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# Adiabatic Decomposition of the $\zeta$-Determinant and Scattering Theory

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## 1. Introduction and Statement of Results

Let $\mathcal{D}\colon C^\infty(M,S) \to C^\infty(M,S)$ be a compatible Dirac operator acting on sections of a Clifford bundle $S$ over a closed manifold $M$ of dimension $n$. The operator $\mathcal{D}$ is a self-adjoint operator with discrete spectrum $\{\lambda_k\}_{k\in\mathbb{Z}}$. The $\zeta$-determinant of the Dirac Laplacian $\mathcal{D}^2$ is given by the formula

$$\det{}_\zeta \mathcal{D}^2 = e^{-\zeta'_{\mathcal{D}^2}(0)}, \tag{1.1}$$

where $\zeta_{\mathcal{D}^2}(s)$ is defined as follows:

$$\zeta_{\mathcal{D}^2}(s) = \sum_{\lambda_k\neq 0}(\lambda_k^2)^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}[\operatorname{Tr}(e^{-t\mathcal{D}^2}) - \dim\ker\mathcal{D}]\,dt. \tag{1.2}$$

This is a holomorphic function of $s$ for $\Re(s) \gg 0$ and has the meromorphic extension to the complex plane with $s=0$ as a regular point.

Let us consider a decomposition of $M$ as $M_1\cup M_2$, where $M_1$ and $M_2$ are compact manifolds with boundaries such that

$$M = M_1\cup M_2, \qquad Y = M_1\cap M_2 = \partial M_1 = \partial M_2. \tag{1.3}$$

In this paper we study the adiabatic decomposition of the $\zeta$-determinant of $\mathcal{D}^2$, which describes the contributions in $\det_\zeta \mathcal{D}^2$ coming from the submanifolds $M_1$ and $M_2$. Throughout the paper, we assume that the manifold $M$ and the operator $\mathcal{D}$ have product structures in a neighborhood of the cutting hypersurface $Y$. Hence, there is a bicollar neighborhood $N\cong[-1,1]_u\times Y$ of $Y\cong\{0\}\times Y$ in $M$ such that the Riemannian structure on $M$ and the Hermitian structure on $S$ are products of the corresponding structures over $[-1,1]_u$ and $Y$ when restricted to $N$, so that $\mathcal{D}$ has the following form:

$$\mathcal{D} = G(\partial_u + B) \ \text{ over } N. \tag{1.4}$$

Here $u$ denotes the normal variable, $G\colon S|_Y \to S|_Y$ is a bundle automorphism, and $B$ is a corresponding Dirac operator on $Y$. Moreover, $G$ and $B$ do not depend on $u$, and they satisfy

$$G^* = -G, \quad G^2 = -\mathrm{Id}, \quad B = B^*, \quad GB = -BG. \tag{1.5}$$

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To prove the adiabatic decomposition formula of $\det_\zeta \mathcal{D}^2$, we follow the original Douglas–Wojciechowski proof of the decomposition formula for the $\eta$-invariant in [10]. However, we face two new problems not present in the case of the $\eta$-invariant. First, $\det_\zeta \mathcal{D}^2$ is a much more *nonlocal* invariant than the $\eta$-invariant; one result is that the value of $\det_\zeta \mathcal{D}^2$ varies with the length of the cylinder. Second, the contribution of $\det_\zeta \mathcal{D}^2$ over the cylindrical part is now nontrivial. We still follow the idea of [10] and stretch our manifold $M$ to separate $M_1$ and $M_2$. For this, let us introduce a manifold $M_R$ equal to the manifold $M$ with $N$ replaced by $N_R \cong [-R, R]_u \times Y$. By assumption of product structures over $N$, we can extend the bundle $S$ to $M_R$. Furthermore, we can use (1.4) to extend $\mathcal{D}$ to the Dirac operator $\mathcal{D}_R$ over $M_R$. Now we decompose $M_R$ by the hypersurface $\{0\} \times Y$ into two submanifolds $M_{1,R}$ and $M_{2,R}$, and we obtain $\mathcal{D}_{1,R}$ and $\mathcal{D}_{2,R}$ by restricting $\mathcal{D}_R$ to $M_{1,R}$ and $M_{2,R}$, respectively.

In order to express the decomposition formula for the $\zeta$-determinant, we must describe the invariant on a manifold with boundary that enters the picture at this point. The tangential operator $B$ has discrete spectrum with infinitely many positive and infinitely many negative eigenvalues. Let $\Pi_>$ and $\Pi_<$ denote the Atiyah–Patodi–Singer (APS) spectral projections onto the subspaces spanned by the eigensections of $B$ corresponding to the positive and negative eigenvalues, respectively. We select two involutions $\sigma_1, \sigma_2$ on the kernel of $B$ that satisfy $G\sigma_i = -\sigma_i G$, and we define $\pi_i = (1 - \sigma_i)/2$ as the orthogonal projections onto $-1$ eigenspaces of $\sigma_i$. Now define

$$P_1 = \Pi_< + \pi_1 \quad \text{and} \quad P_2 = \Pi_> + \pi_2, \tag{1.6}$$

which provides us with the *ideal boundary condition* introduced by Cheeger in [6; 7]. The projection $P_i$ imposes an elliptic boundary condition for $\mathcal{D}_{i,R}$ (see [1]; see [2] for an exposition of the theory of elliptic boundary problems for Dirac operators). This means that the associated operator

$$(\mathcal{D}_{i,R})_{P_i} = \mathcal{D}_{i,R} \colon \operatorname{dom}(\mathcal{D}_{i,R})_{P_i} \to L^2(M_{i,R}, S),$$

where

$$\operatorname{dom}(\mathcal{D}_{i,R})_{P_i} = \{s \in H^1(M_{i,R}, S) \mid P_i(s|_Y) = 0\}$$

is a self-adjoint Fredholm operator with $\ker(\mathcal{D}_{i,R})_{P_i} \subset C^\infty(M_{i,R}, S)$ and discrete spectrum.

The main concern of this paper is to consider the limit of the ratio of the $\zeta$-determinants

$$\frac{\det_\zeta \mathcal{D}_R^2}{\det_\zeta(\mathcal{D}_{1,R})_{P_1}^2 \cdot \det_\zeta(\mathcal{D}_{2,R})_{P_2}^2} \quad \text{as } R \to \infty, \tag{1.7}$$

which we call as the *adiabatic decomposition* of the $\zeta$-determinant of $\mathcal{D}^2$.

The eigenvalues of $\mathcal{D}_R$ fall into three different categories as $R \to \infty$. There are infinitely many large eigenvalues (*l-values*) bounded away from 0 and infinitely many small eigenvalues (*s-values*) of the size $O(R^{-1})$. Besides these, there are finitely many eigenvalues that decay exponentially with $R$ (*e-values*). The number $h_M$ of $e$-values is given by

$$h_M = \dim \ker_{L^2} \mathcal{D}_{1,\infty} + \dim \ker_{L^2} \mathcal{D}_{2,\infty} + \dim L_1 \cap L_2, \tag{1.8}$$

where $\mathcal{D}_{i,\infty}$ is the operator defined from $\mathcal{D}$ in a natural way over the manifold $M_{i,\infty}$, which is equal to $M_i$ with the half-infinite cylinder $[0,\infty) \times Y$ or $(-\infty, 0] \times Y$ attached. More precisely, the operator $\mathcal{D}_i = \mathcal{D}|_{M_i}$ extends in a natural way to the manifold $M_{i,\infty}$; it has a unique closed self-adjoint extension in $L^2(M_{i,\infty}, S)$, which we denote by $\mathcal{D}_{i,\infty}$. The subspaces $L_i \subset \ker B$ are the spaces of limiting values of extended $L^2$-solutions of $\mathcal{D}_{i,\infty}$. The decomposition of the eigenvalues of the operator $\mathcal{D}_R$ into different classes was discussed by Cappell, Lee, and Miller (see [5]). The corresponding analysis for the operator $(\mathcal{D}_{i,R})_{P_i}$ was provided by Müller (see [17]). The spectrum of the operators $(\mathcal{D}_{i,R})_{P_i}$ splits in the same way as the spectrum of $\mathcal{D}_R$; the only differences are that (a) the operators $(\mathcal{D}_{i,R})_{P_i}$ do not have nonzero $e$-values and (b) the dimension of the space of the solutions of $(\mathcal{D}_{i,R})_{P_i}$ is equal to

$$h_i = \dim \ker(\mathcal{D}_{i,R})_{P_i} = \dim \ker_{L^2} \mathcal{D}_{i,\infty} + \dim L_i \cap \ker(\sigma_i - 1). \tag{1.9}$$

In the adiabatic limit process, the different types of eigenvalues make their contribution at different time intervals of the integral representation of $\zeta_{\mathcal{D}^2}(s)$ in (1.2). The contribution made by $l$-values comes from the time interval $[0, R^{2-\varepsilon}]$, where $\varepsilon$ is a sufficiently small positive number; we fix $\varepsilon$ from now on. More precisely, it is not difficult to show that the $l$-values contribution to the adiabatic limit of (1.7) from the time interval $[R^{2-\varepsilon}, \infty]$ disappears as $R \to \infty$ (see Section 2). The contribution made by $l$-values was discussed in [19]. To be more precise, in [19] we discussed the case of the operator $\mathcal{D}$ such that $\mathcal{D}_{i,\infty}$ and $B$ have trivial kernels. These conditions imply that there are no $e$-values or $s$-values. This allows us to reduce the computation of the quotient in (1.7) to the corresponding quotient on the cylinder, from which one can show that the limit of (1.7) as $R \to \infty$ is equal to $2^{-\zeta_{B^2}(0)}$. Actually, even in the presence of $e$-values and $s$-values, we are able to show that in the adiabatic limit the contribution of $l$-values comes only from the time interval $[0, R^{2-\varepsilon}]$ and so we can reduce to the cylinder as in [19]. The method we use to prove this combines Duhamel's principle and the "finite propagation speed" (FPS) property of the wave operators. Details are presented in Section 2.

The $s$-values contribution comes from the time interval $[R^{2-\varepsilon}, R^{2+\varepsilon}]$. The computation of the $s$-values contribution is the main achievement of this paper. We follow Müller (see [17]) and use scattering theory to get a description of the $s$-values. The operators $\mathcal{D}_{i,\infty}$ on $M_{i,\infty}$ determine scattering matrices $C_i(\lambda)$. In fact, the matrix $C_{12} = C_1(0) \circ C_2(0)$ on $\ker B \cap \ker(G + i)$ determines the contribution of $s$-values of the operator $\mathcal{D}_R$ in the adiabatic limit. Similarly, the finite-dimensional unitary matrix $S_{\sigma_i}$ on $\ker(\sigma_i + 1)$, which is defined by the scattering matrix $C_i(0)$ and the involution $\sigma_i$, determines the contribution of $s$-values of the operators $(\mathcal{D}_{i,R})_{P_i}$. The exact correspondence is stated in Section 3.

Finally, we need to discuss $e$-values of $\mathcal{D}_R$. The number of $e$-values is equal to $h_M$, which (as remarked previously) is constant. On the other hand, the set of zero eigenvalues of $\mathcal{D}_R$, which is a subset of $e$-values by definition, is very unstable with respect to $R$. Hence, without making additional assumptions we are not able to control the adiabatic limit of the determinant of $\mathcal{D}_R^2$ owing to the finite number of nonzero $e$-values. Hence, we assume that all the $e$-values are zero eigenvalues in order to avoid the technical difficulty of the nonzero $e$-values. One of the important examples of such situations is the case of the operator

$$d_\rho + d_\rho^*\colon \bigoplus_{i=0}^{n} \Omega^i(M, V_\rho) \to \bigoplus_{i=0}^{n} \Omega^i(M, V_\rho),$$

where $V_\rho$ denotes the flat vector bundle defined by the unitary representation $\rho$ of $\pi_1(M)$ (see Proposition 3.9). For the operator $L\colon W \to W$ acting on a finite-dimensional vector space $W$, we denote by $\det^* L$ the determinant of the operator $L$ restricted to the subspace $(\ker L)^\perp$. Now we are ready to formulate the main result of this paper.

THEOREM 1.1. *When all the e-values of $\mathcal{D}_R$ are zero eigenvalues, the following formula holds*:

$$\lim_{R\to\infty} R^{-2h} \cdot \frac{\det_\zeta \mathcal{D}_R^2}{\det_\zeta (\mathcal{D}_{1,R})^2_{P_1} \cdot \det_\zeta (\mathcal{D}_{2,R})^2_{P_2}}$$
$$= 2^{-\zeta_{B^2}(0) - h_Y + 2h_M} \cdot \det{}^*\left(\frac{2\,\mathrm{Id} - C_{12} - C_{12}^{-1}}{4}\right) \cdot \prod_{i=1}^{2} \det{}^*\left(\frac{2\,\mathrm{Id} - S_{\sigma_i} - S_{\sigma_i}^{-1}}{4}\right)^{-1}, \tag{1.10}$$

*where* $h = h_M - h_1 - h_2$ *and* $h_Y = \dim\ker B$.

REMARK 1.2. In [12] and [11], the reduced normal operators corresponding to our operators $C_{12}, S_{\sigma_i}$ were introduced in the framework of $b$-calculus and used in the analysis of $s$-values for the analytic surgery of the $\eta$-invariant and analytic torsion.

To prove Theorem 1.1, we consider the following relative $\zeta$-function and its derivative at $s = 0$:

$$\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} [\mathrm{Tr}(\exp\{-t\mathcal{D}_R^2\} - \exp\{-t(\mathcal{D}_{1,R})^2_{P_1}\} - \exp\{-t(\mathcal{D}_{2,R})^2_{P_2}\}) - h]\, dt.$$

We decompose this into two parts,

$$\zeta_s^R(s) = \frac{1}{\Gamma(s)} \int_0^{R^{2-\varepsilon}} (\cdot)\, dt \quad \text{and} \quad \zeta_l^R(s) = \frac{1}{\Gamma(s)} \int_{R^{2-\varepsilon}}^\infty (\cdot)\, dt,$$

where $\varepsilon$ is the fixed sufficiently small positive number. The derivatives of $\zeta_s^R(s)$ and $\zeta_l^R(s)$ at $s = 0$ give the small and large time contribution in (1.10).

In Section 2 we deal with the small time contribution and prove that it is equal to $2^{-\zeta_{B^2}(0)}$, which gives the first factor on the right side of (1.10). Section 3 contains a basic description of the small eigenvalues. We follow [17] and use scattering theory in order to get a description of the $s$-values of $\mathcal{D}_R$ and $(\mathcal{D}_{i,R})_{P_i}$, which allows us to make a comparison of $s$-values of those operators with the eigenvalues of certain model operators over $\mathbb{S}^1$. This is the central part of our paper. In Section 4 we use the results of Section 3 to show that, in the adiabatic limit, the large time contribution to the quotient (1.7) is equal to

$$2^{-h_Y+2h_M}\cdot\det{}^*\left(\frac{2\,\mathrm{Id}-C_{12}-C_{12}^{-1}}{4}\right)\cdot\prod_{i=1}^{2}\det{}^*\left(\frac{2\,\mathrm{Id}-S_{\sigma_i}-S_{\sigma_i}^{-1}}{4}\right)^{-1};$$

this is the second factor on the right side of (1.10). The zero eigenvalues make their presence via the factor $R^{-2h}$ on the left side of (1.10).

In Section 5 we review the decomposition formula for the $\eta$-invariant and offer a new proof based on the method developed in order to prove Theorem 1.1. This proof is more complicated than other proofs presented in [3; 4; 9; 12; 13; 16; 18; 23]. However, it is a nice illustration of the differences we encounter when dealing with the $\zeta$-determinant instead of the $\eta$-invariant.

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## 2. Small Time Contribution

In this section we determine the small time contribution, which is accomplished in two steps. First, we use Duhamel's principle and the FPS (finite propagation speed) property of the wave operator to show that we can reduce the problem to computations on the cylinder. Then, we perform the explicit calculations on the cylinder. Both parts are fairly standard. The cylinder contribution has been computed in [19]. Therefore, we discuss only the reduction scheme and refer to [19] for the explicit computation on the cylinder.

Let $\mathcal{E}_R(t;x,y)$ denote the kernel of the operator $e^{-t\mathcal{D}_R^2}$. We introduce the specific parametrix for $\mathcal{E}_R(t;x,y)$, which fits our main purpose to *localize* the contribution coming from the cylinder $[-R,R]_u\times Y$ and the interior of $M_R$. In fact, the interesting point here is that we use $\mathcal{E}_R(t;x,y)$ to construct this parametrix. Let $\mathcal{E}_c(t;x,y)$ denote the kernel of the operator $e^{-t(-\partial_u^2+B^2)}$ on the infinite cylinder $\mathbb{R}\times Y$. We introduce a smooth, increasing function $\rho(a,b)\colon[0,\infty)\to[0,1]$ that is equal to 0 for $0\le u\le a$ and equal to 1 for $b\le u$. We use $\rho(a,b)(u)$ to define

$$\phi_{1,R}=1-\rho\left(\tfrac{5}{7}R,\tfrac{6}{7}R\right),\qquad \psi_{1,R}=1-\psi_{2,R},$$
$$\phi_{2,R}=\rho\left(\tfrac{1}{7}R,\tfrac{2}{7}R\right),\qquad \psi_{2,R}=\rho\left(\tfrac{3}{7}R,\tfrac{4}{7}R\right).$$

We extend these functions to symmetric functions on the whole real line. These functions are constant outside the interval $[-R,R]_u$ and we use them to define the corresponding functions on a manifold $M_R$, which are denoted in the same way. Now we define $Q_R(t;x,y)$ a *parametrix* for the kernel $\mathcal{E}_R(t;x,y)$ by

$$Q_R(t;x,y)=\phi_{1,R}(x)\mathcal{E}_c(t;x,y)\psi_{1,R}(y)+\phi_{2,R}(x)\mathcal{E}_R(t;x,y)\psi_{2,R}(y). \tag{2.1}$$

It follows from Duhamel's principle that

$$\mathcal{E}_R(t;x,y)=Q_R(t;x,y)+(\mathcal{E}_R*\mathcal{C}_R)(t;x,y), \tag{2.2}$$

where $\mathcal{E}_R * \mathcal{C}_R$ is the convolution given by

$$(\mathcal{E}_R * \mathcal{C}_R)(t;x,y) = \int_0^t ds \int_{M_R} dz\, \mathcal{E}_R(s;x,z)\mathcal{C}_R(t-s;z,y)$$

and where the error term $\mathcal{C}_R(t;x,y)$ is given by the formula

$$\begin{aligned}\mathcal{C}_R&(t;x,y)\\ &= -\partial_u^2\phi_{1,R}(x)\mathcal{E}_c(t;x,y)\psi_{1,R}(y) - \partial_u\phi_{1,R}(x)\partial_u\mathcal{E}_c(t;x,y)\psi_{1,R}(y)\\ &\quad - \partial_u^2\phi_{2,R}(x)\mathcal{E}_R(t;x,y)\psi_{2,R}(y) - \partial_u\phi_{2,R}(x)\partial_u\mathcal{E}_R(t;x,y)\psi_{2,R}(y).\end{aligned}$$

The following elementary lemma follows from the construction of $Q_R(t;x,y)$.

LEMMA 2.1. *For a fixed $y$, the support of the error term $\mathcal{C}_R(t;x,y)$ as a function of $x$ is a subset of $\left(\left[-\frac{6}{7}R,-\frac{1}{7}R\right]_u \cup \left[\frac{1}{7}R,\frac{6}{7}R\right]_u\right)\times Y$. Moreover, it is equal to $0$ if the distance between $x$ and $y$ is smaller than $\frac{1}{7}R$.*

Now, following Cheeger, Gromov, and Taylor (see [8]; see also [4, Sec. 3]), we use the FPS property for the wave operator. The technique introduced in [8] allows us to compare the heat kernel of the operator $\mathcal{D}_R^2$ over $M_R$ with the heat kernel of the operator $-\partial_u^2 + B^2$ on the cylinder $\mathbb{R}\times Y$. We describe the case we need in our work. Let $X_1$ and $X_2$ be Riemannian manifolds of dimension $n$, and let $S_i$ be spinor bundles with Dirac operators $\mathcal{D}_i$ over $X_i$. Assume that there exists a decomposition $X_i = K_i \cup U_i$, where $U_i$ is an open subset of $X_i$. Moreover, we assume that there exists an isometry $h\colon U_1 \to U_2$ covered by the unitary bundle isomorphism $\Phi_h\colon S_1|_{U_1} \to S_2|_{U_2}$, which intertwines the Dirac operators $\mathcal{D}_1|_{U_1}$ and $\mathcal{D}_2|_{U_2}$. We identify

$$U \cong U_1 \cong U_2,$$

so that $X_1$ and $X_2$ have a common open subset $U$. Let $\mathcal{E}_i(t;x,y)$ denote the kernel of the operator $e^{-t\mathcal{D}_i^2}$. Then we have the following estimate on the difference of the heat kernels on $U$ (cf. [4, Lemma 3.6]).

PROPOSITION 2.2. *For $x,y\in U$ and $t>0$, there exist positive constants $c_1$ and $c_2$ such that*

$$\|\partial_u^j\mathcal{E}_1(t;x,y) - \partial_u^j\mathcal{E}_2(t;x,y)\| \le c_1 e^{-c_2 r^2/t}, \tag{2.3}$$

*where $j=0,1$ and $r=\min(d(x,K_1),d(y,K_1))$.*

In our situation, $X_1 = M_R$, $X_2 = \mathbb{R}\times Y$, and $U = [-R,R]_u \times Y$. Note that the heat kernel $\mathcal{E}_c(t;x,y)$ over $\mathbb{R}\times Y$ satisfies the standard estimate. More precisely, for $t>0$ we have

$$\|\partial_u^j\mathcal{E}_c(t;(u,w),(v,z))\| \le c_1|u-v|^j t^{-n/2-j} e^{-c_3(u-v)^2/t}, \tag{2.4}$$

where $j=0,1$, $u,v\in\mathbb{R}$, and $w,z\in Y$. This follows from the corresponding estimate for the heat kernel of $B^2$ over the closed manifold $Y$ (see [21, Prop. 4.1]) and the explicit form of the heat kernel of $-\partial_u^2$ over $\mathbb{R}$. We shall use (2.3) and (2.4) in the following proposition.

PROPOSITION 2.3. *There exist constants* $c_1, c_2 > 0$ *such that, for any* $t$ *with* $0 < t < R^{2-\varepsilon}$ *and* $((u,w),(v,z)) \in \operatorname{supp} \mathcal{C}_R(t;\cdot,\cdot)$,

$$\begin{aligned} \|\mathcal{E}_R(t;(u,w),(v,z))\| &\le c_1 e^{-c_2 R^2/t}, \\ \|\mathcal{C}_R(t;(u,w),(v,z))\| &\le c_1 e^{-c_2 R^2/t}. \end{aligned} \tag{2.5}$$

*Proof.* For $j = 0, 1$, we have

$$\begin{aligned} \|\partial_u^j \mathcal{E}_R(t;(u,w),(v,z))\| &\le \|\partial_u^j \mathcal{E}_c(t;(u,w),(v,z))\| \\ &\quad + \|\partial_u^j \mathcal{E}_R(t;(u,w),(v,z)) - \partial_u^j \mathcal{E}_c(t;(u,w),(v,z))\|. \end{aligned}$$

By Lemma 2.1 and inequalities (2.3) and (2.4), there exist some constants $c_1, c_2 > 0$ such that, for $(u,w),(v,z) \in \operatorname{supp} \mathcal{C}_R(t;\cdot,\cdot)$, both summands on the right side satisfy the desired estimate. This estimate for $j = 0$ (resp. $j = 1$) implies the first (resp. second) estimate in (2.5). □

Now we are ready to prove the following technical result.

PROPOSITION 2.4.

$$\lim_{R\to\infty} \frac{d}{ds}\bigg|_{s=0} \frac{1}{\Gamma(s)} \int_0^{R^{2-\varepsilon}} t^{s-1}\, dt \int_{M_R} \operatorname{tr}(\mathcal{E}_R * \mathcal{C}_R)(t;x,x)\, dx = 0. \tag{2.6}$$

*Proof.* By Lemma 2.1 and Proposition 2.3,

$$\begin{aligned} &|\operatorname{tr}(\mathcal{E}_R * \mathcal{C}_R)(t;x,x)| \\ &\quad \le \|(\mathcal{E}_R * \mathcal{C}_R)(t;x,x)\| \\ &\quad \le \int_0^t ds \int_{[-6R/7,6R/7]_u \times Y} \|\mathcal{E}_R(s;x,z)\mathcal{C}_R(t-s;z,x)\|\, dz \\ &\quad \le c_1^2 \cdot \int_0^t ds \int_{[-6R/7,6R/7]_u \times Y} e^{-c_2R^2/s} e^{-c_2R^2/(t-s)}\, dz \\ &\quad \le c_3 R \cdot \int_0^t e^{-c_2 t R^2/s(t-s)}\, ds \le c_3 R \cdot \int_0^{t/2} e^{-2c_2R^2/s}\, ds \le c_3 R \frac{t}{2} e^{-4c_2R^2/t}, \end{aligned}$$

where the last estimate is a consequence of the elementary inequality

$$\int_0^t e^{-c/s}\, ds \le t e^{-c/t}.$$

Hence we have proved that

$$|\operatorname{tr}(\mathcal{E}_R * \mathcal{C}_R)(t;x,x)| \le c_4 R t e^{-c_5 R^2/t}. \tag{2.7}$$

This allows us to estimate as follows:

$$\begin{aligned} &\left| \frac{1}{\Gamma(s)} \int_0^{R^{2-\varepsilon}} t^{s-1}\, dt \int_{M_R} \operatorname{tr}(\mathcal{E}_R * \mathcal{C}_R)(t;x,x)\, dx \right| \\ &\qquad\qquad \le c_6 R^2 \cdot \left| \frac{1}{\Gamma(s)} \right| \int_0^{R^{2-\varepsilon}} |t^s| e^{-c_5R^2/t}\, dt. \end{aligned}$$

As $R \to \infty$, the function of $s$ on the right side uniformly converges to zero for $s$ in any compact set in $\mathbb{C}$. Hence, the derivative at $s = 0$ of the meromorphic function on the left side converges to zero as $R \to \infty$. This completes the proof. $\square$

The corresponding result to Proposition 2.4 for the operator $(\mathcal{D}_{i,R})^2_{P_i}$ can be carried out in exactly the same manner. First, as for $\mathcal{D}_R$ over $M_R$, we can construct the parametrices for the heat kernels of $\exp\{-t(\mathcal{D}_{i,R})^2_{P_i}\}$ using $\mathcal{E}_R(t;x,y)$ and the heat kernels of $(G(\partial_u + B))^2_{P_i}$ over $[0,\infty)_u \times Y$ or $(-\infty,0]_u \times Y$. Second, one can obtain the corresponding estimate to (2.3) using the explicit form of the heat kernel of $(G(\partial_u + B))^2_{P_i}$. Third, one can also derive the corresponding estimate to Proposition 2.3 for $(\mathcal{D}_{i,R})^2_{P_i}$, since the similar estimate as in Proposition 2.2 holds over the support of the error terms (see [4, Lemma 3.6]). All these statements imply that the similar estimate to Proposition 2.4 holds for $(\mathcal{D}_{i,R})^2_{P_i}$. Now we are ready to prove the following main result of this section.

PROPOSITION 2.5.

$$\lim_{R\to\infty}\left(\frac{d}{ds}\bigg|_{s=0}\zeta_s^R(s) + h(\gamma + (2-\varepsilon)\cdot\log R)\right) = \zeta_{B^2}(0)\cdot\log 2. \tag{2.8}$$

*Proof.* First we observe that

$$\frac{d}{ds}\bigg|_{s=0}\left(\frac{h}{\Gamma(s)}\int_0^{R^{2-\varepsilon}} t^{s-1}\,dt\right) = h(\gamma + (2-\varepsilon)\log R). \tag{2.9}$$

Hence we need to compute the limit as $R \to \infty$ of the following remaining part of $\zeta_s^R(s)$:

$$\frac{d}{ds}\bigg|_{s=0}\frac{1}{\Gamma(s)}\int_0^{R^{2-\varepsilon}} t^{s-1}\,\mathrm{Tr}(\exp\{-t\mathcal{D}_R^2\} - \exp\{-t(\mathcal{D}_{1,R})^2_{P_1}\} - \exp\{-t(\mathcal{D}_{2,R})^2_{P_2}\})\,dt.$$

By Proposition 2.4 and corresponding results for $(\mathcal{D}_{i,R})^2_{P_i}$, it is sufficient to consider the limit as $R \to \infty$ of

$$\frac{d}{ds}\bigg|_{s=0}\frac{1}{\Gamma(s)}\int_0^{R^{2-\varepsilon}} t^{s-1}\,dt\int_{M_R}\mathrm{tr}(Q_R(t;x,x) - Q_{1,R}(t;x,x) - Q_{2,R}(t;x,x))\,dx,$$

where $Q_{i,R}(t;x,y)$ denotes the parametrix for $\exp\{-t(\mathcal{D}_{i,R})^2_{P_i}\}$. Now the interior contributions to the different parametrices, all determined by the kernel $\mathcal{E}_R(t;x,y)$, cancel out and we are left only with the cylinder contribution. Hence we have to deal with the limit as $R \to \infty$ of

$$\frac{d}{ds}\bigg|_{s=0}\frac{1}{\Gamma(s)}\int_0^{R^{2-\varepsilon}} t^{s-1}\,dt\int_{M_R}\mathrm{tr}(\psi_{1,R}\mathcal{E}_c(t;x,x) - \psi_{1,R}\mathcal{E}_{c,1}(t;x,x) - \psi_{1,R}\mathcal{E}_{c,2}(t;x,x))\,dx, \tag{2.10}$$

where $\mathcal{E}_{c,i}(t;x,y)$ denotes the heat kernel of $(G(\partial_u + B))^2_{P_i}$ over the half-cylinder. We repeat the computations of [19, Sec. 2], where we assumed that $B$ is invertible and $Y$ is even dimensional. But we can easily derive the same formula following [19, Sec. 2] without these assumptions. Hence we can show that, for $s$ in a compact subset of $\mathbb{C}$, the integral part in (2.10) uniformly converges to the following function as $R \to \infty$:

$$2\left(\frac{\Gamma(s)}{4} - \frac{\Gamma\big(s+\frac{1}{2}\big)}{4s\sqrt{\pi}}\right) \cdot \zeta_{B^2}(s).$$

We thus obtain

$$\begin{aligned}\lim_{R\to\infty} \frac{d}{ds}\bigg|_{s=0} \frac{1}{\Gamma(s)} \int_0^{R^{2-\varepsilon}} & t^{s-1}\,\mathrm{Tr}(\exp\{-t\mathcal{D}_R^2\} \\ & - \exp\{-t(\mathcal{D}_{1,R})_{P_1}^2\} - \exp\{-t(\mathcal{D}_{2,R})_{P_2}^2\})\, dt \\ = \frac{d}{ds}\bigg|_{s=0} \frac{2}{\Gamma(s)} & \left(\frac{\Gamma(s)}{4} - \frac{\Gamma\big(s+\frac{1}{2}\big)}{4s\sqrt{\pi}}\right) \cdot \zeta_{B^2}(s) = \zeta_{B^2}(0) \cdot \log 2. \quad (2.11)\end{aligned}$$

Combining (2.9) and (2.11) completes the proof. □

## 3. Small Eigenvalues and Scattering Matrices

In this section we investigate the relation between the $s$-values of the operators $\mathcal{D}_R$, $(\mathcal{D}_{i,R})_{P_i}$ and the scattering matrices $C_i(\lambda)$ determined by the operators $\mathcal{D}_{i,\infty}$ on $M_{i,\infty}$ for $i = 1, 2$. We refer to Section 4 and Section 8 in [17] for a more detailed exposition of the elements of scattering theory that are used in this paper.

Let us recall that $M_R$ has the cylindrical part $N_R = [-R, R]_u \times Y$. Hence $M_{1,R}$ and $M_{2,R}$ have the cylindrical part $[-R, 0]_u \times Y$ and $[0, R]_u \times Y$, respectively. Yet in order to consider $M_{i,R}$ as a submanifold of $M_{i,\infty}$ that is obtained by attaching $[0,\infty)_v \times Y$ or $(-\infty, 0]_v \times Y$ to $M_i$, we change the variable by $v = u + R$ or $v = u - R$ so that the cylindrical part of $M_{i,R}$ is given by $[0, R]_v \times Y$ or $[-R, 0]_v \times Y$. Throughout this section, we will use this convention when it is needed.

For any $\psi \in \ker B$ and $\lambda \in \mathbb{C} - (-\infty, -\mu_1] \cup [\mu_1, +\infty)$, where $\mu_1$ denotes the lowest positive eigenvalue of the tangential operator $B$, there exists a generalized eigensection $E(\psi, \lambda)$ of $\mathcal{D}_{1,\infty}$ over $M_{1,\infty}$ determined by the couple $(\psi, \lambda)$ (see [17, Sec. 4]) in the following sense:

$$\mathcal{D}_{1,\infty} E(\psi, \lambda) = \lambda E(\psi, \lambda).$$

The section $E(\psi, \lambda)$ has the following form over $[0, \infty)_v \times Y$:

$$E(\psi, \lambda) = e^{-i\lambda v}(\psi - iG\psi) + e^{i\lambda v} C_1(\lambda)(\psi - iG\psi) + \theta(\psi, \lambda), \qquad (3.1)$$

where $\theta$ is a square integrable section such that, for each $v$, $\theta(\psi, \lambda, (v, \cdot))$ is orthogonal to $\ker B$. The operator $C_1(\lambda)\colon \ker B \to \ker B$ is regular and unitary for $|\lambda| < \mu_1$ and equals the scattering matrix such that

$$C_1(\lambda) C_1(-\lambda) = \mathrm{Id} \quad \text{and} \quad C_1(\lambda) G = -G C_1(\lambda),$$

which imply

$$C_1(0)^2 = \mathrm{Id} \quad \text{and} \quad C_1(0) G = -G C_1(0).$$

Therefore, $C_1(0)$ gives a distinguished unitary involution of $\ker B$. In fact, the space of the limiting values of the extended $L^2$-solutions of $\mathcal{D}_{1,\infty}$, $L_1 \subset \ker B$ is equal to the $(+1)$-eigenspace of $C_1(0)$; that is, $L_1 = \ker(C_1(0) - 1)$. The following proposition is a basic tool for dealing with $E(\psi, \lambda)$.

PROPOSITION 3.1 (Maass–Selberg). *The following equality holds*:

$$\begin{aligned}\langle E(\phi, \lambda), E(\psi, \lambda)\rangle_{M_{1,R}} \\ &= 4R\langle \phi, \psi\rangle_Y - i\langle C_1(-\lambda)C_1'(\lambda)(\phi - iG\phi), \psi - iG\psi\rangle_Y + O(e^{-cR}),\end{aligned}$$

*where* $\phi, \psi \in \ker B$.

*Proof.* By Green's formula, we have

$$\begin{aligned}h\langle E(\phi, \lambda + h), E(\psi, \lambda)\rangle_{M_{1,R}} \\ &= \langle \mathcal{D}_{1,R}E(\phi, \lambda + h), E(\psi, \lambda)\rangle_{M_{1,R}} - \langle E(\phi, \lambda + h), \mathcal{D}_{1,R}E(\psi, \lambda)\rangle_{M_{1,R}} \\ &= \langle GE(\phi, \lambda + h)|_{\partial(M_{1,R})}, E(\psi, \lambda)|_{\partial(M_{1,R})}\rangle_{\partial(M_{1,R})}. \qquad (3.2)\end{aligned}$$

Using (3.1), the last line has the following form:

$$\begin{aligned}ie^{-ihR}\langle \phi - iG\phi, \psi - iG\psi\rangle_Y \\ &- ie^{ihR}\langle C_1(\lambda + h)(\phi - iG\phi), C_1(\lambda)(\psi - iG\psi)\rangle_Y + O(e^{-cR}) \\ &= ie^{-ihR}\langle \phi - iG\phi, \psi - iG\psi\rangle_Y - ie^{ihR}\langle \phi - iG\phi, \psi - iG\psi\rangle_Y \\ &+ ie^{ihR}\langle C_1(\lambda)(\phi - iG\phi), C_1(\lambda)(\psi - iG\psi)\rangle_Y \\ &- ie^{ihR}\langle C_1(\lambda + h)(\phi - iG\phi), C_1(\lambda)(\psi - iG\psi)\rangle_Y + O(e^{-cR}).\end{aligned}$$

Now, dividing the right side by $h$ and taking the limit as $h \to 0$, we obtain

$$\begin{aligned}2R\langle \phi - iG\phi, \psi - iG\psi\rangle_Y - i\langle C_1'(\lambda)(\phi - iG\phi), C_1(\lambda)(\psi - iG\psi)\rangle_Y + O(e^{-cR}) \\ &= 4R\langle \phi, \psi\rangle_Y - i\langle C_1(-\lambda)C_1'(\lambda)(\phi - iG\phi), (\psi - iG\psi)\rangle_Y + O(e^{-cR}).\end{aligned}$$

Comparing this with (3.2) (divided by $h$) completes the proof. □

Now we shall analyze the $s$-values of $\mathcal{D}_R$ over $M_R$. Let us consider an $s$-value $\lambda = \lambda(R)$ of $\mathcal{D}_R$ such that

$$|\lambda(R)| \le R^{-\kappa} \quad \text{for sufficiently large } R,$$

where $\kappa$ is a fixed constant with $0 < \kappa < 1$. Let $\Psi_R$ denote a *normalized* eigensection of $\mathcal{D}_R$ corresponding to $s$-value $\lambda$; that is,

$$\mathcal{D}_R\Psi_R = \lambda\Psi_R, \quad \|\Psi_R\| = 1.$$

Over the cylindrical part $[-R, R]_u \times Y$ in $M_R$, the eigensection $\Psi_R$ corresponding to the $s$-value $\lambda$ of $\mathcal{D}_R$ has the form

$$\Psi_R = e^{-i\lambda u}\psi_1 + e^{i\lambda u}\psi_2 + \hat{\Psi}_R, \qquad (3.3)$$

where $\psi_1 \in \ker B \cap \ker(G - i)$, $\psi_2 \in \ker B \cap \ker(G + i)$, and $\hat{\Psi}_R$ is orthogonal to $\ker B$.

LEMMA 3.2. *We have the estimates*

$$\|\hat{\Psi}_R|_{\{u\}\times Y}\|_Y \le c_1 e^{-c_2 R} \quad \textit{for } -\tfrac{3}{4}R \le u \le \tfrac{3}{4}R,$$

*where* $c_1, c_2$ *are positive constants independent of* $R$.

The proof of this lemma is same as the one of Lemma 2.1 in [22]. Now we can prove the following result.

PROPOSITION 3.3. *The zero eigenmode* $e^{-i\lambda u}\psi_1 + e^{i\lambda u}\psi_2$ *of the eigensection* $\Psi_R$ *of* $s$*-value* $\lambda(R)$ *of* $\mathcal{D}_R$ *is nontrivial.*

*Proof.* We follow the proof of [22, Thm. 2.2]; we assume that the zero eigenmode of $\Psi_R$ is trivial, which will contradict the fact that $\lambda(R)$ is a $s$-value. Throughout the proof, we regard $M_{1,R}$ as a submanifold of $M_{1,\infty}$ using the change of variable $v = u + R$. We define a section $\Phi_R$ on $M_{1,\infty}$ by

$$\Phi_R = \begin{cases} h_R(x)\Psi_R(x) & \text{for } x \in M_{1,R}, \\ 0 & \text{for } x \in M_{1,\infty} \setminus M_{1,R}, \end{cases}$$

where $h_R$ is a smooth function on $M_{1,\infty}$, equal to 1 for $x \in M_1 \cup \left[0, \frac{1}{2}R\right]_v \times Y$ and equal to 0 for $x \in \left[\frac{3}{4}R, \infty\right)_v \times Y$ with $|\partial^j h/\partial v^j| \le CR^{-j}$ for a constant $C > 0$. Let $H^1(M_{1,\infty}, S)$ denote the first Sobolev space. For any $a \ge 0$, we introduce a closed subspace of $H^1(M_{1,\infty}, S)$ by

$$\begin{aligned} &H^1_a(M_{1,\infty}, S) \\ &\quad = \{\Phi \in H^1(M_{1,\infty}, S) \mid \langle \Phi(v,\cdot), \phi_k\rangle = 0 \text{ for } v \ge a,\ k = 1, \ldots, h_Y\}, \end{aligned}$$

where $\phi_1, \ldots, \phi_{h_Y}$ denotes an orthonormal basis of $\ker B$. Consider the quadratic form

$$Q(\Phi) = \|D\Phi\|^2 \quad \text{for } \Phi \in H^1_a(M_{1,\infty}, S),$$

where $D$ denotes the differential operator over $M_{1,\infty}$ whose self-adjoint extension is $\mathcal{D}_{1,\infty}$. Then this quadratic form is represented by a positive self-adjoint operator $H_a$ in the closure of $H^1_a(M_{1,\infty}, S)$ in $L^2(M_{1,\infty}, S)$. Hence $H_a$ has pure point spectrum near 0 and, by [17, Prop. 8.7], $\ker H_a = \ker_{L^2} \mathcal{D}_{1,\infty}$ for any $a \ge 0$. Following the proof of [22, Prop. 2.4], we can prove that there exist positive constants $c_1, c_2$ such that

$$|\langle \Phi_R, s\rangle| \le c_1 e^{-c_2 R}\|s\| \tag{3.4}$$

for $s \in \ker H_a$. Now let $\tilde{\Phi}_R := \Phi_R - \sum_{k=1}^{h_{1,\infty}} \langle \Phi_R, s_k\rangle s_k$, where $\{s_k\}_{k=1}^{h_{1,\infty}}$ denotes an orthonormal basis of $\ker H_a$ with $h_{1,\infty} := \dim\ker H_a$. As a result, $\tilde{\Phi}_R$ is orthogonal to $\ker H_a$ and, by (3.4), there is a positive constant $C$ independent of $R$ such that $\|\tilde{\Phi}_R\| \ge \frac{1}{2}\|\Phi_R\| \ge C > 0$ for sufficiently large $R$. Noting that $\tilde{\Phi}_R \in \operatorname{dom} H_a$, by the mini-max principle we have

$$\langle H_a \tilde{\Phi}_R, \tilde{\Phi}_R\rangle \ge \nu^2 C^2, \tag{3.5}$$

where $\nu^2$ is the smallest nonzero eigenvalue of $H_a$. Now

$$\begin{aligned}\lambda(R)^2 &= \langle \mathcal{D}_R^2 \Psi_R, \Psi_R \rangle = \int_{M_R} \|\mathcal{D}_R \Psi_R(x)\|^2 \, dx \\ &\geq \int_{M_{1,R}} \|\mathcal{D}_R \Psi_R(x)\|^2 \, dx = \int_{M_{1,R}} \|\mathcal{D}_R(h_R \Psi_R + (1-h_R)\Psi_R)(x)\|^2 \, dx \\ &\geq \int_{M_{1,\infty}} \|H_a \Phi_R(x)\| \, dx - \int_{M_{1,R}} \|\mathcal{D}_R(1-h_R)\Psi_R(x)\|^2 \, dx.\end{aligned}$$

By (3.5), the first term has the lower bound $\nu^2 C^2$ since $H_a \Phi_R = H_a \tilde{\Phi}_R$. For the second term, we have

$$\begin{aligned}&\int_{M_{1,R}} \|\mathcal{D}_R(1-h_R)\Psi_R(x)\|^2 \, dx \\ &\qquad = \int_{M_{1,R}} \|(1-h_R)(x)\mathcal{D}_R \Psi_R(x) - G(\partial_u h_R)(x)\Psi_R(x)\|^2 \, dx \\ &\qquad \leq 2 \int_{M_{1,R}} \|\lambda(R)(1-h_R)(x)\Psi_R(x)\|^2 + \|G(\partial_u h_R)(x)\Psi_R(x)\|^2 \, dx.\end{aligned}$$

Applying Lemma 3.2 with $v = u + R$ to each term of the last line yields

$$\int_{M_{1,R}} \|\mathcal{D}_R(1-h_R)\Psi_R(x)\|^2 \, dx \leq c_3 e^{-c_4 R}$$

for positive constants $c_3, c_4$. Hence these inequalities imply that $\lambda(R)^2 \geq \frac{1}{2}\nu^2 C^2$ for sufficiently large $R$. This completes the proof. □

Changing to the variable $v = u + R$, we view the cylindrical part $N_R$ of $M_R$ as being given by $[0, 2R]_v \times Y$. In particular, we have the following new expression for $\Psi_R$ from (3.3):

$$\Psi_R = e^{-i\lambda v}\phi_1^1 + e^{i\lambda v}\phi_2^1 + \hat{\Psi}_R, \tag{3.6}$$

where $\phi_1^1 = e^{i\lambda R}\psi_1$ and $\phi_2^1 = e^{-i\lambda R}\psi_2$. Let $(\ker B)_\pm$ denote the $\pm i$ eigenspace of $G\colon \ker B \to \ker B$. We shall need the following lemma.

LEMMA 3.4. *Let $\sigma$ be an involution over* $\ker B$ *such that $G\sigma = -\sigma G$. Then, for any element $\phi \in (\ker B)_\pm$, there exists a unique $\psi \in \mathrm{Im}(\sigma + 1)$ such that*

$$\phi = \psi \mp iG\psi.$$

*Proof.* For a given $\phi \in (\ker B)_+$ let $\psi := \frac{1}{2}(1+\sigma)\phi$, which lies in $\mathrm{Im}(\sigma + 1)$ by definition. Then we have

$$\begin{aligned}\psi - iG\psi &= \tfrac{1}{2}((1-iG)\phi + (\sigma - iG\sigma)\phi) \\ &= \tfrac{1}{2}((1-iG)\phi + (\sigma + i\sigma G)\phi) = \tfrac{1}{2} \cdot 2\phi = \phi.\end{aligned}$$

This completes the proof for the "+" case, and the "−" case can be proved in the same way. □

By Proposition 3.3, one of $\phi_1^1$ and $\phi_2^1$ in (3.6) is nontrivial. First we assume that $\phi_1^1$ is nontrivial. Now, since $L_1 = \mathrm{Im}(C_1(0)+1)$ and since $C_1(0)$ is an involution over ker $B$, by Lemma 3.4 we can choose $\psi \in L_1$ such that $\phi_1^1 = \psi - iG\psi$. Then the generalized eigensection $E(\psi, \lambda)$ over $M_{1,\infty}$ associated to $\psi$ has the expression

$$E(\psi, \lambda) = e^{-i\lambda v}(\psi - iG\psi) + e^{i\lambda v}C_1(\lambda)(\psi - iG\psi) + \theta(\psi, \lambda)$$

over $[0, \infty)_v \times Y$. Following [17], we introduce

$$F = \Psi_R|_{M_{1,R}} - E(\psi, \lambda)|_{M_{1,R}}.$$

Green's formula gives

$$0 = \langle \mathcal{D}_{1,R}F, F\rangle_{M_{1,R}} - \langle F, \mathcal{D}_{1,R}F\rangle_{M_{1,R}} = \int_{\partial(M_{1,R})} \langle GF|_{\partial(M_{1,R})}, F|_{\partial(M_{1,R})}\rangle \, dy.$$

On the other hand, Lemma 3.2 shows that

$$\int_{\partial(M_{1,R})} \langle GF|_{\partial(M_{1,R})}, F|_{\partial(M_{1,R})}\rangle \, dy = -i\,\|C_1(\lambda)\phi_1^1 - \phi_2^1\|^2 + O(e^{-c_3R})$$

for some positive constant $c_3$. This yields the estimate

$$\|C_1(\lambda)\phi_1^1 - \phi_2^1\| \le e^{-cR} \tag{3.7}$$

for a positive constant $c$. Hence, for $R \gg 0$, if $\phi_1^1$ is nontrivial then $\phi_2^1$ is also nontrivial; one can show the converse in the same way. We may therefore conclude that both $\phi_1^1$ and $\phi_2^1$ in (3.6) are nontrivial for $R \gg 0$.

Now we want to obtain the corresponding estimate involving the scattering matrix $C_2(\lambda)$. For this, we change the variable by $v = u - R$ and regard the cylindrical part as $[-2R, 0]_v \times Y$. Then we have the corresponding expression for $\Psi_R$,

$$\Psi_R = e^{-i\lambda v}\phi_1^2 + e^{i\lambda v}\phi_2^2 + \hat{\Psi}_R,$$

where $\phi_1^2 = e^{-i\lambda R}\psi_1$ and $\phi_2^2 = e^{i\lambda R}\psi_2$. For the given $\phi_2^2 \in (\ker B)_-$, we use Lemma 3.4 and choose $\psi \in L_2 = \mathrm{Im}(C_2(0) + 1)$ such that $\phi_2^2 = \psi + iG\psi$. The generalized eigensection $E(\psi, \lambda)$ over $M_{2,\infty}$ attached to the couple $(\psi, \lambda)$ has the expression

$$E(\psi, \lambda) = e^{i\lambda v}(\psi + iG\psi) + e^{-i\lambda v}C_2(\lambda)(\psi + iG\psi) + \theta(\psi, \lambda)$$

over $(-\infty, 0]_v \times Y$. As before, comparing $\Psi_R$ and $E(\psi, \lambda)$ yields

$$\|C_2(\lambda)\phi_2^2 - \phi_1^2\| \le e^{-cR} \tag{3.8}$$

for a positive constant $c$. By definition, we have

$$\phi_1^1 = e^{2i\lambda R}\phi_1^2 \quad \text{and} \quad \phi_2^1 = e^{-2i\lambda R}\phi_2^2. \tag{3.9}$$

Now, combining (3.7), (3.8), and (3.9), we obtain

$$\|e^{4i\lambda R}C_1(\lambda) \circ C_2(\lambda)\phi_2^1 - \phi_2^1\| \le e^{-cR}. \tag{3.10}$$

We define the operator $C_{12}(\lambda)$ by

$$C_{12}(\lambda) := C_1(\lambda) \circ C_2(\lambda)|_{(\ker B)_-} : (\ker B)_- \to (\ker B)_-.$$

The operator $C_{12}(\lambda)$ is a unitary operator and is an analytic function of $\lambda$ for $\lambda \in (-\delta, \delta)$ with a small $\delta > 0$, since the unitary operators $C_1(\lambda), C_2(\lambda)$ are analytic functions of $\lambda$ for $\lambda \in (-\delta, \delta)$. Furthermore, there exist real analytic functions $\alpha_j(\lambda)$ for $1 \leq j \leq h_Y/2$ of $\lambda \in (-\delta, \delta)$ such that $\exp\{i\alpha_j(\lambda)\}$ are the corresponding eigenvalues of $C_{12}(\lambda)$ and $\alpha_j(\lambda)$ has the following expansion at $\lambda = 0$:

$$\alpha_j(\lambda) = \alpha_{j0} + \alpha_{j1}\lambda + \alpha_{j2}\lambda^2 + \alpha_{j3}\lambda^3 + \cdots . \tag{3.11}$$

We now introduce

$$\Omega(R) := \{\rho \in \mathbb{R} - \{0\} \mid \det(e^{4i\rho R}C_{12}(\rho) - \mathrm{Id}) = 0,\ |\rho| \leq R^{-\kappa}\}. \tag{3.12}$$

The following theorem is a main result of this section.

THEOREM 3.5. *Assume that all the e-values of $\mathcal{D}_R$ are zero eigenvalues. Let $\lambda_1(R) \leq \lambda_2(R) \leq \cdots \leq \lambda_{p(R)}(R)$ be the nonzero eigenvalues, counted to multiplicity, of $\mathcal{D}_R$ that satisfy $|\lambda_k(R)| \leq R^{-\kappa}$, and let $\rho_1(R) \leq \rho_2(R) \leq \cdots \leq \rho_{m(R)}(R)$ be the nonzero element, counted to multiplicity, of $\Omega(R)$. Then there exist $R_0$ and $c > 0$, independent of $R$, such that for $R \geq R_0$ we have $p(R) = m(R)$ and*

$$|\lambda_k(R) - \rho_k(R)| \leq e^{-cR} \quad \textit{for } k = 1, \ldots, p(R).$$

*Proof.* The proof of this theorem consists of two steps.

*Step I.* Let $\lambda = \lambda(R)$ be a given $s$-value with multiplicity $m(\lambda)$. By Proposition 3.3, we have $m(\lambda)$ linearly independent vectors $\phi_1, \ldots, \phi_{m(\lambda)}$ in $(\ker B)_-$, which satisfies (3.10). Since $C_{12}(\lambda)$ is unitary, the eigenvalues of $e^{4i\lambda R}C_{12}(\lambda) - \mathrm{Id}$ have the form $e^{i\theta} - 1$ for $\theta \in \mathbb{R}$. Let $0 \leq \zeta$ be the smallest eigenvalue of $(e^{4i\lambda R}C_{12}(\lambda) - \mathrm{Id})(e^{4i\lambda R}C_{12}(\lambda) - \mathrm{Id})^*$; then

$$\zeta = \min_{\phi \in (\ker B)_-} \frac{\|(e^{4i\lambda R}C_{12}(\lambda) - \mathrm{Id})\phi\|^2}{\|\phi\|^2}.$$

Combined with (3.10), this implies that $\zeta \leq e^{-cR}$. Hence $e^{4i\lambda R}C_{12}(\lambda)$ has an eigenvalue $e^{i\theta}$ satisfying $|1 - \cos\theta| \leq e^{-cR}$, and there exists a $k \in \mathbb{Z}$ such that $|2\pi k - \theta| \leq e^{-cR}$. Therefore, by definition of $\alpha_j(\lambda)$,

$$|4\lambda R + \alpha_j(\lambda) - 2\pi k| \leq e^{-cR} \tag{3.13}$$

for pairwise distinct branches $\alpha_1, \ldots, \alpha_{m(\lambda)}$. Now, let us fix $\delta_1$ with $0 < \delta_1 < \delta$ and let

$$m_j = \max_{\lambda \in (-\delta_1, \delta_1)} |\alpha_j'(\lambda)|.$$

Then the function $f(\lambda) = 4\lambda R + \alpha_j(\lambda)$ is strictly increasing for $|\lambda| < \delta_1$ and $R \geq m_j$. Choose $R_1$ such that $R_1 \geq \max(m_j, \delta_1^{-1/\kappa})$ for any $j = 1, \ldots, h_Y/2$. For $R \geq R_1$ and $k \in \mathbb{Z}$, there exists at most one solution $\rho_{j,k}$ of

$$4\lambda R + \alpha_j(\lambda) = 2\pi k, \quad |\lambda| \leq R^{-\kappa}. \tag{3.14}$$

Let $k_{j,\max}$ be the maximal $k$ for which (3.14) has a solution; then, by (3.14),

$$|k_{j,\max}| \leq \frac{2R^{1-\kappa}}{\pi} + C \leq R^{1-\kappa}. \tag{3.15}$$

Now, for $R \geq R_1$, any element $\rho$ in $\Omega(R)$ is given by $\rho = \rho_{j,k}$ for some $1 \leq j \leq h_Y/2$, and $|k| \leq k_{j,\max}$. Hence if $R \geq R_1$ then, for a given $\lambda$ satisfying (3.13) with $|\lambda| \leq R^{-\kappa}$, there is a unique solution $\rho_{j,k}$ of (3.14) such that

$$|\lambda - \rho_{j,k}| \leq e^{-cR}. \tag{3.16}$$

In conclusion: if $R \geq R_1$ then, for a given $s$-value $\lambda = \lambda(R)$ of $\mathcal{D}_R$ with multiplicity $m(\lambda)$ and satisfying $|\lambda| \leq R^{-\kappa}$, there exist $m(\lambda)$ elements $\rho_{j,k}$ in $\Omega(R)$ with the relation (3.16); in particular, $p(R) \leq m(R)$.

*Step II.* To complete the proof, we need to show that $m(R) \leq p(R)$. For $k$ with $1 \leq k \leq m(R)$, we choose $\psi_k \in (\ker B)_-$ with the following properties:

(1) $e^{4i\rho_k R} C_{12}(\rho_k)\psi_k = \psi_k$, $|\rho_k| \leq R^{-\kappa}$;

(2) if $\rho_k = \rho_{k+1} = \cdots = \rho_{k+\ell}$, then $\psi_k, \psi_{k+1}, \ldots, \psi_{k+\ell}$ form an orthonormal system of vectors of $(\ker B)_-$.

For a given pair $(\psi_k, \rho_k)$ for some $k$, we put

$$\phi_k^1 = e^{-i\rho_k R} C_1(-\rho_k)\psi_k \quad \text{and} \quad \phi_k^2 = e^{i\rho_k R}\psi_k. \tag{3.17}$$

Now we consider the generalized eigensections $E(\phi_k^1, \rho_k)$ over $M_{1,\infty}$ and $E(\phi_k^2, \rho_k)$ over $M_{2,\infty}$, which have the following forms:

$$E(\phi_k^1, \rho_k) = e^{-i\rho_k v}\phi_k^1 + e^{i\rho_k v} C_1(\rho_k)\phi_k^1 + O(e^{-cv}) \ \text{ over } [0,\infty)_v \times Y \subset M_{1,\infty};$$

$$E(\phi_k^2, \rho_k) = e^{i\rho_k v}\phi_k^2 + e^{-i\rho_k v} C_2(\rho_k)\phi_k^2 + O(e^{-cv}) \ \text{ over } (-\infty,0]_v \times Y \subset M_{2,\infty}.$$

(Here we use abuse notation for simplicity since for $E(\phi_k^i, \rho_k)$ we should more correctly write $E(\tilde{\phi}_k^i, \rho_k)$, with $\phi_k^i = \tilde{\phi}_k^i + (-1)^i\sqrt{-1}G\tilde{\phi}_k^i$ by Lemma 3.4.) Restricting $E(\phi_k^i, \rho_k)$ to $M_{i,R}$ yields sections over $M_{i,R}$. Let $f_{1,R}$ be the restriction to $M_{1,R}$ of the smooth function $h_R$ over $M_{1,\infty}$ defined in the proof of Proposition 3.3, and let $f_{2,R}$ be a smooth function over $M_{2,R}$ defined in a similar way. These functions have the obvious extension over $M_R$. Denoting by $E_0(\phi_k^i, \rho_k)$ the zero eigenmode of $E(\phi_k^i, \rho_k)$ and using (3.17) and $e^{4i\rho_k R} C_{12}(\rho_k)\psi_k = \psi_k$, we have

$$\begin{aligned} E_0(\phi_k^1, \rho_k) &= e^{-i\rho_k u} e^{-2i\rho_k R} C_1(-\rho_k)\psi_k + e^{i\rho_k u}\psi_k \\ &= e^{i\rho_k u}\psi_k + e^{-i\rho_k u} e^{2i\rho_k R} C_2(\rho_k)\psi_k = E_0(\phi_k^2, \rho_k). \end{aligned} \tag{3.18}$$

Hence we can see that $E_0(\phi_k^1, \rho_k)$ and $E_0(\phi_k^2, \rho_k)$ define a smooth section over $N_R$, which we denote by $E_0(\psi_k, \rho_k)$. Let us define

$$\begin{aligned} \tilde{\Psi}_k &:= f_{1,R}(E(\phi_k^1, \rho_k) - \chi_{[-R,0]_u} E_0(\phi_k^1, \rho_k)) \\ &\quad + f_{2,R}(E(\phi_k^2, \rho_k) - \chi_{[0,R]_u} E_0(\phi_k^2, \rho_k)) + \chi_{[-R,R]_u} E(\psi_k, \rho_k), \end{aligned} \tag{3.19}$$

where $\chi_{[a,b]_u}$ is the characteristic function of the $u$-variable over $[a,b]_u \times Y \subset N_R$. By (3.18), $\tilde{\Psi}_k$ is a smooth section over $M_R$. Put $\Psi_k := \tilde{\Psi}_k / \|\tilde{\Psi}_k\|$ and

$$\hat{\Psi}_k = \Psi_k - \pi_R \Psi_k \quad \text{for } k = 1, \ldots, m(R),$$

where $\pi_R$ denotes the orthogonal projection of $L^2(M_R, S)$ onto $\ker \mathcal{D}_R$. Recall that $\ker \mathcal{D}_R$ equals the space spanned by eigensections of $e$-values by our assumption, so the dimension of this space is constant with respect to $R$. Combining this fact and Lemma 3.6 yields

$$|\langle \hat{\Psi}_k, \hat{\Psi}_\ell \rangle - \delta_{k\ell}| \le e^{-cR} \quad \text{for } k, \ell = 1, \dots, m(R).$$

From this and (3.15), it follows that the $\{\hat{\Psi}_k\}_{k=1}^{m(R)}$ are linearly independent for $R \gg 0$. Now let $0 < \tilde{\lambda}_1 \le \tilde{\lambda}_2 \le \cdots \le \tilde{\lambda}_{p(R)}$ denote those nonzero eigenvalues (counted with multiplicity) of $\mathcal{D}_R^2$ that are $\le R^{-2\kappa}$. Let $k_1, \dots, k_{m(R)}$ be a permutation of $\{1, \dots, m(R)\}$ such that $0 < \rho_{k_1}^2 \le \cdots \le \rho_{k_{m(R)}}^2$. By the mini-max principle, we have

$$\tilde{\lambda}_\ell = \min_W \max_{\phi \in W} \frac{\|\mathcal{D}_R \phi\|^2}{\|\phi\|^2},$$

where $W$ runs over all $\ell$-dimensional subspaces of $L^2(M_R, S)$ that are orthogonal to $\ker(\mathcal{D}_R)$. Let $W_\ell$ be the subspace of $L^2(M_R, S)$ spanned by $\hat{\Psi}_{k_1}, \dots, \hat{\Psi}_{k_\ell}$. Then, by Lemma 3.6,

$$\tilde{\lambda}_\ell \le \max_{\phi \in W_\ell} \frac{\|\mathcal{D}_R \phi\|^2}{\|\phi\|^2} \le \rho_{k_\ell}^2 (1 + Ce^{-cR})$$

for some constants $C, c > 0$. Hence there exists an $R_2$ such that $m(R) \le p(R)$ for $R \ge R_2$. Putting $R_0 = \max(R_1, R_2)$, this completes the proof of Theorem 3.5. □

LEMMA 3.6. *Assume that all the e-values of $\mathcal{D}_R$ are zero eigenvalues. Then there exist $c_1, c_2 > 0$ such that*

$$|\langle \Psi_k, \Psi_\ell \rangle| \le c_1 e^{-c_2 R} \quad \textit{for } k \ne \ell,\ k, \ell = 1, \dots, m(R),$$
$$|\langle \Psi_k, \Psi \rangle| \le c_1 e^{-c_2 R} \quad \textit{for } k = 1, \dots, m(R),\ \Psi \in \ker \mathcal{D}_R \textit{ with } \|\Psi\| = 1.$$

*Proof.* For a couple $(\psi_k, \rho_k)$ and $\phi_k^i$ satisfying (3.17), we put

$$\begin{aligned} E_k^\perp &= E(\phi_k^1, \rho_k)|_{M_{1,R}} - \chi_{[-R,0]_u} E_0(\phi_k^1, \rho_k) \\ &\quad + E(\phi_k^2, \rho)|_{M_{2,R}} - \chi_{[0,R]_u} E_0(\phi_k^2, \rho_k), \\ E_{k,0} &= E(\psi_k, \rho_k) = \chi_{[-R,0]_u} E_0(\phi_k^1, \rho_k) + \chi_{[0,R]_u} E_0(\phi_k^2, \rho_k). \end{aligned}$$

Setting $f_R = f_{1,R} + f_{2,R}$, it is easy to see that the $\tilde{\Psi}_k$ defined in (3.19) has the form $f_R E_k^\perp + \chi_{[-R,R]_u} E_{k,0}$. Now we have

$$\begin{aligned} \langle \tilde{\Psi}_k, \tilde{\Psi}_\ell \rangle &= \langle f E_k^\perp + \chi E_{k,0}, f E_\ell^\perp + \chi E_{\ell,0} \rangle \\ &= \langle f E_k^\perp, f E_\ell^\perp \rangle + \langle \chi E_{k,0}, \chi E_{\ell,0} \rangle \\ &= \langle E_k^\perp - (1-f) E_k^\perp, E_\ell^\perp - (1-f) E_\ell^\perp \rangle + \langle \chi E_{k,0}, \chi E_{\ell,0} \rangle \\ &= \langle E_k^\perp, E_\ell^\perp \rangle - \langle E_k^\perp, (1-f) E_\ell^\perp \rangle - \langle (1-f) E_k^\perp, E_\ell^\perp \rangle \\ &\quad + \langle (1-f) E_k^\perp, (1-f) E_\ell^\perp \rangle + \langle \chi E_{k,0}, \chi E_{\ell,0} \rangle \\ &= \langle E_k, E_\ell \rangle - \langle E_k^\perp, (1-f) E_\ell^\perp \rangle - \langle (1-f) E_k^\perp, E_\ell^\perp \rangle \\ &\quad + \langle (1-f) E_k^\perp, (1-f) E_\ell^\perp \rangle, \end{aligned} \tag{3.20}$$

where $f = f_R$ and $\chi = \chi_{[-R,R]_u}$. Since $\mathrm{supp}(1-f_R) \subset \left[-\frac{1}{2}R, \frac{1}{2}R\right]_u \times Y$, where $E_k^{\perp}, E_\ell^{\perp}$ are $O(e^{-cR})$, it follows that the last three terms in (3.20) are $O(e^{-cR})$. Now we consider the first term in (3.20), which can be written as

$$\langle E_k, E_\ell\rangle = \langle E(\phi_k^1, \rho_k), E(\phi_\ell^1, \rho_\ell)\rangle_{M_{1,R}} + \langle E(\phi_k^2, \rho_k), E(\phi_\ell^2, \rho_\ell)\rangle_{M_{2,R}}. \tag{3.21}$$

When $\rho_k \neq \rho_\ell$, as in the proof of Proposition 3.1, we apply Green's formula to each term on the right side of (3.21); then these equal

$$(\rho_k - \rho_\ell)^{-1}\langle GE(\phi_k^1, \rho_k)|_{\partial(M_{1,R})}, E(\phi_\ell^1, \rho_\ell)|_{\partial(M_{1,R})}\rangle_{\partial(M_{1,R})} \\ - (\rho_k - \rho_\ell)^{-1}\langle GE(\phi_k^2, \rho_k)|_{\partial(M_{2,R})}, E(\phi_\ell^2, \rho_\ell)|_{\partial(M_{2,R})}\rangle_{\partial(M_{2,R})}.$$

Now using (3.18), the restrictions of constant terms over $\partial(M_{i,R})$ cancel each other out and the remaining terms are $O(e^{-cR})$. Hence, in this case, the left side of (3.21) is $O(e^{-cR})$ and so all the terms in (3.20) are $O(e^{-cR})$. When $\rho_k = \rho_\ell$, note that $\langle \phi_k^i, \phi_\ell^i\rangle = 0$ for $i = 1, 2$; hence, applying Proposition 3.1, we can see that all the terms are $O(e^{-cR})$ except for the terms

$$\langle C_1(-\rho_k)C_1'(\rho_k)\phi_k^1, \phi_\ell^1\rangle + \langle C_2(-\rho_k)C_2'(\rho_k)\phi_k^2, \phi_\ell^2\rangle. \tag{3.22}$$

Using the conditions in (3.17) for $\phi_k^i, \phi_\ell^i$ and the relations

$$\begin{aligned} e^{4i\rho_k R}C_2(\rho_k)\psi_k &= C_1(-\rho_k)\psi_k \quad \text{and} \\ e^{4i\rho_k R}C_2(\rho_k)\psi_\ell &= C_1(-\rho_k)\psi_\ell, \end{aligned} \tag{3.23}$$

one can show that the terms in (3.22) equal

$$\langle e^{4i\rho_k R}C_1'(\rho_k)C_2(\rho_k)\psi_k, \psi_\ell\rangle + \langle e^{4i\rho_k R}C_1(\rho_k)C_2'(\rho_k)\psi_k, \psi_\ell\rangle. \tag{3.24}$$

Next we choose a family of sections $\psi_k(t)$ with $\psi_k(0) = \psi_k$ for $t \in (-\varepsilon, \varepsilon)$ such that

$$a(t)C_1(\rho_k + t)C_2(\rho_k + t)\psi_k(t) = \psi_k(t),$$

where $a(0) = e^{4i\rho_k R}$. Taking the derivative of this at $t = 0$, we obtain

$$e^{4i\rho_k R}C_1'(\rho_k)C_2(\rho_k)\psi_k + e^{4i\rho_k R}C_1(\rho_k)C_2'(\rho_k)\psi_k \\ = -a'(0)C_1(\rho_k)C_2(\rho_k)\psi_k - e^{4i\rho_k R}C_1(\rho_k)C_2(\rho_k)\psi_k'(0) + \psi_k'(0).$$

Using this, (3.23), and $\langle \psi_k, \psi_\ell\rangle = 0$, we can see that (3.24) equals

$$\langle -e^{4i\rho_k R}C_1(\rho_k)C_2(\rho_k)\psi_k'(0) + \psi_k'(0), \psi_\ell\rangle \\ = \langle \psi_k'(0), \psi_\ell\rangle - \langle \psi_k'(0), e^{-4i\rho_k R}C_2(-\rho_k)C_1(-\rho_k)\psi_\ell\rangle = 0.$$

Hence, in the case of $\rho_k = \rho_\ell$, all the terms in (3.20) are $O(e^{-cR})$. This completes the proof of the first claim once we recall that $\Psi_k = \tilde{\Psi}_k/\|\tilde{\Psi}_k\|$.

For the second claim, recall that the eigenspaces of the $e$-values are spanned by the sections defined by gluing (as in (3.19)) the elements in $\ker_{L^2}\mathcal{D}_{i,\infty}$ for $i = 1, 2$ or the extended $L^2$-solutions of $\mathcal{D}_{i,\infty}$ whose limiting values lie in $L_1 \cap L_2$. By our assumption, this space is the same as $\ker \mathcal{D}_R$. For a section $\Psi$ given by gluing elements in $\ker_{L^2}\mathcal{D}_{i,\infty}$, the claim follows easily by applying Green's formula as before. For a section $\Psi$ given by gluing the extended $L^2$-solutions of $\mathcal{D}_{i,\infty}$ whose limiting values lie in $L_1 \cap L_2$, we use Theorem 3.8, which implies that such a $\Psi$ is actually given by (3.19) for the couple $(\psi_k, \rho_k)$ with $\rho_k = 0$. Hence, the claim for this case can be proved as in the previous case of $\rho_k \neq \rho_\ell$. This completes the proof of the second claim. $\square$

In general, the map $C_{12} := C_{12}(0)\colon (\ker B)_- \to (\ker B)_-$ does not equal the identity map, but it is not difficult to see that

$$C_1(0) \circ C_2(0)\phi = \phi \iff \phi \in (L_1 \cap L_2) \oplus (GL_1 \cap GL_2).$$

Putting

$$I_+ = 1 + iG\colon \ker B \to (\ker B)_-,$$

we see that $I_+(L_1 \cap L_2)$ and $I_+(GL_1 \cap GL_2)$ are the same subspace in $(\ker B)_-$.

PROPOSITION 3.7. *The map $C_{12}$ equals the identity map when restricted to the subspace $I_+(L_1 \cap L_2)$, and the multiplicity of the eigenvalue* $(+1)$ *of the operator $C_{12}$ is* $\dim(L_1 \cap L_2) = \dim(I_+(L_1 \cap L_2))$.

*Proof.* Using the diagram

$$\begin{tikzcd} L_1 \cap L_2 \arrow[r, "I_+"] \arrow[d, "C_1(0)\circ C_2(0)"'] & (\ker B)_- \arrow[d, "C_1(0)\circ C_2(0)"] \\ L_1 \cap L_2 \arrow[r, "I_+"] & (\ker B)_-, \end{tikzcd}$$

we can easily see that the first claim holds. To complete the proof, it is sufficient to show that if $C_1(0) \circ C_2(0)\phi = \phi$ then $\phi \in (L_1 \cap L_2) \oplus (GL_1 \cap GL_2)$. For this, choose $\phi_+ \in L_1$; then $C_1(0) \circ C_2(0)\phi_+ = \phi_+$ implies $\phi_+ = C_1(0)\phi_+ = C_2(0)\phi_+$, since $C_1(0)^2 = \mathrm{Id}$. Hence, this means that $\phi_+ \in L_2$ and so $\phi_+ \in L_1 \cap L_2$. Repeating the same argument, if $\phi_- \in GL_1$ and $C_1(0) \circ C_2(0)\phi_- = \phi_-$, then $\phi_- \in GL_1 \cap GL_2$. Since $\ker B = L_1 \oplus GL_1$, this completes the proof. $\square$

Now let us consider the eigenvalues $\lambda(R)$, which correspond to $\alpha_j(0) = 0$ and $k = 0$ in the following equality (which is equivalent to (3.13)):

$$4\lambda R + \alpha_j(\lambda) = 2\pi k + O(e^{-cR}).$$

It is easy to see that such eigenvalues must be $e$-values. Hence, by Lemma 3.7 this provides another proof of the following result, originally shown in [5].

THEOREM 3.8. *The space of eigensections corresponding to those e-values not determined by* $\ker_{L^2}(\mathcal{D}_{i,\infty})$, $i = 1, 2$, *is given by the space* $L_1 \cap L_2$.

Now we consider the Dirac-type operator

$$\mathcal{D}_R = d_\rho + d_\rho^*\colon \bigoplus_{i=0}^{n} \Omega^i(M_R, V_\rho) \to \bigoplus_{i=0}^{n} \Omega^i(M_R, V_\rho), \tag{3.25}$$

where $V_\rho$ denotes the flat vector bundle defined by a unitary representation $\rho$ of $\pi_1(M_R)$. The dimension of $\ker \mathcal{D}_R$ is constant with respect to $R$, since $\ker \mathcal{D}_R$ is the space of the twisted harmonic forms over $M_R$ and since this space is always isomorphic to the de Rham cohomology $H^*(M_R, V_\rho)$ by Hodge's theorem. Moreover, one can show that all the $e$-values of the operator $\mathcal{D}_R$ in (3.25) are the zero eigenvalues by using the argument in [11, Sec. 4].

PROPOSITION 3.9. *For the operator $\mathcal{D}_R$ in* (3.25), *all the e-values of $\mathcal{D}_R$ are the zero eigenvalues.*

*Proof.* First observe that $\mathcal{D}_{i,\infty}$ is self-adjoint, so $\ker_{L^2} \mathcal{D}_{i,\infty} = \ker_{L^2} \mathcal{D}^2_{i,\infty}$ and $L_i$ is also the limiting value of extended $L^2$-solutions of $\mathcal{D}^2_{i,\infty}$. Let $\Delta^q_{i,\infty}$ be the restriction of $\mathcal{D}^2_{i,\infty}$ to $\Omega^q(M_R, V_\rho)$ and let

$$h^q_M := \dim \ker_{L^2} \Delta^q_{1,\infty} + \dim \ker_{L^2} \Delta^q_{2,\infty} + \dim L^q_1 \cap L^q_2, \tag{3.26}$$

where the $L^q_i$ are limiting values of the extended $L^2$-solutions of $\Delta^q_{i,\infty}$. Then it is sufficient to show that $\beta^q := \dim(\ker \mathcal{D}^2_R \cap \Omega^q(M_R, V_\rho)) \geq h^q_M$, since $\beta_q \leq h^q_M$ by definition. For this, we use the Mayer–Vietoris sequence

$$\begin{aligned} \cdots \to H^{q-1}(Y) \to H^q(M_R) \\ \to H^q_a(M_{1,R}) \oplus H^q_a(M_{2,R}) \to H^q(Y) \to \cdots, \end{aligned} \tag{3.27}$$

where $H^q_a(M_{i,R})$ denotes the absolute cohomology. (Here, for simplicity, the bundle $V_\rho$ is omitted from the notation.) The space $\bigoplus_{q=0}^n H^q_a(M_{i,R})$ can be identified with the kernel of the operator $\mathcal{D}_{i,R}$ with the absolute boundary condition. In more detail: the operator $\mathcal{D}_R = d_\rho + d^*_\rho$ has the form

$$\mathcal{D}_R = d_\rho + d^*_\rho = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left( \partial_u + \begin{pmatrix} 0 & d_Y + d^*_Y \\ d_Y + d^*_Y & 0 \end{pmatrix} \right) \tag{3.28}$$

over $N_R$ with respect to

$$\Omega^*(N_R) \cong (\Omega^*(Y) \oplus \Omega^*(Y)) \otimes C^\infty([-R, R]_u). \tag{3.29}$$

Here $d_Y$ and $d^*_Y$ denote the restricted operator to $Y$ of $d_\rho$ and of its adjoint, respectively. The operator $\mathcal{D}_{i,R}$ has the same form near the boundary and with respect to (3.29). A section $\Psi$ in $\Omega^*(M_{i,R})$ over the cylinder near the boundary $Y$ has the form

$$\Psi = \Psi_0 + \Psi_1 \wedge du,$$

where $\Psi_i$ has no $du$ factor. Then the absolute boundary condition for $\mathcal{D}_{i,R}$ is given by $\Psi_1 = 0$. Similarly, the relative boundary condition for $\mathcal{D}_{i,R}$ is given by $\Psi_0 = 0$. We denote by $\mathcal{D}^a_{i,R}, \mathcal{D}^r_{i,R}$ the resulting operators. Now recall that the Cauchy data spaces $\mathcal{H}(\mathcal{D}_{i,R})$ of $\mathcal{D}_{i,R}$ are Lagrangian subspaces in $\Omega^*(Y) \oplus \Omega^*(Y)$ with respect to the symplectic form $\langle G, \cdot \rangle$, where $G = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\langle \cdot, \cdot \rangle$ denotes the inner product over $\Omega^*(Y) \oplus \Omega^*(Y)$. This now implies that $\mathcal{H}_0(\mathcal{D}_{i,R}) := \mathcal{H}(\mathcal{D}_{i,R}) \cap (H^*(Y) \oplus H^*(Y))$ are also Lagrangian subspaces in $H^*(Y) \oplus H^*(Y)$. Moreover, the space $\mathcal{H}_0(\mathcal{D}_{i,R})$ has the decomposition

$$\mathcal{H}_0(\mathcal{D}_{i,R}) = A_i \oplus R_i,$$

where $A_i$ and $R_i$ are the spaces spanned by the boundary values of $\ker \mathcal{D}^a_{i,R}$ and $\ker \mathcal{D}^r_{i,R}$ (respectively) in $H^*(Y) \oplus H^*(Y)$. Decomposing $A_i = \bigoplus_{q=0}^{n-1} A_i^q$ and $R_i = \bigoplus_{q=0}^{n-1} R_i^q$, where $A_i^q, R_i^q$ are spaces of $q$-form parts, the Lagrangian subspace property of $\mathcal{H}_0(\mathcal{D}_{i,R})$ in $H^*(Y) \oplus H^*(Y)$ implies that

$$H^q(Y) \cong A_i^q \oplus R_i^q. \tag{3.30}$$

By the exactness of (3.27), we also have

$$\begin{aligned} H^q(M_R) &\cong \operatorname{Im}(H^{q-1}(Y) \to H^q(M_R)) \\ &\qquad \oplus \operatorname{Im}(H^q(M_R) \to H^q_a(M_{1,R}) \oplus H^q_a(M_{2,R})) \\ &\cong (\operatorname{Im} k^{q-1})^\perp \oplus \ker k^q, \end{aligned}$$

where $k^q$ is the boundary map from $H^q_a(M_{1,R}) \oplus H^q_a(M_{2,R})$ to $H^q(Y)$. Now we summarize the consequences of the foregoing considerations. First, by (3.30), we have

$$(\operatorname{Im} k^{q-1})^\perp = (A_1^{q-1} + A_2^{q-1})^\perp = R_1^{q-1} \cap R_2^{q-1}.$$

Second, we note that $\ker k^q$ contains the harmonic sections whose boundary values are lying in $A_1^q \cap A_2^q$ and the harmonic sections that can be extended as $L^2$-solutions of $\Delta^q_{i,\infty}$. Therefore,

$$\dim \ker k^q \ge \dim(A_1^q \cap A_2^q) + \dim \ker_{L^2} \Delta^q_{1,\infty} + \dim \ker_{L^2} \Delta^q_{2,\infty}.$$

By these facts and the equality

$$\dim(L_1^q \cap L_2^q) = \dim(A_1^q \cap A_2^q) + \dim(R_1^{q-1} \cap R_2^{q-1}),$$

we can conclude that $\beta^q \ge h^q_M$ once we recall (3.26). This completes the proof. □

Let $\Psi_R$ be a normalized eigensection of $(\mathcal{D}_{1,R})_{P_1}$ corresponding to the $s$-value $\lambda = \lambda(R)$ with $|\lambda(R)| \le R^{-\kappa}$, where $\kappa$ is a fixed constant such that $0 < \kappa < 1$. Then

$$\mathcal{D}_{1,R}\Psi_R = \lambda\Psi_R, \quad \|\Psi_R\| = 1, \quad P_1(\Psi_R|_{\{v=R\}\times Y}) = 0. \tag{3.31}$$

The section $\Psi_R$ can be represented on $[0, R]_v \times Y \subset M_{1,R}$ as

$$\Psi_R = e^{-i\lambda v}\psi_1 + e^{i\lambda v}\psi_2 + \hat{\Psi}_R, \tag{3.32}$$

where $\psi_1 \in (\ker B)_+$ and $\psi_2 \in (\ker B)_-$ and where $\hat{\Psi}_R$ is a smooth $L^2$-section orthogonal to $\ker B$. The next result corresponds to Proposition 3.3, which can be proved in the same way as [17, Prop. 8.14].

PROPOSITION 3.10. *The zero eigenmode $e^{-i\lambda v}\psi_1 + e^{i\lambda v}\psi_2$ of the eigensection $\Psi_R$ of $s$-value $\lambda(R)$ of $(\mathcal{D}_{1,R})_{P_1}$ is nontrivial.*

Now we define

$$\begin{aligned} I_{\pm} &= 1 \pm iG\colon \ker B \to (\ker B)_{\mp}, \\ I_{\sigma_1} &= I_{-}|_{\ker(\sigma_1+1)}\colon \ker(\sigma_1+1) \to (\ker B)_{+}, \\ P_{\sigma_1} &= \tfrac{1}{2}(1-\sigma_1)\colon \ker B \to \ker(\sigma_1+1), \end{aligned}$$

and

$$S_{\sigma_1}(\lambda) = -P_{\sigma_1} \circ C_1(\lambda) \circ I_{\sigma_1}\colon \ker(\sigma_1+1) \to \ker(\sigma_1+1).$$

For $\psi_1$ in (3.32), by Lemma 3.4 there exists a unique $\phi \in \ker(\sigma_1+1)$ such that $\psi_1 = \phi - iG\phi$. As in the derivation of (3.7), we compare $\Psi_R$ with $E(\phi, \lambda)$ and then use Proposition 3.10 to obtain

$$\|C_1(\lambda)\psi_1 - \psi_2\| \le e^{-cR}. \tag{3.33}$$

By the boundary condition in (3.31), we have

$$e^{-2i\lambda R} P_{\sigma_1}(\psi_1) = -P_{\sigma_1}(\psi_2).$$

Combining this equation and (3.33) yields

$$\|e^{2i\lambda R} S_{\sigma_1}(\lambda)\phi - \phi\| \le e^{-cR}$$

for $\phi \in \ker(\sigma_1+1)$. We also define

$$\begin{aligned} I_{\sigma_2} &= I_{+}|_{\ker(\sigma_2+1)}\colon \ker(\sigma_2+1) \to (\ker B)_{-}, \\ P_{\sigma_2} &= \tfrac{1}{2}(1-\sigma_2)\colon \ker B \to \ker(\sigma_2+1), \end{aligned}$$

and

$$S_{\sigma_2}(\lambda) := -P_{\sigma_2} \circ C_2(\lambda) \circ I_{\sigma_2}\colon \ker(\sigma_2+1) \to \ker(\sigma_2+1),$$

where $C_2(\lambda)$ is the scattering matrix defined from the generalized eigensection over $M_{2,\infty}$. In the same way as before we can derive

$$\|e^{2i\lambda R} S_{\sigma_2}(\lambda)\phi - \phi\| \le e^{-cR}$$

for $\phi \in \ker(\sigma_2+1)$. Now we introduce

$$\Omega_i(R) := \{\rho \in \mathbb{R} - \{0\} \mid \det(e^{2i\rho R} S_{\sigma_i}(\rho) - \mathrm{Id}) = 0,\ |\rho| \le R^{-\kappa}\}$$

for $i = 1, 2$. We repeat the argument used in [17] to prove the corresponding result for $s$-values of $(\mathcal{D}_{i,R})_{P_i}$, noting that all the arguments for the involution $C_i(0)$ in [17] hold for the involution $\sigma_i$; thus we obtain the following result.

THEOREM 3.11. *For $i = 1, 2$, let $\lambda_1(R) \le \lambda_2(R) \le \cdots \le \lambda_{p(R)}(R)$ be the nonzero eigenvalues, counted to multiplicity, of $(\mathcal{D}_{i,R})_{P_i}$ that satisfy $|\lambda_k(R)| \le R^{-\kappa}$, and let $\rho_1(R) \le \rho_2(R) \le \cdots \le \rho_{m(R)}(R)$ be the nonzero element, counted to multiplicity, of $\Omega_i(R)$. Then there exist $R_0$ and $c > 0$, independent of $R$, such that for $R \ge R_0$ we have $p(R) = m(R)$ and*

$$|\lambda_k(R) - \rho_k(R)| \le e^{-cR} \quad \textit{for } k = 1, \ldots, p(R).$$

We now have the following proposition.

PROPOSITION 3.12. *There is a natural isomorphism*

$$\ker(\mathcal{D}_{i,R})_{P_i} \cong \ker_{L^2} \mathcal{D}_{i,\infty} \oplus \ker(\sigma_i - 1) \cap L_i$$

*for* $i = 1, 2$.

*Proof.* Let $\Psi \in \ker(\mathcal{D}_{1,R})_{P_1}$. Then the section $\Psi$ satisfies $G(\partial_v + B)\Psi = 0$ on the cylinder $[0, R]_v \times Y$, and it has the following representation when restricted to this cylinder:

$$\Psi = \phi_0 + \sum_{\mu_j > 0} c_j e^{-\mu_j v} \phi_j,$$

where $(\sigma_i - 1)(\phi_0) = 0$. We use this expansion to extend $\Psi$ to a smooth section $\tilde{\Psi}$ on $M_{1,\infty}$ satisfying $\mathcal{D}_{1,\infty}\tilde{\Psi} = 0$. This means that $\tilde{\Psi}$ belongs to the space of the extended $L^2$-solutions of $\mathcal{D}_{1,\infty}$ and hence $\phi_0$ is an element of $L_1$. Let $E(\phi_0, \lambda)$ be the generalized eigensection attached to $\phi_0$. Then $\tilde{\Psi} - \frac{1}{2}E(\phi_0, 0)$ is square integrable and $\mathcal{D}_{1,\infty}(\tilde{\Psi} - \frac{1}{2}E(\phi_0, 0)) = 0$, and the map

$$\Psi \to (\tilde{\Psi} - \tfrac{1}{2}E(\phi_0, 0), \phi_0)$$

gives the expected isomorphism. □

The restriction of $S_{\sigma_i} := S_{\sigma_i}(0)$ to $\ker(\sigma_i + 1) \cap \ker(C_i(0) + 1)$ is equal to the identity map, and

$$\dim(\ker(\sigma_i + 1) \cap \ker(C_i(0) + 1)) = \dim(\ker(\sigma_i - 1) \cap \ker(C_i(0) - 1)).$$

It follows from Proposition 3.12 that the number of $(+1)$-eigenspaces of $S_{\sigma_i} := S_{\sigma_i}(0)$ is equal to the dimension of the subspace of $\ker(\mathcal{D}_{i,R})_{P_i}$ that is complementary to the subspace $\ker_{L^2} \mathcal{D}_{i,\infty}$ for $i = 1, 2$.

Now we define our model operator. Let $C\colon W \to W$ denote a unitary operator acting on a $d$-dimensional vector space $W$ with eigenvalues $e^{i\alpha_j}$ for $j = 1, \dots, d$. We introduce the operator $D(C)$,

$$D(C) := -i\frac{1}{2}\frac{d}{du}\colon C^\infty(\mathbb{S}^1, E_C) \to C^\infty(\mathbb{S}^1, E_C), \tag{3.34}$$

where $E_C$ is the flat vector bundle over $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ defined by the holonomy $\bar{C}$, the complex conjugate of $C$. The spectrum of $D(C)$ is equal to

$$\{\pi k - \tfrac{1}{2}\alpha_j \mid k \in \mathbb{Z},\ j = 1, \dots, d\}. \tag{3.35}$$

The operators $C_{12}, S_{\sigma_1}, S_{\sigma_2}$ are the unitary operators acting on finite-dimensional vector spaces. Hence we can define self-adjoint elliptic operators $D(C_{12})$, $D(S_{\sigma_1})$, $D(S_{\sigma_2})$ on $\mathbb{S}^1$.

THEOREM 3.13. *Assume that all the e-values of $\mathcal{D}_R$ are zero eigenvalues. Let $\lambda_1(R) \le \lambda_2(R) \le \cdots \le \lambda_{p(R)}(R)$ be the nonzero eigenvalues, counted to multiplicity, of $\mathcal{D}_R$ that satisfy $|\lambda_k(R)| \le R^{-\kappa}$, and let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_{n(R)}$ be the nonzero eigenvalues, counted to multiplicity, of $D(C_{12})$ that satisfy $|\lambda_k| \le 2R^{1-\kappa}$.*

*Then there exist $R_0$ and $C > 0$, independent of $R$, such that for $R \geq R_0$ we have $p(R) = n(R)$ and*

$$|2R\lambda_k(R) - \lambda_k| \leq CR^{-\kappa} \quad \text{for } k = 1, \ldots, p(R).$$

*A similar statement holds for $(\mathcal{D}_{i,R})_{P_i}$ and $D(S_{\sigma_i})$ with the relation*

$$|R\lambda_k(R) - \lambda_k| \leq CR^{-\kappa} \quad \text{for } k = 1, \ldots, p_i(R),$$

*where $p_i(R)$ is the number of s-values of $(\mathcal{D}_{i,R})_{P_i}$ with $|\lambda_k(R)| \leq R^{-\kappa}$.*

*Proof.* First we introduce

$$\Omega^*(R) := \{\rho \in \mathbb{R} - \{0\} \mid \det(e^{4i\rho R}C_{12} - \mathrm{Id}) = 0, \ |\rho| \leq R^{-\kappa}\}.$$

By definition, this set consists of the nonzero solution $\rho^*_{j,k}$ of

$$4\lambda R + \alpha_j(0) = 2\pi k \quad \text{with } |\lambda| \leq R^{-\kappa}, \tag{3.36}$$

where $e^{i\alpha_j(0)}$ for $j = 1, \ldots, h_Y/2$ are the eigenvalues of $C_{12} = C_{12}(0)$. Now, for an element $\rho_{j,k}$ in $\Omega(R)$ defined in (3.12), one can show (as near (3.14)) that if $R \gg 0$ then there is the corresponding solution $\rho^*_{j,k}$ of

$$4\lambda R + \alpha_j(0) = 2\pi k \quad \text{with } |\lambda| \leq R^{-\kappa} + R^{-1-\kappa},$$

noting that $|\alpha_j(\lambda) - \alpha_j(0)| \leq cR^{-\kappa}$ for a positive constant $c$. Since $|\rho_{j,k} - \rho^*_{j,k}| \leq cR^{-1-\kappa}$, this gives a one-to-one correspondence from $\Omega(R)$ to $\Omega^*(R_0)$ with $R_0^{-\kappa} = R^{-\kappa} + R^{-1-\kappa}$ for $R \gg 0$. Now observe that, for any pair of $\rho^*_{j,k} \neq \rho^*_{j',k'}$ in $\Omega^*(R)$, we have $|\rho^*_{j,k} - \rho^*_{j',k'}| \geq a_0 R^{-1}$ for a positive constant $a_0$. Hence, for $R \gg 0$, this implies that $\Omega^*(R) = \Omega^*(R_0)$ with $R_0^{-\kappa} = R^{-\kappa} + R^{-1-\kappa}$. In conclusion, there is a one-to-one correspondence between $\Omega(R)$ and $\Omega^*(R)$ for $R \gg 0$ with the relation

$$|\rho_k - \rho^*_k| \leq cR^{-1-\kappa},$$

where $\rho_1 \leq \cdots \leq \rho_{m(R)}$ (resp. $\rho^*_1 \leq \cdots \leq \rho^*_{n(R)}$) denotes the elements, counted to multiplicity, of $\Omega(R)$ (resp. $\Omega^*(R)$). For $\rho^* \in \Omega^*(R)$, the map $\rho^* \to 2R\rho^*$ gives a one-to-one correspondence from $\Omega^*(R)$ to the subset of the eigenvalues $\lambda_k$ of $D(C_{12})$ with $|\lambda_k| \leq 2R^{1-\kappa}$. Applying Theorem 3.5 now completes the proof for $s$-values for $\mathcal{D}_R$. The case of $(\mathcal{D}_{i,R})_{P_i}$ can be proved in the same way. □

## 4. Large Time Contribution

In this section we prove the following proposition.

PROPOSITION 4.1.

$$\begin{aligned}
&\lim_{R\to\infty} \int_{R^{2-\varepsilon}}^{\infty} t^{-1}[\mathrm{Tr}(\exp\{-t\mathcal{D}_R^2\} - \exp\{-t(\mathcal{D}_{1,R})^2_{P_1}\} \\
&\qquad\qquad - \exp\{-t(\mathcal{D}_{2,R})^2_{P_2}\}) - h]\,dt - h(\gamma - \varepsilon \cdot \log R) \\
&= \frac{d}{ds}\bigg|_{s=0} \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1}[\mathrm{Tr}(\exp\{-t\tfrac{1}{4}D(C_{12})^2\} \\
&\qquad\qquad - \exp\{-tD(S_{\sigma_1})^2\} - \exp\{-tD(S_{\sigma_2})^2\}) - h]\,dt,
\end{aligned}$$

*where $h = \dim(L_1 \cap L_2) - \dim(L_1 \cap \ker(\sigma_1 - 1)) - \dim(L_2 \cap \ker(\sigma_2 - 1))$.*

Recalling that

$$\left.\frac{d}{ds}\right|_{s=0} \zeta_l^R(s)$$
$$= \int_{R^{2-\varepsilon}}^{\infty} t^{-1}[\mathrm{Tr}(\exp\{-t\mathcal{D}_R^2\} - \exp\{-t(\mathcal{D}_{1,R})_{P_1}^2\} - \exp\{-t(\mathcal{D}_{2,R})_{P_2}^2\}) - h]\, dt,$$

Proposition 4.1 immediately implies that the large time contribution to the adiabatic decomposition formula for the $\zeta$-determinant is equal to

$$\frac{\det_\zeta \frac{1}{4} D(C_{12})^2}{\det_\zeta D(S_{\sigma_1})^2 \cdot \det_\zeta D(S_{\sigma_2})^2}.$$

We start with the following result.

PROPOSITION 4.2. *The following equality holds*:

$$\lim_{R\to\infty}\Bigg(\left.\frac{d}{ds}\right|_{s=0} \frac{1}{\Gamma(s)} \int_0^{R^{-\varepsilon}} t^{s-1}[\mathrm{Tr}(\exp\{-t\tfrac{1}{4} D(C_{12})^2\} - \exp\{-tD(S_{\sigma_1})^2\}$$
$$- \exp\{-tD(S_{\sigma_2})^2\}) - h]\, dt$$
$$+ h(\gamma - \varepsilon \cdot \log R)\Bigg) = 0.$$

*Proof*. Recalling the definition of $D(C)$ in (3.34), we can see that if $\mathcal{L}$ is one of $D(C_{12})^2$, $D(S_{\sigma_1})^2$, and $D(S_{\sigma_1})^2$ then

$$\mathrm{Tr}(e^{-t\mathcal{L}}) \sim \sqrt{\frac{\pi}{t}}\,\frac{h_Y}{2} + O\big(\sqrt{t}\big) \text{ near } t = 0,$$

since $h_Y = 2\dim(\ker B)_- = 2\dim\ker(\sigma_i + 1)$. Hence there exists a constant $c_1$ such that

$$|\mathrm{Tr}(\exp\{-t\tfrac{1}{4} D(C_{12})^2\} - \exp\{-tD(S_{\sigma_1})^2\} - \exp\{-tD(S_{\sigma_2})^2\})|$$
$$< c_1\sqrt{t} \text{ near } t = 0. \quad (4.1)$$

This allows us to estimate

$$\Bigg|\left.\frac{d}{ds}\right|_{s=0} \frac{1}{\Gamma(s)} \int_0^{R^{-\varepsilon}} t^{s-1}\, \mathrm{Tr}(\exp\{-t\tfrac{1}{4} D(C_{12})^2\}$$
$$- \exp\{-tD(S_{\sigma_1})^2\} - \exp\{-tD(S_{\sigma_2})^2\})\, dt\Bigg|$$
$$\le c_2 \cdot \int_0^{R^{-\varepsilon}} \frac{dt}{\sqrt{t}} = 2c_2 \cdot R^{-\varepsilon/2}.$$

Combining this inequality with

$$\left.\frac{d}{ds}\right|_{s=0} \frac{h}{\Gamma(s)} \int_0^{R^{-\varepsilon}} t^{s-1}\, dt = h(\gamma - \varepsilon \cdot \log R)$$

completes the proof. $\square$

It follows from Proposition 4.2 that Proposition 4.1 is equivalent to

$$\lim_{R\to\infty}\int_{R^{2-\varepsilon}}^{\infty} t^{-1}[\mathrm{Tr}(\exp\{-t\mathcal{D}_R^2\} - \exp\{-t(\mathcal{D}_{1,R})_{P_1}^2\} - \exp\{-t(\mathcal{D}_{2,R})_{P_2}^2\}) - h]\,dt$$
$$= \lim_{R\to\infty}\frac{d}{ds}\bigg|_{s=0}\frac{1}{\Gamma(s)}\int_{R^{-\varepsilon}}^{\infty} t^{s-1}[\mathrm{Tr}(\exp\{-t\tfrac{1}{4}D(C_{12})^2\} - \exp\{-tD(S_{\sigma_1})^2\}$$
$$- \exp\{-tD(S_{\sigma_2})^2\}) - h]\,dt. \tag{4.2}$$

Now using a change of variables yields

$$\int_{R^{2-\varepsilon}}^{\infty} t^{-1}[\mathrm{Tr}(\exp\{-t\mathcal{D}_R^2\} - \exp\{-t(\mathcal{D}_{1,R})_{P_1}^2\} - \exp\{-t(\mathcal{D}_{2,R})_{P_2}^2\}) - h]\,dt$$
$$= \int_{R^{-\varepsilon}}^{\infty} t^{-1}[\mathrm{Tr}(\exp\{-tR^2\mathcal{D}_R^2\} - \exp\{-tR^2(\mathcal{D}_{1,R})_{P_1}^2\}$$
$$- \exp\{-tR^2(\mathcal{D}_{2,R})_{P_2}^2\}) - h]\,dt.$$

Then (4.2) is equivalent to

$$\lim_{R\to\infty}\int_{R^{-\varepsilon}}^{\infty} t^{-1}[\mathrm{Tr}(\exp\{-tR^2\mathcal{D}_R^2\} - \exp\{-tR^2(\mathcal{D}_{1,R})_{P_1}^2\}$$
$$- \exp\{-tR^2(\mathcal{D}_{2,R})_{P_2}^2\}) - h]\,dt$$
$$= \lim_{R\to\infty}\frac{d}{ds}\bigg|_{s=0}\frac{1}{\Gamma(s)}\int_{R^{-\varepsilon}}^{\infty} t^{s-1}[\mathrm{Tr}(\exp\{-t\tfrac{1}{4}D(C_{12})^2\} - \exp\{-tD(S_{\sigma_1})^2\}$$
$$- \exp\{-tD(S_{\sigma_2})^2\}) - h]\,dt. \tag{4.3}$$

Next we split

$$\mathrm{Tr}(\exp\{-tR^2\mathcal{D}_R^2\} - \exp\{-tR^2(\mathcal{D}_{1,R})_{P_1}^2\} - \exp\{-tR^2(\mathcal{D}_{2,R})_{P_2}^2\}) - h$$

into two parts as follows:

$$\mathrm{Tr}_{I,R}(\exp\{-tR^2\mathcal{D}_R^2\} - \exp\{-tR^2(\mathcal{D}_{1,R})_{P_1}^2\} - \exp\{-tR^2(\mathcal{D}_{2,R})_{P_2}^2\}),$$
$$\mathrm{Tr}_{II,R}(\exp\{-tR^2\mathcal{D}_R^2\} - \exp\{-tR^2(\mathcal{D}_{1,R})_{P_1}^2\} - \exp\{-tR^2(\mathcal{D}_{2,R})_{P_2}^2\}),$$

where $\mathrm{Tr}_{I,R}(\cdot)$ (resp. $\mathrm{Tr}_{II,R}(\cdot)$) is the part of the trace restricted to the eigenvalues of $R^2\mathcal{D}_R^2$, $R^2(\mathcal{D}_{1,R})_{P_1}^2$, and $R^2(\mathcal{D}_{2,R})_{P_2}^2$ that are larger than (resp. smaller than or equal to) $R^{1/2}$. The next proposition shows that $\mathrm{Tr}_{I,R}(\cdot)$ can be neglected as $R \to \infty$.

PROPOSITION 4.3. *We have the estimate*

$$\int_{R^{-\varepsilon}}^{\infty} t^{-1}\,\mathrm{Tr}_{I,R}(\exp\{-tR^2\mathcal{D}_R^2\} - \exp\{-tR^2(\mathcal{D}_{1,R})_{P_1}^2\}$$
$$- \exp\{-tR^2(\mathcal{D}_{2,R})_{P_2}^2\})\,dt \le c_1\exp\{-c_2R^{1/2-\varepsilon}\}$$

*for some positive constants* $c_1, c_2$.

*Proof.* Let $\lambda_{k_0}(R)^2$ denote the smallest *large* eigenvalue of $\mathcal{D}_R^2$, that is, the smallest $l$-value with $\lambda_{k_0}(R)^2 > R^{-3/2}$. We now estimate $\mathrm{Tr}_{I,R}(\exp\{-tR^2\mathcal{D}_R^2\})$ as

$$\begin{aligned}
&\mathrm{Tr}_{I,R}(\exp\{-tR^2\mathcal{D}_R^2\})\\
&\quad = \sum_{\lambda_k^2>R^{-3/2}} \exp\{-tR^2\lambda_k^2\} = \sum_{\lambda_k^2>R^{-3/2}} \exp\{-(tR^2-1)\lambda_k^2\}\exp\{-\lambda_k^2\}\\
&\quad \le \exp\{-(tR^2-1)\lambda_{k_0}^2\} \sum_{\lambda_k^2>R^{-3/2}} \exp\{-\lambda_k^2\}\\
&\quad \le \exp\{-(tR^2-1)\lambda_{k_0}^2\}\,\mathrm{Tr}(\exp\{-\mathcal{D}_R^2\})\\
&\quad \le b_1 R\exp\{-(tR^2-1)R^{-3/2}\} \le b_1 R\exp\left\{-b_2 t\sqrt{R}\,\right\}
\end{aligned}$$

for some positive constants $b_1$ and $b_2$. Hence we have

$$\begin{aligned}
&\int_{R^{-\varepsilon}}^{\infty} t^{-1}\,\mathrm{Tr}_{I,R}(\exp\{-tR^2\mathcal{D}_R^2\})\,dt\\
&\qquad \le \int_{R^{-\varepsilon}}^{\infty} t^{-1} b_1 R\exp\left\{-b_2 t\sqrt{R}\,\right\}dt\\
&\qquad \le \frac{b_1}{b_2}R^{1/2+\varepsilon}\int_{b_2 R^{1/2-\varepsilon}}^{\infty} \exp\{-v\}\,dv \le b_3\exp\{-b_4 R^{1/2-\varepsilon}\}.
\end{aligned}$$

The trace $\mathrm{Tr}_{I,R}(\exp\{-tR^2(\mathcal{D}_{i,R})_{P_i}^2\})$ for $i=1,2$ can be estimated in the same way. This completes the proof. $\square$

We also split $\mathrm{Tr}(\exp\{-t\frac{1}{4}D(C_{12})^2\} - \exp\{-tD(S_{\sigma_1})^2\} - \exp\{-tD(S_{\sigma_2})^2\}) - h$ into two parts:

$$\mathrm{Tr}_{I,R}(\exp\{-t\tfrac{1}{4}D(C_{12})^2\} - \exp\{-tD(S_{\sigma_1})^2\} - \exp\{-tD(S_{\sigma_2})^2\}),$$
$$\mathrm{Tr}_{II,R}(\exp\{-t\tfrac{1}{4}D(C_{12})^2\} - \exp\{-tD(S_{\sigma_1})^2\} - \exp\{-tD(S_{\sigma_2})^2\}),$$

where $\mathrm{Tr}_{I,R}(\cdot)$ (resp. $\mathrm{Tr}_{II,R}(\cdot)$) is taken over the nonzero eigenvalues of $\frac{1}{4}D(C_{12})^2$, $D(S_{\sigma_1})^2$, and $D(S_{\sigma_2})^2$ that are larger than (resp. smaller than or equal to) $R^{1/2}$. The following proposition corresponds to Proposition 4.3, and its proof is essentially the same as the proof of that result.

PROPOSITION 4.4. *There exist positive constants $c_1, c_2$ such that*

$$\begin{aligned}
\int_{R^{-\varepsilon}}^{\infty} t^{-1}\,\mathrm{Tr}_{I,R}(\exp\{-t\tfrac{1}{4}D(C_{12})^2\} - \exp\{-tD(S_{\sigma_1})^2\}&\\
- \exp\{-tD(S_{\sigma_2})^2\})\,dt \le c_1\exp\{-c_2 R^{1/2-\varepsilon}\}&.
\end{aligned}$$

By Propositions 4.3 and 4.4, we can see that the equality (4.3) is equivalent to

$$\begin{aligned}
\lim_{R\to\infty}\Bigg(\int_{R^{-\varepsilon}}^{\infty} t^{-1}\,\mathrm{Tr}_{II,R}(\exp\{-tR^2\mathcal{D}_R^2\} - \exp\{-tR^2(\mathcal{D}_{1,R})_{P_1}^2\}&\\
- \exp\{-tR^2(\mathcal{D}_{2,R})_{P_2}^2\})\,dt&\\
- \int_{R^{-\varepsilon}}^{\infty} t^{-1}\,\mathrm{Tr}_{II,R}(\exp\{-t\tfrac{1}{4}D(C_{12})^2\} - \exp\{-tD(S_{\sigma_1})^2\}&\\
- \exp\{-tD(S_{\sigma_2})^2\})\,dt\Bigg) = 0.&
\end{aligned} \tag{4.4}$$

Equation (4.4) is a consequence of the next result.

PROPOSITION 4.5. *For sufficiently large $R$, there exist positive constants $c_1, c_2$, independent of $R$ and $t$, such that*

$$|\mathrm{Tr}_{II,R}(\exp\{-tR^2\mathcal{D}_R^2\}) - \mathrm{Tr}_{II,R}(\exp\{-t\tfrac{1}{4}D(C_{12})^2\})| \le c_1 t R^{-1/4}\exp\{-c_2 t\}$$

*and*

$$|\mathrm{Tr}_{II,R}(\exp\{-tR^2(\mathcal{D}_{i,R})^2_{P_i}\}) - \mathrm{Tr}_{II,R}(\exp\{-tD(S_{\sigma_i})^2\})| \le c_1 t R^{-1/4}\exp\{-c_2 t\}$$

*for any $t > 0$.*

*Proof.* We use the analysis of $s$-values developed in Section 3 and fix $\kappa = \frac{3}{4}$. It follows from Theorem 3.13 that, for any eigenvalue $\lambda(R)$ of $\mathcal{D}_R$ with $|\lambda(R)| \le R^{-3/4}$, there exists an analytic function $\alpha(\lambda)$ such that

$$R\lambda(R) = \lambda_j + \tfrac{1}{4}\lambda(R)\alpha(\lambda(R)) + O(e^{-cR}),$$

where $\lambda_j$ is an eigenvalue of $\frac{1}{2}D(C_{12})$ with $|\lambda_j| \le R^{1/4}$. Hence there exist functions $c(R), d(R)$ and a constant $C > 0$ such that

$$R^2\lambda(R)^2 = \lambda_j^2 + \lambda_j\frac{c(R)}{R^{3/4}} + \frac{d(R)}{R^{3/2}} \quad \text{with } |c(R)| \le C \text{ and } |d(R)| \le C$$

for any sufficiently large $R$. We use the elementary inequality $|e^{-\lambda} - 1| \le |\lambda|e^{|\lambda|}$ to obtain

$$\begin{aligned}
&|\exp\{-tR^2\lambda(R)^2\} - \exp\{-t\lambda_j^2\}| \\
&\quad = |\exp\{-t\lambda_j^2\}(\exp\{-t[R^2\lambda(R)^2 - \lambda_j^2]\} - 1)| \\
&\quad \le \left(\frac{|\lambda_j c(R)|}{R^{3/4}} + \frac{|d(R)|}{R^{3/2}}\right)t\exp\left\{-\left(\lambda_j^2 - \frac{|\lambda_j c(R)|}{R^{3/4}} - \frac{|d(R)|}{R^{3/2}}\right)t\right\} \\
&\quad \le \frac{C}{R^{1/2}}t\exp\{-\tfrac{1}{2}\lambda_j^2 t\}
\end{aligned}$$

for $R \gg 0$. In the last inequality we used the fact that $|\lambda_j| \le R^{1/4}$. Let us fix a sufficiently large $R$. We take the sum over finitely many eigenvalues $\lambda_j^2$ of $\frac{1}{4}D(C_{12})^2$ with $\lambda_j^2 \le R^{1/2}$ and thus obtain

$$\begin{aligned}
&|\mathrm{Tr}_{II,R}(\exp\{-tR^2\mathcal{D}_R^2\}) - \mathrm{Tr}_{II,R}(\exp\{-t\tfrac{1}{4}D(C_{12})^2\})| \\
&\qquad\qquad \le C\frac{t}{R^{1/2}}\sum_{\lambda_j^2 \le R^{1/2}}\exp\{-\tfrac{1}{2}\lambda_j^2 t\}.
\end{aligned}$$

The operator $\frac{1}{4}D(C_{12})^2$ is a Laplace-type operator over $S^1$, so the number of eigenvalues $\lambda_j^2$ of $\frac{1}{4}D(C_{12})^2$ with $\lambda_j^2 \le R^{1/2}$ can be estimated by $R^{1/4}$. Therefore,

$$C\frac{t}{R^{1/2}}\sum_{\lambda_j^2 \le R^{1/2}}\exp\{-\tfrac{1}{2}\lambda_j^2 t\} \le c_1\frac{t}{R^{1/2}}R^{1/4}\exp\{-\tfrac{1}{2}\lambda_1^2 t\},$$

where $\lambda_1^2$ denotes the first nonzero eigenvalue of $\frac{1}{4}D(C_{12})^2$. Note that $c_1$ and $\lambda_1^2$ are independent of $R$. This proves the first claim once we put $c_2 = \frac{1}{2}\lambda_1^2$; the proof of the second claim goes in the same way. □

The proof of Proposition 4.1 is now complete.

*Proof of Theorem 1.1.* Now Proposition 2.5 and Proposition 4.1 give us the equality

$$\begin{aligned}
&\lim_{R\to\infty}\big((\zeta_s^R)'(0) + h(\gamma + (2-\varepsilon)\cdot\log R) + (\zeta_l^R)'(0) - h(\gamma - \varepsilon\cdot\log R)\big)\\
&\quad = \frac{d}{ds}\bigg|_{s=0}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}[\mathrm{Tr}(\exp\{-t\tfrac{1}{4}D(C_{12})^2\} - \exp\{-tD(S_{\sigma_1})^2\}\\
&\qquad\qquad - \exp\{-tD(S_{\sigma_2})^2\}) - h]\,dt + \zeta_{B^2}(0)\cdot\log 2.
\end{aligned}$$

By an elementary computation (see e.g. [15, Prop. 2.2]) we can derive

$$\begin{aligned}
\det{}_\zeta\, \frac{1}{4}D(C_{12})^2 &= 2^{h_Y+2h_M}\det{}^*\left(\frac{2\,\mathrm{Id} - C_{12} - C_{12}^{-1}}{4}\right),\\
\det{}_\zeta\, D(S_{\sigma_i})^2 &= 2^{h_Y}\det{}^*\left(\frac{2\,\mathrm{Id} - S_{\sigma_i} - S_{\sigma_i}^{-1}}{4}\right).
\end{aligned} \tag{4.5}$$

Combining these three equalities provides us with the final formula (1.10) in Theorem 1.1.

## 5. A Proof of the Decomposition Formula of the $\eta$-Invariant

In this section we offer a new proof of the decomposition formula for the $\eta$-invariant. This formula has been proved by several authors (see [3; 4; 9; 12; 13; 16; 18; 23]), and the proof we discuss in this section is not the simplest one. Still we believe that it is worthwhile to present this "scattering" approach to the decomposition of the $\eta$-invariant. The key in our proof is to show that the scattering data provide us with the contribution given by the boundary conditions in the decomposition formula for the $\eta$-invariant.

We remind the reader that the $\eta$-function of a Dirac operator $\mathcal{D}$ on a closed manifold $M$, introduced in [1], is defined as

$$\eta_{\mathcal{D}}(s) = \sum_{\lambda_k\neq 0}\mathrm{sign}(\lambda_k)|\lambda_k|^{-s},$$

where the sum is taken over all nonzero eigenvalues of $\mathcal{D}$. The $\eta$-function is well-defined for $\Re(s)$ large; it has a meromorphic extension to the whole complex plane and $s=0$ is a regular point, so $\eta_{\mathcal{D}}(0)$ is well-defined. Following [1], we introduce the $\eta$-invariant of $\mathcal{D}$ as

$$\eta(\mathcal{D}) = \tfrac{1}{2}\cdot(\eta_{\mathcal{D}}(0) + \dim\ker\mathcal{D}). \tag{5.1}$$

Now, let us assume that we have a decomposition of a closed odd-dimensional manifold $M$ to $M_1\cup M_2$ as described in Section 1. For $\mathcal{D}_i := \mathcal{D}|_{M_i}$, we impose the boundary conditions given by the generalized APS spectral projections $P_i$ defined in (1.6). Then the $\eta$-function of $(\mathcal{D}_i)_{P_i}$ is also well-defined and has the same properties as the $\eta$-function of the Dirac operator on a closed manifold; in particular, the $\eta$-function of $(\mathcal{D}_i)_{P_i}$ is regular at $s = 0$. Hence, we can define the $\eta$-invariant of $(\mathcal{D}_i)_{P_i}$ as in (5.1). The following result has been proved by several authors.

THEOREM 5.1. *We have*

$$\eta(\mathcal{D}) = \eta((\mathcal{D}_1)_{P_1}) + \eta((\mathcal{D}_2)_{P_2}) + \eta(\mathcal{D}; \sigma_1, \sigma_2) \mod \mathbb{Z}; \tag{5.2}$$

*here $\eta(\mathcal{D}; \sigma_1, \sigma_2)$ denotes the $\eta$-invariant of the operator $\mathcal{D} = G(\partial_u + B)$ over $N \cong [-1, 1] \times Y$, subject to the boundary condition $P_2$ at $u = -1$ and $P_1$ at $u = 1$.*

For the involution $\sigma_i$ that defines $P_i$ in (1.6), observe that

$$U = \sigma_1\sigma_2 \colon \ker B \to \ker B$$

is the unitary operator such that $UG = GU$, $\det U = 1$, and $U^* = \sigma_1 U \sigma_1$. It follows that the spectrum of $U$ is invariant under complex conjugation. Moreover, the maps $U_\pm = U|_{(\ker B)_\pm} \colon (\ker B)_\pm \to (\ker B)_\pm$ are well-defined. The following result, proved in [14, Sec. 2], was the key ingredient in the proof of Theorem 5.1.

PROPOSITION 5.2. *We have*

$$\eta(\mathcal{D}; \sigma_1, \sigma_2) = -\frac{1}{2\pi i} \log\det(-U_+) \mod \mathbb{Z}. \tag{5.3}$$

One way to prove the decomposition formula (5.2) is to use the adiabatic analysis developed in the proof of Theorem 1.1. That analysis easily gives us the following theorem.

THEOREM 5.3. *The following formula for the $\eta$-invariant holds*:

$$\begin{aligned}\eta(\mathcal{D}) - \eta((\mathcal{D}_1)_{P_1}) - \eta((\mathcal{D}_2)_{P_2})& \\ = \eta(D(C_{12})) - \eta(D(S_{\sigma_1})) - \eta(D(S_{\sigma_2})) \mod \mathbb{Z}.&\end{aligned} \tag{5.4}$$

*Proof.* We repeat the corresponding argument used to derive Theorem 1.1 for the $\eta$-invariant in order to obtain the expected formula

$$\begin{aligned}\lim_{R\to\infty} \{\eta(\mathcal{D}_R) - \eta((\mathcal{D}_{1,R})_{P_1}) - \eta((\mathcal{D}_{2,R})_{P_2})\}& \\ = \eta(D(C_{12})) - \eta(D(S_{\sigma_1})) - \eta(D(S_{\sigma_2})) \mod \mathbb{Z}.&\end{aligned}$$

We can now use that $\eta(\mathcal{D}_R)$ and $\eta((\mathcal{D}_{i,R})_{P_i})$ are independent of $R$ modulo the integers (see [17, Prop. 2.16]) to complete the proof. □

Now we need to show that

$$\eta(\mathcal{D}; \sigma_1, \sigma_2) = \eta(D(C_{12})) - \eta(D(S_{\sigma_1})) - \eta(D(S_{\sigma_2})) \mod \mathbb{Z}.$$

For this, note that the scattering matrix $C_i = C_i(0)$ can be represented as

$$C_i = \begin{pmatrix} 0 & C(i)_- \\ C(i)_+ & 0 \end{pmatrix}, \quad \text{where } C(i)_\pm C(i)_\mp = \mathrm{Id},$$

with respect to the decomposition $\ker B = (\ker B)_+ \oplus (\ker B)_-$. We see that

$$C_{12} = C(1)_+ C(2)_- : (\ker B)_- \to (\ker B)_-. \tag{5.5}$$

Similar formulas hold for the involutions $\sigma_i$, and we have

$$S_{\sigma_1} = -P_{\sigma_1} \circ C_1 \circ I_{\sigma_1} = -\frac{1}{2}\begin{pmatrix} \mathrm{Id} & -\sigma(1)_- \\ -\sigma(1)_+ & \mathrm{Id} \end{pmatrix}\begin{pmatrix} 0 & C(1)_- \\ C(1)_+ & 0 \end{pmatrix}\begin{pmatrix} 2\,\mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \sigma(1)_- C(1)_+ & 0 \\ -C(1)_+ & 0 \end{pmatrix}.$$

We can also see that each element of $\ker(\sigma_1+1)$ is represented in the form $\begin{pmatrix} f \\ -\sigma(1)_+ f \end{pmatrix}$ for some $f \in (\ker B)_+$. This allows us to represent the map $S_{\sigma_1}$ over $\ker(\sigma_1+1)$ as

$$S_{\sigma_1}\begin{pmatrix} f \\ -\sigma(1)_+ f \end{pmatrix} = \begin{pmatrix} \sigma(1)_- C(1)_+ & 0 \\ -C(1)_+ & 0 \end{pmatrix}\begin{pmatrix} f \\ -\sigma(1)_+ f \end{pmatrix}$$
$$= \begin{pmatrix} \sigma(1)_- C(1)_+ f \\ -\sigma(1)_+ \sigma(1)_- C(1)_+ f \end{pmatrix}.$$

Therefore, from the spectral point of view, the operator $S_{\sigma_1}$ is equal to the operator

$$\sigma(1)_- C(1)_+ : (\ker B)_+ \to (\ker B)_+$$

or (equivalently) to the operator

$$C(1)_+ \sigma(1)_- : (\ker B)_- \to (\ker B)_-.$$

The corresponding analysis for the operator $S_{\sigma_2}$ implies that $S_{\sigma_2}$ is equivalent to

$$\sigma(2)_+ C(2)_- : (\ker B)_- \to (\ker B)_- \quad \text{or}$$
$$C(2)_- \sigma(2)_+ : (\ker B)_+ \to (\ker B)_+.$$

Combining these with (5.5) yields

$$\frac{\det(C_{12})}{\det(S_{\sigma_1})\det(S_{\sigma_2})} = \det(\sigma(1)_+ \sigma(2)_-). \tag{5.6}$$

For the operator $D(C)$ on $S^1$ defined by a unitary map $C$ in (3.34),

$$\eta(D(C)) = -\frac{1}{2\pi i}\log\det(-\bar{C}) \mod \mathbb{Z} \tag{5.7}$$

(see Theorem 2.1 and Lemma 2.3 in [14]). If we combine (5.6) and (5.7) then

$$\eta(D(C_{12})) - \eta(D(S_{\sigma_1})) - \eta(D(S_{\sigma_2})) = -\frac{1}{2\pi i}\log\det(-\overline{\sigma(1)_+ \sigma(2)_-}) \mod \mathbb{Z}.$$

Noting that $\det(-\overline{\sigma(1)_+ \sigma(2)_-}) = \det(-\sigma(1)_- \sigma(2)_+)$, the foregoing and Proposition 5.2 yield a proof of the following theorem.

THEOREM 5.4.

$$\eta(\mathcal{D}; \sigma_1, \sigma_2) = \eta(D(C_{12})) - \eta(D(S_{\sigma_1})) - \eta(D(S_{\sigma_2})) \mod \mathbb{Z}. \tag{5.8}$$

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