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Maximally Symmetric Stable Curves

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In this paper we prove a sharp bound for the automorphism group of a stable curve of a given genus and describe all curves attaining that bound. All curves are defined over **C** and are projective.

A well-known result of Hurwitz [Hu] states that the maximal order of the automorphism group of a smooth curve of genus g is $42(2g-2)$. This bound is not attained in every genus; for example, the maximal order of the automorphism group of a smooth genus-2 curve is 48, attained by the curve with affine equation $y^2 = x^5 + x$. In genus 3, the bound of 168 is achieved by the famous *Klein quartic* $y^7 = x^3 - x^2$ (given in homogeneous coordinates by $x^3y + y^3z + z^3x$). A curve attaining the Hurwitz bound is known as a *Hurwitz curve*; a great deal is known about these curves and the corresponding automorphism groups.

Exercise 2.26 of [HM] asks if this bound holds for stable curves. A *stable curve* is a curve with nodal singularities and finite automorphism group. Gluing three copies of the elliptic curve with j-invariant 0 (which has six automorphisms that fix a point) to a copy of the projective line yields a genus-3 curve with $6^3 \cdot 6$ automorphisms, breaking the Hurwitz bound. In genus 2, gluing the aforementioned elliptic curve to itself yields a curve with 72 automorphisms—not breaking the Hurwitz bound, but more symmetric than any smooth genus-2 curve.

Our goal in this paper is to give a sharp bound for the automorphism group of a stable curve of genus g and to roughly describe all curves attaining this bound. Since the moduli space of stable curves is locally the quotient of a smooth germ by the automorphism group of a stable curve, this bound gives some measure of how singular this moduli space can be near the boundary.

In order to state the bound, some notation is necessary. We define three functions on positive integers g computed as follows. Write g in binary and, starting from the left-hand side, count pairs (11 or 10) of binary digits that start with 1, ignoring intermediate 0s (there may be an unpaired 1 in the ones place after pairing). Let $k(g)$ be the number of 11s and $l(g)$ the number of 10s, and set $N(g)$ equal to g if there is no unpaired 1 or to $g-1$ if there is an unpaired 1. For example: 215 is written in binary as 11010111. We count the first 11, ignore the next 0, count the 10, another 11, and there is a 1 left over. Hence $k(215) = 2$, $l(215) = 1$, and $N(215) = 214$.

Theorem 0.1 (Main Theorem). *The order of the automorphism group of a stable curve of genus g is less than or equal to $6^g \cdot 2^{N(g)} \cdot \left(\frac{3}{8}\right)^{k(g)} \cdot \left(\frac{1}{2}\right)^{l(g)}$ unless:*

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$g = 3 \cdot 2^n$ $(n \geq 0)$, *in which case the bound is* $6^g \cdot 2^{g-3} \cdot 6$; *or* $g = 5 \cdot 2^n$ $(n \geq 0)$, *in which case the bound is* $6^g \cdot 2^{g-5} \cdot 10$. *In all cases, these bounds are sharp.*

We describe the dual graph (a graph with one vertex, labeled with the genus, for each irreducible component of the curve and where two vertices are joined by an edge if the corresponding components intersect) of a curve attaining the given bounds. The dual graph will be a tree whose leaves correspond to the elliptic curve with j-invariant 0 (which has six automorphisms after fixing a point) and whose interior vertices all correspond to rational curves.

When the genus is $3 \cdot 2^n$, an optimal graph is three binary trees (with 2^n leaves each) attached by their roots to a common vertex. For genus 2^n, an optimal graph is a binary tree with 2^n leaves. A pair 11 counted as before contributes $3 \cdot 2^n$ to the genus, and a pair 10 contributes 2^n. To a chain of rational curves of length $k(g)+l(g)+(N(g)-g)$, attach the corresponding optimal curves of genus $3 \cdot 2^n$ or 2^n just described. Now stabilize the resulting curve (delete valence-2 vertices from the graph). If $g = 5 \cdot 2^n$ then the result is not quite optimal: the optimal graph is five binary trees (with 2^n leaves each) all joined to a vertex. One checks easily that the orders of the automorphism groups are the bounds given in the Main Theorem.

We will be able to do a little better; in fact, any curve attaining the bound must have a tree as dual graph. In the final section, we use this fact to describe all curves with maximal automorphism group.

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1. Geometric Preparation

Denote by E the most symmetric elliptic curve (that with j-invariant 0). We will also use $\mathbf{P}^1$ somewhat abusively to denote a smooth rational curve, and $\mathbf{P}^1$ will be coordinatized as the Riemann sphere. Given a vertex v, we will call the number of edges connecting v to a vertex corresponding to a $\mathbf{P}^1$ the *rational valence* of v; *elliptic valence* is defined similarly.

Automorphisms of stable curves come from two sources: automorphisms of their components that preserve or permute the nodes properly, and certain automorphisms of the dual graph. Not every graph automorphism is induced by an automorphism of the curve. For example, n points on $\mathbf{P}^1$ can be permuted at most dihedrally. Once another point is required to be fixed, n additional points may be permuted at most cyclically. After fixing two points (say, zero and infinity), n points may still be permuted cyclically (the nth roots of unity). All attaching of curves to copies of $\mathbf{P}^1$ will be tacitly done in the most efficient way: placing several isomorphic branches of the curve at roots of unity and then making other attachments at zero and infinity so as not to disrupt the cyclic symmetry.

DEFINITION 1.1. An automorphism of a dual graph G will be called *geometric* if it is induced by an automorphism of the corresponding stable curve. The group of such automorphisms will be denoted $\mathrm{Aut}^0 G$.

We will call the vertices of the dual graph corresponding to the rational curves *rational vertices* and those corresponding to elliptic curves *elliptic vertices.* Finally, a vertex in a graph meeting a single edge will always be called a *leaf,* whether or not the graph in question is a tree. As usual, the corresponding components of the curve are called *tails.*

DEFINITION 1.2. A *maximally symmetric stable curve* is a stable curve whose automorphism group has maximal order among all stable curves of the same genus.

Note that this definition makes sense: a stable curve of genus g has at most $2g-2$ components, each with normalization of genus at most g. Hence there is a bound for the automorphism group of a genus-g curve of $[42(2g-2)]^{2g-2}(2g-2)!$.

In this section we shrink the class of curves that need to be considered. An initial lemma will be essential in what follows.

LEMMA 1.3. *The dual graph of the stabilization of a nodal curve of genus $g > 1$ with no tails has at least as many automorphisms as the dual graph of the original curve.*

Proof. First note that if a vertex is deleted in stabilizing a graph, then all vertices in the orbit of this vertex are also deleted.

We proceed by induction on the number of vertices in the graph. There are no unstable curves of genus 2 or higher with a graph with a single node, so there is nothing to prove in this case.

The only possibility is that there is a vertex v of valence 2 corresponding to a rational curve. Proceed in both directions along geodesics away from v until vertices u and w are reached that are stable (i.e., not deleted in stabilizing). Such vertices exist because the curve has genus ≥ 2, and they may coincide. If v is moved by an automorphism then the arc from u to w is moved with it, and the converse is obviously true. Therefore, replacing the entire arc from u to w with a single edge from u to w (including the case of replacing the arc with a loop from $u = w$ to itself) does not decrease the automorphism group of the curve. □

LEMMA 1.4. *A maximally symmetric curve has only smooth components.*

Proof. Let C be any stable curve, and suppose that C_1 is a component with nodes. Replacing C_1 in C with its normalization drops the genus of C by the number of nodes of C_1. For each pair of points of the normalization lying over a node, choose one and glue a copy of E to it. This makes up the genus deficit. The automorphisms of each copy of E multiply the order of the automorphism group by 6. The automorphisms of C_1 correspond to those of its normalization that permute the nodes appropriately. Therefore, the normalization of C_1 has more automorphisms than C_1, and fewer of these are killed off by gluing elliptic curves to only one of each pair of points over a node than by identifying these points. If the normalization is genus 0 and if C_1 has only one node, then the normalized component will be collapsed to keep the curve stable and E will be re-attached at the point of attachment of the rational curve.

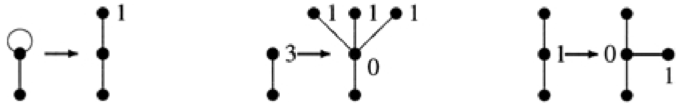

Figure 1 Illustrating Lemmas 1.4–1.6

Finally, if multiple isomorphic copies of C_1 occur in C and result in a symmetry of the dual graph that induces automorphisms of C, then replace *each* copy of C_1 by this construction to maintain the symmetry. We call this *maintaining graph symmetry* and do not explicitly mention it hereafter. □

LEMMA 1.5. *There exists a maximally symmetric stable curve whose components are all* $\mathbf{P}^1$ *or* E.

Proof. By Lemma 1.4, all components may be taken as smooth. It is clear that using E to replace an elliptic curve less symmetric than E can only help.

If C_1 is a component of C with genus $h > 1$ then replace C_1 with a rational curve, attach a rational tail, and glue h copies of E to this second rational curve. Ignoring graph symmetry, C_1 contributed at most $42(2h-2)$ automorphisms to C, whereas the new construction contributes at least $2 \cdot 6^h$ (the factor of 2 is because there are at least two copies of E that may be permuted, since $h > 1$).

There is a technical issue: $2 \cdot 6^2$ is not greater than 84, but we have already seen that there is no smooth genus-2 curve with more than $2 \cdot 6^2$ automorphisms. Also, if C is itself a smooth genus-2 curve then the construction leads to a nonstable curve. This is fine—the maximally symmetric genus-2 curve is two copies of E glued together, which satisfies the conclusion of the lemma. □

LEMMA 1.6. *A maximally symmetric stable curve's components are all copies of* $\mathbf{P}^1$ *or* E. *The dual graph of such a curve has no multiple edges, its leaves are elliptic, and other vertices are rational.*

Proof. Apply the constructions of Lemmas 1.4 and 1.5. Suppose there is a copy of E that is not a tail of the curve. Then E is attached in at least two points, so replacing E with a $\mathbf{P}^1$ and gluing E to this $\mathbf{P}^1$ does not decrease the number of automorphisms; this makes E a tail.

By Lemma 1.4 there are no loops in the graph. Suppose two vertices are connected by n edges. Since we may assume at this point that all elliptic components are tails, these vertices must be rational. Replace the multiple edge by a vertex joined by two edges to the former endpoints of the multiple edge. Add a rational tail to this new vertex, and arrange $n-1$ copies of E as tails around this rational vertex. The curve may need to be stabilized, but we have seen that in this case stabilization will not affect automorphisms.

If $n = 3$ and if the entire curve consists of two $\mathbf{P}^1$ attached in three points, then this construction does not result in a stable curve. But again this is an exceptional case, and we know that the maximally symmetric genus-2 curve satisfies the conclusions of the lemma.

The configuration of two rational curves connected by n nodes and connected some other way to the rest of the curve contributes (excluding graph symmetry) at most $2n$ automorphisms. The construction here replaces this with a configuration contributing 6^{n-1} automorphisms. □

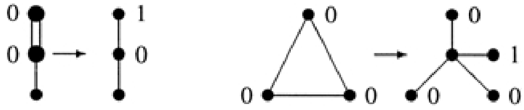

Figure 2 Illustrating Lemmas 1.6 and 2.2

2. Breaking Cycles

This section contains the main part of the reduction: we may assume that the dual graph of a maximally symmetric stable curve is a tree. To achieve this, we need to break cycles in the graph and so produce a new graph with more automorphisms. In one case, this is easy. Throughout this section we assume that all of the reductions from the previous section have been carried out; we are now studying simple graphs whose interior vertices all correspond to smooth rational curves and whose leaves are copies of E.

DEFINITION 2.1. A cycle in a graph is called *isolated* if it shares no edge with any cycle in its orbit under the action of the automorphism group of the graph.

LEMMA 2.2. *The dual graph of a maximally symmetric stable curve C has no isolated edge-transitive cycles.*

Proof. The replacement picture is the right-hand side of Figure 2. Such a cycle of n rational curves contributes 1 to the genus and contributes at most $2n$ automorphisms (dihedral symmetry). Detach the n curves in the cycle from each other; then attach each of them to a new $\mathbf{P}^1$ at the nth roots of unity, attaching a copy of E at 0. This may decrease graph symmetry by a factor of 2 (reflections in the dihedral group are not included in this case because our automorphisms must fix the point of attachment of the spoke), but it multiplies automorphisms by 6 owing to the introduction of a copy of E. □

REMARK 2.3. The assumption that the cycle is isolated is necessary so that the construction can be carried out on every cycle in the orbit.

The following proposition is obvious, but we state it for ease of reference.

PROPOSITION 2.4. *There are at most two orbits of the automorphism group among the vertices of an edge-transitive graph.*

DEFINITION 2.5. Suppose G is a graph and G' is a subgraph. Then $\mathrm{Aut}^0_{G'} G$ will denote the group of geometric automorphisms of G that fix G'.

In what follows, a graph will be called *optimal* if its geometric automorphism group is maximal among dual graphs of stable curves of a given genus.

PROPOSITION 2.6 (Edge-transitive graphs are not optimal). *Let C be a stable curve with dual graph G. If G is an edge-transitive graph with valence at least* 3 *at each of its vertices, then there exists a curve C' whose dual graph is a tree with elliptic leaves such that* $|\mathrm{Aut}\, C'| > |\mathrm{Aut}\, C|$.

Proof. By Propositoin 2.4, there are two orbits of vertices, $O(v)$ and $O(w)$, with respective valences e_v and e_w; set $n_v = |O(v)|$ and $n_w = |O(w)|$. Then $n_v e_v = n_w e_w$ and the genus of G is $g = \frac{n_v}{2}(e_v - 2) + \frac{n_w}{2}(e_w - 2) + 1$. The number of vertices of G is $n = n_v + n_w$. The case in which there exists a single orbit $O(v)$ is dealt with exactly as in the case where $e_v = e_w$.

We will bound $|\mathrm{Aut}^0 G|$ by using a sequence of trees that "grow" and eventually include all the vertices of G (spanning trees). Start with a vertex v, and denote by T_0 the tree consisting of v alone. There are n_v choices for T_0.

For T_1, take T_0 and all vertices (which are in the orbit of w) adjacent with T_0. Then $|\mathrm{Aut}^0_{T_0} T_1| \le 2e_v$ (the automorphisms of T_1 fixing T_0 at most dihedrally permute the e_v edges around v).

Assume the trees $T_0, \dots, T_i$ have been constructed. If T_i spans G, we stop. If not, there is a vertex of T_i (call it x) such that at least one of its neighbors is not in T_i. Let T_{i+1} be the span of T_i and x. We use n_i to denote the number of vertices in the tree T_i and e to denote the valence of x (which equals either e_v or e_w).

We have the following possibilities.

- If $e \ge 4$ and x has at least three neighbors in T_i, or if $e = 3$ and x has two neighbors in T_i, then $|\mathrm{Aut}^0_{T_i} T_{i+1}| = 1$. In both cases, $n_{i+1} \ge n_i + 1$.
- If $e \ge 4$ and x has at most two neighbors in T_i, or if $e = 3$ and x has exactly one neighbor in T_i, then $|\mathrm{Aut}^0_{T_i} T_{i+1}| \le 2$. Here $n_{i+1} \ge n_i + (e - 2)$ (resp. $n_{i+1} = n_i + 2$).

This process will terminate after a finite number of steps, since G has finitely many vertices.

Denoting by s_v the number of times the second possibility occurs with $x \in O(v)$ and by s_w the number of times the second possibility occurs with $x \in O(w)$, we have $n_w \ge e_v + s_v(e_v - 2)$ (resp. $n_w \ge 3 + 2s_v$ when $e_v = 3$) and $n_v \ge 1 + s_w(e_w - 2)$ (resp. $n_v \ge 1 + 2s_w$ when $e_w = 3$). At the same time, it is clear that $|\mathrm{Aut}^0 G| \le n_v \cdot 2e_v \cdot 2^{s_v + s_w}$.

A curve of genus g whose dual graph is a tree with elliptic leaves will have at least 6^g automorphisms. We want to show that $6^g > 3 \cdot 2n_v e_v \cdot 2^{s_v+s_w}$. This means $6^{\frac{n_v}{2}(e_v-2)+\frac{n_w}{2}(e_w-2)} > n_v e_v 2^{s_v+s_w}$.

We have several cases to consider, depending on the valences e_v and e_w.

Case 1: $e_v = e_w = 3$. Then $s_v \le \frac{n_w - 3}{2}$ and $s_w \le \frac{n_v - 1}{2}$, so it is sufficient to prove $6^{\frac{n_v}{2}+\frac{n_w}{2}} > 3n_v 2^{\frac{n_w-3}{2}+\frac{n_v-1}{2}}$ or $\sqrt{6}^n > \frac{3}{4} n \sqrt{2}^n$, which is clearly true.

Case 2: $e_v = 3 < e_w$. Then $s_w \le \frac{n_v-1}{e_w-2}$ and $s_v \le \frac{n_w-3}{2}$; the inequality to prove becomes $6^{\frac{n_v}{2}+\frac{n_w}{2}(e_w-2)} \ge 3n_v 2^{\frac{n_w-3}{2}+\frac{n_v-1}{e_w-2}}$. Now $n_v = n_w \frac{e_w}{3}$, so the inequality becomes (removing the index w) $6^{\frac{ne}{6}+\frac{n}{2}(e-2)} \ge ne2^{\frac{n-3}{2}+\frac{ne-3}{3(e-2)}}$ or $6^{\frac{2ne}{3}-n} \ge ne \cdot 2^{\frac{5n}{6}-\frac{3}{2}+\frac{2n-3}{3(e-2)}}$ $\left(\text{since } \frac{ne-3}{3(e-2)} = \frac{n}{3} + \frac{2n-3}{3(e-2)}\right)$. Since $e > 3$, this claim is true by the inequality $6^{\frac{2ne}{3}-n} \ge ne \cdot 2^{2n}$ or $\left(\frac{3}{2}\right)^n \cdot 6^{\frac{2n(e-3)}{3}} \ge ne$. For $n \ge 2$ we have $\left(\frac{3}{2}\right)^n \ge n$ and $6^{\frac{4(e-3)}{3}} \ge 6^{e-3} > e$ (since $e > 3$). For $n = 1$, $\frac{3}{2} \cdot 6^{\frac{2}{3}(e-3)} > \frac{3}{2}3^{e-3} > e$ since $e > 3$. So in this case also we are done.

Case 3: $e_v, e_w > 3$. Then $s_w \le \frac{n_v-1}{e_w-2}$ and $s_v \le \frac{n_w-e_v}{e_v-2}$. The inequality to prove becomes $6^{\frac{n_v}{2}(e_v-2)+\frac{n_w}{2}(e_w-2)} \ge n_v e_v \cdot 2^{\frac{n_v-1}{e_w-2}+\frac{n_w-e_v}{e_v-2}}$; since $e_v, e_w > 3$, this is implied by $6^{n_v e_v - n_v - n_w} > n_v e_v 2^{\frac{n_v+n_w}{2}}$. Using $n_w = \frac{n_v e_v}{n_w}$ and dropping the index v yields $6^{ne} > ne \cdot (6\sqrt{2})^{n+\frac{ne}{e_w}}$. Now $6\sqrt{2} < 9$ and $e_w \ge 4$, so it is enough to show that $6^{ne} > ne \cdot 3^{2n+\frac{ne}{2}}$ or $(2\sqrt{3})^{ne} > ne \cdot 3^{2n}$. Since $2n \le \frac{ne}{2}$, this is implied by $(2\sqrt{3})^{ne} > ne \cdot (\sqrt{3})^{ne}$, which becomes $2^{ne} > ne$ and so finishes the proof of this final case. $\square$

PROPOSITION 2.7. *Assume that G is a dual graph that consists of an edge-transitive graph H having rational vertices $v_1, \dots, v_n$ and with a tree T_i attached at each vertex v_i that is either degenerate (i.e., consists of v_i only) or has only elliptic leaves and otherwise rational vertices. To avoid trivialities, assume that H is not a tree. Then G is not an optimal graph.*

Proof. The point is that H has at least one cycle, and this forces less than desirable symmetry in G.

By Proposition 2.4 we know that there are at most two orbits of vertices in H. Assume that there are exactly two, $O(v)$ and $O(w)$; the other case is similar (practically identical proof using $e_v = e_w$). Denote by n_v and n_w the order of $O(v)$ and $O(w)$ in H and by e_v and e_w their respective valence in H. Then $n_v e_v = n_w e_w$. If $e_v = e_w = 2$, then H is an isolated edge-transitive cycle in G and thus G cannot be optimal. The genus of H is

$$g(H) = \frac{n_v}{2}(e_v - 2) + \frac{n_w}{2}(e_w - 2) + 1 \ge 2.$$

If all of the T_i are degenerate then G is simply H and, by Proposition 2.6, there exists an optimal tree with strictly more automorphisms than G; hence we are done in this case. If some of the T_i are not degenerate, then the genus of H is smaller than the genus of G and so, by induction, there exists an optimal tree T with $|\mathrm{Aut}^0\, T| > |\mathrm{Aut}^0\, H|$. (Note that, in light of Lemma 1.3, Proposition 2.6 applies to graphs with some vertices of valence 2.) Detach the nondegenerate isomorphic T_i (including the edge that connects their root to the vertex v_i) from the vertices of H and pair them two-by-two around a new root, ultimately connecting these roots of pairs to a new root V. Connecting V to the root of T by an edge does not change the overall genus and yields at least a 2-fold increase in automorphisms (actually a $2^{\frac{n-1}{2}}$-fold increase, where n is the number of the T_i detached); this increase is due to the extra freedom allowed on the T_i, which can be swapped independently of the vertices v_i. The reattaching can decrease the symmetry of T by a factor of at most 2, so as soon as one orbit of isomorphic trees has at least two elements we obtain the desired strict increase in symmetry. In particular, G was not optimal.

If there exists a unique tree T_i that is nondegenerate, there necessarily exists an orbit of vertices of H with a single vertex in it. Since H is edge-transitive, the condition that H not be a tree implies that H is "tree-like" (of diameter 2) but with multiple edges. From the breaking of multiple edges earlier in the proof we may derive at least a 3-fold increase in symmetry by shooting out elliptic tails in place of multiple edges. Again, G was not optimal. □

So now we need only show that—starting with a graph (that is not a tree) with elliptic leaves and otherwise rational vertices—there exists a graph of the same genus satisfying the properties of Lemma 2.2 and with at least as many automorphisms. Then the lemma shows that the original graph could not have been optimal.

LEMMA 2.8. *We may assume that an optimal graph of genus g has the following property. Around each vertex there are at most three orbits of edges and at most one orbit of two or more edges.*

Proof. Assume that in an optimal graph we have a vertex v, necessarily rational (by previous reductions), around which there exist either (a) four or more orbits or (b) at least two orbits of edges, each orbit with at least two edges (ending at v) in it.

Let us establish some notation. Arrange the orbits of edges around v in decreasing order of their sizes so that $O_1, \dots, O_k, O_{k+1}, \dots, O_{k+l}$, where $|O_1| \geq |O_2| \geq \cdots \geq |O_k| \geq 2 > 1 = |O_{k+1}| = \cdots = |O_{k+l}|$ (k or l may be zero). We must show that there exists an optimal graph in which $k \leq 1$ and $k + l \leq 3$. Note that if $k + l \geq 4$ there are no automorphisms permuting the edges around v, and if $k + l = 3$ there is at most one (nontrivial) orbit being cyclically (half-dihedrally) permuted by the automorphisms fixing v.

We will perform the following operations on the given graph:

1. detach all edges from around v, keeping track of their orbits;
2. replace v by a path of rational vertices $v_1 - v_2 - \cdots - v_k - v_{k+1} - \cdots - v_{k+l-1}$;
3. attach the orbit O_i to v_i for $1 \leq i \leq k + l - 2$, and attach the orbits O_{k+l-1} and O_{k+l} to v_{k+l-1}.

Each operation should be done simultaneously at all vertices in the orbit of v so as not to lose the initial graph symmetry. That these vertices are in the same orbit implies that the same partition of edges is repeated around each such vertex, and thus the same insertion of the new rational path can be done everywhere. Note that the genus of the graph has not been changed, as one vertex has been replaced by $k + l - 1$ vertices and $k + l - 2$ edges.

Observe that there is a map from the new graph to the old graph. Prescribing that the newly introduced paths will go (orientation-preserving) only onto another similar (newly introduced) path means that distinct automorphisms of the original graph are lifted to distinct automorphisms of the new graph. Thus we have preserved or increased the order of the automorphism group (i.e., the new graph is also optimal) and so the claim of the lemma is established.

We will need the following lemma when changing graphs into those of the type in Proposition 2.7.

LEMMA 2.9. *Let G be a simple graph of genus $g \geq 1$ with elliptic leaves and otherwise rational vertices. Assume that the valence at each rational vertex is at least* 3*—except possibly at some rational vertices $v_1, \dots, v_k$, where it may be* 2. *Let H be a subgroup of* $\mathrm{Aut}^0 G$ *that fixes the vertices $v_1, \dots, v_k$ and acts at most cyclically* (*i.e., not fully dihedrally*) *on the edges around them. Then* $2|H| < |\mathrm{Aut}^0 T|$ *for some T a tree with g elliptic tails.*

Proof. The lemma is obviously true for $g = 2$, as case-by-case inspection shows (and noting that the vertices of valence 2 do not bring any extra symmetry).

Assume that H fixes only one rational vertex v. The orbit of this vertex by H is trivial. By valence reasons, removing this vertex and the $k \geq 2$ edges around it (which can be in an orbit only by themselves) will yield a graph G' of genus $g - k + 1$ and with valence of at least 2 at any vertex, but the removal will not decrease the automorphism group by a factor of any more than k (because the action on the removed edges is at most cyclic). In other words, the subgroup of H fixing the edges around v has index at most k in H. Moreover, the automorphisms fixing the edges around v will fix their opposite ends in G'. Now either G' has genus 1 (with our assumptions on valences on G this means a cycle, which would be fixed by any automorphism fixing the edges around v) or has genus ≥ 2. In either case one can replace inductively G' by a tree T' with at least twice as many automorphisms (in the case of the cycle, one may simply use another elliptic tail), arrange another $k - 1$ elliptic tails around a root, link this root to the root of T' at infinity and fix zero as well. At most a dihedral symmetry is lost at the root of the tree replacing G' in this manner, but 6^{k-1} is gained through the elliptic tails. Overall one gains at least a factor of $\frac{6^{k-1}}{k} > 2$ in the automorphism group, which is what is needed.

Now, if H fixes several rational vertices $v_1, \dots, v_k$, then we may use the previous case to bound the larger subgroup—fixing only one of the v_i—and thus obtain the desired bound. □

THEOREM 2.10. *The optimal graph of genus g is a tree. More precisely: For any graph G of genus g that is not a tree, there exists a tree T* (*with g elliptic tails*) *such that* $|\mathrm{Aut}^0 T| > |\mathrm{Aut}^0 G|$.

Proof. Begin with an optimal graph that satisfies the conclusions of Lemma 2.8, and recall that an optimal tree should not have isolated cycles.

Choose an edge e such that:

- the order of its orbit is smallest among all orbits of edges; and
- if this order is at least 2, require that the orbit of one of its ends be the smallest among the possible choices for e.

The goal of this choice is to control the valence of the graph left by removing e and the edges in its orbit.

To fix notation for the remainder of the proof, denote by v the end of e with the smallest orbit and by w the other end; denote by n_v the order of the orbit of v and use like notation for w.

LEMMA 2.11. *Assume that e has a unique representative around v. Then one and only one of the following cases can occur*:

(1) *e has a unique representative around w*; *or*
(2) *all the edges around w are in $O(e)$.*

Proof. Assume that e would have at least another representative around w and that there would exist another edge $f \notin O(e)$. Then v and w are not in the same orbit, and $|O(e)| \geq 2n_w$. At the same time, $|O(f)| \leq n_w$ (by Lemma 2.8), contradicting the choice of e. □

We will prove Theorem 2.10 inductively. Let us establish some more notation. The connected components of $G' = G \setminus O(e)$ will be denoted by $G_1, \dots, G_k$; the graph obtained from G by contracting the components $G_1, \dots, G_k$ to vertices will be called G'' (this could have multiple edges—i.e., be nonsimple); and some optimal tree with elliptic tails and the same genus as G_i (resp. G'') will be called T_i (resp. T'').

The basic idea is to look at the connected components of G', replace them inductively (if necessary) by optimal trees, then reconnect the trees at their roots to form a common tree. Care needs to be taken with this procedure because some components may be degenerate (isolated vertices) and because some symmetry might be lost in the individual trees (those with full dihedral symmetry around their root) when they are connected (by an edge ending at their root) to some other trees. However, Lemma 2.9 shows that the latter is not a real concern.

PROPOSITION 2.12. *With the given choice of e, the following statements hold.*

(1) *If around a vertex v there are three edges in $O(e)$, then all edges ending at v are in $O(e)$.*
(2) *Removing the edges in the orbit of e from the graph G cannot leave a vertex with valence* 1.

Proof. (1) If f is an edge ending at v not in $O(e)$, then $|O(e)| \geq \frac{3}{2}n_v$ while $|O(f)| \leq n_v$, contradicting the choice of e.

Now, if $O(e)$ has only one element ending at v and w, then we are done: by stability there must be at least two other edges, not in the orbit of e (unless one or both of these is a tail, in which case the remaining valence is 0), around both

v and w. So assume there are at least two edges in the orbit of e ending at v. If there are at least three such edges, then part (1) says that all the edges around v are in $O(e)$ and so v would have valence 0 in G'. If there are precisely two edges in $O(e)$ around v, then stability of G and Lemma 2.11 show that e is not unique around w, either.

- If $O(e)$ has only two representatives around w, then the existence of an edge-transitive isolated cycle formed by edges in $O(e)$ with vertices in $O(v)$ and $O(w)$ is immediate.
- If $O(e)$ has at least three representatives around w, then part (1) says that all edges around w must be in $O(e)$. Then, using f to denote (one of) the extra edge(s) around v, we have that $|O(e)| \geq \frac{2n_v+3n_w}{2} > n_v \geq |O(f)|$, which would lead to a contradiction.

(2) Note that (a) G'' can be a tree only if e is unique in its orbit around v and (b) all G_i containing a vertex in $O(w)$ contain a unique such vertex but no vertex in $O(v)$. There are two possibilities for G': it is either connected or disconnected.

Case 1: G' is connected. Then there are no isolated vertices left in G' and all vertices have valence ≥ 2; by Proposition 2.12, e must be unique in its orbit around both v and w. The automorphism group of G' has order at least that of G (examining the movement of vertices). The genus of G' is at least one less than that of G. Stabilizing G' does not decrease the number of automorphisms and does preserve the genus—except when G' is a cycle. But then it would be isolated in its orbit and not adjacent to any edge-transitive cycles, so G would not be optimal by an argument similar to the proof of Lemma 2.2.

By induction, there exists an optimal graph that is a tree T' whose genus is (at least) one less than that of G', and $|\mathrm{Aut}^0\, T'| > |\mathrm{Aut}^0\, G'|$; now compensate for the loss of genus by attaching the necessary number of elliptic tails to the root of T'. If T' had dihedral symmetry at the root, then the loss of it (factor of 2) is easily compensated by each elliptic tail yielding a 6-fold increase in automorphisms. But this contradicts the optimality of G.

Case 2: G' is disconnected. If all the components G_i are single vertices (i.e., if all the edges around both v and w are in $O(e)$), then we are done because one of the following holds.

- All these vertices are rational, in which case G is an edge-transitive graph with only rational vertices (so there are at most two orbits of vertices in it). Lemma 2.6 shows that these are not optimal; that is, this case cannot occur with our choice of e.
- All of these vertices are elliptic, in which case G was the dual graph of two elliptic curves meeting in a node and was thus a tree.
- Some of the vertices are elliptic and some rational. In this case it is clear that there can be only one rational vertex and that all the elliptic vertices were connected by e and its translates to it. Hence G was already a tree (of diameter 2).

So we may assume that some component G_i is not an isolated point and thus must be of positive genus; then the genus of G'' is strictly less than the genus of G'

and we are in the situation described in Lemma 2.11 (i.e., $O(e)$ does not exhaust all edges around both ends of e).

Note that we have the following formula bounding $|\mathrm{Aut}^0 G|$:

$$|\mathrm{Aut}^0 G| \leq |\mathrm{Aut}^0_G G''| \cdot \prod_{i=1}^{k} |\mathrm{Aut}^0_{v,w} G_i|,$$

where $\mathrm{Aut}^0_{v,w} G_i$ is the group of automorphisms of G_i fixing the ends of edges in $O(e)$ and $\mathrm{Aut}^0_G G''$ is the group of automorphisms of G'' induced by those of G. This follows because the automorphisms fixing $O(e)$ automatically fix vertices in each component G_i.

If G has no cycles, then it is a tree and we are done. Henceforth we shall assume implicitly that G has a cycle and will show that it is not optimal.

Construct a graph H as follows. Take G'' and attach to its vertices the trees T_i via a new edge, at their roots. Note that the genus of H is the same as the genus of G, which is $\sum_{i=1}^{k} g(G_i) + g(G'')$. When does this construction provide a stable graph with more symmetry than the original G? Or rather: How much is the symmetry of the graph affected by this construction?

Lemma 2.9 shows that no symmetry is lost in replacing the G_i with T_i. It is similarly clear that fixing only the root (as opposed to any other vertex) of a tree yields the maximum number of automorphisms. Let f denote the root of the tree T_i (which may be an edge; see Section 4). Observe that

$$|\mathrm{Aut}^0 H| \geq |\mathrm{Aut}^0_G G''| \cdot \prod_{i=1}^{k} |\mathrm{Aut}^0_f T_i|$$

and so H must also be optimal. Now Proposition 2.7 finishes the proof of the fact that G was not optimal. □

We have shown that the optimal graphs are trees. Hence we may say something about the valence of the interior vertices.

DEFINITION 2.13. If the rational valence at a vertex is r and the elliptic valence is e, then we say the *valence* of this vertex is (r, e). This will cause no confusion with the usual use of the word.

LEMMA 2.14. *In an optimal graph, the valence of each rational vertex may be only one of* $(0,3)$, $(0,4)$, $(0,5)$, $(3,0)$, $(4,0)$, $(5,0)$, $(1,2)$, $(1,3)$, $(2,1)$, *or* $(3,1)$.

Proof. "Smaller" valences are ruled out by stability of the curve. Suppose there is a point of valence $(n, 0)$ with $n \geq 6$. If all of the branches from this vertex are mutually nonisomorphic, then the vertex can be replaced by a chain of rational vertices and various branches distributed in a way that decreases the valence into the allowed range. This will not affect automorphisms. At the other extreme, if all vertices are isomorphic and if $n = 2k$, then we can replace the vertex with a chain of k rational vertices and attach the branches to this pairwise. This replaces dihedral symmetry of order $4k$ with k involutions plus a global involution of the chain—at least 2^{k+1} automorphisms. If $k = 2$ this does not affect the order of the automorphism group, but if $k \geq 3$ the order increases. If n is odd, a similar procedure grouping three branches together and otherwise pairing branches works similarly. In the intermediate cases where there are some isomorphic branches but not all branches are isomorphic, split the vertex into a chain of rational vertices, one for each isomorphism class of branches; reattach the branches; and then apply the preceding arguments. The cases $(0, n)$ are handled similarly.

For a vertex of the form $(1, n)$ with $n > 3$, pair the elliptic leaves as much as possible and connect the resulting two-leaf branches (and possibly a single leaf) to the vertex. This will transfer excess elliptic valence to rational valence, which can then be distributed as before. In this case, there is one branch that cannot possibly be isomorphic to the others; hence, to make symmetry maximal, this one branch (the rational one) should be glued to the origin of $\mathbf{P}^1$, and the elliptic branches should be glued as symmetrically as possible at roots of unity. But an automorphism that fixes the origin can permute the roots of unity only cyclically, so we can reduce the valence further than in the previous cases. The cases $(n, 1)$ are handled similarly.

Finally, vertices of valence (r, e) with r and e both larger than allowed here are dealt with by adding two branches to the vertex in question, distributing the rational branches of the original vertex around one and the elliptic vertices around the other. This reduces to two vertices of types $(1, e)$ and $(r + 1, 0)$, which have already been dealt with. $\square$

Note that we cannot do better than this lemma in general: if we have a vertex of valence $(5, 0)$ and if all the branches are isomorphic, then the contribution to symmetry near this vertex is dihedral of order 10. On the other hand, if we split this into two vertices of valence $(4, 0)$ and $(3, 0)$ connected by an edge, then the connecting edge corresponds to the components of the curve being glued together; the most symmetric option is to glue the two curves at their origins and the branches at roots of unity. But gluing the origins together makes the symmetry of the roots of unity only cyclic, so we have six automorphisms only. Splitting into two branches with two leaves and a single leaf gives at most eight automorphisms.

The following definition will be useful.

Definition 2.15. A stable curve is *simple* if

(1) its components are all copies of E or $\mathbf{P}^1$;
(2) its dual graph is a tree with all leaves elliptic and all other vertices $\mathbf{P}^1$; or
(3) the valence of each vertex is no greater than 5, and the elliptic valence is no greater than 3.

The previous reductions show that there exists a maximally symmetric stable curve of genus g that is simple. The "geometric" contribution to the automorphism group of a genus-g simple curve is 6^g, and the rest comes from certain automorphisms of the graph.

Under the assumption of simplicity, elliptic components are distinguished from rational components by occurring as leaves on the tree, so we need not count automorphisms of the dual graph as a weighted graph. Therefore, the problem of finding a maximally symmetric stable curve has been reduced to finding a maximally geometrically symmetric graph of a certain type.

3. Proof of the Main Theorem

A tree has either an edge or vertex that is invariant under the action of the automorphism group (see [Se, Cor. 2.2.10]). An invariant vertex will be called a *root* of the tree; an invariant edge is called a *virtual root*. If no confusion is possible, either one will be called a *root*. We will actually need something a little stronger, as follows.

LEMMA 3.1. *If G is a tree of finite diameter n, then all geodesics of length n have* (a) *the same middle vertex if n is odd or* (b) *a common middle edge if n is even.*

Proof. This is [Se, Exer. 2.2.3].

Such a vertex (edge) will be called an *absolute* (*virtual*) *root*. Since G is a tree, given any other vertex v of G there is a unique edge adjacent to v along the geodesic to an absolute (virtual) root. Let G_v denote the subtree of G obtained as follows: remove the edge adjacent to v along the geodesic to an absolute root; then G_v is the connected component containing v of what remains.

In the rest of this section, G is always assumed to be the dual graph of a simple stable curve.

DEFINITION 3.2. Given a graph G, let $V(G)$ be the set of vertices of G and let $E(G)$ be the set of edges. Define $o_V : V(G) \to \mathbf{N}$ by $o_V(v) = |O(v)|$. When we speak of an automorphism acting on an edge, the edge is assumed to be oriented (i.e., swapping the endpoints of an edge is considered a nontrivial automorphism of that edge). With this convention, define $o_E : E(G) \to \mathbf{N}$ by $o_E(e) = |O(e)|$.

DEFINITION 3.3. A tree is called *perfect of type n* if

(1) $n = 1$: the tree has a single vertex;
(2) $n = 2$: the tree is a binary tree;
(3) $n > 2$: the tree consists of n binary trees linked to a common root.

LEMMA 3.4. *Suppose G is a tree with an edge e such that $o_E(e) = 1$. Removing e results in two connected trees G_1 and G_2. In this case,* $\mathrm{Aut}^0 G$ *decomposes as* $\mathrm{Aut}^0 G_1 \times \mathrm{Aut}^0 G_2$. *Moreover, if G is optimal then so are G_1 and G_2.*

Proof. The first part is clear: e is not moved by any automorphism, so any nontrivial automorphism must be an automorphism of G_1 or G_2 (or a composition thereof). The second part is also easy: if G_1 is not optimal, then G_1 (as a subgraph of G) could be replaced by an optimal graph with the same number of leaves of G_1, contradicting the optimality of G. □

The following lemma is the most essential part of proving the Main Theorem. It states that if a vertex v is moved by the automorphism group, then its branches should all be isomorphic (except along the geodesic leading to the absolute root).

If not, then the various copies of the branches attached to vertices in the orbit of v should be removed and grouped together to increase symmetry.

LEMMA 3.5. *Suppose v is a vertex with $o_V(v) > 1$. Then the branches of G_v around v are all isomorphic.*

Proof. The strategy is this: if v is a moving vertex and has two nonisomorphic branches, then these branches also move and a more symmetric graph can be created by grouping like branches together.

Suppose that there are two vertices v_1 and v_2 adjacent to v in G_v such that $G_{v_1} \ncong G_{v_2}$. Because v moves, there are other copies of G_{v_i} in the graph G. Denote by G_i' the tree obtained by removing G_{v_i} and its orbit under the automorphism group of G. Then $\mathrm{Aut}^0 G$ surjects onto $\mathrm{Aut}^0 G_i'$ with kernel those automorphisms fixing the vertices of G not in the image of G_{v_i} under the action of $\mathrm{Aut}^0 G$. The order of this kernel is $|\mathrm{Aut}^0 G_{v_i}|^{o_V(v_i)}$. Let G_i denote the stabilization of G_i'. Then $\mathrm{Aut}^0 G_i'$ is isomorphic to $\mathrm{Aut}^0 G_i$.

Now construct a graph G_i° from G by removing everything from G except the orbit of G_{v_i} and the corresponding geodesics to the absolute root (including the root edge if the absolute root is virtual) and stabilizing the result. Since $G_{v_1} \ncong G_{v_2}$, it follows that G_1 and G_1° are nontrivial and that the sum of their numbers of leaves is the number of leaves of G. Join G_1 and G_1° at their roots (making an appropriate construction when the root is virtual or when this makes the valence too high at the new root). The resulting graph has more automorphisms than G, since the order of $\mathrm{Aut}^0 G_1^\circ$ is at least $o_V(v_1)|\mathrm{Aut}^0 G_{v_1}|^{o_V(v_1)}$. This contradicts the optimality of G and thus proves the lemma. □

PROPOSITION 3.6. *Suppose G is optimal. Then* $\min(o_E) \leq 5$ *and* $\min(o_V) \leq 2$. *Moreover, if* $\min(o_E) \geq 3$, *then the graph consists of* $\min(o_E)$ *isomorphic subtrees attached to a common root.*

Proof. First suppose that G has an invariant vertex. Then clearly $\min(o_V) = 1$ is attained at this vertex. Since an optimal graph has valence at most 5, the orbit of an edge with this root has at most five elements. This shows that $\min(o_E) \leq 5$, since these five edges may be permuted at most among themselves.

Now suppose that G has no invariant vertex. Then there must be an invariant edge. This edge is either carried in an oriented way onto itself or the orientation is reversed, so $\min(o_E) \leq 2$. Because the edge is invariant, its endpoints can at most be taken to each other, so $\min(o_V) \leq 2$ in this case as well.

The graph has an absolute root or an absolute virtual root. If there is an absolute virtual root e, then this edge has $o_E(e) = 1$ or $o_E(e) = 2$. If $\min(o_E) \geq 3$ then there is an absolute root v. Consider the isomorphism classes of the branches from v. If any class has a single member then the edge from v to the root of that branch is an invariant edge, which contradicts $\min(o_E) \geq 3$. Now, if there are at least two isomorphism classes (each with more than one member) then, by the proof of Lemma 2.14, we have a contradiction of optimality. It is clear that one of these edges attains the minimum of o_E (since the whole graph rotates around v), which proves the last statement of the proposition. □

PROPOSITION 3.7. *An optimal graph with $2g$ leaves can be obtained from an optimal graph with g leaves by replacing each leaf with a vertex attached to two leaves.*

Proof. It suffices to show that an optimal graph with $2g$ leaves has the property that exactly two leaves are connected to a vertex that is connected to any leaves. Such a graph is certainly obtained by "doubling". Conversely, if such a graph has its leaves removed to obtain a nonoptimal graph with g leaves, then doubling an optimal graph with g leaves will produce a more symmetric graph with $2g$ leaves.

For reasons of valence, the only configurations of leaves other than two per branch are branches with one leaf or branches with three, except for the two cases of four leaves around a root and five leaves around a root. Five is not an even number and so the lemma doesn't apply, and there is an optimal tree with four leaves that is doubled from the optimal tree with two leaves. By Lemma 3.5 and the stability of the curve in question, branches with one leaf are not permuted by the geometric automorphism group: by stability, there must be at least two edges other than the one to the leaf, and removing one on the geodesic to the root leaves a tree with at least one rational branch and one elliptic leaf. An even number of such branches may be combined pairwise to increase the order of the automorphism group (remove one leaf and place it on a branch with another, yielding an involution). Therefore, an *optimal* graph contains at most one such branch.

Now suppose that v_1 is a vertex adjacent to three leaves. We claim that $o_V(v_1) = 1$. Denote by v_0 the vertex one step from v_1 toward the absolute root (if v_1 is the absolute root then the claim is clearly true).

Suppose that $o_V(v_0) > 1$. Then Lemma 3.5 implies that all vertices one unit away from v_0 in the tree G_{v_0} are branches with three leaves. By valence considerations, the only possibilities are that G_{v_0} has two to five branches. If there are two branches, split the six leaves into three branches with two leaves each. If there are three branches, split them into the configuration shown in Figure 3. In both cases, the contribution to automorphisms increases, in the first case from 18 to 24 and in the second from 81 to 128. Similar constructions can be performed on a configuration of four or five branches of three leaves around a root (the answers are given by the Main Theorem) to obtain more automorphisms. This contradicts optimality, so $o_V(v_0) = 1$.

Now, if there is another branch of G_{v_0} adjacent to v_0 that has three leaves, these could be combined as in the previous paragraph with the leaves around v_1, contradicting optimality. Hence v_1 is the only branch of G_{v_0} adjacent to v_0 with three leaves; therefore, if it is moved by some automorphism of G then v_0 will follow. This contradicts $o_V(v_0) = 1$.

Thus, branches with an odd number of leaves do not move around the graph. Hence they may be broken up to increase symmetry: pairing two branches with a single leaf adds an involution, switching the leaves. Pairing a single leaf with a branch with three leaves and splitting into pairs increases the automorphism group by a factor of at least $8/3$. Similarly, two branches with three leaves each can be combined. These constructions contradict optimality of the graph, and

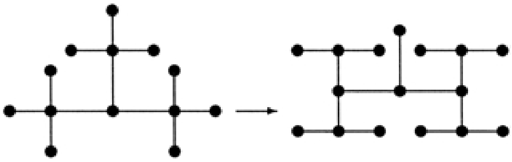

Figure 3 Splitting a 3-3-3 configuration in Lemma 3.7

we conclude that an optimal graph of even order has exactly two leaves on every branch that has any leaves at all. The proposition is proved. □

Replacing each leaf with two according to Proposition 3.7 will be called *doubling* a graph. The following lemma strengthens Lemma 3.5.

LEMMA 3.8. *If* $o_V(v) > 1$ *for some vertex* v*, then the branches of* G_v *around* v *are all isomorphic* perfect *trees.*

Proof. If the number of leaves of one of these branches is even, then Lemma 3.7 allows us to prove the result by induction. If the number of leaves on a branch is not even, then each branch has a subbranch with three leaves—but we have seen that such configurations are not optimal, so the number of leaves must be even. □

Now the preliminaries are in place, and the Main Theorem may be proved.

Proof of Main Theorem. The genus-2 case is easily checked by hand. The base case for part two is that of genus 3, which follows from the fact that there is a unique simple graph among dual graphs of genus-3 curves. In genus 4 there are two simple dual graphs, both maximally symmetric and one satisfying the form of the theorem. In genus 5 it is also easy to find a maximally symmetric graph among the simple graphs, which is the final case of the Main Theorem.

The proof proceeds by induction on the number of binary digits of g. Suppose the result is known for g with m or fewer binary digits. Lemma 3.7 then shows that if g has $m+1$ binary digits and the last digit is 0, the result follows. So we may suppose that g has $m+1$ digits and the last digit is 1. This implies that $\min(o_E) \neq 2$ (otherwise there would be an even number of leaves). If $\min(o_E) \geq 3$, then we are done by Lemma 3.8 and Proposition 3.6.

The only remaining possibility is that g is odd and $\min(o_V) = 1$. That is, there is an invariant edge (there may be several such edges). Let e be an invariant edge where the ratio between the number of vertices on the large side and small side is maximized. Remove this edge and call the larger resulting graph G_1 and the smaller resulting graph G_2. By Lemma 3.4 it follows that $\mathrm{Aut}^0 G \cong \mathrm{Aut}^0 G_1 \times \mathrm{Aut}^0 G_2$ and that G_1 and G_2 are optimal.

If G_2 has only one vertex then G_2 contributes nothing to $\mathrm{Aut}^0 G$. Therefore, G is obtained from an optimal graph by adding a vertex. We may add an edge to an existing appendage so that the resulting new appendage is maximally symmetric. This will yield the wrong answer if the appendage grows too large (i.e., becomes a binary tree). But then the answer has been given by Lemma 3.8, so the case of G_2 a single vertex is done.

Either G_1 or G_2 has an odd number of leaves. Suppose first that G_1 has an odd number. By the induction hypothesis, G_2 is doubled from a graph with half as many leaves and so it has no vertices of valence $(2,1)$, $(3,1)$, or $(1,3)$. On the other hand, G_1 must have an invariant vertex of one of these three types. Since the edge connecting this invariant vertex of valence $(2,1)$ or $(3,1)$ to a leaf must be invariant, vertices of valence $(2,1)$ and $(3,1)$ do not occur (the ratio of the number of vertices in G_1 to that in G_2 was chosen to be maximal, and G_2 has at least two vertices). Thus G_1 has a vertex of valence $(1,3)$, which is unique by induction. Considering the subtree of G rooted at this vertex shows that G_2 has at most two leaves; otherwise, G could be divided at the $(1,3)$ vertex to yield a higher weight ratio. Yet because here the $(1,3)$ vertex of G_1 is invariant, this subtree can be removed and then joined to the branch supporting the two leaves of G_2 with the leaves redistributed to increase the order of the automorphism group.

Hence G_2, the smaller graph, has an odd number of leaves. Previous arguments on G_1 show that it is enough to consider when G_2 has three leaves. By induction, we have one of the following three cases.

Case 1: G_1 has $3 \cdot 2^n + a$ leaves and is of the form given by the theorem. If $a+3 < 2^n$, then G fits the form of the theorem: the three leaves of G_2 are part of an appendage. In any case, the appendage of G_1 is itself a nested collection of maximally symmetric trees of the types given, so G_2 is attached to the last of these. The problem thus reduces to adding a branch with three leaves to an appendage with six leaves. It is easy to see that any such configuration can be rearranged to give more automorphisms; therefore, a G_2 with three leaves does not actually occur in this case.

Case 2: G_1 has $4 \cdot 2^n + b$ leaves and is of the form given by the theorem. If $b+3 < 2^{n+1}$, then G fits as in case 1. An argument similar to that given previously shows that this border crossing does not happen here, either (in this case, adding a branch with three leaves to a branch with two leaves does not give an optimal configuration of five leaves).

Case 3: G_1 has $5 \cdot 2^n$ leaves. It is clear that adding G_2 to a maximal G_1 as given in the theorem will not yield a maximally symmetric curve (the root may be broken), so this case does not occur. □

4. Description of All Maximally Symmetric Curves

In Sections 2 and 3 we described one way of finding a maximally symmetric stable curve of genus g. It is natural to ask if the curve found is unique. The first counterexample occurs in genus 4, where there are two maximally symmetric curves; this happens again in genus 7 (see Figure 4).

As the genus increases, even worse nonuniqueness can occur. However, we can describe all maximally symmetric curves of a given genus.

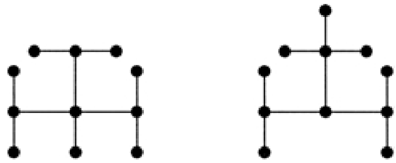

Figure 4 Nonuniqueness in genus 7

PROPOSITION 4.1. *A maximally symmetric curve of genus g has*:

(1) *all components* $\mathbf{P}^1$ *or* E; *and*

(2) *dual graph a tree with all leaves elliptic and other vertices rational.*

Proof. We actually gain automorphisms in all of the reduction steps, with one exception: reducing valence at a vertex where neighboring vertices are nonisomorphic does not necessarily add any automorphisms. Therefore, conditions of valence were omitted from the conditions of simplicity giving the present result. □

An inspection of the proof of Lemma 3.5 shows that the restrictions on valence were not used there. As a result, Lemma 3.5 may be applied to nonsimple curves.

LEMMA 4.2. *Let G_0 indicate the subtree of G that is fixed by all geometric automorphisms of G. If G is optimal, then each leaf of G_0 is the root of a perfect subtree of G. Moreover, trees of types* 4 *and* 5 *may occur only when G_0 is a point.*

Proof. Suppose t_0 is a leaf of G_0 that is not the root of a perfect subtree. Since t_0 is a leaf of the fixed subtree, none of its neighbors outside of the fixed subtree are fixed. Then Lemma 3.8 states that the subtrees whose roots are these neighbors (call them v_i) have all isomorphic branches and that these branches are perfect.

We now claim that (i) the G_{v_i} are all isomorphic and (ii) there are at most five of them. The second claim follows from the first because, if all the branches are isomorphic and greater in number than five, they can be rearranged (by Lemma 2.14) to contradict the optimality of G.

Suppose at least two of the subtrees are nonisomorphic. Then there are at least two orbits of trees around t_0. However, since the point of attachment of t_0 to the rest of the invariant tree must be fixed by any automorphism of the $\mathbf{P}^1$ corresponding to t_0 in the curve, there are not enough automorphisms of the $\mathbf{P}^1$ left to realize every possible graph symmetry (since all orbits must be nontrivial). All of the symmetries can be realized by splitting t_0 and rearranging the branches, which contradicts optimality. Consequently, there are at most five subtrees and they are all isomorphic. The lemma follows, since these subtrees are themselves perfect.

The last part of the lemma follows also because a tree of type 4 or 5 attached to a leaf of a nontrivial G_0 cannot realize its full symmetry group . Then splitting the tree into two trees increases the order of the automorphism group, contradicting the optimality of G. □

The following definition, especially the last condition, serves to isolate the binary pairs 10 and 11 occurring in the proof of the Main Theorem. Note the exception in (4); without it, there is no strict optimal graph in genus 11. The upshot of this definition and the proofs to follow is that the behavior in genus 7 and 11 somehow is the whole picture.

DEFINITION 4.3. Call a tree G *strict optimal* if the following statements hold.

(1) G is the union of the fixed subtree G_0 and k perfect subtrees G_i whose roots are on G_0; index the G_i so that their numbers of leaves N_i are decreasing.
(2) The roots of the G_i are the leaves of G_0.
(3) The valence at interior vertices of G_0 is exactly 3.
(4) $N_i \geq 4N_{i-1}$ for $i = 2, \ldots, k$, except that G_i is type 2 with 2^{s+3} leaves and G_{i-1} is type 3 with $3 \cdot 2^s$ leaves.

LEMMA 4.4. *A strict optimal tree is optimal.*

Proof. This follows from Lemma 3.4 (the automorphism group of a strict optimal tree is the direct product of the automorphism groups of its subtrees rooted at leaves on the invariant tree) and Definition 4.3(4), which allows us to compute the order of the automorphism group of a strict optimal tree and see that it has the value given in the Main Theorem. □

DEFINITION 4.5. If an optimal graph has two perfect subtrees G_i and G_j with 2^{s+2} and $3 \cdot 2^s$ leaves, respectively, then we define a *neutral move of Type I* as follows. Remove a perfect subtree with 2^s leaves from G_j (leaving it a binary tree with 2^{s+1} leaves) and attach it to a different vertex of G_0, splitting an edge with a new vertex if necessary to keep valence low (note in particular that a neutral move for a given tree is not unique). Now attach the rest of G_j (the aforementioned binary tree) at the root of G_i, resulting in a perfect subtree with $3 \cdot 2^{s+1}$ leaves. This process will tacitly be followed by any stabilization of the graph necessitated by bad choices.

We define a *neutral move of Type II* only for an (optimal) tree with precisely 2^n ($n \geq 2$) leaves. In this case, if four perfect binary trees are sharing the common root G_0, then we separate two of the four trees around another rational node that is linked by an edge to the original rational root.

The definition is easier to grasp in light of the examples of nonuniqueness in genus 7. The left-hand example in Figure 4 is a strict optimal tree. Its invariant subtree is the lower central segment, bearing a perfect subtree with six leaves and also one with a single leaf. This satisfies the inequalities in the definition of strictness. The right-hand example has as its most central vertical segment an invariant subtree bearing perfect subtrees with four and three leaves. This violates strictness. Here we are in the situation of the previous definition with $s = 0$. Remove the highest vertical edge in the figure and place it at the lower vertex of the invariant tree. All neutral moves are obtained by "doubling" this move.

PROPOSITION 4.6. *A neutral move preserves the order of the geometric automorphism group of the graph.*

Proof. Using Lemma 3.7 backwards, the situation of the definition of neutral move reduces to the case of subtrees of orders 3 and 4, where the explanation of the genus-7 example clearly shows that the automorphism group does not grow or shrink. □

Our next result shows that the Main Theorem is close to giving all maximally symmetric curves. The motivation is trying to reverse the formula of the Main Theorem: pairing binary digits, we try to reconstruct the tree. A neutral move occurs when a odd-length sequence of 1s (three or more) occurs in the binary expansion. The proof follows slightly different lines.

THEOREM 4.7. *Every maximally symmetric genus-g curve has either a strict optimal dual graph or a dual graph that can be made strict optimal by a sequence of neutral moves, valence reduction, and stabilization.*

Proof. Clearly a maximally symmetric curve must have an optimal graph. As before, denote the subtrees rooted at leaves of the invariant tree G_0 by $G_1, \dots, G_k$, ordered so that the number of leaves in these subtrees is decreasing. Since the G_i are rooted at invariant nodes, no two have the same number of leaves. If they did then, since the G_i are perfect, they would be isomorphic and so the graph could be rearranged to be more symmetric. Therefore, $N_1 > N_2 > \cdots > N_k$. We also have $\mathrm{Aut}^0 G = \prod_{i=1}^{k} \mathrm{Aut}^0 G_i$.

If G itself is a perfect tree then it can have 2^n, $3 \cdot 2^n$, or $5 \cdot 2^n$ leaves. In the last two cases (or in the first when $n = 1$) there are no other optimal graphs possible, by Lemma 3.8. In the first case with $n \geq 2$, there are two possibilities, (i) $\min(o_E) = 4$ or (ii) $\min(o_E) = 2$. For (i) we do a neutral move of Type II to obtain a strictly optimal graph; with (ii), the graph is already strictly optimal.

If G is not a perfect tree then we have at least two distinct leaves in G_0. Moreover, valence considerations ensure that none of the G_i can be perfect of type 4 or 5. By induction, we may remove the subtree G_1 and assume that the tree remaining is optimal and satisfies the conclusion of the theorem.

Let G_1 be perfect of type 3 with $N_1 = 3 \cdot 2^s$ leaves.

1. If G_2 is of type 2 and $N_2 = 2^p$ then $p \leq s + 1$; if $p = s + 1$ or $p = s$, then G is not optimal (“undouble” and rearrange). Therefore, $N_1 \geq 6N_2 > 4N_2$.
2. If G_2 is of type 3 and $N_2 = 3 \cdot 2^p$ then $p \leq s - 1$; an undoubling argument shows that if $p = s - 1$ then G is not optimal, as in the first case. Therefore, again we have $N_1 \geq 4N_2$.

Consequently, if G_1 is of type 3 then G is strict optimal by induction.

Now let G_1 be of type 2 with $N_1 = 2^{s+2}$ leaves (the case $g = 2$ is clear).

1. If G_2 is of type 2 and $N_2 = 2^p$ then $p < s + 1$ (by optimality), so $N_1 \geq 4N_2$.
2. If G_2 is of type 3 and $N_2 = 3 \cdot 2^p$ then we have the following subcases.
   (a) $p < s - 1$: $N_1 \geq 4N_2$.
   (b) $p = s - 1$: $N_1 = \frac{8}{3} N_2$ (this is the “exception” in the condition of strictness).
   (c) $p = s$: do a neutral move to change N_1 to $3 \cdot 2^{s+1}$ and N_2 to 2^s; then, in the new tree, $N_1 \geq 4N_2$.

Having achieved the numerical condition, it is easy to achieve valence 3 at every interior vertex of the fixed tree. A perfect subtree attached to an interior node may be branched out so that it is rooted at a leaf (the new edge will also be invariant). $\square$

REMARK 4.8. In many cases, the number of maximally symmetric curves is finite (in some cases—notably $5 \cdot 2^n$, $3 \cdot 2^n$, and 2^n—it is unique). But there are cases of a positive-dimensional family of maximally symmetric curves (exactly when $k(g) + l(g) + g - N(g) - 3$ is positive, in which case this quantity is the dimension of the family of maximally symmetric stable curves). The easiest example to see is probably in genus $1 + 4 + 16 + 64$. By the Main Theorem, a maximally symmetric curve is constructed by first arranging a binary tree with 64 leaves, then attaching to its root a maximally symmetric curve of genus $1 + 4 + 16$, and so on. In the end, there is a root connected to binary trees with 1, 4, 16, and 64 leaves. As in the previous theorem, this root could be split to yield strict optimal trees (note in particular that strict optimal trees are not unique in a given genus), but this does not affect automorphisms and so we might as well keep all four branches tied to a single root. However, the automorphism group of $\mathbf{P}^1$ is only three-point transitive: after attaching the first three branches, there are infinitely many choices for the point of attachment of the fourth branch.

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