HK}}(R)\ge\max_{1\le i\le r}\left\{\frac{\mathrm{e}}{\mathrm{e}-k_i}\right\}.$$

As a consequence,

$$\mathrm{e}_{\mathrm{HK}}(R)\ge\frac{\mathrm{e}}{\mathrm{e}-\frac{\mathrm{e}-2}{r-1}}\ge\frac{r+1}{r}.$$

Proof. Because R is not F-regular, if u denotes a socle element modulo $(\mathbf{x})$ then $u\in(\mathbf{x})^*$. Thus $(\mathbf{x}):\mathfrak{m}=(u,\mathbf{x})\subseteq(\mathbf{x})^*$. We may then apply Theorem 3.2 with $I=\mathfrak{m}^j+(\mathbf{x})$ and $J=(\mathbf{x})^*:I\supseteq(u,\mathbf{x}):\mathfrak{m}^j=((\mathbf{x}):\mathfrak{m}):\mathfrak{m}^j=(\mathbf{x}):\mathfrak{m}^{j+1}$. In this case, $\lambda(R/I)=\sum_{i=0}^{j-1}k_i$ and $\lambda(R/J)=\mathrm{e}-\lambda(J/(\mathbf{x}))\le\mathrm{e}-\lambda(R/(\mathfrak{m}^{j+1}+(\mathbf{x})))=\mathrm{e}-\left(\sum_{i=0}^{j}k_i\right)$ (Matlis duality and the fact that $J\subset(\mathbf{x}):\mathfrak{m}^{j+1}$ give the inequality). Hence

$$\begin{aligned}\mathrm{e}_{\mathrm{HK}}(R)&\ge\frac{\mathrm{e}}{\lambda^*(R/(\mathfrak{m}^j+(\mathbf{x})))+\lambda^*(R/J)}\\&\ge\frac{\mathrm{e}}{\sum_{i=0}^{j-1}k_i+\mathrm{e}-\left(\sum_{i=0}^{j}k_i\right)}=\frac{\mathrm{e}}{\mathrm{e}-k_j}.\end{aligned}$$

Since $k_0=k_r=1$, there exists at least one $k_i\ge\frac{\mathrm{e}-1-1}{r-1}$ and so

$$\mathrm{e}_{\mathrm{HK}}(R)\ge\frac{\mathrm{e}}{\mathrm{e}-\frac{\mathrm{e}-2}{r-1}}.$$

A bit of algebra shows that

$$\frac{\mathrm{e}}{\mathrm{e}-\frac{\mathrm{e}-2}{r-1}}\ge\frac{r+1}{r}\iff\mathrm{e}\ge r+1.$$

The latter condition always holds. □

COROLLARY 3.10. *Let* $(R,\mathfrak{m})$ *be a non-F-regular Gorenstein ring of dimension* $d>1$. *Then* $\mathrm{e}_{\mathrm{HK}}(R)\ge\frac{d+1}{d}$. *If R is not a hypersurface, then* $\mathrm{e}_{\mathrm{HK}}(R)\ge\frac{d}{d-1}$.

Proof. By Remark 2.10, we may assume that (a) R is complete with infinite residue field and (b) $\mathbf{x}$ is a s.o.p. that is a minimal reduction of $\mathfrak{m}$.

Let G and r be as in the proof of Corollary 3.9. The result of Corollary 3.9 suffices if $r+1\le d$, so we assume that $r\ge d$. By the Briançon–Skoda theorem, $\mathfrak{m}^d\subseteq(\mathbf{x})^*$.

Let $\mathrm{e}=\mathrm{e}(R)$ be the multiplicity. It is easy to see that, for any integer $n\le\mathrm{e}$, we have $\frac{\mathrm{e}}{\mathrm{e}-n}\ge\frac{d}{d-1}$ if and only if $n\ge\mathrm{e}/d$. By Corollary 3.9 we are done if some $k_i\ge\mathrm{e}/d$, so assume that each $k_i<\mathrm{e}/d$.

Let $I=\mathfrak{m}^{d-1}+(\mathbf{x})$. Then $(\mathbf{x})^*:I\supseteq\mathfrak{m}$ (by the Briançon–Skoda theorem) and so, by Theorem 3.2,

$$\mathrm{e}_{\mathrm{HK}}(R)\ge\frac{\mathrm{e}}{\mathrm{e}-(\mathrm{e}-1)+1+k_1+\cdots+k_{d-2}}=\frac{\mathrm{e}}{2+k_1+\cdots+k_{d-2}}.$$

Since each $k_i < \mathrm{e}/d$ we have $\mathrm{e}_{\mathrm{HK}}(R) > \frac{\mathrm{e}}{2+(d-2)(\mathrm{e}/d)}$, and the right-hand side is easily seen to be at least $\frac{d}{d-1}$ provided that $\mathrm{e} \geq 2d$.

The only case left is if $\mathrm{e} < 2d$. Then $2d > e > dk_i$ for all k_i implies that each $k_i = 1$ (i.e., R is a hypersurface) and $\mathrm{e} = r + 1$ (recall that $r \geq d$). Say $\mathfrak{m} = (z, \mathbf{x})$ minimally. By the Briançon–Skoda theorem, $z^d \in (\mathbf{x})^*$ and so $(\mathbf{x})^* : \mathfrak{m} \supseteq (z^d, \mathbf{x}) : z \supseteq (z^{d-1}, \mathbf{x})$. Applying Theorem 3.2 with $I = \mathfrak{m}$ now yields

$$\mathrm{e}_{\mathrm{HK}}(R) \geq \frac{\mathrm{e}}{1 + d - 1} = \frac{\mathrm{e}}{d} \geq \frac{d+1}{d}. \qquad \square$$

4. Radical Extensions and Comparison of Hilbert–Kunz Multiplicities

In this section we develop a technique that, in conjuction with the results obtained so far, will give a lower bound for the Hilbert–Kunz multiplicity of unmixed non-regular local rings of dimension d that depends only on d and is strictly greater than 1, thus showing that $\varepsilon(d) > 0$. This answers one of the open questions mentioned in the Introduction.

We will need to use a result of Watanabe and Yoshida [15, Thm. 2.7; 17, Thm. 1.6]. For a domain R we use $Q(R)$ for the fraction field of R as well as R^+ for the absolute integral closure of R (i.e., an integral closure of R in an algebraic closure of $Q(R)$).

THEOREM 4.1. *Let $(R, \mathfrak{m}) \hookrightarrow (S, \mathfrak{n})$ be a module-finite extension of local domains. Then, for every $\mathfrak{m}$-primary ideal I of R,*

$$\mathrm{e}_{\mathrm{HK}}(I) = \frac{\mathrm{e}_{\mathrm{HK}}(IS)}{[Q(S) : Q(R)]} \cdot [S/\mathfrak{n} : R/\mathfrak{m}]. \tag{4.1}$$

We need the following definition.

DEFINITION 4.2. Let $(R, \mathfrak{m})$ be a domain. Let $z \in \mathfrak{m}$, and let n be a positive integer. Let $v \in R^+$ be any root of $f(X) = X^n - z$. We call $S = R[v]$ a *radical extension* for the pair R and z.

REMARK 4.3. If S is radical for R and z, then $b := [Q(S) : Q(R)] \leq n$. Assume also that R is normal and that z is a minimal generator of $\mathfrak{m}$. Then, in fact, $b = n$. To see this we need to show that $f(X) = X^n - z$ is the minimal polynomial for $v = z^{1/n}$ over R. Let $g(X)$ be the minimal polynomial of v over $Q(R)$. Because R is normal, $g(X) \in R[X]$. The constant term of $g(X)$ is in $\mathfrak{m}$, since z is not a unit. Then $g(X) | f(X)$ in $R[X]$. Say $f(X) = g(X)h(X)$. Then the constant term of $h(X)$ is a unit (or else $z \in \mathfrak{m}^2$). But modulo $\mathfrak{m}$, $g(X)h(X) = X^n$ and so, in fact, $h(X)$ is a unit constant.

In what follows, $\mathfrak{n}$ will denote the maximal ideal of S whenever S is local. We remark that if R is a complete local domain and $z \in \mathfrak{m}$, then S must be local.

THEOREM 4.4. *Let $(R, \mathfrak{m})$ be a complete local domain of positive prime characteristic with algebraically closed residue field. Let $\mathbf{x} = x_1, \dots, x_d$ be a system of parameters, and set $\mathrm{e} = \mathrm{e}_{\mathrm{HK}}((\mathbf{x})) = \mathrm{e}((\mathbf{x}))$ and $a = \lambda(R/(\mathbf{x})^*)$.*

Let $z \in \mathfrak{m} - (\mathbf{x})^*$ be a minimal generator and let $v \in R^+$ be any nth root of z. Let $S = R[v]$ be a radical extension for R and z, and denote the maximal ideal of S by $\mathfrak{n}$. Let $b = [Q(S) : Q(R)]$. Then

$$\mathrm{e}_{\mathrm{HK}}(R) \geq \frac{b(n-1)\,\mathrm{e} + n\,\mathrm{e}_{\mathrm{HK}}(S)}{b(a(n-1)+1)}.$$

For $b = n$, this inequality simplifies to

$$\mathrm{e}_{\mathrm{HK}}(R) \geq \frac{(b-1)\,\mathrm{e} + \mathrm{e}_{\mathrm{HK}}(S)}{a(b-1)+1}.$$

REMARK 4.5. If we put $\mathrm{e}_{\mathrm{HK}}(R) = 1 + \delta_R$ and $\mathrm{e}_{\mathrm{HK}}(S) = 1 + \delta_S$, then the preceding inequalities are equivalent to

$$\delta_R \geq \frac{b(n-1)(\mathrm{e}-a) + n - b + n\delta_S}{b(a(n-1)+1)}$$

and, if $b = n$,

$$\delta_R \geq \frac{(b-1)(\mathrm{e}-a) + \delta_S}{a(b-1)+1}.$$

For the proof of Theorem 4.4, the following remark is helpful.

REMARK 4.6. Let $I \subseteq R$ be an ideal in a local ring $(R, \mathfrak{m})$, and let $v \in \mathfrak{m}$ be an element such that (I, v) is $\mathfrak{m}$-primary. Then, for all $n \geq 1$,

$$\mathrm{e}_{\mathrm{HK}}((I, v^n), (I, v^{n-1})) \geq \mathrm{e}_{\mathrm{HK}}((I, v^{n+1}), (I, v^n)).$$

To see this, we observe that $(I, v^n)^{[q]} : v^{(n-1)q} \subseteq (I, v^{n+1})^{[q]} : v^{nq}$ for all q; hence

$$\begin{aligned}
&\mathrm{e}_{\mathrm{HK}}((I, v^n), (I, v^{n-1})) \\
&\quad = \lim_{q\to\infty} \frac{1}{q^d}\lambda\left(\frac{(I, v^{n-1})^{[q]}}{(I, v^n)^{[q]}}\right) = \lim_{q\to\infty} \frac{1}{q^d}\lambda\left(\frac{R}{(I, v^n)^{[q]} : v^{(n-1)q}}\right) \\
&\quad \geq \lim_{q\to\infty} \frac{1}{q^d}\lambda\left(\frac{R}{(I, v^{n+1})^{[q]} : v^{nq}}\right) = \mathrm{e}_{\mathrm{HK}}((I, v^{n+1}), (I, v^n)).
\end{aligned}$$

Proof of Theorem 4.4. Let $(\mathbf{x})^* = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_{a-2} \subsetneq (I_{a-2}, z) = I_{a-1} = \mathfrak{m} \subsetneq R$ be a saturated filtration, and let $w_i \in R$ be an element whose image generates I_i/I_{i-1} (in particular, take $w_{a-1} = z$). We can then filter $(\mathbf{x})^*S \subseteq S$ by filling in each $I_{i-1}S \subseteq I_iS$ with

$$I_{i-1}S \subseteq (I_{i-1}, v^{n-1}w_i)S \subseteq \cdots \subseteq (I_{i-1}, vw_i)S \subseteq I_iS$$

(where we allow that some of the containments may be equalities).

From Theorem 4.1 and the fact that $[S/\mathfrak{n} : R/\mathfrak{m}] = 1$ ($R/\mathfrak{m}$ is algebraically closed), we have that $\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}S) = b\,\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}R)$. Thus, $\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}S, \mathfrak{n}) = b\,\mathrm{e}_{\mathrm{HK}}(R) - \mathrm{e}_{\mathrm{HK}}(S)$. By Remark 4.6, for each $1 \leq j < n$,

$$\mathrm{e}_{\mathrm{HK}}((v^j, \mathfrak{m}S), (v^{j-1}, \mathfrak{m}S)) \geq \mathrm{e}_{\mathrm{HK}}((v^{j+1}, \mathfrak{m}S), (v^j, \mathfrak{m}S)).$$

Therefore,

$$\mathrm{e}_{\mathrm{HK}}((\mathfrak{m}S), (v^{n-1}, \mathfrak{m}S)) \leq \frac{\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}S, \mathfrak{n})}{n-1}.$$

Set $y := \mathrm{e}_{\mathrm{HK}}((\mathfrak{m}S), (v^{n-1}, \mathfrak{m}S))$, and consider the filtration

$$\begin{aligned} \mathfrak{m}S = (z, I_{a-2})S \supseteq (zv, I_{a-2})S \supseteq (zv^2, I_{a-2})S \\ \supseteq \cdots \supseteq (zv^{n-1}, I_{a-2})S \supseteq I_{a-2}S. \end{aligned} \tag{4.2}$$

Remark 4.6 applies to each containment in equation (4.2), so each relative Hilbert–Kunz multiplicity is at most

$$\mathrm{e}_{\mathrm{HK}}((zv, I_{a-2})S, \mathfrak{m}S) = \mathrm{e}_{\mathrm{HK}}((v^{n+1}, I_{a-2})S, (v^n, I_{a-2})S) \leq y.$$

Adding them all up yields $\mathrm{e}_{\mathrm{HK}}(I_{a-2}S, \mathfrak{m}S) \leq ny$, and from this it follows that

$$\mathrm{e}_{\mathrm{HK}}(I_{a-2}S, \mathfrak{m}S) \leq n \cdot \frac{\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}S, \mathfrak{n})}{n-1}.$$

Using Theorem 4.1 to return to R, we have

$$\mathrm{e}_{\mathrm{HK}}(I_{a-2}, \mathfrak{m}) \leq n \frac{\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}S, \mathfrak{n})}{b(n-1)}.$$

Each of the other $a-1$ terms in the filtration of $(\mathbf{x})^* \subseteq R$ has relative Hilbert–Kunz multiplicity at most $\mathrm{e}_{\mathrm{HK}}(R)$, so we obtain the inequality

$$\left(n \frac{\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}S, \mathfrak{n})}{b(n-1)} \right) + (a-1)\, \mathrm{e}_{\mathrm{HK}}(R) \geq \mathrm{e}_{\mathrm{HK}}((\mathbf{x})^*) = \mathrm{e}. \tag{4.3}$$

But this yields

$$\mathrm{e}_{\mathrm{HK}}(R) \geq \frac{b(n-1)\,\mathrm{e} + n\, \mathrm{e}_{\mathrm{HK}}(S)}{b(a(n-1)+1)}. \qquad \square$$

COROLLARY 4.7. *Let $(R, \mathfrak{m})$ be an F-rational complete nonregular local ring of positive prime characteristic with algebraically closed residue field. Let $\mathbf{x} = x_1, \ldots, x_d$ be a system of parameters and a minimal reduction for $\mathfrak{m}$, and let $\mathrm{e} = \mathrm{e}(R) = \mathrm{e}_{\mathrm{HK}}((\mathbf{x})) = \mathrm{e}((\mathbf{x}))$.*

Let $z \in \mathfrak{m} - (\mathbf{x})$ be a minimal generator and let $v \in R^+$ be any nth root of z. Let $S = R[v]$ be a radical extension for R and z, and denote its maximal ideal of S by $\mathfrak{n}$. Then

$$\mathrm{e}_{\mathrm{HK}}(R) \geq \frac{(n-1)\,\mathrm{e} + \mathrm{e}_{\mathrm{HK}}(S)}{\mathrm{e}(n-1)+1}.$$

Proof. By Remark 4.3, $b = [Q(S) : Q(R)] = n$. Since R is F-rational, $(\mathbf{x}) = (\mathbf{x})^*$. Hence one can apply Theorem 4.4 together with the observation that $a = \mathrm{e}$. $\square$

REMARK 4.8. Corollary 4.7 can be substantially improved, but the proof is considerably more difficult. We will give improved versions in a later paper along with improved estimates of lower bounds for $\varepsilon(d)$.

COROLLARY 4.9. *Let $(R, \mathfrak{m})$ be a complete local domain of positive prime characteristic with algebraically closed residue field. Let $\mathbf{x} = x_1, \ldots, x_d$ be a system of parameters and a minimal reduction for $\mathfrak{m}$, and set $\mathrm{e} = \mathrm{e}_{\mathrm{HK}}((\mathbf{x})) = \mathrm{e}((\mathbf{x}))$ and $a = \lambda(R/(\mathbf{x})^*)$. Then*

$$\mathrm{e}_{\mathrm{HK}}(R) \geq \frac{\mathrm{e}+1}{a+1}.$$

Proof. If $\mathfrak{m} = (\mathbf{x})^*$, then $a = 1$ and $\mathrm{e_{HK}}(R) = \mathrm{e_{HK}}((\mathbf{x})) = \mathrm{e} \geq \frac{\mathrm{e}+1}{2}$.

Otherwise, take any minimal generator of $\mathfrak{m}$ not in $(\mathbf{x})^*$ and adjoin a square root of it from R^+. Then apply Theorem 4.4 and note that, since $2 = n \geq b$ and $\mathrm{e_{HK}}(S) \geq 1$,

$$\mathrm{e_{HK}}(R) \geq \frac{b(n-1)\,\mathrm{e} + b}{b(a(n-1)+1)} = \frac{(n-1)\,\mathrm{e}+1}{a(n-1)+1} = \frac{\mathrm{e}+1}{a+1}. \qquad \square$$

REMARK 4.10. Assume that $(R, \mathfrak{m})$ is CM of type t, and let I be a parameter ideal and minimal reduction for $\mathfrak{m}$ such that $I \subsetneq I^* \subsetneq \mathfrak{m}$. Then $e = \lambda(R/I)$ and $t = \lambda((I : \mathfrak{m})/I)$.

The two ideals I^* and $(I : \mathfrak{m})$ are incomparable in many cases. However, in the special case when $(I : \mathfrak{m}) \subseteq I^*$ (the Gorenstein case, for example), $t \leq e - a$ and $\frac{e}{e-t+1} \leq \frac{e+1}{a+1}$. So Corollary 4.9 improves an earlier result of ours in this case.

We now begin a construction that will yield a lower bound for the Hilbert–Kunz multiplicity of Gorenstein, F-regular, nonregular local rings.

Assume that $(R, \mathfrak{m})$ is a Gorenstein F-regular local ring of multiplicity $\mathrm{e} = \mathrm{e}(R) > 1$. Note that R must be a normal domain. We may complete and (by [1, Thm. 3.4]) extend the residue field to assume that it is algebraically closed. Let $\mathbf{x} = x_1, \dots, x_d$ be a minimal reduction of $\mathfrak{m}$, so that $\lambda(R/(\mathbf{x})) = \mathrm{e}$.

LEMMA 4.11. *Let R and $\mathbf{x}$ be as just described, and suppose that z, $v = z^{1/n}$, and S are as in Corollary 4.7. Assume, moreover, that $x_1, \dots, x_{d-1}, z$ is also a minimal reduction of $\mathfrak{m}$. Let $u \in \mathfrak{m}$ denote a socle element modulo $(x_1, \dots, x_{d-1}, z)$. Then:*

(a) *$x_1, \dots, x_{d-1}, v$ is a minimal reduction of $\mathfrak{n}$ (the maximal ideal of S);*

(b) *u is still a socle element modulo $(x_1, \dots, x_{d-1}, v)S$; and*

(c) *S is Gorenstein and $\mathrm{e}(S) = \mathrm{e}(R)$.*

Proof. Let $\mathbf{x}_{d-1} = x_1, \dots, x_{d-1}$.

(a) If $\mathfrak{m} = (\mathbf{x}_{d-1}, z) + J$, where $\mu(J) = \mu(\mathfrak{m}) - d$, then $\mathfrak{n} = (\mathbf{x}_{d-1}, v)S + JS$. Since J is integral over $(\mathbf{x}_{d-1}, z)R$, it follows that the ideal JS is integral over $(\mathbf{x}_{d-1}, z)S$ and hence over the larger ideal $(\mathbf{x}_{d-1}, v)S$. This suffices to show (a).

(b) If $u \in (\mathbf{x}_{d-1}, v)S$ then $u \in (\mathbf{x}_{d-1}, z)S \cap R \subseteq ((\mathbf{x}_{d-1}, z)R)^* = (\mathbf{x}_{d-1}, z)R$, a contradiction. With J as in part (a), we have

$$\begin{aligned}\mathfrak{n}u = ((\mathbf{x}_{d-1}, v)S + JS)u &\subseteq JuS + (\mathbf{x}_{d-1}, v)S \\ &\subseteq (\mathbf{x}_{d-1}, z)S + (\mathbf{x}_{d-1}, v)S \subseteq (\mathbf{x}_{d-1}, v)S.\end{aligned}$$

Thus u is a socle element.

(c) By Remark 4.3, $X^n - z$ is the minimal polynomial of v over R. Therefore, S is R-free and hence flat with Gorenstein closed fiber. Thus S is Gorenstein. Then

$$\begin{aligned}\mathrm{e}(S) = \lambda_S(S/(\mathbf{x}_{d-1}, v)) &= \frac{1}{n}\lambda_S(S/(\mathbf{x}_{d-1}, v^n)) \\ &= \frac{1}{n}\lambda_S(S/(\mathbf{x}_{d-1}, z)) = \lambda_R(R/(\mathbf{x}_{d-1}, z)) = \mathrm{e}(R). \qquad \square\end{aligned}$$

Let $d = \dim R$ and $k = \mu(\mathfrak{m}) - d > 1$. Observe that $\mathrm{e}_{\mathrm{HK}}(R) \geq \frac{\mathrm{e}(R)}{d!}$. Hence, whenever $\mathrm{e}(R) \geq d! + 1$, we have $\mathrm{e}_{\mathrm{HK}}(R) \geq 1 + \frac{1}{d!}$.

Therefore, if we want to produce a lower bound for $\mathrm{e}_{\mathrm{HK}}(R)$ in terms of only d, then there is no harm in fixing $\mathrm{e}(R) = \mathrm{e}$ as well. This is so because we can take the minimum of the lower bounds obtained for fixed d, e while letting e vary between 2 and $d!$.

The residue field of R is infinite, so we may pick $y_1, \dots, y_{d+1} \in \mathfrak{m} - \mathfrak{m}^2$ in general position and thus assume that each d-element subset is a minimal reduction of $\mathfrak{m}$ (see e.g. Theorem 8.6.6 of [10] and the comment after it). Let u denote a socle element modulo $(y_1, \dots, y_d)R$, and let $r = \max\{i \mid u \in \mathfrak{m}^i + (y_1, \dots, y_d)R\}$. Set $n = \lceil d/r \rceil$ (so $nr \geq d$). Let $R_0 = R$, and for each $i \geq 1$ let $v_i = y_i^{1/n}$ and set $R_i = R_{i-1}[v_i]$. For each i, write $\mathrm{e}_{\mathrm{HK}}(R_i) = 1 + \delta_i$.

For a given $i \geq 1$, if R_{i-1} is F-regular then we may apply Corollary 4.7 to $R_{i-1} \subseteq R_i$ with $\mathbf{x} = v_1, \dots, v_{i-1}, y_{i+1}, \dots, y_{d+1}$ and $z = y_i$ ($\mathbf{x}$ is a minimal reduction of R_{i-1} by Lemma 4.11(a)). Also, by Lemma 4.11(b), u is a socle element modulo $(v_1, \dots, v_i, y_{i+1}, \dots, y_d)R_i$. Noting that the multiplicity stays the same, we obtain

$$1 + \delta_{i-1} \geq 1 + \frac{1}{\mathrm{e}(R_{i-1})(n-1)+1}\delta_i = 1 + \frac{1}{\mathrm{e}(R_0)(n-1)+1}\delta_i. \tag{4.4}$$

We claim that, for some $i \leq d$, R_i is not F-regular. If not, then R_d is F-regular. Let $\mathfrak{m}_{R_0} = (y_1, \dots, y_d) + J$ with $\mu(J) = \mu(\mathfrak{m}) - d$. It is then clear that $\mathfrak{m}_{R_d} = (v_1, \dots, v_d) + JR_d$. By the Briançon–Skoda theorem, $\mathfrak{m}_{R_d}^d \subseteq ((v_1, \dots, v_d)R_d)^*$; hence

$$\begin{aligned} u \in (JR_0)^r &\subseteq \overline{(y_1, \dots, y_d)^r R_d} = \overline{(y_1^r, \dots, y_d^r)R_d} = \overline{(v_1^{rn}, \dots, v_d^{rn})R_d} \\ &\subseteq \overline{(v_1^d, \dots, v_d^d)R_d} \subseteq ((v_1, \dots, v_d)R_d)^* = (v_1, \dots, v_d)R_d, \end{aligned}$$

in contradiction to Lemma 4.11(b).

Assume, then, that $i_0 = \min\{i \mid R_i \text{ is not F-regular}\}$. By Corollary 3.10, it follows that $\mathrm{e}_{\mathrm{HK}}(R_i) \geq \frac{d+1}{d} = 1 + \frac{1}{d}$. Repeated application of equation (4.4) yields

$$\mathrm{e}_{\mathrm{HK}}(R) = \mathrm{e}_{\mathrm{HK}}(R_0) \geq 1 + \left(\frac{1}{\mathrm{e}(R)(n-1)+1}\right)^{i_0} \frac{1}{d}.$$

We are now in position to state and prove the main result of the paper.

THEOREM 4.12. *Let $(R, \mathfrak{m}, k)$ be a formally unmixed local ring of positive characteristic p and dimension $d \geq 2$. If R is not regular, then*

$$\mathrm{e}_{\mathrm{HK}}(R) \geq 1 + \frac{1}{d \cdot (d!(d-1)+1)^d}.$$

Proof. Because we can make a faithfully flat extension, we may assume that k is algebraically closed and also that R is complete.

We can assume that $\mathrm{e}_{\mathrm{HK}}(R) < 1 + \frac{1}{d!}$ and so, by Corollary 3.6, R is Gorenstein and F-rational and hence is strongly F-regular.

If $\mathrm{e} \geq d! + 1$, then $\mathrm{e}_{\mathrm{HK}}(R) \geq \frac{\mathrm{e}(R)}{d!} \geq 1 + \frac{1}{d!}$. We can thus assume that $\mathrm{e} \leq d!$.

Now we apply the technique described just before our statement of the theorem and, noting that $n \leq d$, obtain that

$$\mathrm{e}_{\mathrm{HK}}(R) \geq 1 + \frac{1}{(d \cdot (\mathrm{e}(d-1)+1)^{d}} \geq 1 + \frac{1}{d \cdot (d!(d-1)+1)^{d}}. \qquad \square$$

References

[1] I. M. Aberbach, *Extension of weakly and strongly F-regular rings by flat maps,* J. Algebra 241 (2001), 799–807.

[2] I. M. Aberbach and G. Leuschke, *The F-signature and strong F-regularity,* Math. Res. Lett. 10 (2003), 51–56.

[3] M. Blickle and F. Enescu, *On rings with small Hilbert–Kunz multiplicity,* Proc. Amer. Math. Soc. 132 (2004), 2505–2509.

[4] R. O. Buchweitz and Q. Chen, *Hilbert–Kunz functions of cubic curves and surfaces,* J. Algebra 197 (1997), 246–267.

[5] F. Enescu and K. Shimomoto, *On the upper semi-continuity of the Hilbert–Kunz multiplicity,* J. Algebra 285 (2005), 222–237.

[6] D. Hanes, *Bounds on multiplicities of local rings,* Comm. Algebra 30 (2002), 3789–3812.

[7] ———, *Notes on the Hilbert–Kunz function,* J. Algebra 265 (2003), 619–630.

[8] M. Hochster and C. Huneke, *Tight closure, invariant theory, and the Briançon–Skoda theorem,* J. Amer. Math. Soc. 3 (1990), 31–116.

[9] C. Huneke, *Tight closure and its applications,* with an appendix by Melvin Hochster. CBMS Reg. Conf. Ser. Math., 88, Amer. Math. Soc., Providence, RI, 1996.

[10] C. Huneke and I. Swanson, *Integral closure of ideals, rings, and modules,* London Math. Soc. Lecture Note Ser., 336, Cambridge Univ. Press, Cambridge, 2006.

[11] C. Huneke and Y. Yao, *Unmixed local rings with minimal Hilbert–Kunz multiplicity are regular,* Proc. Amer. Math. Soc. 130 (2002), 661–665.

[12] S. Loepp and C. Rotthaus, *Some results on tight closure and completion,* J. Algebra 246 (2001), 859–880.

[13] J. Sally, *Cohen–Macaulay local rings of maximal embedding dimension,* J. Algebra 56 (1979), 168–183.

[14] K. E. Smith, *Tight closure of parameter ideals and F-rationality,* Ph.D. thesis, Univ. of Michigan, 1993.

[15] K.-i. Watanabe and K. Yoshida, *Hilbert–Kunz multiplicity and an inequality between multiplicity and colength,* J. Algebra 230 (2000), 295–317.

[16] ———, *Hilbert–Kunz multiplicity of two-dimensional local rings,* Nagoya Math. J. 162 (2001), 87–110.

[17] ———, *Hilbert–Kunz multiplicity of three-dimensional local rings,* Nagoya Math. J. 177 (2005), 47–75.

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