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Hölder Estimates on Lineally Convex Domains of Finite Type

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1. Introduction

In [DFo1] Diederich and Fornæss constructed support functions for convex domains of finite type. This result together with a good knowledge of the local geometry of convex domains of finite type has been used in [DFFo] to prove optimal Hölder estimates for a solution of the Cauchy–Riemann equation in such domains. Hefer [H] has extended and refined this result by using information on the multitype of the domain. Further results on solution operators based on the support functions of [DFo1] can be found in [A; DM1; DM2; F1].

A different approach is attributable to Cumenge: in [Cu1] and [Cu2] she uses the Bergman kernel to construct solution operators for the Cauchy–Riemann equation in convex domains of finite type. Using these operators and some precise estimates (due to McNeal [Mc]) of the Bergman kernel at the boundaries of the domains, she also proves optimal Hölder estimates and some other results.

All results mentioned so far have one feature in common. The hypotheses on the given functions and forms and the conclusions of the papers are isotropic, whereas the proofs are quite nonisotropic in that the respective estimates necessarily take into account the different behavior of the geometry of the domains in different directions, even of the holomorphic tangent spaces. Introducing nonisotropy into the conclusions was first considered in [F2], where optimal nonisotropic Hölder estimates for certain solutions of the $\bar\partial$-equation are proved for bounded data.

Recently the larger class of lineally convex domains of finite type has received much attention. A smooth lineally convex domain differs from a (linearly) convex domain in that, with the former, only the complex tangent space at each boundary point is supposed to lie outside of the domain.

Diederich and Fornæss [DFo2] have constructed a smooth family of holomorphic support functions with best possible nonisotropic estimates for lineally convex domains of finite type. It has been shown by Conrad [C] that the local geometry of such domains shares all essential properties with convex domains of finite type.

In this paper we shall use these support functions to establish both isotropic and nonisotropic Hölder estimates on lineally convex domains of finite type. In doing so we can follow rather closely the proofs given in [DFFo; F2] for the corresponding results in the linearly convex case.

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THEOREM 1.1. *Let $D \subset\subset \mathbb{C}^n$ be a lineally convex domain with $\mathcal{C}^\infty$-smooth boundary of finite type m. We denote by $C^0_{(0,q)}(\bar{D})$ the Banach space of $(0,q)$-forms with continuous coefficients on $\bar{D}$ and by $\Lambda^{1/m}_{(0,q)}(D)$ the Banach space of $(0,q)$-forms whose coefficients are uniformly Hölder continuous of order $1/m$ on D. Then there are bounded linear operators*

$$T_q : C^0_{(0,q+1)}(\bar{D}) \to \Lambda^{1/m}_{(0,q)}(D)$$

such that $\bar{\partial} T_q f = f$ for all $f \in C^0_{(0,q+1)}(\bar{D})$ with $\bar{\partial} f = 0$.

It is more appropriate to work with the nonisotropic Hölder norm that comes from the pseudodistance d associated to the domain D. (See Section 2 for an exact definition of d.) We can prove that the solution operators of Theorem 1.1 are $(1/m)$-Hölder continuous with respect to this pseudodistance.

THEOREM 1.2. *Let D, T_q, and f be as in Theorem 1.1. Then, for every $\varepsilon > 0$, there exists a constant C such that the solution $u := T_q f$ of the Cauchy–Riemann equation $\bar{\partial} u = f$ satisfies the following nonisotropic Hölder estimate:*

$$|T_q f(z_0) - T_q f(z_1)| \le C \|f\|_\infty \max\{d(z_0, z_1)^{1/m}, |z_0 - z_1|^{1-\varepsilon}\}.$$

We also want to mention that both results are optimal. It is well known that Theorem 1.1 gives the best possible isotropic Hölder estimates for finite-type domains, and an example for the optimality of the nonisotropic Hölder estimates in Theorem 1.2 is given in [F2].

This paper is organized as follows. In Section 2 we give the definition of the solution operators T_q. We split them into several parts and first formulate their required estimates as Lemma 2.1; then we use these estimates to prove Theorem 1.1 and Theorem 1.2. The proof of Lemma 2.1 will be given in the remaining sections. In Section 3 we first describe several properties of the local geometry of lineally convex domains of finite type; then we use the result from [DFo2] to prove an appropriate estimate for the support function. Finally, in Section 4 the proof of Lemma 2.1 is completed in the same way as in [F2].

2. Solution Operators

In this paper we use exactly the same integral operator as in [DFFo]. (Of course, the definition of the support function and hence also of the solution operator depends on the given domain.) In [DFFo] we used the support function from [DFo1], which was defined only for convex domains of finite type. Using the results from [DFo2], we will see that the same definition also makes sense on lineally convex domains of finite type once we put in the support function of [DFo2].

Throughout this paper we will assume that $D := \{z \in \mathbb{C}^n : \varrho(z) < 0\}$ and that the defining function ϱ is chosen in such a way that, for all $-2\varepsilon_0 < t < 2\varepsilon_0$, the domains $D_t := \{z \in \mathbb{C}^n : \varrho(z) < t\}$ are lineally convex and of finite type $\le m$. We will also use the notation $U := \{z \in \mathbb{C}^n : |\varrho(z)| < \varepsilon_0\}$.

We write $l_\zeta(z) = \Phi(\zeta)(z - \zeta)$, where $\Phi(\zeta)$ is a unitary matrix depending smoothly on $\zeta \in \partial D$ such that the unit outer normal vector to ∂D will be turned into $(1, 0, \ldots, 0)$. As in [DFo2], we define

$$r_\zeta(w) := \varrho(l_\zeta^{-1}(w)), \qquad a_\alpha(\zeta) := \frac{1}{\alpha!}\frac{\partial^{|\alpha|} r_\zeta}{\partial w^\alpha}(0),$$

$$S_\zeta(w) := 3w_1 + K w_1^2 - c\sum_{j=2}^{m} M^{2^j}\sigma_j \sum_{\substack{|\alpha|=j\\ \alpha_1=0}} a_\alpha(\zeta) w^\alpha, \tag{1}$$

where M and K are large constants, c is a small constant, and $\sigma_j = \operatorname{Re} i^j$. Finally, we set

$$S(z,\zeta) := S_\zeta(l_\zeta(z)).$$

Next we give a decomposition of $S(z,\zeta)$ such that

$$S(z,\zeta) = \sum_{j=1}^{n} Q_j(z,\zeta)(z_j - \zeta_j).$$

For this we simply define

$$Q_\zeta^1(w) := 3 + K w_1 \tag{2}$$

and (for $k > 1$)

$$Q_\zeta^k(w) := -c\sum_{j=2}^{m} M^{2^j}\sigma_j \sum_{\substack{|\alpha|=j\\ \alpha_1=0,\, \alpha_k>0}} \frac{\alpha_k}{j} a_\alpha(\zeta)\frac{w^\alpha}{w_k} \tag{3}$$

and then set

$$Q(z,\zeta) := \Phi^T(\zeta) Q_\zeta(l_\zeta(z)),$$

where we use the notation $Q_\zeta = (Q_\zeta^1, \ldots, Q_\zeta^n)$ and $Q = (Q_1, \ldots, Q_n)$.

Now we introduce Cauchy–Fantappiè integral operators R_q based on the support function S and its Leray decomposition $Q(z,\zeta)$. We define the Cauchy–Fantappiè form

$$W(z,\zeta) := \sum_i \frac{Q_i(z,\zeta)}{S(z,\zeta)}\, d\zeta_i.$$

Let

$$B = \frac{b}{|\zeta - z|^2} = \sum_i \frac{\bar\zeta_i - \bar z_i}{|\zeta - z|^2}\, d\zeta_i$$

be the usual Martinelli–Bochner form and let K_q be the corresponding Martinelli–Bochner operator. Furthermore, put

$$\begin{aligned} R_q f &:= \sum_{k=0}^{n-q-2} c_k^q \int_{\zeta\in\partial D} f \wedge W \wedge B \wedge (\bar\partial_\zeta W)^k \wedge (\bar\partial_\zeta B)^{n-q-k-2} \wedge (\bar\partial_z B)^q \\ &= \sum_{k=0}^{n-q-2} c_k^q \int_{\zeta\in\partial D} f \wedge \frac{Q \wedge b \wedge (\bar\partial_\zeta Q)^k \wedge (\bar\partial_\zeta b)^{n-q-k-2} \wedge (\bar\partial_z b)^q}{S^{k+1}|\zeta - z|^{2(n-k-1)}}. \end{aligned} \tag{4}$$

In the last line we used the convention of denoting the $(1,0)$-form $\sum_i Q_i(z,\zeta)\,d\zeta_i$ again by Q. It is well known (see e.g. [R] or [DFoW]) that the operators $T_q = R_q + K_q$ are solution operators, meaning that $\bar\partial T_q f = f$ for all $\bar\partial$-closed forms $f \in C^0_{(0,q+1)}(\bar D)$.

While estimating our solution operators we must make use of the special local geometry of the given domain D. For this we let $\zeta \in U$ and $\varepsilon < \varepsilon_0$, choose some unit vector γ, and define the complex directional level distances by

$$\tau(\zeta,\gamma,\varepsilon) := \max\{c : |\varrho(\zeta+\lambda\gamma) - \varrho(\zeta)| < \varepsilon \text{ for all } \lambda \in \mathbb{C},\ |\lambda| < c\}.$$

For a fixed point ζ and a fixed radius ε we define the ε-extremal basis $(v_1,\dots,v_n)$ or, more precisely, $(v_1(\zeta,\varepsilon),\dots,v_n(\zeta,\varepsilon))$ centered at ζ as in [C], which is basically the same construction as in [Mc, Prop. 2.1] and [H, Def. 2.5]. The first vector v_1 is always the unit outward normal at ζ. After that, the unit vectors v_k are recursively chosen in the orthogonal complement of $v_1,\dots,v_{k-1}$ in such a way that they minimize (in [Mc], maximize) the function $\tau(\zeta,\cdot,\varepsilon)$. Using the abbreviation $\tau_k(\zeta,\varepsilon) := \tau(\zeta,v_k,\varepsilon)$, we see immediately that $\tau_1(\zeta,\varepsilon) \approx \varepsilon$ and $\tau_k(\zeta,\varepsilon) \gtrsim \tau_1(\zeta,\varepsilon)$ for all $k = 2,\dots,n$ and small enough values of ε. We now can define the ε-distinguished polydiscs

$$AP_\varepsilon(\zeta) := \left\{ z = \zeta + \sum \lambda_k v_k(\zeta,\varepsilon) : |\lambda_k| \le A\tau_k(\zeta,\varepsilon) \text{ for } k = 1,\dots,n \right\}$$

and the pseudodistance

$$d(\zeta,z) := \inf\{\varepsilon : z \in P_\varepsilon(\zeta)\}.$$

Note that these definitions are exactly analogous to those given for convex domains of finite type. The fundamental properties of these objects on lineally convex domains of finite type will be listed in the next section. Here we will continue the investigation of our solution operator T_q.

The Martinelli–Bochner operator K_q is known to satisfy (isotropic) α-Hölder estimates for all $\alpha < 1$. This is good enough for both Theorems 1.1 and 1.2, so it remains to estimate $R_q f$. In order to do so we consider z-derivatives of this form. For a z-derivative δ_γ in the direction of γ it is easy to see that $\delta_\gamma \bar\partial_\zeta b = \delta_\gamma \bar\partial_z b = 0$. Thus $\delta_\gamma R_q f$ can be written as a sum of integrals of the form

$$\int_{\partial D} f \wedge \frac{\delta_\gamma Q \wedge b \wedge (\bar\partial_\zeta Q)^k \wedge (\bar\partial_\zeta b)^{n-q-k-2} \wedge (\bar\partial_z b)^q}{S^{k+1}|\zeta - z|^{2(n-k-1)}},$$

$$\int_{\partial D} f \wedge \frac{Q \wedge \delta_\gamma b \wedge (\bar\partial_\zeta Q)^k \wedge (\bar\partial_\zeta b)^{n-q-k-2} \wedge (\bar\partial_z b)^q}{S^{k+1}|\zeta - z|^{2(n-k-1)}},$$

$$\int_{\partial D} f \wedge \frac{Q \wedge b \wedge \delta_\gamma(\bar\partial_\zeta Q) \wedge (\bar\partial_\zeta Q)^{k-1} \wedge (\bar\partial_\zeta b)^{n-q-k-2} \wedge (\bar\partial_z b)^q}{S^{k+1}|\zeta - z|^{2(n-k-1)}},$$

$$\int_{\partial D} f \wedge \frac{Q \wedge b \wedge (\bar\partial_\zeta Q)^k \wedge (\bar\partial_\zeta b)^{n-q-k-2} \wedge (\bar\partial_z b)^q}{S^{k+2}|\zeta - z|^{2(n-k-1)}}\, \delta_\gamma S,$$

$$\int_{\partial D} f \wedge \frac{Q \wedge b \wedge (\bar\partial_\zeta Q)^k \wedge (\bar\partial_\zeta b)^{n-q-k-2} \wedge (\bar\partial_z b)^q}{S^{k+1}|\zeta - z|^{2(n-k-1)+1}}\, \delta_\gamma|\zeta - z|,$$

where the third integral appears only for $k > 0$.

These integrals must be estimated. We majorize f by $\|f\|_\infty$ and observe that $\delta_\gamma b \lesssim 1$ and $\delta_\gamma|\zeta - z| \lesssim 1$. Hence the second and the fifth integral can each be replaced by $\|f\|_\infty I_a$ with

$$I_a := \int_{\partial D} \frac{|[Q \wedge (\bar{\partial}_\zeta Q)^k \wedge \beta]_t|}{|S|^{k+1}|\zeta - z|^{2(n-k-1)}}\, d\sigma_{2n-1},$$

where β is a differential form that contains all the remaining $d\zeta_j$ and $d\bar{\zeta}_j$ such that $Q \wedge (\bar{\partial}_\zeta Q)^k \wedge \beta$ is of bidegree $(n, n-1)$ in ζ. Here $[Q \wedge (\bar{\partial}_\zeta Q)^k \wedge \beta]_t$ denotes the tangential part of the form $Q \wedge (\bar{\partial}_\zeta Q)^k \wedge \beta$, which is the only part of the form that contributes to the integral over ∂D.

For the other three integrals we must consider

$$I_b := \int_{\partial D} \frac{|[\delta_\gamma Q \wedge (\bar{\partial}_\zeta Q)^k \wedge \beta]_t|}{|S|^{k+1}|\zeta - z|^{2(n-k-1)-1}}\, d\sigma_{2n-1},$$

$$I_c := \int_{\partial D} \frac{|[Q \wedge \delta_\gamma(\bar{\partial}_\zeta Q) \wedge (\bar{\partial}_\zeta Q)^{k-1} \wedge \beta]_t|}{|S|^{k+1}|\zeta - z|^{2(n-k-1)-1}}\, d\sigma_{2n-1},$$

$$I_d := \int_{\partial D} \frac{|[Q \wedge (\bar{\partial}_\zeta Q)^k \wedge \beta]_t||\delta_\gamma S|}{|S|^{k+2}|\zeta - z|^{2(n-k-1)-1}}\, d\sigma_{2n-1},$$

where again β is an appropriate differential form and $[\cdot]_t$ denotes the tangential part of the form in brackets.

We proceed by formulating the estimates for these integrals, which are needed in the proof of Theorem 1.1 and Theorem 1.2.

LEMMA 2.1. *If* $\varrho = |\varrho(z)|$ *and* $0 < \sigma \leq 1$, *then*

$$|I_a| \lesssim \varrho^{\sigma(1/m-1)}. \tag{5}$$

The remaining integrals—where the derivative in the γ*-direction is applied to either* Q, $\bar{\partial}_\zeta Q$, *or* S*—satisfy the estimate*

$$|I_b|, |I_c|, |I_d| \lesssim \frac{\varrho^{1/m}}{\tau(z, \gamma, \varrho)}. \tag{6}$$

The proof of the main theorems follows from these estimates and some basic facts concerning the pseudodistance, which will be discussed in Section 3. Note that Lemma 2.1 will be proved only for z close to the boundary. It could be generalized for all $z \in D$ by some compactness argument, but in fact it is needed only for z in a neighborhood of the boundary (and small values of ϱ) because it is enough to establish the Hölder estimates there. In the proof of Theorem 1.2 we can also assume that z_0 and z_1 are close to each other and in a given neighborhood of the boundary.

Proof of Theorem 1.1. The theorem follows from Theorem 1.2 because we have the estimate $d(z, \zeta) \lesssim |z - \zeta|$. Alternatively, the theorem can be proved directly as follows. Using the relation $\varepsilon \lesssim \tau(z, \gamma, \varepsilon)$, which is a consequence of Lemma 2.1 and the definition of the ε-extremal basis (see also Proposition 3.1(v)), we obtain $|I_*| \lesssim \varrho^{1/m-1}$ for all the integrals $I_a, \ldots, I_d$. Since $\varrho = |\varrho(z)|$ is comparable to $\operatorname{dist}(z, \partial D)$ we have $|\delta_\gamma R_q f| \lesssim \|f\|_\infty \operatorname{dist}(z, \partial D)^{1/m-1}$, and the $(1/m)$-Hölder continuity of $R_q f$ follows from the Hardy–Littlewood lemma. □

Proof of Theorem 1.2. First let u be that part of $R_q f$ whose derivative we have shown to be majorized by a multiple of I_a. Let $\alpha < 1$ and choose σ so small that $\sigma(1/m-1) > \alpha-1$. Then Lemma 2.1 implies that $|\delta_\gamma u(z)| \lesssim \|f\|_\infty \operatorname{dist}(z, \partial D)^{\alpha-1}$ and hence the Hardy–Littlewood lemma implies that u is isotropically α-Hölder continuous for every $\alpha < 1$.

Next let u be one of those parts of $R_q f$ whose δ_γ-derivative has been majorized by a multiple of one of the integrals I_b, I_c, or I_d. Let $A = d(z_0, z_1)$ and $\gamma = (z_1 - z_0)/|z_1 - z_0|$, and let ν be the inward normal direction at $\zeta_0 = \pi(z_0)$. Consider the additional points $\tilde{z}_0 = z_0 + A\nu$ and $\tilde{z}_1 = z_1 + A\nu$. Then we estimate

$$\begin{aligned} |u(z_0) - u(z_1)| &\le |u(z_0) - u(\tilde{z}_0)| + |u(\tilde{z}_0) - u(\tilde{z}_1)| + |u(\tilde{z}_1) - u(z_1)| \\ &\le \int_{z_0}^{\tilde{z}_0} |\delta_\nu u(t)|\, dt + \int_{\tilde{z}_0}^{\tilde{z}_1} |\delta_\gamma u(t)|\, dt + \int_{z_1}^{\tilde{z}_1} |\delta_\nu u(t)|\, dt. \end{aligned}$$

In the first and the third integral of the right side we have the worst case, because ν is approximately the normal direction and thus Proposition 3.1(v) implies that $\tau(t, \nu, |\varrho(t)|) \approx |\varrho(t)|$. Nevertheless, from (6) we derive the estimate

$$\int_{z_0}^{\tilde{z}_0} |\delta_\nu u(t)|\, dt \le \int_0^A |\delta_\nu u(z_0 + s\nu)|\, ds \lesssim \int_0^A s^{1/m-1}\, ds \lesssim A^{1/m}.$$

(The same is true for the third integral.)

To estimate the second integral, first observe that d is a pseudodistance (see Proposition 3.1(x)) and therefore satisfies the approximate triangle inequality $d(\tilde{z}_0, \tilde{z}_1) \lesssim d(\tilde{z}_0, z_0) + d(z_0, z_1) + d(z_1, \tilde{z}_1) \lesssim A$. Thus $\tilde{z}_1$ and the whole line from $\tilde{z}_0$ to $\tilde{z}_1$ belong to some polydisc $P_{CA}(\tilde{z}_0)$. Proposition 3.1(viii) then gives the relation $\tau(\zeta, \gamma, A) \approx \tau(\tilde{z}_0, \gamma, A)$ for all those ζ. On the other hand, $|\varrho(\tilde{z}_0)| \gtrsim A$ and so $|\delta_\gamma u(\zeta)| \lesssim A^{1/m}/\tau(\tilde{z}_0, \gamma, A)$ for all ζ on the line to $\tilde{z}_1$. Thus by (6) it follows that

$$\begin{aligned} \int_{\tilde{z}_0}^{\tilde{z}_1} |\delta_\gamma u(t)|\, dt &\lesssim \int_0^{\tau(\tilde{z}_0, \gamma, A)} |\delta_\gamma u(\tilde{z}_0 + s\gamma)|\, ds \\ &\lesssim \int_0^{\tau(\tilde{z}_0, \gamma, A)} \frac{A^{1/m}}{\tau(\tilde{z}_0, \gamma, A)}\, ds \lesssim A^{1/m}, \end{aligned}$$

and together with the estimates for the first and third integral this yields

$$|u(z_0) - u(z_1)| \le d(z_0, z_1)^{1/m}.$$

Now recalling that the other parts of $R_q f$ are isotropic α-Hölder continuous for every $\alpha < 1$ completes the proof of the theorem. □

3. The Pseudodistance and Estimates for the Support Function

We will start this section by listing some properties of $\tau(z, \gamma, \varepsilon)$, $P_\varepsilon(z)$, and $d(z, \zeta)$ as defined previously (some of them have already been used in the proofs of Theorem 1.1 and Theorem 1.2). The following results have been proved in [C].

PROPOSITION 3.1. *Let $D = \{z : \varrho(z) < 0\}$ be a lineally convex domain of finite type m. Then the following statements hold.*

(i) *There is a constant c such that $cP_{|\varrho(\zeta)|}(\zeta) \subset D$ for all $\zeta \in D$.*

(ii) *Let w be any orthonormal coordinate system centered at z and let v_j be the unit vector in the w_j-direction; then*

$$\left|\frac{\partial^{|\alpha+\beta|}\varrho(z)}{\partial w^\alpha \partial \bar{w}^\beta}\right| \lesssim \frac{\varepsilon}{\prod_j \tau(z, v_j, \varepsilon)^{\alpha_j+\beta_j}}$$

for all multi-indices α and β with $|\alpha+\beta| \geq 1$.

(iii) *If $\gamma = \sum_{j=1}^n a_j v_j$, where $(v_1, \dots, v_n)$ is the ε-extremal basis at ζ, then*

$$\frac{1}{\tau(\zeta, \gamma, \varepsilon)} \approx \sum_{j=1}^n \frac{|a_j|}{\tau_j(\zeta, \varepsilon)};$$

in particular, for every unit vector γ we have $\tau(\zeta, \gamma, \varepsilon) \lesssim \tau_k(\zeta, \varepsilon)/|a_k|$ for all k.

(iv) *Let γ be a unit vector and let*

$$a^\gamma_{\alpha\beta}(\zeta) := \frac{\partial^{\alpha+\beta}}{\partial\lambda^\alpha \partial\bar{\lambda}^\beta} \varrho(\zeta + \lambda\gamma)|_{\lambda=0};$$

then

$$\sum_{1\leq\alpha+\beta\leq m} |a^\gamma_{\alpha\beta}(\zeta)| \tau(\zeta, \gamma, \varepsilon)^{\alpha+\beta} \approx \varepsilon$$

uniformly for all ζ, γ, and ε.

(v) $\tau_1(\zeta, \varepsilon) \approx \varepsilon$ *and* $\tau(\zeta, \gamma, \varepsilon) \lesssim \varepsilon^{1/m}$ *for every unit vector γ, and if γ is a unit vector in complex tangential direction then also $\varepsilon^{1/2} \lesssim \tau(\zeta, \gamma, \varepsilon)$.*

(vi) *If γ is a complex directional unit vector and $0 < \varepsilon < \delta$, then*

$$\left(\frac{\varepsilon}{\delta}\right)^{1/2} \tau(\zeta, \gamma, \delta) \lesssim \tau(\zeta, \gamma, \varepsilon) \lesssim \left(\frac{\varepsilon}{\delta}\right)^{1/m} \tau(\zeta, \gamma, \delta).$$

(vii) *For every $k > 0$ there exist constants $c(k)$ and $C(k)$ such that*

$$c(k)P_\varepsilon(\zeta) \subset P_{k\varepsilon}(\zeta) \subset C(k)P_\varepsilon(\zeta) \quad \text{and} \quad P_{c(k)\varepsilon}(\zeta) \subset kP_\varepsilon(\zeta) \subset P_{C(k)\varepsilon}(\zeta).$$

(viii) *For every $z \in P_\varepsilon(\zeta)$ we have $\tau(\zeta, \gamma, \varepsilon) \approx \tau(z, \gamma, \varepsilon)$.*

(ix) *There is a constant C such that $P_\varepsilon(\zeta) \cap P_\varepsilon(z) \neq \emptyset$ implies*

$$P_\varepsilon(\zeta) \subset CP_\varepsilon(z) \quad \text{and} \quad P_\varepsilon(z) \subset CP_\varepsilon(\zeta).$$

(x) *The pseudodistance $d(z, \zeta)$ satisfies the properties*

$$d(z, \zeta) \approx d(\zeta, z) \quad \text{and}$$
$$d(z, \zeta) \lesssim d(z, w) + d(w, \zeta).$$

Note that Proposition 3.1(vii) implies that there exist constants $C > 1$ and $0 < c < 1$ such that $CP_{\varepsilon/2}(\zeta) \supset \frac{1}{2}P_\varepsilon(\zeta)$ and $CP_t(\zeta) \subset P_\varepsilon(\zeta)$ for all $t < c\varepsilon$. We can therefore use the polyannuli

$$P^i_\varepsilon(\zeta) := CP_{2^{-i}\varepsilon}(\zeta) \setminus \tfrac{1}{2}P_{2^{-i}\varepsilon}(\zeta)$$

to construct two coverings. This yields

$$\bigcup_{i=0}^{\infty} P_\varepsilon^i(\zeta) \supset P_\varepsilon(\zeta) \setminus \{\zeta\} \quad \text{and} \quad \bigcup_{i=0}^{i_0(\varepsilon)} P_1^i(\zeta) \supset P_1(\zeta) \setminus P_\varepsilon(\zeta), \tag{7}$$

where $i_0(\varepsilon)$ is a finite number depending only on ε and satisfying $i_0(\varepsilon) < -\log_2(c\varepsilon)$ for another small constant c.

Now we recall the support function of [DFo2]; more precisely, we want to make use of the explicit support function given in [DFo2, Thm. 2.6]. For this we must introduce another transformation of the form $\hat{w} = \lambda_\zeta(w)$ with $\hat{w}_1(1 - A_\zeta(\hat{w})) = w_1$ and $\hat{w}_j = w_j$ for $j > 1$. Here $A_\zeta(\hat{w})$ is a smooth family of holomorphic polynomials depending only on $\hat{w}_2, \dots, \hat{w}_n$ and satisfying $A_\zeta(0) = 0$. Writing $\hat{l}_\zeta = \lambda_\zeta \circ l_\zeta$, we then have

$$\hat{r}_\zeta(\hat{w}) := \varrho(\hat{l}_\zeta^{-1}(\hat{w})) - \varrho(\zeta), \qquad \hat{a}_\alpha(\zeta) := \frac{1}{\alpha!} \frac{\partial^{|\alpha|} \hat{r}_\zeta(\hat{w})}{\partial \hat{w}^\alpha}\bigg|_{\hat{w}=0}$$

$$\hat{S}_\zeta(\hat{w}) := \hat{w}_1 + K\hat{w}_1^2 - c\sum_{j=2}^{m} M^{2^j}\sigma_j \sum_{|\alpha|=j,\, \alpha_1=0} \hat{a}_\alpha(\zeta)\hat{w}^\alpha,$$

and finally $\hat{S}(z,\zeta) = \hat{S}_\zeta(\hat{l}_\zeta(z))$. Theorem 2.6 of [DFo2] supplies the following estimate.

PROPOSITION 3.2. *Let γ be a unit vector in complex tangential direction in ζ, and define*

$$\hat{a}_j^\gamma(\zeta) := \sum_{\alpha+\beta=j} \left| \frac{\partial^{\alpha+\beta}}{\partial\lambda^\alpha \partial\bar{\lambda}^\beta} \hat{r}_\zeta(\lambda\gamma)|_{\lambda=0} \right|.$$

Then there exists a smooth function $\hat{h}(\zeta,\hat{w})$ that is positive and bounded away from 0 *such that, for every $\hat{w} = \mu n_\zeta + \lambda\gamma$ (where n_ζ is the unit normal at ζ),*

$$\operatorname{Re} \hat{S}_\zeta(\hat{w}) \le \hat{r}_\zeta(\hat{w})\hat{h}(\zeta,\hat{w}) - \frac{K}{2}(\operatorname{Im}\mu)^2 - \hat{c}c\sum_{j=2}^{m} \hat{a}_j^\gamma(\zeta)|\lambda|^j.$$

We use this result to derive some estimates for the support function S.

LEMMA 3.3. *Let z be in $D \cap U$ and assume that ε is smaller than ε_0. Then*

$$|S(z,\zeta)| \gtrsim \varepsilon \qquad \textit{for all } \zeta \in \partial D \cap P_\varepsilon^0(\pi(z)), \tag{8}$$

$$|S(z,\zeta)| \gtrsim |\varrho(z)| \quad \textit{for all } \zeta \in \partial D \cap P_{|\varrho(z)|}(\pi(z)). \tag{9}$$

Proof. To prove the lemma we must consider the difference between $S_\zeta(w)$ and $\hat{S}_\zeta(\lambda_\zeta(w))$. Recall that $\hat{w}_j = w_j$ for $j > 1$. Moreover, since derivatives are only in complex tangential directions and evaluation is at $\hat{w} = w = 0$, it follows that $\hat{a}_\alpha(\zeta) = a_\alpha(\zeta)$. Thus we have

$$S_\zeta(w) - \hat{S}_\zeta(\lambda_\zeta(w)) = w_1\left(\frac{-A_\zeta(w)}{1 - A_\zeta(w)}\right) + Kw_1^2\left(\frac{-2A_\zeta(w) + A_\zeta^2(w)}{(1 - A_\zeta(w))^2}\right)$$

and, for every $\tilde{c} > 0$, there exists some $\tilde{\varepsilon}$ such that

$$|S_\zeta(w) - \hat{S}_\zeta(\lambda_\zeta(w))| \le \tilde{c}(|\mathrm{Re}\, w_1| + |\mathrm{Im}\, w_1|)$$

for all $z \in P_{\tilde{\varepsilon}}(\zeta)$. Therefore,

$$\mathrm{Re}\, S_\zeta(w) \le r_\zeta(w)h(\zeta, w) + \tilde{c}|\mathrm{Re}\,\mu| - \frac{K}{2}(\mathrm{Im}\,\mu)^2 + \tilde{c}|\mathrm{Im}\,\mu| - \hat{c}c\sum_{j=2}^{m} a_j^\gamma(\zeta)|\lambda|^j.$$

Now we observe (a) that $\partial r_\zeta(\mu n_\zeta + \lambda\gamma)/\partial\,\mathrm{Re}\,\mu$ is positive and bounded away from zero and (b) that Proposition 3.1(iv) and the definition of $\tau(\zeta, \gamma, \varepsilon)$ imply that $r_\zeta(\lambda\gamma) \lesssim \sum a_j^\gamma(\zeta)|\lambda|^j$. Hence, for every point $w = \mu n_\zeta + \lambda\gamma$ in $D \cap P_{\tilde{\varepsilon}}(\zeta)$,

$$\begin{aligned}\mathrm{Re}\, S_\zeta(w) &\le c' r_\zeta(w) - \frac{K}{2}(\mathrm{Im}\,\mu)^2 + \tilde{c}|\mathrm{Im}\,\mu| - c''\sum_{j=2}^{m} a_j^\gamma(\zeta)|\lambda|^j \\ &\le \tilde{c}|\mathrm{Im}\,\mu| - c''\sum_{j=2}^{m} a_j^\gamma(\zeta)|\lambda|^j. \qquad (10)\end{aligned}$$

Here $\tilde{c}$ still depends only on $\tilde{\varepsilon}$, and after fixing $\tilde{c}$ we can simply assume that $\varepsilon_0 < \tilde{\varepsilon}$.

Now let $\zeta \in \partial D$ and ε be fixed, let $0 < \ell < L$ be some fixed constants with ℓ_0 a small constant to be chosen later. As before, we write $z = \mu n_\zeta + \lambda\gamma$ and define

$$\tilde{P}_\varepsilon^0(\zeta) := \{z : |\mathrm{Re}\,\mu| < \ell_0,\ (z - \mathrm{Re}\,\mu n_\zeta) \in LP_\varepsilon(\zeta),\ (z - \mathrm{Re}\,\mu n_\zeta) \notin \ell P_\varepsilon(\zeta)\}.$$

We will show that, for every pair $0 < \ell < L$, there exist constants ℓ_0 and c_0 such that

$$|S(z, \zeta)| \ge c_0\varepsilon \quad \text{for all } z \in \tilde{P}_\varepsilon^0(\zeta). \qquad (11)$$

Here ℓ_0 and c_0 do depend on ℓ and L but not on ζ or ε.

By Proposition 3.1(iii), there clearly exists a constant k_1' such that $|\mathrm{Im}\,\mu| \le k_1'\tau(\zeta, n_\zeta, \varepsilon)$ and $|\lambda| \le k_1'\tau(\zeta, \gamma, \varepsilon)$ imply $(z - \mathrm{Re}\,\mu n_\zeta) \in \ell P_\varepsilon(\zeta)$. Thus we have either $|\lambda| \ge k_1'\tau(\zeta, \gamma, \varepsilon)$ or $|\mathrm{Im}\,\mu| \ge k_1'\tau(\zeta, n_\zeta, \varepsilon)$ or both. Let $k_1 < k_1'$ be a constant to be chosen later. If $|\lambda| \ge k_1\tau(\zeta, \gamma, \varepsilon)$ then we can use estimate (10), Proposition 3.1(iv), and $|\mathrm{Im}\,\mu| \le L\tau(\zeta, n_\zeta, \varepsilon) \le CL\varepsilon$ to obtain

$$|S(z, \zeta)| \ge -\mathrm{Re}\, S_\zeta(w) \ge c''\sum_{j=2}^{m} a_j^\gamma(\zeta)|\lambda|^j - \tilde{c}|\mathrm{Im}\,\mu| \ge c'''\varepsilon - \tilde{c}CL\varepsilon.$$

Choosing $\tilde{c}$ small enough we finally get $|S(z, \zeta)| \gtrsim \varepsilon$ for the case $|\lambda| \ge k_1\tau(\zeta, \gamma, \varepsilon)$.

If $|\lambda| \le k_1\tau(\zeta, \gamma, \varepsilon)$ then necessarily $|\mathrm{Im}\,\mu| \ge k_1'\tau(\zeta, n_\zeta, \varepsilon) \ge k_2\varepsilon$. Considering now the imaginary part of S yields

$$\begin{aligned}|S(z, \zeta)| &\ge |\mathrm{Im}\, S(z, \zeta)| \\ &\ge |3\,\mathrm{Im}\,\mu| - |2K\,\mathrm{Re}\,\mu\,\mathrm{Im}\,\mu| - c\sum_{j=2}^{m} a_j^\gamma(\zeta)|\lambda|^j.\end{aligned}$$

Using the estimate for λ and again Proposition 3.1(iv), the last term of this inequality can be estimated by $k_1^2 c\varepsilon$. Now we can choose k_1 so small that $k_1^2 c\varepsilon < k_2\varepsilon$. By the definition of $\tilde{P}_\varepsilon^0(\zeta)$, we also have $|\mathrm{Im}\,\mu| < L\tau(\zeta, n_\zeta, \varepsilon) \le CL\varepsilon$. Hence the second term can be estimated by $2Kk_0CL\varepsilon$ and k_0 can be chosen so small that $2Kk_0CL\varepsilon < k_2\varepsilon$. Altogether we have

$$|S(z,\zeta)| \geq 3k_2\varepsilon - k_2\varepsilon - k_2\varepsilon \gtrsim \varepsilon,$$

and the proof of (11) is complete.

To prove (8) we need only observe that $\zeta \in P_\varepsilon^0(\pi(z))$ implies both $\zeta \in CP_\varepsilon(\pi(z))$ and $\zeta \notin \frac{1}{2}P_\varepsilon(\pi(z))$. Using Proposition 3.1(ix) and (vii) then yields $\pi(z) \in CP_\varepsilon(\zeta)$ and $\pi(z) \notin cP_\varepsilon(\zeta)$ for certain constants c and C. If z is close enough to the boundary and ε is small enough, this implies $z \in \tilde{P}_\varepsilon^0(\zeta)$ for still some other constants ℓ and L. The first statement of the lemma now follows from (11).

The estimate (9) also follows from (11) in a similar way. First, Proposition 3.1(i) implies that $\zeta \in \partial D$ does not belong to some $cP_{|\varrho(z)|}(z)$ and therefore $z \notin \ell P_{|\varrho(z)|}(\zeta)$. On the other hand, $\zeta \in P_{|\varrho(z)|}(\pi(z))$ and $\pi(z) \in P_{2|\varrho(z)|}(z)$; using Proposition 3.1(ix) and (vii), this implies $z \in LP_{|\varrho(z)|}(\zeta)$. So z belongs to $\tilde{P}^0_{|\varrho(z)|}(\zeta)$ for certain constants $0 < \ell < L$, and the second statement of the lemma follows from (11). □

4. Proof of Lemma 2.1

The rest of the proof of Lemma 2.1 is exactly the same as in [F2]. First we fix some $\zeta_0 \in \partial D$ and some $\varepsilon < \varepsilon_0$ and then define Φ^* to be the matrix that transforms our coordinates to the ε-extremal coordinates at ζ_0. Thus Φ^* is a constant unitary matrix. Further, we define $w^* = \Phi^*(z - \zeta_0)$ and $\eta^* = \Phi^*(\zeta - \zeta_0)$. We also make use of a condition $(*)$, which says that $|\eta_k^*| \leq C^*\tau_k(\zeta_0, \varepsilon)$ for $k = 1, \dots, n$ and that $|w_1^*| \leq C^*$ and $|w_k^*| \leq C^*\tau_k(\zeta_0, \varepsilon)$ for $k > 1$. Notice that condition $(*)$ is satisfied if $\zeta_0 = \pi(z)$ and $\zeta \in P_\varepsilon(\zeta_0)$.

Next we use the fact that Q is invariant under additional rotations of the complex tangent space and provide a special family of smooth unitary matrices $\Phi(\zeta) := \Psi(\Phi^*(\zeta - \zeta_0))\Phi^*$, where $\Psi(\eta^*)$ is a unitary matrix smoothly depending on η^* and with entries that are obtained directly from derivatives of the defining function (see [DFFo] for details). Using some estimates for $\Psi(\eta^*)$ and writing Q with respect to the ε-extremal coordinates at ζ_0 as

$$Q^*(w^*, \eta^*) = \bar{\Phi}^* Q(\zeta_0 + (\bar{\Phi}^*)^T w^*, \zeta_0 + (\bar{\Phi}^*)^T \eta^*)$$

yields the following lemma.

LEMMA 4.1 [F2, Lemma 3.1]. *For all w^* and η^* satisfying condition* $(*)$:

$$|Q_k^*(w^*, \eta^*)| \lesssim \frac{\varepsilon}{\tau_k(\zeta_0, \varepsilon)},$$

$$\left|\frac{\partial}{\partial w_i^*} Q_k^*(w^*, \eta^*)\right| \lesssim \frac{\varepsilon}{\tau_k(\zeta_0, \varepsilon)\tau_i(\zeta_0, \varepsilon)},$$

$$\left|\frac{\partial}{\partial \eta_j^*} Q_k^*(w^*, \eta^*)\right| \lesssim \frac{\varepsilon}{\tau_k(\zeta_0, \varepsilon)\tau_j(\zeta_0, \varepsilon)},$$

$$\left|\frac{\partial^2}{\partial w_i^* \partial \eta_j^*} Q_k^*(w^*, \eta^*)\right| \lesssim \frac{\varepsilon}{\tau_k(\zeta_0, \varepsilon)\tau_i(\zeta_0, \varepsilon)\tau_j(\zeta_0, \varepsilon)}.$$

This lemma leads to the following integral estimates.

LEMMA 4.2 [F2, Lemma 3.14]. *For every σ with $0 < \sigma \leq 1$ we have the estimate*

$$\int_{\partial D\cap P_\varepsilon(\zeta_0)} \frac{|[Q \wedge (\bar\partial_\zeta Q)^k \wedge \beta]_t|}{|\zeta - z|^{2n-2k-3+(1-\sigma)}}\, d\sigma_{2n-1} \lesssim \varepsilon^{\sigma/m+k+1}.$$

Moreover,

$$\int_{\partial D\cap P_\varepsilon(\zeta_0)} \frac{|[\delta_\gamma Q \wedge (\bar\partial_\zeta Q)^k \wedge \beta]_t|}{|\zeta - z|^{2n-2k-3}}\, d\sigma_{2n-1} \lesssim \frac{\varepsilon^{1/m+k+1}}{\tau(\zeta_0,\gamma,\varepsilon)},$$

$$\int_{\partial D\cap P_\varepsilon(\zeta_0)} \frac{|[Q \wedge \delta_\gamma(\bar\partial_\zeta Q) \wedge (\bar\partial_\zeta Q)^{k-1} \wedge \beta]_t|}{|\zeta - z|^{2n-2k-3}}\, d\sigma_{2n-1} \lesssim \frac{\varepsilon^{1/m+k+1}}{\tau(\zeta_0,\gamma,\varepsilon)}.$$

For derivatives of S we have another lemma as follows.

LEMMA 4.3 [F2, Lemma 3.10]. *Let δ_γ be the z-derivative in the γ-direction. Then, for all w^* and η^* satisfying condition* $(*)$,

$$|\delta_\gamma S^*(w^*,\eta^*)| \lesssim \frac{\varepsilon}{\tau(\zeta_0,\gamma,\varepsilon)}.$$

Finally we make use of the fact that a small neighborhood of ζ_0 can be covered by a family of polyannuli as in (7). In each of those polyannuli we can use Lemma 3.3, which together with the estimates from Lemma 4.2 and Lemma 4.3 gives the desired estimates for Lemma 2.1 in exactly the same way as in [F2].

5. Additional Remarks

In this paper we have seen that the proof of optimal nonisotropic Hölder estimates for solutions of the Cauchy–Riemann equation in lineally convex domains of finite type is very close to the corresponding proof in convex finite type domains. This is due to the domain's geometry entering at two places only. First we need a smooth family of holomorphic support functions with the right estimates; second, the domain's boundary must admit a pseudodistance satisfying certain properties. Once these ingredients have been provided, everything else carries over almost automatically.

Thus, considering the nonisotropic Hölder estimates of Theorem 1.2 as a test case, it can be predicted that other estimates (e.g., those in [A; DM1; DM2; F1; H]) will also be true on lineally convex domains of finite type.

On the other hand, it is not clear whether the method of Cumenge can modified to work also for lineally convex domains; it seems that convexity is used in a more essential way in the construction of her solution operators. In any case, the main estimates from [Mc] would first have to be carried over to the case of lineally convex domains of finite type.

References

[A] W. Alexandre, *Estimées C^k pour les domaines convexes de type fini de $\mathbb{C}^n$*, C. R. Math. Acad. Sci. Paris Sér. I Math. 335 (2002), 23–26.

[C] M. Conrad, *Nicht-isotrope Abschätzungen für lineal konvexe Gebiete endlichen Typs,* Dissertation, BUGH Wuppertal, 2002, ⟨http://elpub.bib.uni-wuppertal.de/edocs/dokumente/fb07/diss2002/conrad⟩.

[Cu1] A. Cumenge, *Estimées Lipschitz optimales dans les convexes de type fini,* C. R. Acad. Sci. Paris Sér. I Math. 325 (1997), 1077–1080.

[Cu2] ———, *Sharp estimates for $\bar{\partial}$ on convex domains of finite type,* Ark. Mat. 39 (2001), 1–25.

[DFFo] K. Diederich, B. Fischer, and J. E. Fornæss, *Hölder estimates on convex domains of finite type,* Math. Z. 232 (1999), 43–61.

[DFo1] K. Diederich and J. E. Fornæss, *Support functions for convex domains of finite type,* Math. Z. 230 (1999), 145–164.

[DFo2] ———, *Lineally convex domains of finite type: Holomorphic support functions,* Manuscripta Math. 112 (2003), 403–431.

[DFoW] K. Diederich, J. E. Fornæss, and J. Wiegerinck, *Sharp Hölder estimates for $\bar{\partial}$ on ellipsoids,* Manuscripta Math. 56 (1986), 399–417.

[DM1] K. Diederich and E. Mazzilli, *Zero varieties for the Nevanlinna class on all convex domains of finite type,* Nagoya Math. J. 163 (2001), 215–227.

[DM2] ———, *Extension of bounded holomorphic functions in convex domains,* Manuscripta Math. 105 (2001), 1–12.

[F1] B. Fischer, *L^p estimates on convex domains of finite type,* Math. Z. 236 (2001), 401–418.

[F2] ———, *Nonisotropic Hölder estimates on convex domains of finite type,* Michigan Math. J. 52 (2004), 219–239.

[H] T. Hefer, *Hölder and L^p estimates for $\bar{\partial}$ on convex domains of finite type depending on Catlin's multitype,* Math. Z. 242 (2002), 367–398.

[Mc] J. D. McNeal, *Estimates on the Bergman kernels of convex domains,* Adv. Math. 109 (1994), 108–139.

[R] R. M. Range, *Holomorphic functions and integral representations in several complex variables,* Grad. Texts in Math., 108, Springer-Verlag, New York, 1986.

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