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Nonremovable Sets for Hölder Continuous Quasiregular Mappings in the Plane

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1. Introduction

Let $\alpha \in (0,1)$. A function $f : \mathbb{C} \to \mathbb{C}$ is said to be *locally* α*-Hölder continuous,* that is, $f \in \mathrm{Lip}_\alpha(\mathbb{C})$, if

$$|f(z) - f(w)| \le C|z - w|^\alpha \tag{1}$$

whenever $z, w \in \mathbb{C}$ and $|z - w| < 1$. A set $E \subset \mathbb{C}$ is said to be *removable* for α-Hölder continuous analytic functions if every function $f \in \mathrm{Lip}_\alpha(\mathbb{C})$, holomorphic on $\mathbb{C} \setminus E$, is actually an entire function. It turns out that there is a characterization of these sets E in terms of Hausdorff measures. For $\alpha \in (0,1)$, Dolženko [7] proved that a set E is removable for α-Hölder continuous analytic functions if and only if $\mathcal{H}^{1+\alpha}(E) = 0$. When $\alpha = 1$, we deal with the class of Lipschitz continuous analytic functions. Although the same characterization holds, a more involved argument, due to Uy [12], is needed to show that sets of positive area are not removable.

The same question may be asked in the more general setting of K-quasiregular mappings. Given a domain $\Omega \subset \mathbb{C}$ and $K \ge 1$, one says that a mapping $f : \Omega \to \mathbb{C}$ is *K-quasiregular in* Ω if f is a $W^{1,2}_{\mathrm{loc}}(\Omega)$ solution of the Beltrami equation

$$\bar\partial f(z) = \mu(z)\partial f(z)$$

for almost every $z \in \Omega$; here μ, the Beltrami coefficient, is a measurable function such that $|\mu(z)| \le \frac{K-1}{K+1}$ at almost every $z \in \Omega$. If f is a homeomorphism, then f is said to be *K-quasiconformal*. When $\mu = 0$, we recover the classes of analytic functions and conformal mappings on Ω, respectively.

It was shown in [6] that if E is a compact set satisfying $\mathcal{H}^d(E) = 0$ for $d = 2\frac{1+\alpha K}{1+K}$, then E is removable for α-Hölder continuous K-quasiregular mappings. This means that any function $f \in \mathrm{Lip}_\alpha(\mathbb{C})$, K-quasiregular in $\mathbb{C} \setminus E$, is actually K-quasiregular on the whole plane. To look for results in the converse direction, one observes that any compact set E with $\mathcal{H}^{1+\alpha}(E) > 0$ is nonremovable for holomorphic functions and hence also for K-quasiregular mappings in Lip_α. We are thus interested in dimensions between d and $1+\alpha$. In this paper we show that the index d is sharp in the following sense: Given $\alpha \in (0,1)$ and $K \ge 1$, for any $t > d$ there exist (i) a compact set E of dimension t and (ii) a function $f \in \mathrm{Lip}_\alpha(\mathbb{C})$

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that is K-quasiregular in $\mathbb{C} \setminus E$ yet has no K-quasiregular extension to $\mathbb{C}$. In other words, we will construct nonremovable sets of any dimension exceeding d.

We first have a look at the case $K = 1$. Given a compact set E with $\mathcal{H}^{1+\alpha}(E) > 0$, by Frostman's lemma (see e.g. [10, p. 112]) there exists a positive Radon measure ν supported on E such that $\nu(B(z,r)) \le Cr^{1+\alpha}$ for any $z \in E$. Thus, the function $h = \frac{1}{\pi z} * \nu$ is α-Hölder continuous everywhere, is holomorphic outside the support of ν, and has no entire extension.

A similar situation is found in the limiting case $\alpha = 0$, where $\mathrm{Lip}_\alpha(\mathbb{C})$ should be replaced by $\mathrm{BMO}(\mathbb{C})$. In this case, a set E is called *removable* for BMO K-quasiregular mappings if every $\mathrm{BMO}(\mathbb{C})$ function f that is K-quasiregular on $\mathbb{C} \setminus E$ is actually K-quasiregular on the whole plane. When $K = 1$, Král [9] characterized these sets as those with zero length. When $K > 1$, it is known [3; 5] that sets with $\mathcal{H}^{2/(K+1)}(E) = 0$ are removable for BMO K-quasiregular mappings. In fact, the appearance of this index $\frac{2}{K+1}$ is not strange. Astala [2] has shown that, for any K-quasiconformal mapping ϕ and any compact set E,

$$\frac{1}{K}\left(\frac{1}{\dim(E)} - \frac{1}{2}\right) \le \frac{1}{\dim(\phi(E))} - \frac{1}{2} \le K\left(\frac{1}{\dim(E)} - \frac{1}{2}\right). \tag{2}$$

Furthermore, both equalities are always attainable. In particular, sets of dimension $\frac{2}{K+1}$ are K-quasiconformally mapped to sets of dimension at most 1, which is the critical point for the analytic BMO situation. Hence, from equality at (2), for any $t > \frac{2}{K+1}$ there exist a compact set E of dimension t and a K-quasiconformal mapping ϕ that maps E to a compact set $\phi(E)$ with dimension

$$t' = \frac{2Kt}{2 + (K-1)t} > 1.$$

In particular, $\mathcal{H}^1(\phi(E)) > 0$. As before, we have a positive Radon measure ν supported on $\phi(E)$, with linear growth, whose Cauchy transform $h = \frac{1}{\pi z} * \nu$ is holomorphic on $\mathbb{C} \setminus E$ and has a $\mathrm{BMO}(\mathbb{C})$ extension that is not entire. Now, since $\mathrm{BMO}(\mathbb{C})$ is invariant under quasiconformal changes of variables [11], the composition $g = h \circ \phi$ is a $\mathrm{BMO}(\mathbb{C})$ K-quasiregular mapping on $\mathbb{C} \setminus E$ that has no K-quasiregular extension to $\mathbb{C}$. In other words, the set E is not removable for BMO K-quasiregular mappings. This argument shows that the index $\frac{2}{K+1}$ is somewhat critical for the BMO K-quasiregular problem.

Our plan is to repeat the foregoing argument after first replacing $\mathrm{BMO}(\mathbb{C})$ with $\mathrm{Lip}_\alpha(\mathbb{C})$. That is, given any dimension $t > 2\frac{1+\alpha K}{1+K}$, we will construct a compact set E of dimension t and a $\mathrm{Lip}_\alpha(\mathbb{C})$ function that is K-quasiregular on $\mathbb{C} \setminus E$ but not on $\mathbb{C}$. We will start with a compact set E of dimension t and a K-quasiconformal mapping ϕ such that $\dim(\phi(E)) = t' = \frac{2Kt}{2+(K-1)t}$. Then, we will show that there are $\mathrm{Lip}_\beta(\mathbb{C})$ functions for some $\beta > 0$, analytic outside of $\phi(E)$, that in turn induce (by composition) K-quasiregular functions on $\mathbb{C} \setminus E$ with some global Hölder continuity exponent. This construction will encounter two obstacles. First, the extremal dimension distortion of sets of dimension $2\frac{1+\alpha K}{1+K}$ through K-quasiconformal mappings is not exactly $1 + \alpha$, the critical number in the analytic setting (this was so for $\alpha = 0$). Second, the composition of β-Hölder continuous functions with

K-quasiconformal mappings is only in $\mathrm{Lip}_{\beta/K}(\mathbb{C})$, so there is some loss of regularity that might be critical. To avoid these troubles, we will construct in an explicit way the mapping ϕ. This concrete construction allows us to show that ϕ exhibits an exponent of Hölder continuity given by

$$\frac{t}{t'} = \frac{1}{K} + \frac{K-1}{2K}t,$$

which is larger than the usual $\frac{1}{K}$ obtained from Mori's theorem. This regularity will be sufficient for our purposes. On the other hand, if $\dim(E) = t$ and $\dim(\phi(E)) = t'$ then it is natural to expect ϕ to be $\mathrm{Lip}_{t/t'}$.

2. Extremal Distortion

Throughout this section, $D(z, r)$ will denote the open disk of center z and radius r. By $\operatorname{diam}(D)$ we mean the diameter of the disk D, and λD will denote the disk concentric with D having diameter $\operatorname{diam}(\lambda D) = |\lambda| \operatorname{diam}(D)$. By $\mathbb{D}$ we will mean the unit disk, and Jf will denote the Jacobian determinant of the function f.

Recall that a Cantor-type set E of m components is the only compact set that is invariant under a fixed family of m similitudes,

$$\begin{aligned} \varphi_j : \mathbb{D} &\to \mathbb{D}, \\ z &\mapsto \varphi_j(z) = a_j + b_j z, \end{aligned}$$

with $a_j, b_j \in \mathbb{C}$ for all $j = 1, \ldots, m$ and such that $D_i = \varphi_i(\bar{\mathbb{D}})$ are disjoint disks and $D_i \subset \mathbb{D}$. In other words, $E \subset \mathbb{D}$ is the only solution to the equation

$$E = \bigcup_{j=1}^{m} \varphi_j(E).$$

Constructively, we have

$$E = \bigcap_{N=1}^{\infty} \Bigg(\bigcup_{\ell(J)=N} \varphi_J(\mathbb{D}) \Bigg),$$

where $\varphi_J = \varphi_{j_1} \circ \cdots \circ \varphi_{j_N}$ for any chain $J = (j_1, \ldots, j_N)$ of length $\ell(J) = N$ of members of $\{1, \ldots, m\}$. The Hausdorff dimension of E is the only solution d to the equation

$$\sum_{j=1}^{m} |\varphi_j'|^d = 1.$$

Under the additional assumption $|\varphi_i'| = r$ for all i, one easily obtains

$$\dim(E) = d = \frac{\log m}{\log(1/r)}.$$

Another typical situation appears when the image sets D_i are uniformly distributed in $\mathbb{D}$; then there is a constant C such that, for any disk D,

$$\sum_{D_i \cap D \neq \emptyset} \operatorname{diam}(D_i)^d \leq \operatorname{diam}(D)^d,$$

where the sum runs over all disks D_i that intersect D. In this case, E is said to be a *regular* Cantor-type set. For these sets, $0 < \mathcal{H}^d(E) < \infty$.

One of the main results in [2] is the sharpness of the dimension distortion equation (2). To obtain the equality there, the author distorted holomorphically a fixed Cantor-type set F. This deformation defined actually a *holomorphic motion* on F. An interesting extension result, known as the λ-lemma, allows this motion to be extended quasiconformally from F to the whole plane. This procedure avoided most of the technicalities and gave the desired result in a surprisingly direct way. However, since we look both for extremal dimension distortion and higher Hölder continuity, ϕ must be constructed explicitly. Thus, let $t \in (0,2)$ and $K \geq 1$ be fixed numbers, and denote $t' = \frac{2Kt}{2+(K-1)t}$. As in [2], we first give a K-quasiconformal mapping ϕ that maps a regular Cantor set E of dimension t to another regular Cantor set $\phi(E)$ for which $\dim(\phi(E))$ is as close as we want to t'. Later, again as in [2], we will glue a suitable sequence of such mappings in the convenient way.

PROPOSITION 1. *Given $t \in (0,2)$, $K \geq 1$, and $\varepsilon > 0$, there exist a compact $E \subset \mathbb{D}$ and a K-quasiconformal mapping $\phi\colon \mathbb{C} \to \mathbb{C}$ with the following properties*:

1. *ϕ is the identity mapping on $\mathbb{C} \setminus \mathbb{D}$*;
2. *E is a self-similar Cantor set, constructed with $m = m(\varepsilon)$ similarities*;
3. $\dim(\phi(E)) \geq t' - \varepsilon$;
4. *$J\phi \in L^p_{\mathrm{loc}}(\mathbb{C})$ if and only if $p \leq \frac{K}{K-1}$*;
5. $|\phi(z) - \phi(w)| \leq C m^{1/t - 1/t'} |z - w|^{t/t'}$ *whenever* $|z - w| < 1$.

Proof. Our construction follows the scheme in [4]. Thus, we will obtain ϕ as a limit of a sequence of K-quasiconformal mappings

$$\phi = \lim_{N \to \infty} \phi_N,$$

where every ϕ_N will act at the Nth step of the construction of E. More precisely, both E and $\phi(E)$ will be regular Cantor sets associated to two fixed families of similitudes $(\varphi_j)_{j=1,\ldots,m}$ and $(\psi_j)_{j=1,\ldots,m}$. At the Nth step, ϕ_N will map each *generating disk* of E, $\varphi_{j_1\ldots j_N}(\mathbb{D})$, to the corresponding generating disk of the image set, $\psi_{j_1\ldots j_N}(\mathbb{D})$. Since ϕ is supposed to be K-quasiconformal and to give extremal distortion of dimension, we think about using a typical radial stretching,

$$f(z) = z|z|^{1/K-1},$$

conveniently modified. It turns out that this radial stetching f is extremal for some basic properties of K-quasiconformal mappings, such as Hölder continuity. In order to find ϕ in a better Hölder space (this fails for f), we will replace f by a linear mapping in a small neighborhood of its singularity. This change will not affect the exponent of integrability but will enable some improvement on the Hölder exponent.

Take $m \geq 100$ and consider m disjoint disks inside of $\mathbb{D}$, $D(z_i, r)$, uniformly distributed and all with the same radius $r = r_m$. By taking m large enough, we may always assume that $c_m = mr^2 \geq \frac{1}{2}$. Given any $\sigma \in (0,1)$ to be determined later, we can consider m similitudes

$$\varphi_i(z) = z_i + \sigma r z, \quad z \in \mathbb{D},$$

and denote, for every $i = 1, \ldots, m$,

$$D_i = \frac{1}{\sigma}\varphi_i(\mathbb{D}) = D(z_i, r_1),$$
$$D_i' = \varphi_i(\mathbb{D}) = D(z_i, \sigma r_1);$$

here we have written $r_1 = r$. We define

$$g_1(z) = \begin{cases} \sigma^{1/K-1}(z - z_i) + z_i & \text{if } z \in D_i', \\ \left|\dfrac{z - z_i}{r_1}\right|^{1/K-1}(z - z_i) + z_i & \text{if } z \in D_i \setminus D_i', \\ z & \text{otherwise.} \end{cases}$$

It may be easily seen that g_1 defines a K-quasiconformal mapping that is conformal everywhere except on each ring $D_i \setminus D_i'$. Moreover, if we put

$$\psi_i(z) = z_i + \sigma^{1/K} r z, \quad z \in \mathbb{D},$$

then g_1 maps every D_i to itself while each D_i' is mapped to $D_i'' = \psi_i(\mathbb{D})$; see Figure 1. Now we denote $\phi_1 = g_1$.

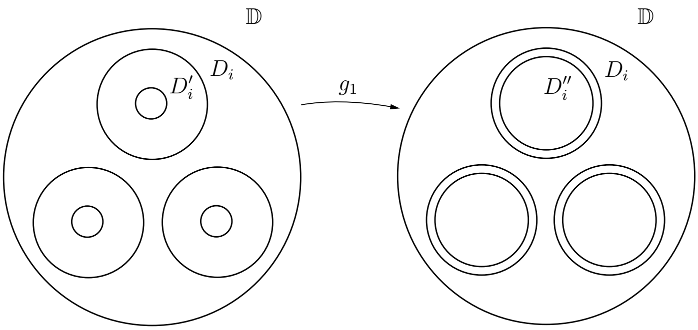

Figure 1

At the second step, we repeat this procedure inside of every D_i'' and leave the rest fixed. That is, we define g_2 on the target set of ϕ_1 and then construct ϕ_2 as

$$\phi_2 = g_2 \circ \phi_1.$$

To do this more explicitly, we denote

$$D_{ij} = \frac{1}{\sigma}\phi_1(\varphi_{ij}(\mathbb{D})) = D(z_{ij}, r_2),$$
$$D_{ij}' = \phi_1(\varphi_{ij}(\mathbb{D})) = D(z_{ij}, \sigma r_2);$$

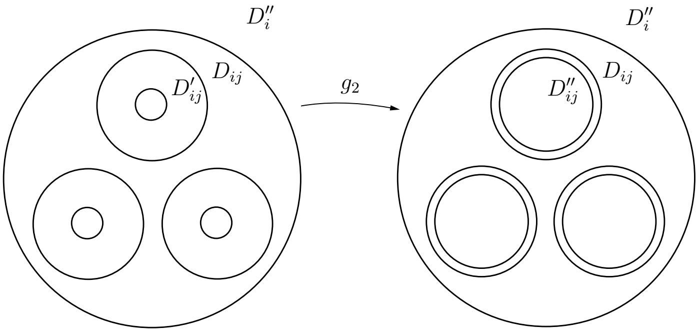

Figure 2

a computation shows that $r_2 = \sigma^{1/K} r r_1$. Now we define

$$g_2(z) = \begin{cases} \sigma^{1/K-1}(z - z_{ij}) + z_{ij} & \text{if } z \in D'_{ij}, \\ \left|\dfrac{z - z_{ij}}{r_2}\right|^{1/K-1}(z - z_{ij}) + z_{ij} & \text{if } z \in D_{ij} \setminus D'_{ij}, \\ z & \text{otherwise.} \end{cases}$$

By construction, g_2 is K-quasiconformal on $\mathbb{C}$, is conformal outside a union of m^2 rings, and maps D'_{ij} to $D''_{ij} = \psi_{ij}(\mathbb{D})$ while every point outside of the disks D_{ij} remains fixed under g_2; see Figure 2. Thus, the composition $\phi_2 = g_2 \circ \phi_1$ (see Figure 3) is still K-quasiconformal and agrees with the identity outside of $\mathbb{D}$; moreover,

$$\phi_2(\varphi_{ij}(\mathbb{D})) = \psi_{ij}(\mathbb{D})$$

for any $i, j = 1, \ldots, m$.

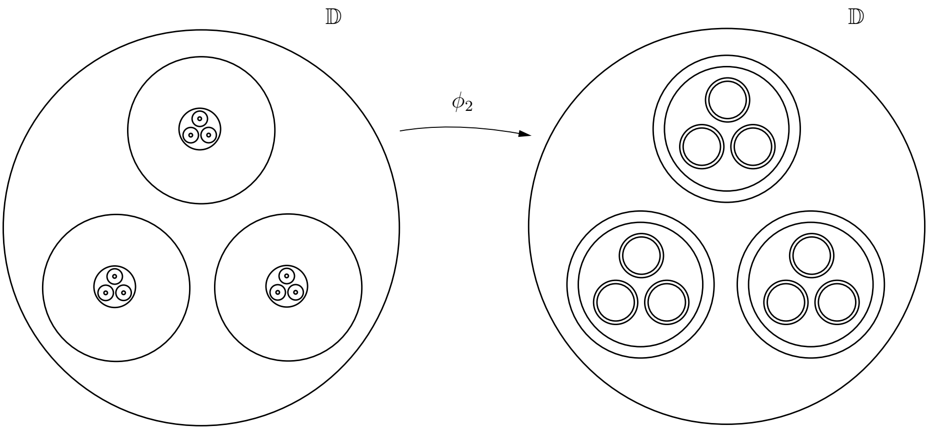

Figure 3

After $N-1$ steps, we will define g_N on the target side of ϕ_{N-1}. For each multi-index $J=(j_1,\dots,j_N)$ of length $\ell(J)=N$, we denote

$$D_J=\frac{1}{\sigma}\phi_{N-1}(\varphi_J(\mathbb{D}))=D(z_J,r_N),$$
$$D'_J=\phi_{N-1}(\varphi_J(\mathbb{D}))=D(z_J,\sigma r_N);$$

now $r_N=\sigma^{1/K}rr_{N-1}$. Then the mapping

$$g_N(z)=\begin{cases}\sigma^{1/K-1}(z-z_J)+z_J & \text{if } z\in D'_J,\\ \left|\dfrac{z-z_J}{r_N}\right|^{1/K-1}(z-z_J)+z_J & \text{if } z\in D_J\setminus D'_J,\\ z & \text{otherwise,}\end{cases}$$

is K-quasiconformal on the plane and conformal outside a union of m^N rings. Furthermore, $g_N(D_J)=D_J$ and $g_N(D'_J)=D''_J$, where $D''_J=\psi_J(\mathbb{D})$.

As a consequence, the composition $\phi_N=g_N\circ\phi_{N-1}$ is also K-quasiconformal and

$$\phi_N(\varphi_J(\mathbb{D}))=\psi_J(\mathbb{D}).$$

With this procedure, it is clear that the sequence ϕ_N is uniformly convergent to a homeomorphism ϕ. It is also clear that ϕ has distortion bounded by K almost everywhere and, in fact, that ϕ is a K-quasiconformal mapping. By construction, ϕ maps the regular Cantor set

$$E=\bigcap_{N=1}^{\infty}\left(\bigcup_{\ell(J)=N}\varphi_J(\mathbb{D})\right)$$

to

$$\phi(E)=\bigcap_{N=1}^{\infty}\left(\bigcup_{\ell(J)=N}\psi_J(\mathbb{D})\right),$$

which obviously is also a regular Cantor set. If now we choose σ so that

$$m(\sigma r)^t=1$$

we directly obtain $0<\mathcal{H}^t(E)<\infty$ as well as

$$\frac{1}{\dim(\phi(E))}=\frac{1}{t'}+\frac{K-1}{2K}\frac{\log(1/mr^2)}{\log m}.$$

Since $c_m\geq\frac{1}{2}$ for all m, we can always get

$$\dim(\phi(E))\geq t'-\varepsilon$$

simply by increasing m if needed.

Now we must look at the regularity properties of our mapping ϕ. To do so, we introduce the following notation. Put $G^0=\mathbb{D}$, and denote by P_J^N and G_J^N (respectively) the peripheral and generating disks of generation N. That is, for any chain $J=(j_1,\dots,j_N)$,

$$P_J^N = \frac{1}{\sigma}\varphi_J(\mathbb{D}),$$
$$G_J^N = \varphi_J(\mathbb{D}).$$

With this notation, $D_J = \phi_{N-1}(P_J^N)$, $D_J' = \phi_{N-1}(G_J^N)$, and $D_J'' = \phi_N(G_J^N)$.

Now take any p such that $J\phi \in L^p_{\mathrm{loc}}(\mathbb{C})$. Of course, we can assume $p \geq 1$. Then, one may decompose the p-mass of $J\phi$ over $\mathbb{D}$ as follows:

$$\int_{\mathbb{D}} J\phi(z)^p\,dA(z) = \int_{\mathbb{D}\setminus\bigcup_i P_i^1} J\phi(z)^p\,dA(z) + \sum_{i=1}^m \int_{P_i^1\setminus G_i^1} J\phi(z)^p\,dA(z) + \sum_{i=1}^m \int_{G_i^1} J\phi(z)^p\,dA(z).$$

Since $\phi = \phi_1$ on $\mathbb{C}\setminus\bigcup_i G_i^1$, it follows that

$$\int_{\mathbb{D}} J\phi(z)^p\,dA(z) = \int_{\mathbb{D}\setminus\bigcup_i P_i^1} J\phi_1(z)^p\,dA(z) + m\int_{P^1\setminus G^1} J\phi_1(z)^p\,dA(z) + m\int_{G^1} J\phi(z)^p\,dA(z),$$

where P^1 and G^1 denote (respectively) any of the first-generation peripheral and generating disks. One may repeat this argument for the last integral, which by a recursive argument yields

$$\int_{\mathbb{D}} J\phi(z)^p\,dA(z) = \sum_{N=0}^{\infty} m^N \int_{G^N\setminus\bigcup_i P_i^{N+1}} J\phi_{N+1}(z)^p\,dA(z) + \sum_{N=1}^{\infty} m^N \int_{P^N\setminus G^N} J\phi_N(z)^p\,dA(z);$$

here, as before, P^N and G^N denote (respectively) any Nth-generation peripheral or generating disk.

Now we compute separately the integrals in both sums. On one hand, if $J = (j_1,\dots,j_N)$ then

$$\begin{aligned}\int_{P_J^N\setminus G_J^N} J\phi_N(z)^p\,dA(z) &= \int_{P_J^N\setminus G_J^N} Jg_N(\phi_{N-1}(z))^p J\phi_{N-1}(z)^p\,dA(z)\\ &= \int_{D_J\setminus D_J'} Jg_N(w)^p J\phi_{N-1}(\phi_{N-1}^{-1}(w))^{p-1}\,dA(w)\\ &= (\sigma^{1/K-1})^{2(N-1)(p-1)} \int_{D_J\setminus D_J'} Jg_N(w)^p\,dA(w)\\ &= r^{2N}\sigma^{(N-1)\gamma}\frac{2\pi}{K^p}\left|\frac{1-\sigma^\gamma}{\gamma}\right|\end{aligned}$$

under the additional assumption $p \neq \frac{K}{K-1}$; here $\gamma = 2p\left(\frac{1}{K}-1\right)+2$. If $p = \frac{K}{K-1}$, then

$$\int_{P_J^N \setminus G_J^N} J\phi_N(z)^{K/(K-1)}\, dA(z) = r^{2N} \frac{2\pi}{K^{K/(K-1)}} \log \frac{1}{\sigma}.$$

On the other hand, for any value of p,

$$\begin{aligned}
\int_{G_J^N \setminus \bigcup_i P_{(Ji)}^{N+1}} J\phi_{N+1}(z)^p\, dA(z) &= \int_{G_J^N \setminus \bigcup_i P_{(Ji)}^{N+1}} Jg_{N+1}(\phi_N(z))^p J\phi_N(z)^p\, dA(z) \\
&= \int_{D_J'' \setminus \bigcup_i D_{(Ji)}} Jg_{N+1}(w)^p J\phi_N(\phi_N^{-1}(w))^{p-1}\, dA(w) \\
&= (\sigma^{1/K-1})^{2N(p-1)} \int_{D_J'' \setminus \bigcup_i D_{(Ji)}} Jg_{N+1}(w)^p\, dA(w) \\
&= (\sigma^{1/K-1})^{2N(p-1)} \int_{D_J'' \setminus \bigcup_i D_{(Ji)}} 1\, dA(w) \\
&= (\sigma^{1/K-1})^{2N(p-1)} \left| D_J'' \setminus \textstyle\bigcup_i D_{(Ji)} \right| \\
&= r^{2N} \sigma^{N\gamma} \pi (1 - c_m).
\end{aligned}$$

Thus, for any $p \neq \frac{K}{K-1}$,

$$\int_{\mathbb{D}} J\phi(z)^p\, dA(z) = \left(\pi(1 - c_m) + c_m \frac{2\pi}{K^p} \left| \frac{1 - \sigma^\gamma}{\gamma} \right| \right) \sum_{N=0}^{\infty} (c_m \sigma^\gamma)^N.$$

Since p is such that $J\phi \in L^p_{\mathrm{loc}}(\mathbb{C})$, we necessarily have $\sigma^\gamma < 1/c_m$. For m large enough, this is equivalent to $\gamma > 0$; that is, $p < \frac{K}{K-1}$. At the critical point $p = \frac{K}{K-1}$,

$$\int_{\mathbb{D}} J\phi(z)^{K/(K-1)}\, dA(z) = \left(\pi(1 - c_m) + c_m \frac{2\pi}{K^{K/(K-1)}} \log \frac{1}{\sigma} \right) \sum_{N=0}^{\infty} (c_m)^N,$$

which will always converge for any fixed value of m. This shows that we can choose m large enough so that $J\phi \in L^p_{\mathrm{loc}}(\mathbb{D})$ if and only if $p \leq \frac{K}{K-1}$.

Finally, it remains only to check that ϕ is Hölder continuous with exponent $\gamma = t/t'$. By means of Poincaré inequality together with the quasiconformality of ϕ, it is enough [8, p. 64] to show that, for any disk D,

$$\int_D J\phi(z)\, dA(z) \leq C \operatorname{diam}(D)^{2t/t'}.$$

Hence, for some fixed disk D, take N such that $(\sigma r)^N \leq \frac{1}{2} \operatorname{diam}(D) < (\sigma r)^{N-1}$. Then

$$\int_D J\phi(z)\, dA(z) \leq \int_{D \setminus \bigcup G_J^N} J\phi(z)\, dA(z) + \int_{\bigcup G_J^N} J\phi(z)\, dA(z),$$

where the union $\bigcup G_J^N$ runs over all disks G_J^N such that $G_J^N \cap D \neq \emptyset$. On $D \setminus \bigcup G_J^N$, we easily see that

$$J\phi = J\phi_N \leq \frac{1}{K} (\sigma^{1/K-1})^{2N}.$$

Thus,

$$\int_{D\setminus\bigcup G_J^N} J\phi(z)\,dA(z) \le \frac{1}{K}(\sigma^{1/K-1})^{2N}\pi\left(\frac{1}{2}\operatorname{diam}(D)\right)^2$$
$$\le \frac{\pi}{K}(c_m^{(K-1)/2K})^{2N} m^{2(1/t-1/t')}\left(\frac{1}{2}\operatorname{diam}(D)\right)^{2t/t'}.$$

On the other hand, recall that $\phi(G_J^N) = \phi_N(G_J^N)$ are disks of radius $(\sigma^{1/K}r)^N$. Hence,

$$\int_{\bigcup_J G_J^N} J\phi(z)\,dA(z) = \sum_J \int_{G_J^N} J\phi(z)\,dA(z) = \sum_J |\phi(G_J^N)|$$
$$= \sum_J |\phi_N(G_J^N)| = \sum_J \pi(\sigma^{1/K}r)^{2N}$$
$$= \pi(c_m^{(K-1)/2K})^{2N}\left(\frac{1}{2}\operatorname{diam}(D)\right)^{2t/t'} \sum_J\left(\frac{(\sigma r)^N}{\frac{1}{2}\operatorname{diam}(D)}\right)^{2t/t'}$$

and it just remains to bound $\sum_J\left(\frac{(\sigma r)^N}{\frac{1}{2}\operatorname{diam}(D)}\right)^{2t/t'}$. Actually, this is equivalent to finding some constant C such that

$$\sum_{G_J^N\cap D\neq\emptyset} \operatorname{diam}(G_J^N)^{2t/t'} \le C\operatorname{diam}(D)^{2t/t'}.$$

But the disks G_J^N come from a self-similar construction that is said to give a regular Cantor set of dimension t. In particular, they may be chosen uniformly distributed so that the *t-dimensional packing condition* is satisfied:

$$\sum_{G_J^N\cap D\neq\emptyset} \operatorname{diam}(G_J^N)^{t} \le C\operatorname{diam}(D)^{t}.$$

It is easy to show that this condition implies the s-dimensional one for all $s > t$ (in particular, for $s = 2t/t'$). Hence, the constant C exists and is independent of m. Thus, what we finally obtain is that

$$\int_D J\phi(z)\,dA(z) \le C m^{1/t-1/t'}(c_m^{(K-1)/2K})^{2N}\left(\frac{1}{2}\operatorname{diam}(D)\right)^{2t/t'},$$

and the result follows. □

COROLLARY 2. *Let $K \ge 1$ and $t \in (0,2)$, and denote $t' = \frac{2Kt}{2+(K-1)t}$. There exist a t-dimensional compact set E and a K-quasiconformal mapping $\phi\colon \mathbb{C}\to\mathbb{C}$ such that*

1. *$\mathcal{H}^t(E)$ is σ-finite,*
2. *$\dim(\phi(E)) = t'$, and*
3. *$|\phi(z)-\phi(w)| \le C|z-w|^{t/t'}$ whenever $|z-w| < 1$.*

Proof. Given $\varepsilon > 0$, $K \geq 1$, and $t \in (0,2)$, let $\phi\colon \mathbb{C} \to \mathbb{C}$ and E be as in Proposition 1. Then, for any fixed $r > 0$, the mapping

$$\psi_r(z) = r\phi(z/r)$$

and the set $E_r = rE$ exhibit the same properties as ϕ and E, since neither K-quasiconformality nor Hausdorff dimension is modified through dilations. However, when computing the new $\mathrm{Lip}_{t/t'}$ constant, if $|z-w| < r$ then

$$|\psi_r(z) - \psi_r(w)| = r|\phi(z/r) - \phi(w/r)| \leq C m^{1/t-1/t'} r^{1-t/t'} |z-w|^{t/t'}.$$

Thus, as in [2], let $D_j = D(z_j, r_j)$ be a countable disjoint family of disks inside of $\mathbb{D}$, and let ε_j be a sequence of positive numbers, $\varepsilon_j \to 0$ as $j \to \infty$. For each j, let ϕ_j and E_j be as in Proposition 1, so that $\dim(\phi_j(E_j)) \geq t' - \varepsilon_j$. In particular, each E_j is a regular Cantor set of m_j components. Denote then $\psi_j(z) = r_j\phi_j\left(\frac{z-z_j}{r_j}\right)$ and $F_j = z_j + r_jE_j$, and define

$$\psi(z) = \begin{cases} \psi_j(z) & \text{if } z \in D_j, \\ z & \text{otherwise.} \end{cases}$$

By construction, ψ is a K-quasiconformal mapping: it maps the set $F = \bigcup_j F_j$ to the set $\psi(F) = \bigcup_j \psi_j(F_j)$. Moreover, $\mathcal{H}^t(F)$ is σ-finite and

$$\dim(\phi(F)) = \sup_j \dim(\psi_j(F_j)) = t'.$$

Finally, assume that z lies inside some fixed D_k and that $w \in \mathbb{D} \setminus \bigcup_j D_j$. Then, consider the line segment L between z and w, and denote $\{z_k\} = L \cap \partial D_k$. Then both z_k and w are fixed points for ψ, so that

$$\begin{aligned} |\psi(z) - \psi(w)| &\leq |\psi(z) - \psi(z_k)| + |\psi(z_k) - \psi(w)| \\ &\leq C m_k^{1/t-1/t'} r_k^{1-t/t'} |z - z_k|^{t/t'} + |z_k - w|. \end{aligned}$$

Since we are still free to choose radii r_j, we may do so such that

$$m_k^{1/t-1/t'} r_k^{1-t/t'} < 1$$

or, equivalently, $m_j r_j^t < 1$. Under this assumption, we finally get

$$|\psi(z) - \psi(w)| \leq (C+1)|z-w|^{t/t'}$$

whenever $|z-w| < 1$. This clearly shows that $\psi \in \mathrm{Lip}_{t/t'}(\mathbb{C})$. □

Although the set in Corollary 2 is more critical than the one we constructed in Proposition 1 (in the sense that the first gives precisely the extremal dimension distortion), both do the same work when studying nonremovable sets for Hölder continuous quasiregular mappings.

COROLLARY 3. *Let $K \geq 1$ and $\alpha \in (0,1)$. Then, for any $t > 2\frac{1+\alpha K}{1+K}$, there exists a compact set E with $0 < \mathcal{H}^t(E) < \infty$ that is nonremovable for K-quasiregular mappings in Lip_α.*

Proof. Let E and ϕ be such that $\dim \phi(E) \geq t' - \varepsilon > 1$ for some sufficiently small ε. Hence, by Frostman's lemma, we can construct a positive Radon measure μ supported on $\phi(E)$ with growth $t' - 2\varepsilon$. Its Cauchy transform $g = \mathcal{C}\mu$ defines a holomorphic function on $\mathbb{C} \setminus \phi(E)$, not entire and with a Hölder continuous extension to the whole plane, with exponent $t' - 2\varepsilon - 1$. Set

$$f = g \circ \phi.$$

Clearly, f is K-quasiregular on $\mathbb{C} \setminus E$ and has no K-quasiregular extension to $\mathbb{C}$. Indeed, if $\tilde{f}$ extends f K-quasiregularly to $\mathbb{C}$ then $\tilde{g} = \tilde{f} \circ \phi^{-1}$ would provide an entire extension of g, which is impossible. Furthermore, f is Hölder continuous with exponent

$$(t' - 2\varepsilon - 1)\frac{t}{t'} = t - (2\varepsilon + 1)\frac{t}{t'}.$$

Thus, we just need $\varepsilon > 0$ small enough that

$$t - (2\varepsilon + 1)\frac{t}{t'} \geq \alpha;$$

but this inequality is equivalent to

$$\left(t - 2\frac{1 + \alpha K}{1 + K}\right) \geq \varepsilon \frac{2}{K + 1}(2 + (K - 1)t)$$

and so the proof is complete. □

Something similar may be said when dealing with finite distortion mappings. Recall that if $\Omega \subset \mathbb{C}$ is an open set then a *mapping of finite distortion* on Ω is a function $f : \Omega \to \mathbb{C}$ in the Sobolev class $W^{1,1}_{\mathrm{loc}}(\mathbb{C})$ with locally integrable Jacobian, $Jf \in L^1_{\mathrm{loc}}(\mathbb{C})$, and such that there exists a measurable function $K_f : \Omega \to [1, \infty]$, called the *distortion function* of f, that is finite almost everywhere and for which

$$|Df(z)|^2 \leq K_f(z) Jf(z)$$

at almost every $z \in \Omega$. When $K_f \in L^\infty$ and $\|K_f\|_\infty = K$, one recovers the class of K-quasiregular mappings. However, weaker assumptions on K_f also give interesting results. The most typical situation appears when we ask the distortion function K_f to be such that

$$\exp\{K_f\} \in L^p_{\mathrm{loc}}(\mathbb{C})$$

for some p large enough. Then we say that K_f is *exponentially integrable* and that f is a *mapping of exponentially integrable distortion*. In [6], it was shown that compact sets E with σ-finite $\mathcal{H}^{2\alpha}(E)$ are removable for α-Hölder continuous mappings of exponentially integrable distortion.

COROLLARY 4. *Let $\alpha \in (0, 1)$. For any $t > 2\alpha$ there exist a compact set E of dimension t and a function $f \in \mathrm{Lip}_\alpha(\mathbb{C})$ that defines a mapping of exponentially integrable distortion $\mathbb{C} \setminus E$ and has no finite distortion extension to $\mathbb{C}$.*

Proof. If $t > 2\alpha$, then there exists $K \geq 1$ such that $t > 2\frac{1+\alpha K}{1+K}$. Thus, we have a compact set E of dimension t and a $\mathrm{Lip}_\alpha(\mathbb{C})$ function f that is K-quasiregular on $\mathbb{C} \setminus E$ but not on $\mathbb{C}$. Of course, f is a mapping of exponentially integrable distortion on $\mathbb{C} \setminus E$, with distortion function K_f essentially bounded by K. If f extended to a mapping of finite distortion on $\mathbb{C}$, then in particular we would have $Jf \in L^1_{\mathrm{loc}}(\mathbb{C})$. But then, since $K_f \leq K$ at almost every point, this would imply that actually f extends K-quasiregularly. $\square$

At this point, it should be said that above the critical index $2\frac{1+\alpha K}{1+K}$ one might find also some removable set. For instance, an unpublished result of S. Smirnov shows that if $E = \partial\mathbb{D}$ and ϕ is a K-quasiconformal mapping then

$$\dim(\phi(E)) \leq 1 + \left(\frac{K-1}{K+1}\right)^2,$$

which is better than the usual dimension distortion equation (2). Hence, if we choose $K \geq 1$ small enough then there exists an α that satisfies

$$K\left(\frac{K-1}{K+1}\right)^2 < \alpha < \frac{K-1}{2K}.$$

For those values of α, the set $E = \partial\mathbb{D}$ is removable for α-Hölder continuous K-quasiregular mappings, although

$$2\frac{1+\alpha K}{1+K} < \dim(E).$$

This suggests that everything could happen between $2\frac{1+\alpha K}{1+K}$ and $1+\alpha$.

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