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Fixed Points and Determining Sets for Holomorphic Self-Maps of a Hyperbolic Manifold

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0. Introduction

Let M be a complex manifold. Then $H(M, M)$ is the set of holomorphic maps from M to M, that is, the set of endomorphisms of M. A special case of endomorphisms are automorphisms of M, $\text{Aut}(M) \subset H(M, M)$.

Definition 0.1. A set $K \subset M$ is called a *determining subset* of M with respect to $\text{Aut}(D)$ ($H(M, M)$ resp.) if, whenever g is an automorphism (endomorphism resp.) such that $g(k) = k$ for all $k \in K$, then g is the identity map of M.

The notion of a determining set was first introduced in a paper written by the first two authors in collaboration with Steven G. Krantz and Kang-Tae Kim [F+1]. That paper was an attempt to find a higher-dimensional analog of the following result of classical function theory [PL]: If $f: M \to M$ is a conformal self-mapping of a plane domain M that fixes three distinct points, then $f(\zeta) = \zeta$.

This one-dimensional result is true even for endomorphisms of a bounded domain $D \subset\subset \mathbb{C}$. To prove this, one may first use the well-known theorem stating that if an endomorphism of D fixes two distinct points then it is an automorphism; then use the [PL] theorem. In the n-dimensional case, determining sets (for automorphisms and endomorphisms) of bounded domains in $\mathbb{C}^n$ have been investigated in [F+2; FMa; KiKr; V1; V2].

Let $W_s(M)$ denote the set of s-tuples $(x_1, \dots, x_s)$, where $x_j \in M$, such that $\{x_1, \dots, x_s\}$ is a determining set with respect to $\text{Aut}(M)$. Similarly, $\hat{W}_s(M)$ denotes the set of s-tuples $(x_1, \dots, x_s)$ such that $\{x_1, \dots, x_s\}$ is a determining set with respect to $H(M, M)$. Hence $\hat{W}_s(M) \subseteq W_s(M) \subseteq M^s$. We now introduce two values $s_0(M)$ and $\hat{s}_0(M)$. If $\text{Aut}(M) = \text{id}$ then $s_0(M) = 0$; otherwise, $s_0(M)$ is the least integer s such that $W_s(M) \neq \emptyset$. If $W_s(M) = \emptyset$ for all s then $s_0(M) = \infty$. Analogously, $\hat{s}_0(M)$ denotes the least integer s such that $\hat{W}_s(M) \neq \emptyset$; if no such integer exists (i.e., if $\hat{W}_s(M) = \emptyset$ for all s) then $\hat{s}_0(M) = \infty$. In all cases $s_0(M) \le \hat{s}_0(M)$.

The main objectives of this paper are (1) to generalize the results for bounded domains in $\mathbb{C}^n$ to hyperbolic manifolds and (2) to illustrate that the results are quite different for the nonhyperbolic manifolds.

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The Bergman metric on a bounded domain in $\mathbb{C}^n$ proved quite useful for the investigation related to determining sets. However, such a Riemannian metric is not always available on a hyperbolic manifold M. In order to overcome this obstacle, we construct for any point $x \in M$ an invariant (with respect to $\mathrm{Aut}(M)$) Hermitian metric in a neighborhood (open but not necessarily connected) of that point.

The paper is roughly divided into three parts. First we introduce the Hermitian metric just mentioned. Second, we completely resolve the case for a hyperbolic manifold. Third, we prove (two) theorems to show that the case of nonhyperbolic manifolds is remarkably different.

Here is a brief description of the second and third parts of the paper. In [V2] the estimate $\hat{s}_0(D) \le n + 1$ was established for all bounded domains in $\mathbb{C}^n$. In Section 2 we generalize this result by proving the same inequality for hyperbolic manifolds of dimension n. This certainly implies the same inequality for *automorphisms* of a hyperbolic manifold M, $s_0(M) \le n+1$. Yet we can provide much more information for automorphisms. The value of $s_0(M)$ depends on how large the group $\mathrm{Aut}(M)$ is, and corresponding estimates on $s_0(M)$ are given in Section 3. In Section 4 we show that if $\dim(M) = n$ then the general estimate ($s_0(M) \le n + 1$) can be refined to $s_0(M) \le n$ for domains that are not biholomorphic to the unit ball $B^n \subset \mathbb{C}^n$ (i.e., the only hyperbolic manifolds for which $s_0(M) = n + 1$ are those biholomorphic to the ball).

For any positive integer $s \ge s_0(M)$ we have $W_s(M) \ne \emptyset$, so there are s points such that if an automorphism of M fixes these points it will fix any point of M. Now the question arises as to whether the choice of these s points is generic. The answer is positive for any hyperbolic manifold M: $W_s(M) \subseteq M^s$ is open and dense if not empty (Section 5). Similar topological properties for the determining sets of *endomorphisms* of a general hyperbolic manifold do not hold. We address related questions in the concluding part of Section 5.

Section 6 is devoted to examining the situation for *nonhyperbolic* manifolds. We first give a complete description of the value for $\hat{s}_0(M)$ for a one-dimensional manifold M (Theorem 6.1). Then for higher-dimensional manifolds we prove that $\hat{s}_0(M) = \infty$ for a general Stein manifold M that has the following property: Any finite number of points lie in a one-dimensional submanifold (for the precise statement see Theorem 6.2).

1. Construction of a Locally Invariant Hermitian Metric

Our main effort in this section will be the construction of a locally invariant (with respect to the automorphism group) metric in a neighborhood of any point in a general hyperbolic manifold. Throughout this section, M denotes a hyperbolic manifold of finite dimension, and $\mathrm{Aut}(M)$ is its group of holomorphic automorphisms. First we present some preliminary statements.

LEMMA 1.1. $\mathrm{Aut}(M)$ *is a normal family.*

Various versions of this statement have been used before. However, we cannot find a direct reference to this result in the literature, so a brief proof is presented here.

Proof of Lemma 1.1. It suffices to prove that if $x_0 \in M$, if $f_j \in \mathrm{Aut}(M)$ is a sequence such that the closure Q of the set $\{f_j(x_0) : j \in \mathbb{N}\}$ is compact, and if K is a compact subset of M, then

$$S := \bigcup_{j=1}^{\infty} f_j(K) \subset\subset M.$$

Let $d(\cdot,\cdot)$ denote the Kobayashi distance. For $x \in M$ and $r > 0$, let $b(x,r) = \{y \in M : d(x,y) < r\}$. Let $\psi(x) = \sup\{r > 0 : b(x,r) \subset\subset M\}$. Now we set

$$m = \max\{d(x_0,x) : x \in K\}, \qquad \delta = \min\{\psi(x) : x \in K\},$$

and

$$P = \{x \in M : d(x,Q) \le m, \psi(x) \ge \delta\}.$$

Then P is compact and $S \subset P$. □

Now we note the following. Let $a \in M$ and let $f : M \to M$ be a holomorphic map such that $f(a) = a$. Consider a small Kobayashi ball $b = b(a,\varepsilon)$ that is biholomorphic to a bounded domain in $\mathbb{C}^n$ and whose closure is compact in M. Since the Kobayashi distance is nonincreasing under holomorphic maps, we have $f : b \to b$. If $f \in \mathrm{Aut}(M)$, then $f|_b \in \mathrm{Aut}(b)$. The following three statements (cf. [V1]) hold for bounded domains in $\mathbb{C}^n$; by using this remark one can prove them for any hyperbolic manifold.

LEMMA 1.2. *Let $a \in M$ and let $f : M \to M$ be a holomorphic map such that $f(a) = a$ and $f'(a) = \mathrm{id}$. Then $f = \mathrm{id}$.*

LEMMA 1.3. *Let $a \in M$, $f \in \mathrm{Aut}(M)$, and $f(a) = a$. Then all the eigenvalues of $f'(a)$ are of modulus 1 and the matrix $f'(a)$ is diagonalizable.*

COROLLARY 1.4. *In the assumption of Lemma 1.3, if $f \ne \mathrm{id}$ then one can find an appropriate power k such that the kth iteration of f, $f^k = h \in \mathrm{Aut}(M)$, will have the properties that $h(a) = a$ and $h'(a)$ has at least one eigenvalue with nonpositive real part.*

Let $z \in M$. Hereafter we use the notion of an isotropy group $I_z(M) = \{g \in \mathrm{Aut}(M) : g(z) = z\}$.

LEMMA 1.5 [C1, p. 80]. *Let $D \subset\subset \mathbb{C}^n$, let $z \in D$, and let $I_z = I_z(D)$ be the isotropy subgroup at z of the automorphism group of D. Then there exists a holomorphic map $\phi : D \to \mathbb{C}^n$ such that $\phi(z) = 0$ and $\phi'(z) = \mathrm{id}$ and such that, for all $f \in I_z$, one has $\phi \circ f = f'(z) \circ \phi$.*

As in [V1, Thm. 2.3], for the proof of this lemma we define $\phi : D \to \mathbb{C}^n$ by

$$\phi(\zeta) = \int_{G_z} f'(z)^{-1}(f(\zeta) - z)\, d\mu(f),$$

where $d\mu$ is the Haar measure on I_z. Then $\phi(z) = 0$, $\phi'(z) = \mathrm{id}$ (and so ϕ is locally biholomorphic), and $\phi \circ g = g'(z) \circ \phi$ for each $g \in I_z$.

Let M again be a hyperbolic manifold. For $x \in M$, let $T_x M$ be the tangent space of M at x and let $I_x = I_x(M)$ be the isotropy subgroup fixing x. The compact group I_x acts on T as differential maps: for $g \in I_x$ and $v \in T$, we have $g_*(v) = dg(x)v$. Since Lemma 1.5 can be considered in a small neighborhood of x and since T is isomorphic to $\mathbb{C}^n$, the following statement holds.

LEMMA 1.6. *For any point $x \in M$, there exists a small neighborhood $V \ni x$ such that there is an injective holomorphic map $\phi: V \to T$ with $g_* \circ \phi = \phi \circ g$ for $g \in I_x$ and $d\phi(x) = \mathrm{id}$, the identity map of $T = T_x M$.*

Finally we introduce a Hermitian invariant metric on a neighborhood of any point in M.

LEMMA 1.7. *Let M be a hyperbolic manifold, let $G = \mathrm{Aut}(M)$, and let $x \in M$. Then there exist a neighborhood U of x such that $G(U) = U$ and a C^∞ Hermitian metric on U that is invariant under G.*

Proof. Since M is hyperbolic, the automorphism group G is a Lie group (see [Ko]) and the isotropy group I_x is a compact subgroup of G. The orbit $G(x)$ is an embedded submanifold of M. Let $T = T_x M$ be the tangent space of M at x. Then T is a complex vector space and is isomorphic to $\mathbb{C}^n$. The elements of the compact group I_x act on T as differential maps: for $g \in I_x$, $g_*(v) = dg(x)v$. Let h be a Hermitian metric on T invariant under I_x. By Lemma 1.6, there exist a small neighborhood V of x in M and an injective holomorphic map $\phi: V \to T$ such that $g_* \circ \phi = \phi \circ g$ for $g \in I_x$ and $d\phi(x) = \mathrm{id}$, the identity map of $T = T_x M$. The real subspace P of T consisting of vectors tangent to $G(x)$ is invariant under I_x. Hence the orthogonal complement (with respect to the real part of h) Q of P is also invariant under I_x.

Let $S_1 = \{v \in Q : \|v\| < \delta\}$, where $\|\cdot\|$ is the norm induced by the Hermitian metric h, and choose $\delta > 0$ so small that $S_1 \subset\subset \phi(V)$. Note that S_1 is invariant under I_x. Let $S = \phi^{-1}(S_1)$; then $I_x(S) = S$. Furthermore, for $g \in G$, we have $g(S) \cap S \neq \emptyset$ if and only if $g \in I_x$. The tube $G(S)$ is diffeomorphic to the normal bundle of $G(x)$ in M and to the twisted product $G \times_{I_s} S$. The pull-back $h_0 = (\phi|_S)^* h$ is a Hermitian metric on the restriction to S of the tangent bundle TM. Now we define a Hermitian metric h_1 on $U = G(S)$ as follows. If $y \in U$ and $u, v \in T_y$, then there is a $g \in G$ such that $g(y) \in S$ and we define $h_1(u, v) = h_0(g_* u, g_* v)$. One can see that h_1 is well-defined, since if $g(y), g'(y) \in S$ then $g' g^{-1} \in I_x$. Now h_1 is a C^∞ metric on U that is invariant under G. □

2. An Estimate for $\hat{s}_0(M)$

We need the following lemma [V2, Thm. 5.2].

LEMMA 2.1. *Let D be a bounded domain in $\mathbb{C}^n$. If $a \in D$ then there is an open $U \subset D^n$ such that $(a, \dots, a) \in \bar{U}$ and, for all $(z_1, \dots, z_n) \in U$, $(a, z_1, \dots, z_n) \in \hat{W}_{n+1}(D)$.*

THEOREM 2.2. *Let M be a hyperbolic manifold of complex dimension n. Then $\hat{s}_0(M) \le n + 1$.*

Proof. Pick a point $a \in M$. Let $f\colon M \to M$ be a holomorphic map such that $f(a) = a$. Consider a small Kobayashi ball $b = b(a, \varepsilon)$ whose closure is compact in M and such that b is biholomorphic to a bounded domain D in $\mathbb{C}^n$; let $h\colon b \to D$ be such a biholomorphic map. Observe that, since the Kobayashi distance is nonincreasing under holomorphic maps, we have $f\colon b \to b$ and therefore $g = h \circ f \circ h^{-1}\colon D \to D$. Using Lemma 2.1, we can pick n points $z_1, \ldots, z_n \in D$ such that $Z = (h(a), z_1, \ldots, z_n) \in \hat{W}_{n+1}(D)$. Consider the set of $n+1$ points $h^{-1}(Z) = (a, h^{-1}(z_1), \ldots, h^{-1}(z_n)) \subset b$. If our function $f \in H(M, M)$ (in addition to a) is also fixing all points $h^{-1}(z_j)$ (i.e., if $f|_{h^{-1}(Z)} = \mathrm{id}$), then $g|_Z = \mathrm{id}$ and therefore $g = \mathrm{id}$. We conclude that $f|_b = \mathrm{id}$ and consequently $f = \mathrm{id}$. Hence, $h^{-1}(Z) \in \hat{W}_{n+1}(M)$ and so $\hat{s}_0(M) \leq n+1$. □

3. Estimates for $s_0(M)$

The goal of this section is to provide estimates for $s_0(M)$ for a hyperbolic manifold M, $\dim(M) = n$. Since $s_0(M) \leq \hat{s}_0(M)$, Theorem 2.2 implies the following statement.

For any hyperbolic manifold M of complex dimension n, $s_0(M) \leq n+1$.

Remark. In the next section we prove a refined inequality $s_0(M) \leq n$ for M not biholomorphic to the unit ball in $\mathbb{C}^n$.

If H is (isomorphic to) a subgroup of the unitary group $U(n)$, let $k(H)$ denote the least number k of vectors $u_1, \ldots, u_k$ such that, if $h \in H$ and if $h(u_j) = u_j$ for $j = 1, \ldots, k$, then $h = \mathrm{id}$. For $x \in M$ the isotropy group $I_x(M)$ is isomorphic to the group of its differentials at x, and these differentials are unitary with respect to the locally defined Hermitian inner product (whose existence was proved in Lemma 1.7) on the tangent space $T_x(M)$. So $I_x(M)$ is isomorphic to a subgroup of $U(n)$.

Theorem 3.1. $s_0(M) \leq 1 + \min\{k(I_x(M)) : x \in M\}$.

Proof. Choose $x \in M$ so that $k(I_x(M)) = \min\{k(I_x(M)) : x \in M\}$. Denote that number by k. Let $u_1, \ldots, u_k$ be vectors in $T_x M$ such that, if $h \in I_x(M)$ and if $dh(x)(u_j) = u_j$ for $j = 1, \ldots, k$, then $dh = \mathrm{id}$ (hence $h = \mathrm{id}$). For each u_j, let x_j be a point on the geodesic through x in the direction u_j and so close to x that the geodesic is the unique length-minimizing geodesic from x to x_j. Let f be an automorphism of M fixing $x, x_1, \ldots, x_k$. Then $df(x)$ fixes $u_1, \ldots, u_k$. It follows that $df(z) = \mathrm{id}$ and $f = \mathrm{id}$. Therefore, $s_0(M) \leq 1 + \min\{k(I_x(M)) : x \in M\}$. □

Let G be a subgroup of $\mathrm{Aut}(M)$. By $s_0(M, G)$ we denote the minimum number of distinct points in M such that, if $g \in G$ and g fixes all these points, then $g = \mathrm{id}$. Thus, $s_0(M) = s_0(M, \mathrm{Aut}(M))$.

Lemma 3.2. *Let M be a hyperbolic manifold, let G be a subgroup of* $\mathrm{Aut}(M)$, *and let $q = \dim G$. If $q \geq 1$ then $s_0(M, G) \leq q$; if $q = 0$ then $s_0(M, G) \leq 1$.*

Proof. First we consider the case where $q \leq 1$. Let e denote the identity element of G, and let $Q = G \backslash \{e\}$. For each $g \in Q$, the set $\{x \in M : g(x) = x\}$ is an analytic set of M of dimension $\leq 2n - 2$. The set $W_1 := \{(g, x) \in Q \times M : g(x) = x\}$ is an analytic set of $Q \times M$ of dimension $\leq (2n - 2) + q \leq 2n - 1 < \dim M$. Let W denote the set of fixed points of nontrivial elements of G. Since $W = \pi(W_1)$, where $\pi : Q \times M \to M$ is the projection, and since $\dim W_1 < \dim M$, we see that $W \neq M$. Therefore, $s_0(M, G) \leq 1$.

Now assume that $q \geq 2$. There must be an orbit Q of G of positive dimension. Let $x \in Q$, and let $H := G_x$ be the subgroup of G consisting of elements g satisfying $g(x) = x$. Then $\dim H < \dim G$. By the induction hypothesis, it follows that $s_0(M, H) \leq \dim G - 1$. Therefore, $s_0(M, G) \leq 1 + s_0(M, H) \leq \dim G$. □

As a corollary we obtain our next theorem.

THEOREM 3.3.

$$\dim(\mathrm{Aut}(M)) \geq 1 \implies s_0(M) \leq \dim(\mathrm{Aut}(M));$$
$$\dim(\mathrm{Aut}(M)) = 0 \implies s_0(M) \leq 1.$$

4. A Characterization of the Ball in $\mathbb{C}^n$

This section is devoted to the proof of the following statement, which is a generalization of [FMa, Thm. 1.1].

THEOREM 4.1. *Let M be a hyperbolic manifold of dimension n. Then $s_0(M) = n + 1$ if and only if M is biholomorphic to the unit ball B^n in $\mathbb{C}^n$.*

The estimate $s_0(B^n) = n + 1$ can be easily verified (see e.g. [FMa]).

The rest of this section will be devoted to proving that $s_0(M) = n + 1$ implies that M is biholomorphic to the unit ball. For this we shall need the following two lemmas.

LEMMA 4.2. *Let M be a hyperbolic manifold and let $x \in M$. Suppose that the isotropy group I_x is transitive on the (real) directions at x. Then M is biholomorphic to the unit ball in $\mathbb{C}^n$.*

Proof. Since I_x is transitive on the directions at x, the group $\mathrm{Aut}(M)$ is not finite. Since the automorphism group of a compact hyperbolic manifold must be finite (see [Ko, p. 70]), we see that M is noncompact. By the main theorem in [GrKr], M is biholomorphic to $\mathbb{C}^n$. □

For a subgroup H of the unitary group $U(n)$ we use the notion $k(H)$ introduced at the beginning of Section 3. The following lemma was proved as Lemma 1.4 in [FMa].

LEMMA 4.3. *If H is a subgroup of $U(n)$ with $n \geq 2$ and if H is not transitive on S^{2n-1}, then $k(H) \leq n - 1$.*

We are now ready to prove the remaining portion of Theorem 4.1 (i.e., $s_0(M) = n + 1$ implies that M is biholomorphic to the unit ball).

Proof of Theorem 4.1. Let $s_0(M) = n + 1$. If $n = 1$ then the statement (M is biholomorphic to the unit disc B^1) is true. Indeed, if M is not biholomorphic to the disc or the annulus then its automorphism group is discrete. For each element $g \in \mathrm{Aut}(M)$, $g \neq \mathrm{id}$, the set of fixed points is discrete. Hence there is a point $x \in M$ that is not a fixed point of any nontrivial automorphism. This point will then form a determining set, so $s_0(M) \leq 1$. For the annulus, $s_0(M) = 1$. Therefore, if $s_0(M) = 2$ then M is biholomorphic to the unit disc.

Consider now the case where $n \geq 2$. Let $z \in M$ and suppose that M is not biholomorphic to B^n. Then $I_z(M)$ is not transitive on the directions at z, by Lemma 4.2. Since $I_z(M)$ is (isomorphic to) a subgroup of $U(n)$, by Lemma 4.3 we have $k(I_z(M)) \leq n - 1$. It follows (see Theorem 3.1) that $s_0(M) \leq 1 + k(I_z(M)) \leq n$ if M is not biholomorphic to B^n. □

5. Determining Sets $W_s(M)$ Are Open and Dense

Our aim in this section is to prove the following theorem.

THEOREM 5.1. *Let M be a hyperbolic manifold and let $s \geq 1$. Then $W_s(M) \subset M^s$ is open. If, in addition, $W_s(M) \neq \emptyset$, then $W_s(M)$ is dense in M^s.*

Denote $W = W_s(M)$. First we prove that $W \subset M^s$ is open.

Proof of Theorem 5.1 (first part). Suppose W is not open. Then one can find a sequence of s-tuples $Z_j = (x_1, \dots, x_s^j) \in M^s$ that converges to $Z = (x_1, \dots, x_s) \in M^s$ and such that Z_j is not a determining set for M but Z is. For each j there is an $f_j \in \mathrm{Aut}(M)$ with $f_j|_{Z_j} = \mathrm{id}$ but $f_j \neq \mathrm{id}$. By Corollary 1.4 (replacing f_j by an appropriate iteration of f_j if needed) we may assume that the real part of at least one eigenvalue of $f_j'(x_1^j)$ is nonpositive. Switching again to a subsequence if necessary, we find a sequence of automorphisms whose limit (see Lemma 1.1) is $g \in \mathrm{Aut}(M)$ and such that $g|_Z = \mathrm{id}$ and one of the eigenvalues of $g'(x_1)$ is nonpositive. Therefore $g \neq \mathrm{id}$, which contradicts the original assumption that Z is a determining set for M. □

REMARK. The foregoing proof of the theorem for a bounded domain is given in [V1, Thm. 3.1]. One can also prove Theorem 5.1 by using the idea of [FMa, Lemma 2.3].

Now suppose that $W \neq \emptyset$. We need to prove that W is dense in M^s.

First we introduce some notation. If G is a subgroup of $\mathrm{Aut}(M)$ then $W_s(M, G)$ denotes the set of s-tuples $(x_1, \dots, x_s)$, where $x_j \in M$, such that each element $g \in M$ satisfying $g(x_j) = x_j$ for $j = 1, \dots, s$ must be the identity. Let $\rho_x(\cdot,\cdot)$ denote the metric introduced in Lemma 1.7 for a point $x \in M$. Let $b(x, r)$ denote the ball with center x and radius r in that metric, and let $\bar{b}(x, r)$ be the closure of $b(z, r)$ in M.

LEMMA 5.2. *Suppose that G is a subgroup of* $\mathrm{Aut}(M)$. *If $W_1(M, G) \neq \emptyset$, then $W_1(M, G)$ is dense in M.*

Proof. In this proof we put $W = W_1(M, G)$. Suppose that W is not dense in M; then the closure K of W in M is not equal to M. Let p be a boundary point of K in M and put $\rho(\cdot,\cdot) = \rho_p(\cdot,\cdot)$. Choose $r > 0$ such that the closure of $b(p, 4r)$ in U is compact, where U is a neighborhood from Lemma 1.7 (chosen for the point p) and such that each pair of points of $b(p, 4r)$ is connected by a unique length-minimizing geodesic segment in that metric. Then there exist points z, w such that $\rho(z, p) < r$, $\rho(w, p) < r$, $w \in W$, and $z \notin K$. Observe that $G(w)$, the orbit of w, is a subset of W.

Let $Q = G(w) \cap \bar{b}(p, 4r)$. Then Q is compact and $Q \subset W$. Let u be a point of Q nearest to z. Then u is also a point of $G(w)$ nearest to z, and $R := \rho(z, u) \le \rho(z, w) < 2r$. Choose a point y on the unique length-minimizing geodesic segment from z to u such that $y \notin K$ and $y \ne z$. For each point x of $G(w)$, we see that

$$\rho(z, y) + \rho(y, x) \ge \rho(z, x) \ge \rho(z, u)$$

and that the two equalities hold simultaneously only if $x = u$. Hence, $\rho(z, y) + \rho(y, x) > \rho(z, u) = R$ for each $x \in G(w)$ with $x \ne u$. It follows that $\rho(y, x) > R - \rho(z, y) = \rho(y, u)$ for each $x \in G(w)$ with $x \ne u$. Therefore, u is the unique point of $G(w)$ nearest to y. Since $y \notin K$, there is a nontrivial $g \in G$ such that $g(y) = y$. Now $\rho(y, u) = \rho(g(y), g(u)) = \rho(y, g(u))$ forces $g(u) = u$. Since $u \in W$ it follows that the map g must be the identity, contradicting the nontriviality of g. Therefore, $W_1(M, G)$ is dense in M. □

Proof of Theorem 5.1 (second part). We have already proved that $W_s(M)$ is open in M^s. Suppose now that $W_s(M) \ne \emptyset$. For $g \in \mathrm{Aut}(M)$, let $Q_s(g)$ denote the mapping

$$Q_s(g) \colon M^s \to M^s, \qquad Q_s(g)(z_1, \dots, z_s) = (g(z_1), \dots, g(z_s)).$$

Let $G = \{Q_s(g) : g \in \mathrm{Aut}(M)\}$. Then $G \subset \mathrm{Aut}(M^s)$ and $W_1(M^s, G) = W_s(M)$. By Lemma 5.2, $W_s(M)$ is dense in M^s. □

Using the same approach as in [V2, Thm. 5.1] allows us to establish the following result.

THEOREM 5.3. *If M is a taut manifold, then $\hat{W}_s(M)$ is open in M^s for all $s \ge 1$.*

In general $\hat{W}_s(M)$ need not be open in M^s (see [FMa]) nor be dense in M^s (cf. [FMa; V2]).

6. Results Concerning the Nonhyperbolic Case

6.1. One-Dimensional Manifolds

THEOREM 6.1. *For a one-dimensional complex manifold M, $\hat{s}_0(M) = 2$ or ∞. More precisely, if M is holomorphically equivalent to the complex plane $\mathbb{C}$, the truncated complex plane $\mathbb{C}^*$, or the Riemann sphere $\mathbb{P}$, then $\hat{s}_0(M) = \infty$; otherwise, $\hat{s}_0(M) = 2$.*

Proof. It is well known that either M is biholomorphic to $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{P}$, or a torus or else it is a hyperbolic manifold.

Suppose $S = \{x_1, \dots, x_k\}$ is a finite set in $\mathbb{C}$, and choose $y \in \mathbb{C} \setminus S$. Let f be a polynomial such that $f(x_j) = x_j$ and $f(y) = y + 1$. Then f is a nonidentity holomorphic self-map of $\mathbb{C}$ fixing each point of S. Therefore, $\hat{s}_0(\mathbb{C}) = \infty$.

We now consider $\mathbb{P}$. Note that the map f in the previous paragraph is also a holomorphic self-map of $\mathbb{P}$ fixing the point at infinity. Thus we see that $\hat{s}_0(\mathbb{P}) = \infty$.

Suppose $S = \{x_1, \dots, x_k\}$ is a finite set in $\mathbb{C}^*$. Choose y_j so that $x_j = \exp y_j$. Let g be a polynomial such that $g(x_j) = y_j$ and let $f(z) = \exp g(z)$. Then f is a nonidentity holomorphic self-map of $\mathbb{C}^*$ fixing each point of S. Hence $\hat{s}_0(\mathbb{C}^*) = \infty$.

Consider a torus T corresponding to a lattice L in the complex plane. Let $\pi : \mathbb{C} \to T$ be the projection. It is well known that each holomorphic self-map of T has the form $f(\pi(z)) = \pi(\lambda z + b)$, where $b \in \mathbb{C}$ and $\lambda \in \Lambda := \{x \in \mathbb{C} : xL \subset L\}$. Clearly f is the identity if and only if $\lambda = 1$ and $b \in L$. Let F be the field generated by $L \cup \Lambda$; then F is countable. Choose $r \in \mathbb{C} \setminus F$. Let $x = \pi(0)$ and $y = \pi(r)$, and suppose that $f(x) = x$ and $f(y) = y$. Then

$$\lambda \cdot 0 + b = 0 + p \quad \text{and} \quad \lambda r + b = r + q$$

for some $p, q \in L$. It follows that $b \in L$ and

$$(\lambda - 1)r = q - p.$$

Now $(\lambda - 1) \in F$ and $(q - p) \in F$ but $r \notin F$. It follows that $\lambda - 1 = 0$ and $f =$ id. Therefore, $\hat{s}_0(T) = 2$.

If M is a hyperbolic manifold M of dimension 1, then $\hat{s}_0(M) = 2$ by Theorem 2.2. □

6.2. Higher-Dimensional Manifolds

The main statement in this section is the following theorem.

THEOREM 6.2. *Let M be a Stein manifold with* $\dim(M) \geq 2$ *and such that, for any k distinct points $\{x_1, \dots, x_k\} \in M$, there is a holomorphic map $g : \mathbb{C} \to M$ such that $g(\mathbb{C}) \supset \{x_1, \dots, x_k\}$. Then $\hat{s}_0(M) = \infty$.*

To prove this we need the following two lemmas.

LEMMA 6.3. *Suppose M is a complex manifold,* $\dim(M) \geq 2$. *Suppose also that, for any distinct k points $\{x_1, \dots, x_k\} \in M$, the following statements are true:*

1. *there is a holomorphic map $f : M \to \mathbb{C}$ such that $f(x_i) \neq f(x_j)$ if $i \neq j$;*
2. *there is a holomorphic map $g : \mathbb{C} \to M$ such that $g(\mathbb{C}) \supset \{x_1, \dots, x_k\}$.*

Then $\hat{s}_0(M) = \infty$.

Proof. For any given k points $\{x_1, \dots, x_k\} \in M$, fix $w_j \in g^{-1}(x_j)$. Now consider $\Psi = g \circ \varphi \circ f : M \to M$, where $\varphi : \mathbb{C} \to \mathbb{C}$ is the Lagrange polynomial, such that $\varphi(f(x_j)) = w_j$. Then $\Psi \neq$ id is a holomorphic endomorphism of M fixing the given points. □

LEMMA 6.4. *Let M be a complex manifold such that, for any two points $p \neq q \in M$, there exists a holomorphic function $h\colon M \to \mathbb{C}$ such that $h(p) \neq h(q)$. Then, for any finite number of distinct points $\{x_1, \dots, x_k\} \in M$, there exists a holomorphic function $f\colon M \to \mathbb{C}$ that separates these points: $f(x_i) \neq f(x_j)$ for $i \neq j$.*

Proof. We proceed by induction. Suppose the statement holds for $k \geq 2$. Note that we may assume without loss of generality that, if a function separates k given points, then its values at these points can be preassigned as we please. Now let the points $\{x_1, \dots, x_{k+1}\} \in M$ be given. For $m = 1, \dots, k+1$, consider functions $f_m\colon M \to \mathbb{C}$ such that $f_m(x_s) = s$ for $s \neq m$. If no f_m separates all $k+1$ points, then for all m the value $f_m(x_m)$ must be an integer (moreover, $f_m(x_m) \in \{1, \dots, k+1\} \backslash \{m\}$). Let $\alpha_1, \dots, \alpha_{k+1}$ be a set of linearly independent numbers over $\mathbb{Z}$, the ring of integers. Consider $f = \sum_{m=1}^{k+1} \alpha_m f_m$. We claim that f does the trick. Indeed, for $i \neq j$ we have

$$f(x_i) - f(x_j) = \sum_{m \neq i, j} \alpha_m (i - j) + \alpha_i (f_i(x_i) - j) + \alpha_j (i - f_j(x_j)) \neq 0,$$

since the number of nonzero coefficients (equal to $(i - j)$) is at least $k - 1 \geq 1$. □

By the definition of a Stein manifold, any two points can be separated by a holomorphic function. Hence, by Lemma 6.4, any finite number of points in such a manifold can be separated. The proof of Theorem 6.2 now follows from Lemma 6.3.

REMARK. The property described in Theorem 6.2 leads to the following natural question, which seems to be open. Find the necessary and sufficient conditions for a complex manifold M to have the geometric property that any finite number of points can be connected by an analytic curve on M.

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