Michigan Math. J. 54 (2006)

Domains in Almost Complex Manifolds with an Automorphism Orbit Accumulating at a Strongly Pseudoconvex Boundary Point

Kang-Hyurk Lee

1. Introduction

Let (M, J) be an almost complex manifold and let Ω be a domain in M. Call $p \in \partial\Omega$ a *strongly J-pseudoconvex boundary point* if there is a C^2 local defining function whose Levi form is positive definite for the *J-complex tangent vector space* $T_p^J\partial\Omega = T_p\partial\Omega \cap JT_p\partial\Omega$ of $\partial\Omega$ at p. For $p \in \Omega$ and a sequence $\varphi^\nu \in \operatorname{Aut}(\Omega, J)$, call the sequence $\{\varphi^\nu(p) : \nu = 1, 2, \dots\}$ an *automorphism orbit* of Ω. This paper pertains to the following problem.

> *Classify the domains Ω in an almost complex manifold (M, J) that admit an automorphism orbit accumulating at a strongly J-pseudoconvex boundary point.*

In the complex case, the Wong–Rosay theorem states that such domains are biholomorphically equivalent to the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$ (see [3; 5; 10; 19; 22]). For the real 4-dimensional almost complex case, Gaussier and Sukhov [7] have shown that under a certain restriction such (Ω, J) is biholomorphic to the unit ball $\mathbb{B}_2$ in $\mathbb{C}^2$. But when $\dim M \geq 6$ it turns out that there are infinitely many biholomorphically distinct domains, as the following example shows.

Example 1.1. Let $z_j = x_j + iy_j$ be the standard coordinate functions of $\mathbb{C}^3 \simeq \mathbb{R}^6$. Set $z' = (z_2, z_3)$ and $z = (z_1, z')$. Let $\rho_t(z) = \operatorname{Re} z_1 + t|z'|^2$ and let

$$J_t(x) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & tx_2 \\ 1 & 0 & 0 & 0 & tx_2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

for $t \in \mathbb{R}$. Consider the domain $\mathbb{H}_t = \{z \in \mathbb{C}^3 : \rho_t(z) < 0\}$ equipped with the almost complex structure J_1. It turns out that $(\mathbb{H}_t, J_1)$ with $t > 1/8$ has automorphisms $\Lambda_k(z) = \left(z_1/k, z'/\sqrt{k}\right)$, which induces an orbit accumulating at 0 that is

Received November 17, 2004. Revision received March 15, 2005.
Research supported in part by Grant no. KRF-2002-070-C00005 (PI: K. T. Kim) from the Korea Research Foundation.

strongly J_1-pseudoconvex. We show in this paper that $(\mathbb{H}_t, J_1)$ and $(\mathbb{H}_s, J_1)$ are biholomorphically distinct whenever $t \neq s$.

In fact, our main theorem is that these manifolds constitute the complete list for $n = 3$. More precisely, we have the following result.

THEOREM 1.2. *Let (M^{2n}, J) be an almost complex manifold equipped with the almost complex structure J of Hölder class $C^{1,\alpha}$. Suppose that a domain Ω in M has a strongly J-pseudoconvex boundary point $q_0 \in \partial\Omega$ admitting a sequence $\varphi^\nu \in \mathrm{Aut}(\Omega, J)$ such that $\varphi^\nu(p_0) \to q_0$ as $\nu \to \infty$ for some $p_0 \in \Omega$. Then (Ω, J) is biholomorphic to one of the models $(\hat{\Omega}, \hat{J})$ in Definition 4.7. Moreover, (Ω, J) is biholomorphic to $(\mathbb{B}_2, J_{st})$ when $n = 2$, and (Ω, J) is biholomorphic to one of $(\mathbb{H}_1, J_t)$ for $0 \le t < 8$ when $n = 3$.*

We use the scaling technique in Section 4 to show that such a (Ω, J) is biholomorphic to some *model domain* $(\hat{\Omega}, \hat{J})$ (see Theorem 4.6) after introducing the basic terminology and presenting some preparations for the scaling method in Sections 2 and 3. We then simplify the model structure $\hat{J}$ (Section 5) and classify the models in the case of real dimension 6 (Sections 6 and 7).

At the time of this writing, we were informed that Gaussier and Sukhov have obtained a similar result independently. We also have results in all dimensions. However, identifying the moduli of all such domains in terms of geometric-analytic invariants remains difficult when $n \geq 4$.

ACKNOWLEDGMENT. This work is part of the author's dissertation for his doctoral degree at the Pohang University of Science and Technology. He would like to express his gratitude to his advisor Kang-Tae Kim for guidance. He also would like to thank the referee for valuable comments.

2. Preliminaries

A pair (M, J) is called an *almost complex manifold* if M is a C^∞-smooth real manifold and J is a field of endomorphisms of the tangent bundle TM satisfying $J^2 = -\mathrm{Id}$. We call J an *almost complex structure* on M.

The canonical example of the almost complex manifold is the complex Euclidean space $\mathbb{C}^n$ with the standard complex structure $J_{st}^{(n)}$ (or simply J_{st} when there is no danger of confusion), which is given by $J_{st}^{(n)}(\partial/\partial x_j) = \partial/\partial y_j$ for $j = 1, \ldots, n$. An almost complex manifold (M^{2n}, J) is said to be *integrable* if J is induced from the standard complex structure $J_{st}^{(n)}$ of $\mathbb{C}^n$ in a local coordinate system about p for each point $p \in M$. The Newlander–Nirenberg theorem [16] says that an almost complex manifold (M, J) is integrable if and only if N_J is vanishing on M, where the *Nijenhuis tensor* N_J of J is defined by

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$$

for all $X, Y \in TM$ with the same base point.

2.1. Pseudoholomorphic Mappings between Almost Complex Manifolds

Given two almost complex manifolds (M, J) and $(\tilde{M}, \tilde{J})$, a mapping f from M to $\tilde{M}$ of class C^1 is said to be $(J, \tilde{J})$*-holomorphic* (or simply *pseudoholomorphic*) if its differential $df: TM \to T\tilde{M}$ satisfies the condition

$$\tilde{J} \circ df = df \circ J$$

on TM. We denote by $\mathcal{O}_{(J,\tilde{J})}(M, \tilde{M})$ the space of $(J, \tilde{J})$-holomorphic mappings from M to $\tilde{M}$. For the standard r-disc $\mathbf{D}_r = \{z \in \mathbb{C} : |z| < r\}$ (simply $\mathbf{D}_1 = \mathbf{D}$), an element of $\mathcal{O}_{(J_{st},J)}(\mathbf{D}_r, M)$ is called a pseudoholomorphic disc in M.

A bijective mapping $f: (M, J) \to (\tilde{M}, \tilde{J})$ is called a *biholomorphism* if $f \in \mathcal{O}_{(J,\tilde{J})}(M, \tilde{M})$ and $f^{-1} \in \mathcal{O}_{(\tilde{J},J)}(\tilde{M}, M)$. For the case $(M, J) = (\tilde{M}, \tilde{J})$, we call f an *automorphism* of (M, J). We denote by $\mathrm{Aut}(M, J)$ the set of all automorphisms of (M, J).

Sikorav [21, Prop. 2.3.6] gave an estimate for pseudoholomorphic discs in a small neighborhood of a given point. His theorem gives rise to the following proposition (see [15]).

Proposition 2.1. *Let J be a $C^{1,\alpha}$ almost complex structure of $\mathbb{R}^{2n}$ and let $\tilde{J}$ be a C^1 almost complex structure of $\mathbb{R}^{2m}$. Then there is a bounded neighborhood U of* 0 *in $\mathbb{R}^{2m}$ with the following property: For a given domain Ω in $\mathbb{R}^{2n}$ and its compact subset K, there exists a positive constant C such that*

$$\|f\|_{C^1(K)} \le C\|f\|_{C^0(\Omega)}$$

whenever $f: \Omega \to U$ is a $(J, \tilde{J})$-holomorphic mapping. Moreover, this estimate holds for sufficiently small C^1 perturbations of J and $\tilde{J}$.

Let J and $\tilde{J}$ be almost complex structures of class C^1 on $\mathbb{R}^{2n}$ and $\mathbb{R}^{2m}$, respectively. Regard J and $\tilde{J}$ as matrix-valued functions expressed by $J = (J_k^j)$ and $\tilde{J} = (\tilde{J}_\mu^\lambda)$. In this section, we use $x = (x_1, x_2, \ldots, x_{2n})$ as the standard real coordinate in $\mathbb{R}^{2n}$.

For a bounded domain Ω in $\mathbb{R}^{2n}$, let $f = (f_1, f_2, \ldots, f_{2m}): \Omega \to \mathbb{R}^{2m}$ be a pseudoholomorphic mapping of class $C^1(\overline{\Omega})$. By [15, Sec. 2], each f_λ satisfies the partial differential equation

$$\mathsf{H}^J f_\lambda = \mathsf{C}(J, \tilde{J}; f)_\lambda \tag{2.1}$$

in the weak sense, where H^J is the linear partial differential operator expressed by

$$\mathsf{H}^J = \sum_{j=1}^{2n} \frac{\partial^2}{\partial x_j \partial x_j} + \sum_{j,k,l=1}^{2n} J_j^k J_j^l \frac{\partial^2}{\partial x_k \partial x_l}$$

and $\mathsf{C}(J, \tilde{J}; f)_\lambda$ is defined by

$$\mathsf{C}(J,\tilde{J};f)_\lambda = -\sum_{j,k=1}^{2n}\sum_{\mu=1}^{2m}\frac{\partial f_\mu}{\partial x_k}\frac{\partial}{\partial x_j}(J_j^k\tilde{J}_\mu^\lambda(f))$$
$$+\sum_{j,k,l=1}^{2n}\sum_{\mu,\nu=1}^{2m}J_j^k\tilde{J}_\mu^\lambda(f)\frac{\partial f_\nu}{\partial x_l}\frac{\partial}{\partial x_k}(J_j^l\tilde{J}_\nu^\mu(f)).$$

The coefficients of H^J have the same regularity with J. The symbol of H^J is $\sum_j \zeta_j^2 + \sum_{j,k,l}\zeta_k J_j^k J_j^l \zeta_l = |\zeta|^2 + |J\zeta|^2$, so H^J is strictly elliptic on Ω.

Let $p > 2n$. By the elliptic regularity theorem, the function f_λ is in $W^{2,p}_{\mathrm{loc}}(\Omega)$ and in the strong solution of (2.1) for each λ.

LEMMA 2.2. *Let $\{J^\nu\}$ and $\{\tilde{J}^\nu\}$ be sequences of $C^{1,\alpha}$ almost complex structures on $\mathbb{R}^{2n}$ and $\mathbb{R}^{2m}$, respectively. Suppose that $\|J^\nu - J\|_{C^1(\bar{\Omega})} \to 0$ for a bounded domain Ω in $\mathbb{R}^{2n}$ and that $\|\tilde{J}^\nu - \tilde{J}\|_{C^1(K)} \to 0$ for any compact subset K of $\mathbb{R}^{2m}$. If a sequence $\{f^\nu \in \mathcal{O}_{(J^\nu,\tilde{J}^\nu)}(\Omega,\mathbb{R}^{2m}) : \nu = 1,2,\dots\}$ converges to f in the compact-open topology, then f is $(J,\tilde{J})$-holomorphic.*

Proof. Because this problem is local, we shall prove the lemma on a relatively compact neighborhood Ω' of a given point in Ω whose boundary is of class C^∞. For $0 < \beta < 1 - 2n/p$, the Sobolev space $W^{2,p}(\Omega')$ is compactly embedded in $C^{1,\beta}(\overline{\Omega'})$ (see [8, Thm. 7.26]). Since $f^\nu \in W^{2,p}(\Omega')$, it suffices to show that $\|f^\nu\|_{W^{2,p}(\Omega')}$ is uniformly bounded. Then f^ν has a subsequence converging to f in $C^{1,\beta}(\overline{\Omega'})$; hence the limiting of the equation $\tilde{J}^\nu \circ df^\nu = df^\nu \circ J^\nu$ shows that f is $(J,\tilde{J})$-holomorphic on Ω'.

The C^1-convergence of J^ν implies that the coefficients of H^{J^ν} converge to those of H^J in $C^1(\Omega)$. Let U be a relatively compact neighborhood of Ω' in Ω. By the L^p estimates of an elliptic equation [8, Thm. 9.11], there exists a constant C such that

$$\|f_\lambda^\nu\|_{W^{2,p}(\Omega')} \le C(\|f_\lambda^\nu\|_{L^p(U)} + \|\mathsf{C}(J^\nu,\tilde{J}^\nu;f^\nu)_\lambda\|_{L^p(U)})$$

for sufficiently large ν and for any λ. We know that $\|f^\nu\|_{L^p(U)}$ is uniformly bounded. Applying Proposition 2.1, one obtains that the gradient of f^ν is locally bounded on Ω and uniformly bounded on $\bar{U}$. Since $\tilde{J}^\nu \to \tilde{J}$ in the C^1 sense, it follows that $\|\mathsf{C}(J^\nu,\tilde{J}^\nu;f^\nu)_\lambda\|_{C^0(U)}$ is uniformly bounded. We thus have that $\|f^\nu\|_{W^{2,p}(\Omega')}$ is uniformly bounded, which proves the lemma. □

Consider the pseudoholomorphic disc $u\colon (\mathbf{D}, J_{st}) \to (\mathbb{R}^{2m}, J)$. Since the operator $\frac{1}{2}\mathsf{H}^J_{st}$ is the same as the standard Laplacian Δ, equation (2.1) can be written as

$$\Delta u_\lambda = \frac{1}{2}\mathsf{C}(J_{st},J;u)_\lambda, \tag{2.2}$$

where

$$\frac{1}{2}\mathsf{C}(J_{st},J;u)_\lambda = \sum_{\mu=1}^{2m}\frac{\partial u_\mu}{\partial x_1}\frac{\partial}{\partial x_2}J_\mu^\lambda(u) - \sum_{\mu=1}^{2m}\frac{\partial u_\mu}{\partial x_2}\frac{\partial}{\partial x_1}J_\mu^\lambda(u). \tag{2.3}$$

2.2. Kobayashi–Royden Pseudometric

Let (M, J) be an almost complex manifold and let J be of class $C^{1,\alpha}$. By the existence theorem of pseudoholomorphic discs (see [17]), we can define the *Kobayashi–Royden pseudometric* $F_{(M,J)}$ that is the same as the one for the integrable case (Royden [20]) as

$$F_{(M,J)}(p, v) = \inf\left\{\frac{1}{|a|} : u \in \mathcal{O}_{(J_{st},J)}(\mathbf{D}, M) \text{ with } u(0) = p,\ du(\mathbf{e}) = av\right\},$$

where $\mathbf{e}$ is the unit vector in $T_0\mathbf{D}$ and where $p \in M$ and $v \in T_pM$. Because $F_{(M,J)}$ is upper semicontinuous on TM (see [9]), the *Kobayashi pseudodistance* $d_{(M,J)}$ may be defined as

$$d_{(M,J)}(p, q) = \inf \int_0^1 F_{(M,J)}(\gamma(t), \gamma'(t))\, dt,$$

where the infimum is taken over all piecewise smooth paths $\gamma : [0,1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$. Since $F_{(M,J)}$ is locally bounded on TM, its integrated pseudodistance $d_{(M,J)}$ is continuous on $M \times M$. As in the integrable case (see [12; 20]), this metric and distance have the usual distance-decreasing property for pseudoholomorphic mappings.

We say that (M, J) is (*Kobayashi*) *hyperbolic* if the Kobayashi pseudodistance $d_{(M,J)}$ is a proper distance. When the Kobayashi ball $B^K_{(M,J)}(p,r) = \{q \in M : d_{(M,J)}(p,q) < r\}$ is always relatively compact in M for any $p \in M$ and any $r > 0$, we call (M, J) *complete hyperbolic*. We present a normal family theorem for the complete hyperbolic almost complex manifolds (cf. [13, Cor. 5.1.2]).

PROPOSITION 2.3. *Suppose that a manifold M admits a sequence J^ν of $C^{1,\alpha}$ almost complex structures that converges to J in the C^1 sense on any compact subset of M. Let $(\tilde{M}, \tilde{J})$ be a complete hyperbolic almost complex manifold. Then a sequence $\{f^\nu : f^\nu \in \mathcal{O}_{(J^\nu,\tilde{J})}(M, \tilde{M})\}$ has a subsequence converging to an element of $\mathcal{O}_{(J,\tilde{J})}(M, \tilde{M})$ whenever $\{f^\nu(p_0)\}$ is relatively compact in $\tilde{M}$ for some $p_0 \in M$.*

Proof. Let us assume that $f^\nu(p_0)$ converges to $q_0 \in \tilde{M}$. It suffices to show that f^ν has a convergent subsequence on any compact subset K of M containing p_0. Let V be a relatively compact neighborhood of K in M and let h be a Hermitian metric on V that is smooth up to $\bar{V}$. We denote by d_h the distance function on V induced by h and let $B_h(p,r) = \{q \in V : d_h(p,q) < r\}$. By Lemma 2.4 in [4], there exists a positive constant C such that

$$F_{(M,J^\nu)}(p, v) \le C\|v\|_h$$

for any $p \in V$ and any $v \in T_pM$ and for sufficiently large ν. Hence we have $d_{(M,J^\nu)}(p,q) \le C d_h(p,q)$ for any p and q in V, so that

$$B_h(p,r) \subset B^K_{(M,J^\nu)}(p, Cr)$$

for any r. For given $p \in V$ and $\varepsilon > 0$, any point $q \in B_h(p, \varepsilon/C)$ satisfies $d_{(\tilde{M},\tilde{J})}(f^\nu(p), f^\nu(q)) \leq \varepsilon$; this implies that $\{f^\nu\}$ is equicontinuous on V. Choose a positive constant R with $K \subset B_h(p_0, R)$. Then, by the distance-decreasing property of the Kobayashi pseudodistance, we conclude that

$$f^\nu(K) \subset f^\nu(B_h(p_0, R)) \subset f^\nu(B^K_{(M,J^\nu)}(p_0, CR)) \subset B^K_{(\tilde{M},\tilde{J})}(q_0, 2CR)$$

for sufficiently large ν. From the complete hyperbolicity of $(\tilde{M}, \tilde{J})$, it follows that $B^K_{(\tilde{M},\tilde{J})}(q_0, 2CR) \subset\subset \tilde{M}$. Hence, by the Arzela–Ascoli theorem there is a convergent subsequence in the compact-open topology. By Lemma 2.2, this proves the proposition. $\square$

2.3. *J-Pseudoconvexity and J-Plurisubharmonic Functions*

For an almost complex manifold (M, J), let $\rho : M \to \mathbb{R}$ be an upper semicontinuous function. Call ρ *J-plurisubharmonic* when, for any $u \in \mathcal{O}_{(J_{st},J)}(\mathbf{D}, M)$, the composition $\rho \circ u$ is always subharmonic. For any ρ of class C^2, one can determine the J-plurisubharmonicity of ρ by the Levi form.

For any 1-form ω on M, $J^*\omega$ is defined by $J^*\omega(v) = \omega(Jv)$. The *Levi form* of ρ at $p \in M$ is defined by

$$\mathcal{L}^J_p \rho(v) = -d(J^* d\rho)(v, Jv)$$

for $v \in T_pM$. For the case $\rho \in C^2$, it is known that ρ is J-plurisubharmonic on M if and only if $\mathcal{L}^J_p \rho(v)$ is nonnegative for any $p \in M$ and any $v \in T_pM$. When the Levi form is positive definite, ρ is said to be *strictly J-plurisubharmonic.*

Suppose that Ω is strongly J-pseudoconvex at $p \in \Omega$ with a defining function ρ on a neighborhood U of p. Then there exist a positive constant A and a small neighborhood V of p in U such that $\rho + A\rho^2$ is strictly J-plurisubharmonic on V and $\Omega \cap V = \{\rho + A\rho^2 < 0\}$. Therefore Ω has a local, strictly J-plurisubharmonic defining function.

3. Boundary Behavior of Pseudoholomorphic Discs

In this section, we investigate the behavior of the pseudoholomorphic discs whose origins are sufficiently close to the strongly J-pseudoconvex boundary point. Ivashkovich and Rosay have given a localization lemma for pseudoholomorphic discs as follows.

LEMMA 3.1 [9, Lemma 2.2]. *Let (M, J) be an almost complex manifold with $J \in C^1$, and let Ω be a domain in M with a strongly J-pseudoconvex boundary point $q_0 \in \partial\Omega$. For every $r_0 \in [0, 1)$ there exist positive constants C_0 and δ_0 such that, for every pseudoholomorphic disc $u \in \mathcal{O}_{(J_{st},J)}(\mathbf{D}, \Omega)$ with* $\operatorname{dist}(u(0), q_0) < \delta_0$,

$$\operatorname{dist}(u(0), u(\zeta)) \leq C_0 \sqrt{\operatorname{dist}(u(0), \partial\Omega)}$$

if $|\zeta| < r_0$, where dist *is the distance induced by a Riemannian metric of M.*

For the scaling technique of Section 4, we need more information about pseudo-holomorphic discs in a perturbed situation.

Let U be a bounded neighborhood of 0 in $\mathbb{R}^{2n}$. We consider the following situation.

(1) There is a sequence $\{J^\nu\}_{\nu=1,2,\dots,\infty}$ of C^1 almost complex structures on $\mathbb{R}^{2n}$ such that $\|J^\nu - J^\infty\|_{C^1(\bar{U})} \to 0$ as $\nu \to \infty$. Moreover, we have

$$J^\infty(0) = J_{st} \quad \text{and} \quad J^\nu(0) = \begin{pmatrix} J^\nu_{(1,1)} & 0 \\ J^\nu_{(2,1)} & J^\nu_{(2,2)} \end{pmatrix}, \tag{3.1}$$

where $J^\nu_{(1,1)}$ and $J^\nu_{(2,2)}$ are 2×2 and $(2n-2) \times (2n-2)$ matrices, respectively. When $J^\nu(z) = J^\nu(0) + E^\nu(z)$, there is an $A_1 > 0$ such that $|E^\nu(z)| < A_1|z|$ for small z and for any $\nu = 1, 2, \dots, \infty$.

(2) Let $\{\rho^\nu\}_{\nu=1,2,\dots,\infty}$ be a sequence of C^2 strictly J^ν-plurisubharmonic functions defined on a neighborhood of U such that $\|\rho^\nu - \rho^\infty\|_{C^2(\bar{U})} \to 0$ as $\nu \to \infty$. Furthermore, $\rho^\nu(z) = \operatorname{Re} z_1 + O(|z|^2)$ uniformly for $\nu = 1, 2, \dots, \infty$, where $z = (z_1, \dots, z_n)$ is a standard coordinate of $\mathbb{C}^n$. This means that $|\rho^\nu(z) - \operatorname{Re} z_1| < A_2|z|^2$ for small z. Let Ω^ν be a domain in $\mathbb{R}^{2n}$ for each $\nu = 1, 2, \dots, \infty$ with $\Omega^\nu \cap U = \{z \in U : \rho^\nu(z) < 0\}$.

(3) For a fixed $0 < r_0 < 1$, there are positive constants C_0 and δ_0 such that

$$\operatorname{dist}(u(0), u(\zeta)) \le C_0 \sqrt{\operatorname{dist}(u(0), \partial\Omega^\nu)}$$

for any $|\zeta| \le r_0$ and for any $u \in \mathcal{O}_{(J_{st}, J^\nu)}(\mathbf{D}, \Omega^\nu)$ with $|u(0)| < \delta_0$.

Define $Q(0, \delta) = \left\{(z_1, z') \in \mathbb{C} \times \mathbb{C}^{n-1} : |z_1| \le \delta,\, |z'| \le \sqrt{\delta}\right\}$. Then we have the following result (see [7, Lemma 5]).

Proposition 3.2. *Let $0 < r < r_0$. Then there are positive constants C_r and δ_r such that, if $u \in \mathcal{O}_{(J_{st}, J^\nu)}(\mathbf{D}, \Omega^\nu)$ and $0 < \delta < \delta_r$, then*

$$u(0) \in Q(0, \delta) \implies u(\mathbf{D}_r) \subset Q(0, C_r\delta)$$

for sufficiently large ν containing ∞.

Observe that if $w \in Q(0, \delta)$ for a sufficiently small $\delta < 1$, then $|w| \le \sqrt{2\delta}$ and $\operatorname{dist}(w, \partial\Omega^\nu) < L\delta$ for large ν. We thus have that if $u \in \mathcal{O}_{(J_{st}, J^\nu)}(\mathbf{D}, \Omega^\nu)$ with $u(0) \in Q(0, \delta)$ then

$$\begin{aligned} |u(\zeta)| &\le |u(0)| + |u(0) - u(\zeta)| \\ &\le |u(0)| + C\sqrt{\operatorname{dist}(u(0), \partial\Omega^\nu)} \\ &\le \sqrt{2\delta} + C_0\sqrt{L\delta} \quad \left(\text{let} = C_1\sqrt{\delta}\right) \end{aligned} \tag{3.2}$$

for $|\zeta| \le r_0$. This suggests that we need to study u_1, denoting $u = (u_1, \dots, u_n)$ as the standard complex coordinate of $\mathbb{C}^n$.

We first look at $\operatorname{Re} u_1$.

LEMMA 3.3. *Suppose that $|z|^2$ is strictly J^ν-plurisubharmonic on U for any ν. Then there are positive constants C_r' and δ_r' such that the following statement holds for sufficiently large ν: If a pseudoholomorphic disc $u \in \mathcal{O}_{(J_{st}, J^\nu)}(\mathbf{D}, \Omega^\nu)$ satisfies $u(0) \in Q(0,\delta)$ with $\delta < \delta_r'$, then*

$$\operatorname{Re} u_1(\zeta) > -C_r'\delta$$

for any $|\zeta| < r$.

Proof. Since $\|\rho^\nu - \rho^\infty\|_{C^2(\bar{U})} \to 0$, we may assume that $|z|^2 - \varepsilon\rho^\nu(z)$ is J^ν-plurisubharmonic on U for some positive ε. For any $u \in \mathcal{O}_{(J_{st}, J^\nu)}(\mathbf{D}, \Omega^\nu)$ whose origin is sufficiently close to 0, it follows that $u(\bar{\mathbf{D}}_{r_0}) \subset U$ and that $|u|^2 - \varepsilon\rho^\nu \circ u$ is a positive-valued subharmonic function. Applying the Poisson integral formula yields a constant C_2 such that

$$\begin{aligned} -\varepsilon\rho^\nu(u(\zeta)) &\le |u(\zeta)|^2 - \varepsilon\rho^\nu(u(\zeta)) \\ &\le C_2 \int_0^{2\pi} \big(|u(r_0 e^{i\theta})|^2 - \varepsilon\rho^\nu(u(r_0 e^{i\theta}))\big)\, d\theta \end{aligned}$$

for $|\zeta| < r$. Since $-\rho^\nu \circ u$ is superharmonic, it follows that if $u(0) \in Q(0,\delta)$ and $|\zeta| < r$ then

$$-\varepsilon\rho^\nu(u(\zeta)) \le 2\pi C_2\big(C_1^2\delta - \varepsilon\rho^\nu(u(0))\big), \tag{3.3}$$

where C_1 is the constant in (3.2).

Expecting a contradiction, assume that there exist sequences

$$u^\nu \in \mathcal{O}_{(J_{st}, J^\nu)}(\mathbf{D}, \Omega^\nu) \quad \text{and} \quad \zeta_\nu \in \mathbf{D}_r$$

such that $u^\nu(0) \in Q(0,\delta_\nu)$ and $\operatorname{Re} u_1^\nu(\zeta_\nu)/\delta_\nu \to -\infty$ as $\nu \to \infty$ when $\delta_\nu \to 0$ as $\nu \to \infty$. Since

$$\begin{aligned} \frac{|\rho^\nu(u^\nu(\zeta_\nu)) - \operatorname{Re} u_1^\nu(\zeta_\nu)|}{\delta_\nu} &\le A_2 \frac{|u^\nu(\zeta_\nu)|^2}{\delta_\nu} \\ &\le A_2 \frac{C_1^2\delta_\nu}{\delta_\nu} \\ &= A_2 C_1^2 \end{aligned}$$

for large ν, we conclude that $\rho^\nu(u^\nu(\zeta_\nu))/\delta_\nu \to -\infty$. From (3.3) it follows that

$$\frac{-\varepsilon\rho^\nu(u^\nu(\zeta_\nu))}{\delta_\nu} \le 2\pi C_2\left(C_1^2 - \varepsilon\frac{\rho^\nu(u^\nu(0))}{\delta_\nu}\right) \to \infty \ \text{ as } \nu \to \infty.$$

But $|\operatorname{Re} u_1^\nu(0)|/\delta_\nu \le 1$ and $|\rho^\nu(u^\nu(0)) - \operatorname{Re} u_1^\nu(0)|/\delta_\nu \le A_2|u^\nu(0)|^2/\delta_\nu \le 2A_2$. Thus $\rho^\nu(u^\nu(0))/\delta_\nu$ is bounded, which is a contradiction. This proves the lemma. □

Suppose that $w \in Q(0,\delta) \cap \Omega^\nu$ with $\operatorname{Re} w_1 > 0$ for sufficiently small δ. Then $\operatorname{Re} w_1 \le |\operatorname{Re} w_1 - \rho^\nu(w)| < A_2|w|^2 < 2A_2\delta$. Choosing a large C_r', we may assume for any ν that, if $u \in \mathcal{O}_{(J_{st}, J^\nu)}(\mathbf{D}, \Omega^\nu)$ and $u(0) \in Q(0,\delta)$ with $\delta < \delta_r'$, then

$$|\operatorname{Re} u_1(\zeta)| < C_r'\delta$$

for $|\zeta| < r$.

From this we obtain the following lemma, which implies Proposition 3.2.

LEMMA 3.4. *There are positive constants C_r and δ_r such that*

$$\|u_1\|_{C^1(\mathbf{D}_r)} < C_r\delta \quad \text{and}$$
$$\|u_j\|_{C^1(\mathbf{D}_r)} < \sqrt{C_r\delta} \quad (j = 2, \dots, n)$$

for any $u \in \mathcal{O}_{(J_{st}, J^\nu)}(\mathbf{D}, \Omega^\nu)$ with $u(0) \in Q(0,\delta)$ and $\delta < \delta_r$.

Proof. Given r, choose r_1 with $r < r_1 < r_0$. Since J^ν converges to J^∞ in the C^1 sense, let us assume that there is a neighborhood V of 0 in Proposition 2.1 such that $\|u\|_{C^1(\mathbf{D}_{r_1})} \le K_1\|u\|_{C^0(\mathbf{D}_{r_0})}$ for any $u \in \mathcal{O}_{(J_{st}, J^\nu)}(\mathbf{D}_{r_0}, V)$ and for any ν. Now we have a constant δ' such that

$$u(0) \in Q(0,\delta) \implies u(\mathbf{D}_{r_0}) \subset B\big(0, C_1\sqrt{\delta}\big) \subset V$$

for any $u \in \mathcal{O}_{(J_{st}, J^\nu)}(\mathbf{D}, \Omega^\nu)$ and for any $\delta < \delta'$. We therefore have

$$\|u\|_{C^1(\mathbf{D}_{r_1})} \le K_1 C_1\sqrt{\delta}. \tag{3.4}$$

From (2.2), $\operatorname{Re} u_1$ is the solution of the equation $\Delta \operatorname{Re} u_1 = \frac{1}{2}\mathsf{C}(J_{st}, J; u)_1$. We may assume that $\|J^\nu\|_{C^1(V)} \le K_2$ for some K_2 and for any ν. Then from (2.3) and (3.4) we obtain that $|\mathsf{C}(J_{st}, J; u)_1| \le 4nK_2(K_1C_1)^2\delta$ on $\mathbf{D}_{r_1}$. Using the gradient estimates for Poisson's equation [8, Thm. 3.9, Thm. 8.32], we may conclude that

$$\begin{aligned}\|\operatorname{Re} u_1\|_{C^1(\mathbf{D}_r)} &\le K_3\Big(\sup_{\mathbf{D}_{r_1}}|\operatorname{Re} u_1| + \sup_{\mathbf{D}_{r_1}}|\mathsf{C}(J_{st}, J; u)_1|\Big)\\ &\le K_3(C'_{r_1} + 4nK_2(K_1C_1)^2)\delta\end{aligned} \tag{3.5}$$

whenever $u(0) \in Q(0,\delta)$.

It remains to analyze $\operatorname{Im} u_1$. Since $u(0) \in Q(0,\delta)$ implies that $|\operatorname{Im} u_1(0)| \le \delta$, it suffices to show that $|\nabla \operatorname{Im} u_1| < C\delta$ on $\mathbf{D}_r$ for some C. We can write $J^\nu_{(1,1)}$ in (3.1) as

$$J^\nu_{(1,1)} = \begin{pmatrix} a_\nu & b_\nu \\ c_\nu & -a_\nu \end{pmatrix},$$

where $a_\nu \to 0$, $b_\nu \to -1$, and $c_\nu = -(1 + a_\nu^2)/b_\nu \to 1$. By this, we can rewrite the $(1,1)$th and $(1,2)$th elements of the equation $du \circ J^{(1)}_{st} = J^\nu \circ du = J^\nu(0) \circ du + E^\nu \circ du$ as

$$\begin{aligned}-b_\nu \frac{\partial \operatorname{Im} u_1}{\partial x_1}(\zeta) &= -\frac{\partial \operatorname{Re} u_1}{\partial x_2}(\zeta) + a_\nu \frac{\partial \operatorname{Re} u_1}{\partial x_1}(\zeta) + \varepsilon_1^\nu(\zeta),\\ -b_\nu \frac{\partial \operatorname{Im} u_1}{\partial x_2}(\zeta) &= \frac{\partial \operatorname{Re} u_1}{\partial x_1}(\zeta) + a_\nu \frac{\partial \operatorname{Re} u_1}{\partial x_2}(\zeta) + \varepsilon_2^\nu(\zeta),\end{aligned}$$

where ε_1^ν and ε_2^ν are (respectively) the $(1,1)$th and $(1,2)$th elements of the matrix $E^\nu \circ du$. Note that $a_\nu \to 0$ and $b_\nu \to -1$ as $\nu \to \infty$. Owing to (3.5), it remains only to establish a bound for $|\varepsilon_j^\nu|$ on $\mathbf{D}_r$. By our assumption, $|E^\nu(u(\zeta))| \le A_1|u(\zeta)| \le A_1C_1\sqrt{\delta}$ for $|\zeta| < r$ if $u(0) \in Q(0,\delta)$ for sufficiently small δ. By the definition of ε_j^ν and equation (3.4), we have

$$|\varepsilon_j^\nu(\zeta)| \le 2nA_1K_1C_1^2\delta$$

for $j = 1, 2$ and $|\zeta| < r$. This establishes the lemma. □

This result leads to the following lemma on complete hyperbolicity; the proof is based on the methods in [9; 11]. The author would like to express deep thanks to K. T. Kim for permitting him to use this unpublished result.

LEMMA 3.5. *Let $\Omega \subset (M, J)$ be a domain with a strongly J-pseudoconvex boundary point q_0, and assume that J is of class $C^{1,\alpha}$. Then the following statements hold.*

(1) *For any $R > 0$, there exists a neighborhood V_R of q_0 such that $B^K_{(\Omega,J)}(p, R)$ is relatively compact in Ω for any $p \in V_R \cap \Omega$.*
(2) *If there is a sequence $\varphi^\nu \in \mathrm{Aut}(\Omega, J)$ such that $\varphi^\nu(p_0) \to q_0$ for some $p_0 \in \Omega$, then (Ω, J) is complete hyperbolic.*

Proof. Take a coordinate system $\Phi\colon (U, 0) \to (M, q_0)$. We identify $q_0 = 0$ and $\Phi(U) = U$. We may assume that Ω is strongly J-pseudoconvex at every point in $\partial\Omega \cap U$. By [9, Prop. 2.1], every point $q \in \partial\Omega \cap U$ is indefinitely far from any point in Ω with respect to the Kobayashi distance. It follows that $B^K_{(\Omega,J)}(p, r) \cap U \subset\subset \Omega$ for any $p \in \Omega$ and any r. It remains to show that if p is sufficiently close to 0 then $B^K_{(\Omega,J)}(p, R) \subset U$.

We estimate the Kobayashi metric in a small neighborhood of 0. Let us define the C^∞-smooth function χ by

$$\chi(z) = |z_1|^2 + |z'|^4$$

on U. It follows that $z \in Q\big(0, \sqrt{\chi(z)}\big)$ for any z. Fix r_0 and r with $0 < r < r_0 < 1$. Applying Lemma 3.1 for r_0 and Lemma 3.4 for r, we have that if $u \in \mathcal{O}_{(J_{st},J)}(\mathbf{D}, \Omega)$ and $u(0)$ is sufficiently close to 0 then

$$\|u_1\|_{C^1(\mathbf{D}_r)} < C_r\sqrt{(\chi\circ u)(0)} \quad \text{and} \quad \|u_j\|_{C^1(\mathbf{D}_r)} < \sqrt{C_r\sqrt{(\chi\circ u)(0)}}$$

for $j = 2, \dots, n$. Set $u_j = g_{2j-1} + ig_{2j}$ for each j. It follows that if $u(0)$ is close to 0 then

$$\begin{aligned}|\nabla(\chi\circ u)(0)| &\le 2\sum_{j=1}^{2}|g_j(0)||\nabla g_j(0)| + 4\sum_{j=3}^{2n}|g_j(0)|^3|\nabla g_j(0)|\\ &\le 8C_r(\chi\circ u)(0) + 16(n-1)\sqrt{C_r}(\chi\circ u)(0)\\ &\le C(\chi\circ u)(0)\end{aligned}$$

for some constant C. Let $B_\chi(r) = \{z \in \mathbb{R}^{2n} : \chi(z) < r\}$ and let R_0 be a constant with $B_\chi(R_0) \subset B(0, \delta_0)$. For a piecewise smooth path $\gamma\colon [0, 1] \to \Omega$ with $\gamma(0) \in B_\chi(R_1)$ and $\gamma(1) \in \Omega \setminus B_\chi(R_0)$ for $R_1 < R_0$, there is a segment $[a, b]$ such that $\chi(\gamma(a)) = R_1$, $\chi(\gamma(b)) = R_0$, and $\gamma([a, b]) \subset B(0, \delta_0)$. Then

$$\int_0^1 F_{(M,J)}(\gamma(t), \gamma'(t))\,dt \ge \int_a^b F_{(M,J)}(\gamma(t), \gamma'(t))\,dt \ge \frac{1}{2}\int_{R_1}^{R_0}\frac{1}{Ct}\,dt$$

by the proof of Lemma 1.1 in [9]. It follows that

$$d_{(M,J)}(p_1, p_2) > \frac{1}{2C} \log \frac{R_0}{R_1}$$

for any $p_1 \in B_\chi(R_1) \cap U$ and $p_2 \in U \setminus B_\chi(R_0)$. Given R, we have a small R_1 such that $\log(R_0/R_1) > 2CR$. Hence $B^K_{(\Omega,J)}(p, R) \subset B_\chi(R_0) \subset U$ for $p \in B_\chi(R_1) \cap U$. This proves (1).

In order to prove (2), choose any point $p \in \Omega$ and any positive real number R. For $R' = d_{(\Omega,J)}(p_0, p)$ there exists a ν_0 such that $\varphi^{\nu_0}(p_0) \in V_{R+2R'}$. Since $\varphi^\nu \in \mathrm{Aut}(\Omega, J)$, the distance-decreasing property of the Kobayashi distance means that $d_{(\Omega,J)}(\varphi^{\nu_0}(p_0), \varphi^{\nu_0}(p)) = d_{(\Omega,J)}(p_0, p) = R'$ and

$$\varphi^{\nu_0}(B^K_{(\Omega,J)}(p, R)) \subset B^K_{(\Omega,J)}(\varphi^{\nu_0}(p_0), R + 2R') \subset\subset \Omega.$$

Therefore, $B^K_{(\Omega,J)}(p, R)$ is relatively compact in Ω and so (Ω, J) is complete in the sense of Kobayashi.

4. Scaling Method

The scaling method used in this section was initiated by Pinchuk [18].

Let (M, J) be an almost complex manifold with $J \in C^{1,\alpha}$ and let Ω be a domain in M. Suppose that, for some point $p_0 \in \Omega$, there is a sequence of automorphisms $\varphi^\nu \in \mathrm{Aut}(\Omega, J)$ such that $\varphi^\nu(p_0)$ converges to the strongly J-pseudoconvex boundary point $q_0 \in \partial\Omega$.

Choosing a coordinate system $\Phi : U \to M$ about q_0 with $\Phi(0) = q_0$, we make the following identifications: $q_0 = 0$; $\Phi(U) = U$, a bounded domain in $\mathbb{R}^{2n}$; and $\Phi^* J = J$, an induced almost complex structure on U. For a suitable Φ, we may assume that:

- $J(0) = J^{(n)}_{st}$;
- $U \cap \Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ for some C^2 strictly J-plurisubharmonic function ρ on U and $T_0\partial\Omega = \{\mathrm{Re}\, z_1 = 0\}$; and
- the defining function ρ can be expressed as

$$\rho(z) = \mathrm{Re}\, z_1 + \sum_{j,k} (\mathrm{Re}\, \rho_{j,k} z_j z_k) + \sum_{j,k} \rho_{j,\bar{k}} z_j \bar{z}_k + \rho_\varepsilon(z),$$

  where $\rho_{j,k}$ and $\rho_{j,\bar{k}}$ are constants with $\rho_{j,k} = \rho_{k,j}$ and $\rho_{j,\bar{k}} = \bar{\rho}_{k,\bar{j}}$ and where $\rho_\varepsilon(z) = o(|z|^2)$.

We shall consider only φ^ν with $\varphi^\nu(p_0) \in U$. For each $p_\nu = \varphi^\nu(p_0)$, there is a point $p^*_\nu \in U \cap \partial\Omega$ with

$$\mathrm{dist}(p_\nu, \partial\Omega) = \mathrm{dist}(p_\nu, p^*_\nu) = \tau_\nu$$

as well as a rigid motion $L^\nu : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with the following properties.

(1) $L^\nu(p^*_\nu) = 0$ and $L^\nu(p_\nu) = (-\tau_\nu, 0, \dots, 0)$.
(2) If we let $\Omega^\nu = L^\nu(U \cap \Omega)$ and $J^\nu = dL^\nu \circ J \circ (dL^\nu)^{-1}$, then the tangent space of $\partial\Omega^\nu$ at 0 is $\{\mathrm{Re}\, z_1 = 0\}$ and each $J^\nu(0)$ carries $\{0\} \times \mathbb{C}^{n-1}$, the complex tangent space at 0, into itself. This means that $J^\nu(0)$ satisfies (3.1).

(3) L^ν converges to the identity mapping on any compact subset of $\mathbb{R}^{2n}$ in the C^2 topology.

It then follows that $\rho^\nu = \rho \circ (L^\nu)^{-1} \to \rho$ in the C^2 sense and that $J^\nu \to J$ in the C^1 sense. Multiplying each ρ^ν by a suitable positive number, we can replace ρ^ν with

$$\rho^\nu(z) = \operatorname{Re} z_1 + \sum_{j,k} (\operatorname{Re} \rho^\nu_{j,k} z_j z_k) + \sum_{j,k} \rho^\nu_{j,\bar{k}} z_j \bar{z}_k + \rho^\nu_\varepsilon(z), \tag{4.1}$$

where $\rho^\nu_{j,k} = \rho^\nu_{k,j} \to \rho_{j,k}$ and $\rho^\nu_{j,\bar{k}} = \bar{\rho}^\nu_{k,\bar{j}} \to \rho_{j,\bar{k}}$ as $\nu \to \infty$ and where $\rho^\nu_\varepsilon(z) = o(|z|^2)$ uniformly for ν.

By Lemma 3.1, for a fixed R_0 with $0 < R_0 < 1$ we have that $u(\mathbf{D}_{R_0}) \subset U \cap \Omega$ if $u \in \mathcal{O}_{(J_{st},J)}(\mathbf{D}, \Omega)$ and if $u(0)$ is sufficiently close to 0. Now we regard u only as its restriction on $\mathbf{D}_{R_0}$. For this u, $L^\nu \circ u|_{\mathbf{D}_{R_0}} \in \mathcal{O}_{(J_{st},J^\nu)}(\mathbf{D}_{R_0}, \Omega^\nu)$.

PROPOSITION 4.1. *For a fixed $0 < r_0 < R_0$, there are positive constants C_0 and δ_0 such that*

$$\operatorname{dist}(u(0), u(\zeta)) \le C_0 \sqrt{\operatorname{dist}(u(0), \partial\Omega^\nu)}$$

for any $|\zeta| \le r_0$ and for any $u \in \mathcal{O}_{(J_{st},J^\nu)}(\mathbf{D}_{R_0}, \Omega^\nu)$ with $|u(0)| < \delta_0$.

Proof. By Lemma 3.1, we have constants C_1 and δ_1 such that $\operatorname{dist}(u(0), u(\zeta)) \le C_1\sqrt{\operatorname{dist}(u(0), \partial\Omega)}$ for any $|\zeta| < r_0$ and for any $u \in \mathcal{O}_{(J_{st},J)}(\mathbf{D}_{R_0}, \Omega)$ with $|u(0)| < \delta_1$. Choose a small δ_0 and a positive integer N_1 such that

$$|(L^\nu)^{-1}(z)| < \delta_1 \quad \text{and} \quad \operatorname{dist}((L^\nu)^{-1}(z), \partial\Omega) < 2\operatorname{dist}(z, \partial\Omega^\nu)$$

for $z \in B(0, \delta_0) \cap \Omega^\nu$ and $\nu > N_1$. We also have that

$$\operatorname{dist}(p, q) < 2\operatorname{dist}((L^\nu)^{-1}(p), (L^\nu)^{-1}(q))$$

for any $p, q \in U$ and $\nu > N_2$. If $u \in \mathcal{O}_{(J_{st},J^\nu)}(\mathbf{D}_{R_0}, \Omega^\nu)$ with $|u(0)| < \delta_0$ for $\nu > \max\{N_1, N_2\}$, then $(L^\nu)^{-1} \circ u \in \mathcal{O}_{(J_{st},J)}(\mathbf{D}_{R_0}, \Omega)$ and $|(L^\nu)^{-1} \circ u(0)| < \delta_1$. Hence it follows that

$$\begin{aligned}
\operatorname{dist}(u(0), u(\zeta)) &< 2\operatorname{dist}((L^\nu)^{-1} \circ u(0), (L^\nu)^{-1} \circ u(\zeta)) \\
&< 2C_1\sqrt{\operatorname{dist}((L^\nu)^{-1} \circ u(0), \partial\Omega)} \\
&< 2\sqrt{2}C_1\sqrt{\operatorname{dist}(u(0), \partial\Omega^\nu)}
\end{aligned}$$

for $|\zeta| < r_0$. This proves the proposition. □

We can choose a small neighborhood V of 0 in U such that $V \cap \Omega^\nu = \{\rho^\nu < 0\}$ and $|z|^2$ is strictly J^ν-plurisubharmonic on $\bar{V}$ for sufficiently large ν. Now we can rewrite Proposition 3.2 and Lemma 3.4 for pseudoholomorphic discs defined on $\mathbf{D}_{R_0}$. Thus there are positive constants C_r and δ_r for each $0 < r < r_0$ such that

$$u(0) \in Q(0, \delta) \implies \begin{cases} u(\mathbf{D}_r) \subset Q(0, C_r\delta), \\ \|u_1\|_{C^1(\mathbf{D}_r)} < C_r\delta, \\ \|u_j\|_{C^1(\mathbf{D}_r)} < \sqrt{C_r\delta} \quad (j = 2, \ldots, n) \end{cases} \tag{4.2}$$

for any $u \in \mathcal{O}_{(J_{st},J^\nu)}(\mathbf{D}_{R_0}, \Omega^\nu)$ and for any $\delta < \delta_r$.

PROPOSITION 4.2. *For each compact subset K of Ω, there is a constant C_K such that*

$$L^\nu \circ \varphi^\nu(K) \subset Q(0, C_K \tau_\nu)$$

for large ν.

Proof. For each point $p \in \Omega$, there exist a neighborhood U_p of p and a family $\mathcal{F}_p$ of pseudoholomorphic discs passing p at the origin such that $U_p \subset \bigcup_{u \in \mathcal{F}_p} u(\mathbf{D}_{r(p)})$, where $r(p) < r_0$ (see [2; 9; 14]). Hence there is a finite covering $\{U_{q_j} : j = 0, \ldots, k\}$ of K with related constants $r(q_j)$ such that $q_0 = p_0$ and $U_{q_j} \cap U_{q_{j+1}} \neq \emptyset$. Let $r = \max\{r(q_j)\} < r_0$. Since $L^\nu \circ \varphi^\nu(q_0) \in Q(0, \tau_\nu)$, Proposition 3.2 implies that $L^\nu \circ \varphi^\nu \circ u(\mathbf{D}_r) \subset Q(0, C_r \tau_\nu)$ for any $u \in \mathcal{F}_{q_0}$. Hence we have $L^\nu \circ \varphi^\nu(U_{q_0}) \subset Q(0, C_r \tau_\nu)$. For some $u \in \mathcal{F}_{q_1}$ there is a $w \in \mathbf{D}_r$ such that $u(w) \in U_{q_0} \cap U_{q_1}$. The new pseudoholomorphic disc $g(\zeta) = u\left(\frac{\zeta + w}{1 + \bar{w}\zeta}\right)$ satisfies both $g(0) = u(w) \in Q(0, C_r \tau_\nu)$ and $g(-w) = u(0)$. Now we have $L^\nu \circ \varphi^\nu(q_1) \in Q(0, C_r^2 \tau_\nu)$, so that $L^\nu \circ \varphi^\nu(U_{q_1}) \subset Q(0, C_r^3 \tau_\nu)$. Inductively, then, $L^\nu \circ \varphi^\nu(U_{q_k}) \subset Q(0, C_r^{2k+1} \tau_\nu)$. This proves the proposition. □

Now we introduce Pinchuk's scaling mapping. For a positive real number τ, define the biholomorphism Λ_τ of $\mathbb{C}^n$ by

$$\Lambda_\tau(z) = \left(\frac{z_1}{\tau}, \frac{z_2}{\sqrt{\tau}}, \ldots, \frac{z_n}{\sqrt{\tau}}\right). \tag{4.3}$$

For simplicity we use Λ^ν to denote Λ_{τ_ν}. Let $F^\nu = \Lambda^\nu \circ L^\nu \circ \varphi^\nu$. It follows that $F^\nu(p_0) = (-1, 0, \ldots) = -\mathbf{1}$. For any compact subset K of Ω, we already know that $L^\nu \circ \varphi^\nu(K) \subset Q(0, C_K \tau_\nu)$. Since $\Lambda^\nu(Q(0, C_K \tau_\nu)) = Q(0, C_K)$, the family $\{F^\nu\}$ is uniformly bounded on K. In order to obtain a convergence of F^ν on Ω, we need the following result.

PROPOSITION 4.3. *Let h be a J-Hermitian metric on M. Then, for each compact subset $K \subset \Omega$, there exists a constant C_K' such that*

$$|dF^\nu(v)| \leq C_K' \|v\|_h$$

for each $v \in T\Omega$ based on K.

Proof. For any $u \in \mathcal{O}_{(J_{st}, J)}(\mathbf{D}, \Omega)$ with $u(0) \in K$, it follows from Proposition 4.2 that $L^\nu \circ \varphi^\nu \circ u(0) \in Q(0, C_K \tau_\nu)$. Hence, by (4.2) we have

$$\|L_1^\nu \circ \varphi^\nu \circ u\|_{C^1(\mathbf{D}_r)} \leq C_r C_K \tau_\nu \quad \text{and} \quad \|L_j^\nu \circ \varphi^\nu \circ u\|_{C^1(\mathbf{D}_r)} \leq \sqrt{C_r C_K \tau_\nu}$$

for $j = 2, \ldots, n$. Therefore,

$$|d(F^\nu \circ u)(\mathbf{e})| < C = \max\left\{C_r C_K, \sqrt{C_r C_K}\right\}.$$

By [17, 5.4a] there is a positive number R such that, for any $v \in T\Omega$ based on K with $\|v\|_h \leq R$, there exists a pseudoholomorphic disc $u \in \mathcal{O}_{(J_{st}, J)}(\mathbf{D}, \Omega)$ satisfying $du(\mathbf{e}) = v$. Hence, for any $v \in T\Omega$ based on K, we can take u such that $du(\mathbf{e}) = (R/\|v\|_h)v$. Then

$$|dF^\nu(v)| = \frac{\|v\|_h}{R}|d(F^\nu \circ u)(\mathbf{e})|$$
$$\leq \frac{C}{R}\|v\|_h.$$

The proposition follows. □

Let $\tilde{J}^\nu = d\Lambda^\nu \circ J^\nu \circ (d\Lambda^\nu)^{-1}$ and $\tilde{\Omega}^\nu = \Lambda^\nu(\Omega^\nu)$. Notice, for each compact subset K of Ω, that $F^\nu\colon K \to \Lambda^\nu(\Omega^\nu)$ is $(J, \tilde{J}^\nu)$-holomorphic for large ν.

Now we go to the limits of $\tilde{J}^\nu$ and $\tilde{\Omega}^\nu$. Write J and J^ν as the matrix-valued functions on V:

$$J(z) = J(0) + E(z) = \begin{pmatrix} J_{st}^{(1)} + A(z) & B(z) \\ C(z) & J_{st}^{(n-1)} + D(z) \end{pmatrix},$$
$$J^\nu(z) = J^\nu(0) + E^\nu(z) = \begin{pmatrix} J^\nu_{(1,1)} + A^\nu(z) & B^\nu(z) \\ J^\nu_{(2,1)} + C^\nu(z) & J^\nu_{(2,2)} + D^\nu(z) \end{pmatrix},$$

where $A^\nu \to A$, $B^\nu \to B$, $C^\nu \to C$, and $D^\nu \to D$ in the C^1 sense. Then $\tilde{J}^\nu$ can be expressed as

$$\tilde{J}^\nu(z) = \begin{pmatrix} I/\tau_\nu & 0 \\ 0 & I/\sqrt{\tau_\nu} \end{pmatrix} J^\nu((\Lambda^\nu)^{-1}(z)) \begin{pmatrix} \tau_\nu I & 0 \\ 0 & \sqrt{\tau_\nu} I \end{pmatrix}$$
$$= \begin{pmatrix} J^\nu_{(1,1)} + A^\nu((\Lambda^\nu)^{-1}(z)) & \left(B^\nu/\sqrt{\tau_\nu}\right)((\Lambda^\nu)^{-1}(z)) \\ \sqrt{\tau_\nu} J^\nu_{(2,1)} + \sqrt{\tau_\nu} C^\nu((\Lambda^\nu)^{-1}(z)) & J^\nu_{(2,2)} + D^\nu((\Lambda^\nu)^{-1}(z)) \end{pmatrix}.$$

Since $(\Lambda^\nu)^{-1}(z)$ converges uniformly to 0 on any compact subset of $\mathbb{C}^n$ and since J^ν converges uniformly to J on V, it follows that

$$J^\nu_{(1,1)} + A^\nu((\Lambda^\nu)^{-1}(z)) \to J_{st}^{(1)},$$
$$\sqrt{\tau_\nu} J^\nu_{(2,1)} + \sqrt{\tau_\nu} C^\nu((\Lambda^\nu)^{-1}(z)) \to 0, \quad \text{and}$$
$$J^\nu_{(2,2)} + D^\nu((\Lambda^\nu)^{-1}(z)) \to J_{st}^{(n-1)}$$

on any compact subset of $\mathbb{R}^{2n}$ in the C^1 sense. Write $B^\nu(z)$ and $B(z)$ as

$$B^\nu(z) = \sum_{j=1}^n (B^\nu_{2j-1} x_j + B^\nu_{2j} y_j) + B^\nu_\varepsilon(z),$$
$$B(z) = \sum_{j=1}^n (B_{2j-1} x_j + B_{2j} y_j) + B_\varepsilon(z),$$

where B^ν_j is a sequence of constant matrices that converges to B_j as $\nu \to \infty$, $B^\nu_\varepsilon \to B_\varepsilon$ in the C^1 sense, and $B^\nu_\varepsilon(z) = o(|z|)$. Then we have

$$\frac{1}{\sqrt{\tau_\nu}} B^\nu((\Lambda^\nu)^{-1}(z)) = \sqrt{\tau_\nu}(B^\nu_1 x_1 + B^\nu_2 y_1)$$
$$+ \sum_{j=2}^n (B^\nu_{2j-1} x_j + B^\nu_{2j} y_j) + \frac{1}{\sqrt{\tau_\nu}} B^\nu_\varepsilon\left(\tau_\nu z_1, \sqrt{\tau_\nu} z'\right)$$
$$\to \sum_{j=2}^n (B_{2j-1} x_j + B_{2j} y_j) \ \text{ as } \nu \to \infty.$$

Now we obtain that $\tilde{J}^\nu$ converges to

$$\hat{J}(z) = \begin{pmatrix} J_{st}^{(1)} & \hat{B}(z') \\ 0 & J_{st}^{(n-1)} \end{pmatrix} \quad \text{where } \hat{B}(z') = \sum_{j=2}^{n} (B_{2j-1}x_j + B_{2j}y_j) \tag{4.4}$$

on any compact subset of $\mathbb{R}^{2n}$ in the C^1 sense.

After scaling ρ^ν, we have

$$\begin{aligned}\tilde{\rho}^\nu &= \rho^\nu \circ (\Lambda^\nu)^{-1}(z) \\ &= \tau_\nu \left(\operatorname{Re} z_1 + \sum_{j,k=2}^{n} (\operatorname{Re} \rho^\nu_{j,k} z_j z_k) + \sum_{j,k=2}^{n} \rho^\nu_{j,\bar{k}} z_j \bar{z}_k \right) \\ &\quad + \tau_\nu^2 (\operatorname{Re} \rho^\nu_{1,1} z_1^2 + \rho^\nu_{1,\bar{1}} z_1 \bar{z}_1) \\ &\quad + \tau_\nu \sqrt{\tau_\nu} \times \text{remaining terms in the summation of (4.1)} \\ &\quad + \rho^\nu_\varepsilon \big(\tau_\nu z_1, \sqrt{\tau_\nu} z' \big).\end{aligned}$$

Therefore the sequence $\tilde{\rho}^\nu/\tau_\nu$ converges to $\hat{\rho}$ defined by

$$\hat{\rho}(z) = \operatorname{Re} z_1 + \sum_{j,k=2}^{n} (\operatorname{Re} \rho_{j,k} z_j z_k) + \sum_{j,k=2}^{n} \rho_{j,\bar{k}} z_j \bar{z}_k, \tag{4.5}$$

and $\tilde{\Omega}^\nu$ converges to $\hat{\Omega} = \{z \in \mathbb{R}^{2n} : \hat{\rho}(z) < 0\}$ in the sense of local Hausdorff set convergence.

PROPOSITION 4.4 (see [6]). *The domain $\hat{\Omega}$ is strongly $\hat{J}$-pseudoconvex at* 0.

Proof. Let $\check{\rho}^\nu = \rho \circ (\Lambda^\nu)^{-1}$ and $\check{J}^\nu = d\Lambda^\nu \circ J \circ (d\Lambda^\nu)^{-1}$. By the same reasons as given for $\tilde{\rho}^\nu$ and $\tilde{J}^\nu$, the sequence $\check{\rho}^\nu/\tau_\nu$ converges to $\hat{\rho}$ in the C^2 sense and $\check{J}^\nu$ converges to $\hat{J}$ in the C^1 sense. Hence

$$\mathcal{L}_0^{\check{J}^\nu} \check{\rho}^\nu/\tau_\nu(v) \to \mathcal{L}_0^{\hat{J}} \hat{\rho}(v)$$

for any vector v. Note that the Levi form is invariant under the pseudoholomorphic mappings. Since each Λ^ν is $(J, \check{J}^\nu)$-holomorphic, $\mathcal{L}_0^J \rho(v) = \mathcal{L}_0^{\check{J}^\nu} \check{\rho}^\nu(d\Lambda^\nu(v))$. From $\check{J}^\nu(0) = J_{st}$ it follows that every complex tangent vector of the domain defined by $\check{\rho}^\nu$ is of the form $v = (0, v')$ and so $d\Lambda^\nu(v) = v/\sqrt{\tau_\nu}$. For this v, we have $\mathcal{L}_0^{\check{J}^\nu} \check{\rho}^\nu(d\Lambda^\nu(v)) = \mathcal{L}_0^{\check{J}^\nu} \check{\rho}^\nu\big(v/\sqrt{\tau_\nu}\big) = \mathcal{L}_0^{\check{J}^\nu} \check{\rho}^\nu/\tau_\nu(v)$. After limiting, one obtains that $\mathcal{L}_0^{\hat{J}} \hat{\rho}(v) > 0$ for any $v \in T_0^{\hat{J}} \partial \hat{\Omega}$. This proves the proposition. □

Now we finish the limiting procedure of F^ν. For each compact subset K of Ω, Propositions 4.2 and 4.3 imply that $F^\nu|_K$ has a convergent subsequence in the compact-open topology. By the convergence of $\tilde{J}^\nu$ and Lemma 2.2, the limit of this subsequence is a $(J, \hat{J})$-holomorphic mapping from the interior of K to the closure of $\hat{\Omega}$. Using a compact exhaustion of Ω yields the following result.

PROPOSITION 4.5. *The sequence F^ν has a subsequence that converges to a $(J, \hat{J})$-holomorphic mapping F from Ω to the closure of $\hat{\Omega}$.*

We now prove our main theorem.

THEOREM 4.6. (Ω, J) *is biholomorphic to* $(\hat{\Omega}, \hat{J})$.

Proof. By Lemma 3.5, (Ω, J) is complete hyperbolic. Since $\Lambda_\tau \in \mathrm{Aut}(\hat{\Omega}, \hat{J})$ and $\Lambda_\tau(-\mathbf{1}) \to 0$ as $\tau \to \infty$, the domain $(\hat{\Omega}, \hat{J})$ is also complete hyperbolic.

Consider the $(\tilde{J}^\nu, J)$-holomorphic mapping $G^\nu = (F^\nu)^{-1} \colon \tilde{\Omega}^\nu \to \Omega$. For each relatively compact neighborhood Ω' of $-\mathbf{1}$ in $\hat{\Omega}$, we have $\Omega' \subset \tilde{\Omega}^\nu$ for sufficiently large ν. Since $G^\nu(-\mathbf{1}) = p_0$, it follows from Proposition 2.3 that $G^\nu|_{\Omega'}$ has a subsequence converging to an element of $\mathcal{O}_{(\hat{J}, J)}(\Omega', \Omega)$ in the compact-open topology. Thus we have a pseudoholomorphic mapping $G \colon (\hat{\Omega}, \hat{J}) \to (\Omega, J)$ that is a subsequential limit of G^ν on each compact exhaustion of $\hat{\Omega}$.

It is easy to see that $F \circ G = \mathrm{Id}_{\hat{\Omega}}$ and $G \circ F|_{F^{-1}(\hat{\Omega})} = \mathrm{Id}_{F^{-1}(\hat{\Omega})}$. Hence it remains only to show that $F^{-1}(\hat{\Omega}) = \Omega$. Take any point $x_0 \in \Omega \cap \partial F^{-1}(\hat{\Omega}) \subset F^{-1}(\partial \hat{\Omega})$ and a sequence $x^\nu \in F^{-1}(\hat{\Omega})$ such that $x^\nu \to x_0$. Since $\lim_{\nu\to\infty} F(x^\nu) \in \partial\hat{\Omega}$, we obtain that $\lim_{\nu\to\infty} d_{(\hat{\Omega}, \hat{J})}(-\mathbf{1}, F(x^\nu)) = \infty$. However, then

$$d_{(\hat{\Omega}, \hat{J})}(-\mathbf{1}, F(x^\nu)) \le d_{(\Omega, J)}(p_0, x^\nu) \to d_{(\Omega, J)}(p_0, x_0) < \infty$$

as $\nu \to \infty$. This is a contradiction, hence $F^{-1}(\hat{\Omega})$ is closed in Ω. The set Ω is connected and so $F^{-1}(\hat{\Omega}) = \Omega$, proving the theorem. □

DEFINITION 4.7. Let $\hat{\Omega} \subset \mathbb{C}^n$ be a domain defined by $\hat{\rho}$ in the form (4.5) and let $\hat{J}$ be an almost complex structure on $\mathbb{C}^n$ as in (4.4). A pair $(\hat{\Omega}, \hat{J})$ is called a *model domain* if $\hat{\Omega}$ is strongly $\hat{J}$-pseudoconvex at 0.

5. Simplification of $\hat{J}$

In order to classify the model domains $(\hat{\Omega}, \hat{J})$, we need to simplify the almost complex structure $\hat{J}$ on $\mathbb{R}^{2n}$. We shall introduce some notation.

A $2n \times 2m$ real matrix $A = (A^j_k)$ is called *anticomplex linear* if $J^{(n)}_{st} \circ A = -A \circ J^{(m)}_{st}$; if $J^{(n)}_{st} \circ A = A \circ J^{(m)}_{st}$ then we call A *complex linear.* For a complex or anticomplex linear matrix A, let $\langle A\rangle = (\langle A\rangle^j_k)$ be a $n \times m$ complex matrix where $\langle A\rangle^j_k = A^{2j-1}_{2k-1} + iA^{2j}_{2k-1}$. The corresponding linear transformation of the complex (resp. anticomplex) linear 2×2 matrix A is $z \mapsto \langle A\rangle z$ (resp. $z \mapsto \langle A\rangle \bar{z}$). It is easy to see that two complex or two anticomplex linear matrices A and B are same if and only if $\langle A\rangle = \langle B\rangle$. If both A and B are either complex linear or anticomplex linear, then AB is complex linear. If A is anticomplex linear and B is complex linear, then AB is anticomplex linear and $\langle AB\rangle = \langle A\rangle\overline{\langle B\rangle}$.

In this paper, by a *shear mapping* we mean a mapping $\Phi \colon \mathbb{C}^n \to \mathbb{C}^n$ defined as

$$\Phi(z) = (z_1 + f(z'), z_2, \dots, z_n), \tag{5.1}$$

where $f\colon \mathbb{C}^{n-1} \to \mathbb{C}$ is a C^1-smooth function. If f is holomorphic in z' then we call Φ *complex shear.* It is easy to see that the shear mapping Φ is a C^1 diffeomorphism of $\mathbb{C}^n$ and that the Jacobian matrices of Φ and its inverse Φ^{-1} can be expressed (respectively) as

$$d\Phi = \begin{pmatrix} I & df \\ 0 & I \end{pmatrix} \quad \text{and} \quad d\Phi^{-1} = \begin{pmatrix} I & -df \\ 0 & I \end{pmatrix}.$$

Now we move on to the simplification of $\hat{J}$ (denoted simply by J). For each model J, let $B^J(z') = \hat{B}(z')$ in (4.4).

Given J, let $B_j^J = (B_{2,j}^J, \dots, B_{n,j}^J)$ for each $B_{k,j}^J$ a 2×2 square matrix. Then

$$B^J(z') = \left(\sum_{j=2}^{n} (B_{2,2j-1}^J x_j + B_{2,2j}^J y_j) \ \cdots \ \sum_{j=2}^{n} (B_{n,2j-1}^J x_j + B_{n,2j}^J y_j) \right).$$

Since $J \circ J = -\mathrm{Id}$, it follows that $J_{st}^{(1)} \circ B^J + B^J \circ J_{st}^{(n-1)} = 0$. So B^J is anticomplex linear. Hence, for each $\sum (B_{k,2j-1}^J x_j + B_{k,2j}^J y_j)$ we can write

$$\begin{aligned} \left\langle \sum_{j=2}^{n} (B_{k,2j-1}^J x_j + B_{k,2j}^J y_j) \right\rangle &= \sum_{j=2}^{n} (\langle B_{k,2j-1}^J \rangle x_j + \langle B_{k,2j}^J \rangle y_j) \\ &= \sum_{j=2}^{n} (a_{k,j}^J z_j + b_{k,j}^J \bar{z}_j), \end{aligned}$$

where

$$a_{k,j}^J = \tfrac{1}{2}(\langle B_{k,2j-1}^J \rangle - i \langle B_{k,2j}^J \rangle) \quad \text{and} \quad b_{k,j}^J = \tfrac{1}{2}(\langle B_{k,2j-1}^J \rangle + i \langle B_{k,2j}^J \rangle).$$

Let $\Phi\colon \mathbb{C}^n \to \mathbb{C}^n$ be a shear mapping as in (5.1). Because $J \circ \Phi^{-1} = J$ on $\mathbb{C}^n$, the induced structure J' by Φ can be written as

$$J' = d\Phi \circ J \circ d\Phi^{-1} = \begin{pmatrix} J_{st}^{(1)} & B^J(z) - J_{st}^{(1)} \circ df + df \circ J_{st}^{(n-1)} \\ 0 & J_{st}^{(n-1)} \end{pmatrix}. \tag{5.2}$$

We shall therefore simplify $B^J(z) - J_{st}^{(1)} \circ df + df \circ J_{st}^{(n-1)}$, which is anticomplex linear. Observe that $J_{st}^{(1)} \circ df - df \circ J_{st}^{(n-1)}$ is also anticomplex linear and that its corresponding matrix is

$$\langle J_{st}^{(1)} \circ df - df \circ J_{st}^{(n-1)} \rangle = 2i \left(\frac{\partial f}{\partial \bar{z}_2}, \dots, \frac{\partial f}{\partial \bar{z}_n} \right). \tag{5.3}$$

One may thereby obtain that every complex shear mapping is an automorphism of $(\mathbb{R}^{2n}, J)$.

Set

$$f(z') = -i \left(\frac{1}{2} a_{2,2}^J z_2 \bar{z}_2 + \frac{1}{4} b_{2,2}^J \bar{z}_2^2 + \frac{1}{2} \sum_{j=3}^{n} (a_{2,j}^J z_j + b_{2,j}^J \bar{z}_j) \bar{z}_2 \right);$$

then $2i \partial f / \partial \bar{z}_2 = \sum_{j=2}^{n} (a_{2,j}^J z_j + b_{2,j}^J \bar{z}_j)$. Hence the induced J' of (5.2) satisfies $B_{2,j}^{J'} = 0$ for $j = 3, \dots, 2n$.

Now we use simply J to denote J'. For this J, let

$$f(z') = -i\left(\frac{1}{2}a^J_{3,3}z_3\bar{z}_3 + \frac{1}{4}b^J_{3,3}\bar{z}_3^2 + \frac{1}{2}\sum_{j=4}^{n}(a^J_{3,j}z_j + b^J_{3,j}\bar{z}_j)\bar{z}_3\right).$$

This f has no term containing z_2 or $\bar{z}_2$, so it follows that $\partial f/\partial\bar{z}_2 = 0$ and $2i\partial f/\partial\bar{z}_3 = \sum_{j=3}^{n}(a^J_{3,j}z_j + b^J_{3,j}\bar{z}_j)$. Hence for newly induced J' we have $B^{J'}_{2,j} = 0$ for $j = 3,\ldots,2n$ and $B^{J'}_{3,j} = 0$ for $j = 5,\ldots,2n$.

Inductively we have that the first J is diffeomorphically equivalent to the J' satisfying $B^{J'}_{k,j} = 0$ for $j \geq 2k-1$. More precisely,

$$B^{J'}(z') = \left(0 \quad B^{J'}_{3,3}x_2 + B^{J'}_{3,4}y_2 \quad \cdots \quad \sum_{j=2}^{n-1}(B^{J'}_{n,2j-1}x_j + B^{J'}_{n,2j}y_j)\right).$$

By this procedure, we conclude that $(\mathbb{R}^4, \hat{J})$ is biholomorphic to $(\mathbb{C}^2, J_{st})$. In fact, the Nijenhuis tensor $N_{\hat{J}}$ is always vanishing on $\mathbb{R}^4$. We thus have the following generalization of the Wong–Rosay theorem for the case of real dimension 4.

PROPOSITION 5.1. *If a domain Ω in an almost complex manifold (M^4, J) admits an automorphism orbit accumulating at a strongly J-pseudoconvex boundary point, then (Ω, J) is biholomorphic to $(\mathbb{B}_2, J_{st})$.*

In $\mathbb{R}^6$, we have more simplification of $\hat{J}$ to J_1 (as in Example 1.1 for the nonintegrable case).

PROPOSITION 5.2. *$(\mathbb{R}^6, \hat{J})$ is biholomorphic to $(\mathbb{C}^3, J_{st})$ or $(\mathbb{R}^6, J_1)$.*

Proof. We already know that $(\mathbb{R}^6, \hat{J})$ is biholomorphic to $(\mathbb{R}^6, J)$ with $B^J(z') = (0,\ B^J_{3,3}x_2 + B^J_{3,4}y_2)$. Suppose there is a shear mapping Φ as in (5.1) such that f is holomorphic in z_2 and $\mathrm{Re}(2i\partial f/\partial\bar{z}_3) = \mathrm{Re}(a^J_{3,2}z_2 + b^J_{3,2}\bar{z}_2)$. Then the J' induced by Φ satisfies

$$B^{J'}(z') = \begin{pmatrix} 0 & 0 & 0 & ax_2 + by_2 \\ 0 & 0 & ax_2 + by_2 & 0 \end{pmatrix}$$

by (5.2) and (5.3). Let $g = \mathrm{Re}(a^J_{3,2}z_2 + b^J_{3,2}\bar{z}_2)$; this is a linear function in x_2 and y_2. There is a harmonic conjugate h of g on all of the z_2-plane such that $h - ig$ is holomorphic in z_2. Then the function $f = (h - ig)\bar{z}_3/2$ satisfies our condition.

Let $w = a - bi$. It follows that $J' = J_{st}$ when $w = 0$. Suppose that $w \neq 0$. Setting $\Phi(z) = (z_1, wz_2, z_3)$, we obtain $d\Phi \circ J' \circ d\Phi^{-1} = J_1$. □

Note that the shear mappings used in this section change our model defining functions. But the induced defining functions are always in the form (4.5).

6. Nijenhuis Tensor and Pseudoholomorphic Mappings in $(\mathbb{R}^6, J_1)$

Computing the Nijenhuis tensor N_{J_1}, we have

$$N_{J_1}\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) = -N_{J_1}\left(\frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}\right) = \frac{\partial}{\partial x_1},$$
$$N_{J_1}\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_3}\right) = N_{J_1}\left(\frac{\partial}{\partial x_3}, \frac{\partial}{\partial y_2}\right) = -\frac{\partial}{\partial y_1},$$
$$N_{J_1}\left(\frac{\partial}{\partial x_1}, \cdot\right) = N_{J_1}\left(\frac{\partial}{\partial y_1}, \cdot\right) = 0.$$

Hence $N_{J_1}(X, Y) \in \langle \partial/\partial x_1, \partial/\partial y_1\rangle$ for any $X, Y \in T\mathbb{R}^6$ with the same base point.

Let $\mathbf{D}^3$ be the polydisc in $\mathbb{C}^3$ and let $\Phi \in \mathcal{O}_{(J_1, J_1)}(\mathbf{D}^3, \mathbb{R}^6)$; this Φ satisfies

$$d\Phi(N_{J_1}(X, Y)) = N_{J_1}(d\Phi(X), d\Phi(Y))$$

for any X and Y. Now $d\Phi(N_{J_1}(\partial/\partial x_2, \partial/\partial x_3)) = d\Phi(\partial/\partial x_1) \in \langle \partial/\partial x_1, \partial/\partial y_1\rangle$ and $d\Phi(N_{J_1}(\partial/\partial x_2, \partial/\partial y_3)) = d\Phi(-\partial/\partial y_1) \in \langle \partial/\partial x_1, \partial/\partial y_1\rangle$. Then $d\Phi'(\partial/\partial x_1) = d\Phi'(\partial/\partial y_1) = 0$, where $\Phi = (\Phi_1, \Phi')$. This means that Φ' is independent of the variable z_1 (precisely x_1 and y_1). Let

$$d\Phi = \begin{pmatrix} d\Phi_{1,z'} & d\Phi_{1,z_1} \\ 0 & d\Phi' \end{pmatrix},$$

where $\Phi_{1,z'}(\zeta) = \Phi_1(\zeta, z')$ and $\Phi_{1,z_1}(\zeta') = \Phi_1(z_1, \zeta')$. The $(1,1)$th and $(2,2)$th parts of the equation $J_1 \circ d\Phi = d\Phi \circ J_1$ are

$$J_{st}^{(1)} \circ d\Phi_{1,z'} = d\Phi_{1,z'} \circ J_{st}^{(1)} \quad \text{and} \quad J_{st}^{(2)} \circ d\Phi' = d\Phi' \circ J_{st}^{(2)},$$

respectively. As a result, $\Phi_{1,z'} \colon \mathbf{D} \to \mathbb{C}$ and $\Phi' \colon \mathbf{D}^2 \to \mathbb{C}^2$ are (standard) holomorphic.

Let $\Omega = \{\rho < 0\}$ and $\Omega' = \{\rho' < 0\}$ be our model domains. We define the slice of Ω at $z' \in \mathbb{C}^2$ by $\Omega_{z'} = \{z_1 \in \mathbb{C} : \rho(z_1, z') < 0\}$, which is connected.

Proposition 6.1. *Suppose there is a biholomorphism $\Phi \colon (\Omega, J_1) \to (\Omega', J_1)$. Then Φ' is an automorphism of $(\mathbb{C}^2, J_{st})$, and $\Phi_{1,z'} \colon \Omega_{z'} \to \Omega'_{\Phi'(z')}$ is a biholomorphism for each $z' \in \mathbb{C}^2$.*

Proof. For each $w' \in \mathbb{C}^2$ there is a $w_1 \in \mathbb{C}$ with $(w_1, w') \in \Omega$. (This inclusion holds also for Ω'.) Hence Φ' is defined on $\mathbb{C}^2$ and is surjective to $\mathbb{C}^2$. Now suppose that $(\Phi')^{-1}(w')$ is not single for some $w' \in \mathbb{C}^2$. Then

$$\Omega'_{w'} = \bigcup_{z' \in (\Phi')^{-1}(w')} \Phi_{1,z'}(\Omega_{z'}).$$

Note that this union is disjoint. For each $z' \in (\Phi')^{-1}(w')$, the holomorphic function $\Phi_{1,z'} \colon \Omega_{z'} \to \Omega'_{w'}$ is nonconstant. By the open mapping theorem, each $\Phi_{1,z'}(\Omega_{z'})$

is open; hence $\Omega'_{w'}$ is the disjoint union of open sets. But since $\Omega'_{w'}$ is connected, this is a contradiction. We conclude that Φ' is injective and $\Phi_{1,z'}\colon \Omega_{z'} \to \Omega'_{\Phi'(z')}$ is biholomorphic. □

Let us extend N_{J_1} as complex linear. In this case, $N_{J_1}(\partial/\partial z_2, \partial/\partial z_3) = \partial/\partial \bar{z}_1$. It follows that

$$d\Phi\left(N_{J_1}\left(\frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}\right)\right) = d\Phi\left(\frac{\partial}{\partial \bar{z}_1}\right) = \left(\overline{\frac{\partial \Phi_1}{\partial z_1}}\right)\frac{\partial}{\partial \bar{z}_1}$$

and

$$N_{J_1}\left(d\Phi\left(\frac{\partial}{\partial z_2}\right), d\Phi\left(\frac{\partial}{\partial z_3}\right)\right) = \left(\frac{\partial \Phi_2}{\partial z_2}\frac{\partial \Phi_3}{\partial z_3} - \frac{\partial \Phi_2}{\partial z_3}\frac{\partial \Phi_3}{\partial z_2}\right)\frac{\partial}{\partial \bar{z}_1},$$

and this implies

$$\left(\overline{\frac{\partial \Phi_1}{\partial z_1}}\right) = \frac{\partial \Phi_2}{\partial z_2}\frac{\partial \Phi_3}{\partial z_3} - \frac{\partial \Phi_2}{\partial z_3}\frac{\partial \Phi_3}{\partial z_2}. \tag{6.1}$$

LEMMA 6.2. *Let Ω and Ω' be model domains. Let $\rho(z) = \operatorname{Re} z_1 + Q(z')$ and $\rho'(z) = \operatorname{Re} z_1 + Q'(z')$ be the defining functions of Ω and Ω', respectively. A C^1 mapping $\Phi\colon (\Omega, J_1) \to (\Omega', J_1)$ is a biholomorphism if and only if Φ satisfies the following.*

(1) *Φ' is an automorphism of $\mathbb{C}^2$ and $\det\langle d\Phi'\rangle = r$ on $\mathbb{C}^2$ for some positive real constant r.*

(2) *$\Phi_1(z) = rz_1 + f(z')$, where $f_1 + if_2 = f\colon \mathbb{C}^2 \to \mathbb{C}$ is of class C^∞. Moreover, $f_1(z') = rQ(z') - Q'(\Phi'(z'))$ and*

$$\begin{aligned} 2i\frac{\partial f_2}{\partial \bar{z}_2} &= -2\frac{\partial f_1}{\partial \bar{z}_2} - \phi_2\left(\overline{\frac{\partial \Phi_3}{\partial z_2}}\right), \\ 2i\frac{\partial f_2}{\partial \bar{z}_3} &= -2\frac{\partial f_1}{\partial \bar{z}_3} - \phi_2\left(\overline{\frac{\partial \Phi_3}{\partial z_3}}\right) + rx_2, \end{aligned} \tag{6.2}$$

where $\phi_2 = \operatorname{Re} \Phi_2$.

Proof. By Proposition 6.1,

$$\Phi_{1,z'}\colon \Omega_{z'} = \{\operatorname{Re} z_1 < -Q(z')\} \to \Omega'_{\Phi'(z')} = \{\operatorname{Re} z_1 < -Q'(\Phi'(z'))\}$$

is a biholomorphism. Equation (6.1) implies that $\partial\Phi_1/\partial z_1 = \partial\Phi_{1,z'}/\partial z_1 = \det\overline{\langle d\Phi'\rangle}$ and that this is independent in z_1 and antiholomorphic in z'. Therefore, $\Phi_{1,z'}$ must be linear in z_1 for each z'. Hence we can write

$$\Phi_{1,z'}(\zeta) = \frac{\partial \Phi_1}{\partial z_1}(z')\zeta + f(z')$$

for each z'. Since $\Omega_{z'}$ and $\Omega'_{\Phi'(z')}$ are left half-planes in $\mathbb{C}$, $(\partial\Phi_1/\partial z_1)(z')$ must be a positive real number $r_{z'}$ and also $\operatorname{Re} f(z') = r_{z'}Q(z') - Q'(\Phi'(z'))$ for each z'. Now the antiholomorphic function $\partial\Phi_1/\partial z_1$ is positive real valued, so it is a positive real constant r throughout $\mathbb{C}^2$. Hence $\Phi_1 = rz_1 + rQ(z') - Q'(\Phi'(z')) + if_2$.

Since J_1 is of class C^∞, we obtain that Φ is C^∞-smooth (see [15]); thus f is also of class C^∞.

Consider the equation $J_1 \circ d\Phi = d\Phi \circ J_1$. Since $d\Phi = \begin{pmatrix} rI & df \\ 0 & d\Phi' \end{pmatrix}$, the $(1,2)$th part of this equation is $J_{st}^{(1)} \circ df + B(\Phi'(z')) \circ d\Phi' = rB(z') + df \circ J_{st}^{(2)}$. We therefore have

$$\begin{aligned}\langle J_{st}^{(1)} \circ df - df \circ J_{st}^{(2)} \rangle &= \langle rB(z') \rangle - \langle B(\Phi'(z')) \rangle \overline{\langle d\Phi' \rangle} \\ &= (0, rx_2 i) - (0, \phi_2 i) \overline{\begin{pmatrix} \frac{\partial \Phi_2}{\partial z_2} & \frac{\partial \Phi_2}{\partial z_3} \\ \frac{\partial \Phi_3}{\partial z_2} & \frac{\partial \Phi_3}{\partial z_3} \end{pmatrix}} \\ &= \left(-\phi_2 i \left(\overline{\frac{\partial \Phi_3}{\partial z_2}} \right), rx_2 i - \phi_2 i \left(\overline{\frac{\partial \Phi_3}{\partial z_3}} \right) \right).\end{aligned}$$

Applying (5.3), one obtains (6.2).

Suppose that $\Phi \colon \Omega \to \Omega'$ satisfies conditions (1) and (2) of the lemma. Then Φ is a bijective pseudoholomorphic mapping from (Ω, J_1) to (Ω', J_1). In order to prove that Φ is biholomorphic, it suffices to show that $d\Phi$ is nonsingular on Ω. From (6.1), we know that $(\partial \Phi_2/\partial z_2)(\partial \Phi_3/\partial z_3) - (\partial \Phi_2/\partial z_3)(\partial \Phi_3/\partial z_2) = r$. The determinant of the Jacobian matrix $d\Phi$ is

$$\begin{aligned}\det \begin{pmatrix} rI & df \\ 0 & d\Phi' \end{pmatrix} = \det \begin{pmatrix} rI & 0 \\ 0 & d\Phi' \end{pmatrix} &= \left| \det \begin{pmatrix} r & 0 & 0 \\ 0 & \frac{\partial \Phi_2}{\partial z_2} & \frac{\partial \Phi_2}{\partial z_3} \\ 0 & \frac{\partial \Phi_3}{\partial z_2} & \frac{\partial \Phi_3}{\partial z_3} \end{pmatrix} \right|^2 \\ &= r^4.\end{aligned}$$

This proves the sufficiency. □

By a similar argument as in the proof of Lemma 6.2, we also obtain the complete description of the (J_1, J_1)-holomorphic mappings as follows.

Proposition 6.3. *A mapping* $\Phi = (\Phi_1, \Phi_2, \Phi_3) \colon \mathbf{D}^3 \to \mathbb{C}^3$ *is* (J_1, J_1)*-holomorphic if and only if*:

(1) Φ_2 *and* Φ_3 *are holomorphic in* z_2 *and* z_3, *independent of* z_1;
(2) $\Phi_1(z) = \overline{r(z')} z_1 + f(z')$, *where*

$$r(z') = \left(\frac{\partial \Phi_2}{\partial z_2} \frac{\partial \Phi_3}{\partial z_3} - \frac{\partial \Phi_2}{\partial z_3} \frac{\partial \Phi_3}{\partial z_2} \right)(z')$$

and $f \colon \mathbf{D}^2 \to \mathbb{C}$; *and*
(3) f *satisfies*

$$\begin{aligned}4 \frac{\partial f}{\partial \bar{z}_2} &= -(\Phi_2 + \bar{\Phi}_2) \left(\overline{\frac{\partial \Phi_3}{\partial z_2}} \right) \quad \textit{and} \\ 4 \frac{\partial f}{\partial \bar{z}_3} &= -(\Phi_2 + \bar{\Phi}_2) \left(\overline{\frac{\partial \Phi_3}{\partial z_3}} \right) + (z_2 + \bar{z}_2) \overline{r(z')}.\end{aligned}$$

7. Classification of Model Domains $(\hat{\Omega}, \hat{J})$ in Real Dimension 6

One can show that every model domain $(\hat{\Omega}, \hat{J})$ of real dimension 6 is biholomorphic to (Ω, J_1) or (Ω, J_{st}), where $\Omega = \{\rho < 0\}$ and is strongly J_1-pseudoconvex or strongly J_{st}-pseudoconvex at 0 and where ρ is in the form (4.5). Since (Ω, J_{st}) is biholomorphically equivalent to $(\mathbb{B}_3, J_{st})$, it remains to classify the domains (Ω, J_1). The complex shear mapping is in $\mathrm{Aut}(\mathbb{R}^6, J_1)$; we may assume that

$$\rho(z) = \operatorname{Re} z_1 + \sum_{j,k=2}^{3} \rho_{j,\bar{k}} z_j \bar{z}_k.$$

Let us compute the Levi form of ρ.

Computing J_1, one obtains

$$J_1^* dz_j = \begin{cases} idz_1 + x_2 id\bar{z}_3 & \text{if } j = 1, \\ idz_j & \text{if } j = 2, 3; \end{cases}$$

$$J_1^* d\bar{z}_j = \begin{cases} -id\bar{z}_1 - x_2 idz_3 & \text{if } j = 1, \\ -id\bar{z}_j & \text{if } j = 2, 3. \end{cases}$$

Now we have

$$J_1^* d\rho = \frac{1}{2}(idz_1 - id\bar{z}_1 + x_2 id\bar{z}_3 - x_2 idz_3) + \sum \rho_{j,\bar{k}}(i\bar{z}_k dz_j - iz_j d\bar{z}_k).$$

The Levi form of ρ is expressed as

$$\begin{aligned} -d(J_1^* d\rho) &= 2\rho_{2,\bar{2}} dz_2 \wedge d\bar{z}_2 + 2\rho_{3,\bar{3}} dz_3 \wedge d\bar{z}_3 \\ &\quad + \left(2\rho_{2,\bar{3}} - \frac{1}{4}\right) idz_2 \wedge d\bar{z}_3 + \left(2\rho_{3,\bar{2}} - \frac{1}{4}\right) idz_3 \wedge d\bar{z}_2 \\ &\quad + \frac{i}{4}(dz_2 \wedge dz_3 - d\bar{z}_2 \wedge d\bar{z}_3). \end{aligned}$$

Since $J_1(0) = J_{st}$, we have $T_0^{J_1} \partial\Omega = \{z_1 = 0\}$. For $w = \sum_{j=2}^{3} \bigl(w_j \frac{\partial}{\partial z_j} + \bar{w}_j \frac{\partial}{\partial \bar{z}_j}\bigr) \in T_0^{J_1} \partial\Omega$, it follows that

$$\mathcal{L}_0^{J_1} \rho(w) = 4 \sum_{j=2}^{3} \rho_{j,\bar{j}} |w_j|^2 + \left(4\rho_{2,\bar{3}} - \frac{1}{2}\right) w_2 \bar{w}_3 + \left(4\rho_{3,\bar{2}} - \frac{1}{2}\right) w_3 \bar{w}_2.$$

The associated matrix of $\mathcal{L}_0^{J_1} \rho$ on $T_0^{J_1} \partial\Omega$ is

$$\begin{pmatrix} 4\rho_{2,\bar{2}} & 4\rho_{2,\bar{3}} - \frac{1}{2} \\ 4\rho_{3,\bar{2}} - \frac{1}{2} & 4\rho_{3,\bar{3}} \end{pmatrix}; \tag{7.1}$$

we call this the *tangential Levi matrix* of ρ at 0. For the domain $\mathbb{H}_t$ in Example 1.1, we have our next proposition.

PROPOSITION 7.1. *The domain $\mathbb{H}_t$ is strongly J_1-pseudoconvex at* 0 *if and only if* $t > 1/8$.

Proof. For $w = \sum_{j=2}^{3}\big(w_j \frac{\partial}{\partial z_j} + \bar{w}_j \frac{\partial}{\partial \bar{z}_j}\big) \in T_0^{J_1}\partial\Omega$, we have

$$\begin{aligned}\mathcal{L}_0^{J_1}\rho_t(w) &= 4t(|w_2|^2 + |w_3|^2) - \tfrac{1}{2}(w_2\bar{w}_3 + \bar{w}_2 w_3)\\ &\geq 4t(|w_2|^2 + |w_3|^2) - |w_2||w_3|,\end{aligned}$$

where equality holds when w_2/w_3 is positive real. Hence the last term of the foregoing inequality is always positive if and only if $1 - 4(4t)^2 < 0$. This proves the proposition. □

Next we address the classification of (Ω, J_1).

An almost complex manifold (M, J) is called *homogeneous* if, for any points p and q in M, there exists an automorphism $\varphi \in \mathrm{Aut}(M, J)$ with $\varphi(p) = q$.

LEMMA 7.2. *(Ω, J_1) is homogeneous.*

Proof. We know that Λ_τ and $\Psi_s(z) = (z_1 + si, z')$ for any positive τ and any real s are automorphisms of (Ω, J_1). It thus suffices to prove that there exists a $\Phi_{w'} \in \mathrm{Aut}(\Omega, J_1)$ with $\Phi'_{w'} = z' + w'$ for any $w' = (w_2, w_3) \in \mathbb{C}^2$. More precisely,

$$\Phi_{w'}(z) = (rz_1 + f(z'), z_2 + w_2, z_3 + w_3)$$

for some $f : \mathbb{C}^2 \to \mathbb{C}$. For this $\Phi'_{w'}$ we have $\langle d\Phi'_{w'}\rangle = \mathrm{Id}$, so Lemma 6.2 implies that $r = 1$ and

$$\begin{aligned}f_1(z') &= \sum_{j,k=2}^{3} \rho_{j,\bar{k}} z_j \bar{z}_k - \sum_{j,k=3}^{2} (z_j + w_j)(\bar{w}_k + \bar{w}_k)\\ &= \sum_{j,k=2}^{3} \rho_{j,\bar{k}}(-z_j\bar{w}_k - \bar{z}_k w_j - w_j\bar{w}_k).\end{aligned}$$

It only remains to find f_2 satisfying the two equations in (6.2), expressed by

$$\begin{aligned}\frac{\partial f_2}{\partial \bar{z}_2} &= -i\rho_{2,\bar{2}}w_2 - i\rho_{3,\bar{2}}w_3 \quad \text{and}\\ \frac{\partial f_2}{\partial \bar{z}_3} &= -i\rho_{2,\bar{3}}w_2 - i\rho_{3,\bar{3}}w_3 + \frac{i}{2}\operatorname{Re} w_2.\end{aligned}$$

Observe that $\partial(\operatorname{Re} a\bar{z})/\partial\bar{z} = a/2$. Let us define the real-valued function f_2 by

$$\begin{aligned}f_2(z') = {}& \operatorname{Re}(-2i\rho_{2,\bar{2}}w_2 - 2i\rho_{3,\bar{2}}w_3)\bar{z}_2\\ &+ \operatorname{Re}(-2i\rho_{2,\bar{3}}w_2 - 2i\rho_{3,\bar{3}}w_3 + i\operatorname{Re} w_2)\bar{z}_3.\end{aligned}$$

Then this f_2 is our desired function. □

Given this lemma, we have our main result as follows.

THEOREM 7.3. *(Ω, J_1) is biholomorphic to (Ω', J_1) if and only if the determinant of the tangential Levi matrix of ρ at 0 is the same as that of ρ'.*

Proof. By Lemma 7.2, the existence of this biholomorphism is equivalent to the existence of a biholomorphism with fixed point $-\mathbf{1} = (-1, 0, 0) \in \mathbb{C}^3$.

Suppose there is a biholomorphism $\Phi\colon (\Omega, J_1) \to (\Omega', J_1)$ with $\Phi(-\mathbf{1}) = -\mathbf{1}$. Proposition 6.1 implies that $\Phi'\colon \mathbb{C}^2 \to \mathbb{C}^2$ and $\Phi_{1,0'}\colon \{\operatorname{Re} z_1 < 0\} \to \{\operatorname{Re} z_1 < 0\}$ are biholomorphisms with $\Phi'(0) = 0$ and $\Phi_{1,0'}(-1) = -1$, respectively. This means that the constant r in Lemma 6.2 is exactly 1. It is easy to see that $\Phi_1 = z_1 + f(z')$, $f(0) = 0$, and $df_0 = 0$. Now we have $d\Phi_0(v) = d\Phi'_0(v')$ for any complex tangent vector $v = (0, v')$ of $\partial\Omega$ at 0. Note that Φ is the (J_1, J_1)-holomorphic mapping defined on all of $\mathbb{C}^3$ and that $\rho = \rho' \circ \Phi$ (Lemma 6.2 and Proposition 6.3). Thus it follows that $\mathcal{L}_0^{J_1}\rho(v) = \mathcal{L}_0^{J_1}\rho'(d\Phi_0(v)) = \mathcal{L}_0^{J_1}\rho'(d\Phi'_0(v'))$ for any $v = (0, v') \in T_0^{J_1}\partial\Omega$. This equation can be expressed as

$$\begin{pmatrix} 4\rho_{2,\bar 2} & 4\rho_{2,\bar 3} - \frac{1}{2} \\ 4\rho_{3,\bar 2} - \frac{1}{2} & 4\rho_{3,\bar 3} \end{pmatrix} = \langle d\Phi'_0\rangle^t \begin{pmatrix} 4\rho'_{2,\bar 2} & 4\rho'_{2,\bar 3} - \frac{1}{2} \\ 4\rho'_{3,\bar 2} - \frac{1}{2} & 4\rho'_{3,\bar 3} \end{pmatrix} \overline{\langle d\Phi'_0\rangle}.$$

Applying $\det\langle d\Phi'\rangle = 1$, one obtains the necessity.

In order to prove the sufficiency, we need only consider the case $\Omega' = \mathbb{H}_t$. Suppose the tangential Levi matrix of ρ is the same as that of ρ_t. We will find complex numbers $\alpha, \beta, \gamma, \delta$ ("our Greek letters") such that there exists a biholomorphism $\Phi\colon (\Omega, J_1) \to (\mathbb{H}_t, J_1)$ with

$$\Phi(-\mathbf{1}) = -\mathbf{1} \quad \text{and} \quad \Phi'(z') = (\alpha z_2 + \beta z_3, \gamma z_2 + \delta z_3).$$

By Lemma 6.2, $\Phi_1(z) = z_1 + f(z')$ must hold where $f = f_1 + i f_2$ and

$$f_1(z') = \sum_{j,k=2}^{3} \rho_{j,\bar k} z_j \bar z_k - t|\alpha z_2 + \beta z_3|^2 - t|\gamma z_2 + \delta z_3|^2.$$

It remains to find $\alpha, \beta, \gamma, \delta$ such that there is a function $f_2\colon \mathbb{C}^2 \to \mathbb{R}$ satisfying equation (6.2). It is easy to see that the existence of such an f_2 is equivalent to the partial derivatives of (6.2) satisfying

$$\frac{\partial^2 f_2}{\partial \bar z_3 \partial \bar z_2} = \frac{\partial^2 f_2}{\partial \bar z_2 \partial \bar z_3}, \qquad \frac{\partial^2 f_2}{\partial z_3 \partial \bar z_2} = \overline{\frac{\partial^2 f_2}{\partial z_2 \partial \bar z_3}},$$
$$\frac{\partial^2 f_2}{\partial z_2 \partial \bar z_2} = \overline{\frac{\partial^2 f_2}{\partial z_2 \partial \bar z_2}}, \qquad \frac{\partial^2 f_2}{\partial z_3 \partial \bar z_3} = \overline{\frac{\partial^2 f_2}{\partial z_3 \partial \bar z_3}}.$$

Because the right-hand sides of (6.2) are already determined, we can rewrite the previous four equations as (respectively)

$$\beta\gamma - \alpha\delta = -1, \tag{7.2}$$
$$\left(4t\bar\alpha - \tfrac{1}{2}\bar\gamma\right)\beta + \left(4t\bar\gamma - \tfrac{1}{2}\bar\alpha\right)\delta = 4\rho_{3,\bar 2} - \tfrac{1}{2}, \tag{7.3}$$
$$4t\alpha\bar\alpha + 4t\gamma\bar\gamma - \tfrac{1}{2}\alpha\bar\gamma - \tfrac{1}{2}\bar\alpha\gamma = 4\rho_{2,\bar 2}, \tag{7.4}$$
$$4t\beta\bar\beta + 4t\delta\bar\delta - \tfrac{1}{2}\beta\bar\delta - \tfrac{1}{2}\bar\beta\delta = 4\rho_{3,\bar 3}. \tag{7.5}$$

Now our problem is to find the solution of (7.2)–(7.5). It is possible to choose α and γ satisfying (7.4). Then β and δ are automatically determined by (7.2) and (7.3). In particular, (7.2) and (7.3) can be expressed as

$$\begin{pmatrix} \gamma & -\alpha \\ 4t\bar{\alpha} - \frac{1}{2}\bar{\gamma} & 4t\bar{\gamma} - \frac{1}{2}\bar{\alpha} \end{pmatrix} \begin{pmatrix} \beta \\ \delta \end{pmatrix} = \begin{pmatrix} -1 \\ 4\rho_{3,\bar{2}} - \frac{1}{2} \end{pmatrix}. \tag{7.6}$$

The determinant of the square matrix in (7.6) is the same as the left-hand side of (7.4). Since Ω is strongly J_1-pseudoconvex at 0, the number $4\rho_{2,\bar{2}}$ must be positive. For chosen α and γ, we can find the solution β and δ of (7.6) via

$$\begin{aligned} \begin{pmatrix} \beta \\ \delta \end{pmatrix} &= \frac{1}{4\rho_{2,\bar{2}}} \begin{pmatrix} 4t\bar{\gamma} - \frac{1}{2}\bar{\alpha} & \alpha \\ -4t\bar{\alpha} + \frac{1}{2}\bar{\gamma} & \gamma \end{pmatrix} \begin{pmatrix} -1 \\ \kappa \end{pmatrix} \\ &= \frac{1}{4\rho_{2,\bar{2}}} \begin{pmatrix} -4t\bar{\gamma} + \frac{1}{2}\bar{\alpha} + \kappa\alpha \\ 4t\bar{\alpha} - \frac{1}{2}\bar{\gamma} + \kappa\gamma \end{pmatrix}, \end{aligned}$$

where $\kappa = 4\rho_{3,\bar{2}} - \frac{1}{2}$.

Now our Greek letters satisfy (7.2)–(7.4), so it remains to test (7.5). Before doing so, we compute that

$$\begin{aligned} 4t\bar{\beta} - \frac{1}{2}\bar{\delta} &= \frac{1}{4\rho_{2,\bar{2}}}\left(\left(-16t^2 + \frac{1}{4}\right)\gamma + 4t\bar{\kappa}\bar{\alpha} - \frac{1}{2}\bar{\kappa}\bar{\gamma}\right), \\ 4t\bar{\delta} - \frac{1}{2}\bar{\beta} &= \frac{1}{4\rho_{2,\bar{2}}}\left(\left(16t^2 - \frac{1}{4}\right)\alpha + 4t\bar{\kappa}\bar{\gamma} - \frac{1}{2}\bar{\kappa}\bar{\alpha}\right). \end{aligned}$$

Observe that $16t^2 - \frac{1}{4}$ is the determinant of the tangential Levi matrix of ρ_t at 0, and set $\mu = 16t^2 - \frac{1}{4}$. Then (7.5) can be written as

$$\begin{aligned} 4\rho_{3,\bar{3}} &= \beta\left(4t\bar{\beta} - \frac{1}{2}\bar{\delta}\right) + \delta\left(4t\bar{\delta} - \frac{1}{2}\bar{\beta}\right) \\ &= \left(\frac{1}{4\rho_{2,\bar{2}}}\right)^2 \Big(4t\mu\gamma\bar{\gamma} - \frac{1}{2}\mu\bar{\alpha}\gamma - \kappa\mu\alpha\gamma - 16t^2\bar{\kappa}\bar{\alpha}\bar{\gamma} + 2t\bar{\kappa}\bar{\alpha}^2 \\ &\qquad + 4t\kappa\bar{\kappa}\alpha\bar{\alpha} + 2t\bar{\kappa}\bar{\gamma}^2 - \frac{1}{4}\bar{\kappa}\bar{\alpha}\bar{\gamma} - \frac{1}{2}\kappa\bar{\kappa}\alpha\bar{\gamma}\Big) \\ &\quad + \left(\frac{1}{4\rho_{2,\bar{2}}}\right)^2 \Big(4t\mu\alpha\bar{\alpha} - \frac{1}{2}\mu\alpha\bar{\gamma} + \kappa\mu\alpha\gamma + 16t^2\bar{\kappa}\bar{\alpha}\bar{\gamma} - 2t\bar{\kappa}\bar{\gamma}^2 \\ &\qquad + 4t\kappa\bar{\kappa}\gamma\bar{\gamma} - 2t\bar{\kappa}\bar{\alpha}^2 + \frac{1}{4}\bar{\kappa}\bar{\alpha}\bar{\gamma} - \frac{1}{2}\kappa\bar{\kappa}\bar{\alpha}\gamma\Big) \\ &= \left(\frac{1}{4\rho_{2,\bar{2}}}\right)^2 (\mu + \kappa\bar{\kappa})\left(4t\alpha\bar{\alpha} + 4t\gamma\bar{\gamma} - \frac{1}{2}\alpha\bar{\gamma} - \frac{1}{2}\bar{\alpha}\gamma\right). \end{aligned}$$

From equation (7.4) it follows that

$$16\rho_{2,\bar{2}}\rho_{3,\bar{3}} - \kappa\bar{\kappa} = \mu.$$

The left-hand side of this equation is the same as the determinant of the tangential Levi matrix of ρ at 0 (see (7.1)).

One may thus conclude that the existence of the solution of (7.2)–(7.5) corresponds to the equivalence of determinants of two tangential Levi matrices. This proves the theorem. □

Now we return to Theorem 4.6. Since the mapping $\Psi(z) = \big(z_1, \sqrt{t}z'\big)$ is the biholomorphism from $(\mathbb{H}_t, J_1)$ to $(\mathbb{H}_1, J_{1/t})$, we have the following result.

COROLLARY 7.4. *$(\hat{\Omega}, \hat{J})$ is biholomorphic to $(\mathbb{H}_1, J_t)$ for some $0 \le t < 8$.*

REMARK 7.5 (The automorphism group of $(\mathbb{H}_1, J_t)$). Since $(\mathbb{H}_1, J_0)$ is biholomorphically equivalent to $(\mathbb{B}_3, J_{st})$, its automorphism group $\mathrm{Aut}(\mathbb{H}_1, J_0)$ is the Lie group of real dimension 15. If $t \neq 0$, then $(\mathbb{H}_1, J_t)$ is biholomorphic to $(\mathbb{H}_{1/t}, J_1)$. Let us compute $\mathrm{Aut}(\mathbb{H}_t, J_1)$ for $t > 1/8$. The topological transformation group $\mathrm{Aut}(\mathbb{H}_t, J_1)$ under the compact-open topology can be decomposed as

$$\mathrm{Aut}(\mathbb{H}_t, J_1) = H \oplus \mathrm{Aut}_{-\mathbf{1}}(\mathbb{H}_t, J_1),$$

where

- H is generated by Λ_τ $(\tau > 0)$, Ψ_s $(s \in \mathbb{R})$, and $\Phi_{w'}$ $(w' \in \mathbb{C}^2)$ as introduced in Lemma 7.2; this H acts on $\mathbb{H}_1$ transitively.
- $\mathrm{Aut}_{-\mathbf{1}}(\mathbb{H}_t, J_1)$ is the isotropy subgroup at $-\mathbf{1} = (-1, 0, 0)$.

Let $\Phi \in \mathrm{Aut}_{-\mathbf{1}}(\mathbb{H}_t, J_1)$. Then $\Phi_1 = z_1 + f(z')$, $f(0) = 0$, and $df_0 = 0$. Hence the differential of Φ at $-\mathbf{1}$ is complex linear and the corresponding complex matrix must be

$$\langle d\Phi_{-\mathbf{1}} \rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \gamma & \delta \end{pmatrix}, \tag{7.7}$$

where the Greek letters are the solutions of (7.2)–(7.5) for $\rho_{2,\bar{2}} = \rho_{3,\bar{3}} = t$ and $\rho_{3,\bar{2}} = 0$. By the argument of the proof of Theorem 7.3, there exists an automorphism $(\mathbb{H}_t, J_1)$ with (7.7). By Cartan's uniqueness theorem (see [15]), such an automorphism is unique for each solution of (7.2)–(7.5). It is easy to see that the solution space of (7.2)–(7.5) is in a one-to-one correspondence with the solution space of (7.4). Therefore, $\mathrm{Aut}_{-\mathbf{1}}(\mathbb{H}_t, J_1)$ is of dimension 3.

References

[1] M. Audin and J. Lafontaine (eds.), *Holomorphic curves in symplectic geometry,* Progr. Math., 117, Birkhäuser, Basel, 1994.

[2] R. Debalme, *Kobayashi hyperbolicity of almost complex manifolds,* preprint, University of Lille, IRMA 50 (1999), math.CV/9805130.

[3] A. M. Efimov, *A generalization of the Wong–Rosay theorem for the unbounded case,* Mat. Sb. (N.S.) 186 (1995), 41–50; translation in Sb. Math. 186 (1995), 967–976.

[4] H. Gaussier and K. T. Kim, *Compactness of certain families of pseudo-holomorphic mappings into $\mathbb{C}^n$*, Internat. J. Math. 15 (2004), 1–12.

[5] H. Gaussier, K. T. Kim, and S. G. Krantz, *A note on the Wong–Rosay theorem in complex manifolds,* Complex Variables Theory Appl. 47 (2002), 761–768.

[6] H. Gaussier and A. Sukhov, *Estimates of the Kobayashi metric on almost complex manifolds,* Bull. Soc. Math. France 133 (2005), 259–273.
[7] ———, *Wong–Rosay theorem in almost complex manifolds,* preprint, math.CV/0307335.
[8] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order,* 2nd ed., Grundlehren Math. Wiss., 224, Springer-Verlag, Berlin, 1983.
[9] S. Ivashkovich and J.-P. Rosay, *Schwarz-type lemmas for solutions of $\bar{\partial}$-inequalities and complete hyperbolicity of almost complex manifolds,* Ann. Inst. Fourier (Grenoble) 54 (2004), 2387–2435.
[10] S.-J. Kan and D. Ma, *On rigidity of Grauert tubes over locally symmetric spaces,* J. Reine Angew. Math. 524 (2000), 205–225.
[11] K. T. Kim, unpublished private note.
[12] S. Kobayashi, *Hyperbolic manifolds and holomorphic mappings,* Pure Appl. Math., 2, Marcel Dekker, New York, 1970.
[13] ———, *Hyperbolic complex spaces,* Grundlehren Math. Wiss., 318, Springer-Verlag, Berlin, 1998.
[14] B. S. Kruglikov and M. Overholt, *Pseudoholomorphic mappings and Kobayashi hyperbolicity,* Differential Geom. Appl. 11 (1999), 265–277.
[15] K. H. Lee, *Almost complex manifolds and Cartan's uniqueness theorem,* Trans. Amer. Math. Soc. 358 (2006), 2057–2069.
[16] A. Newlander and L. Nirenberg, *Complex analytic coordinates in almost complex manifolds,* Ann. of Math. (2) 65 (1957), 391–404.
[17] A. Nijenhuis and W. Woolf, *Some integration problems in almost-complex and complex manifolds,* Ann. of Math. (2) 77 (1963), 424–489.
[18] S. I. Pinchuk, *Holomorphic inequivalence of certain classes of domains in $\mathbb{C}^n$* (Russian), Mat. Sb. (N.S.) 111 (1980), 67–94, 159.
[19] J.-P. Rosay, *Sur une caracterisation de la boule parmi les domaines de $\mathbb{C}^n$ par son groupe d'automorphismes,* Ann. Inst. Fourier (Grenoble) 29 (1979), 91–97.
[20] H. L. Royden, *Remarks on the Kobayashi metric,* Several complex variables, II (Proc. Internat. Conf., Univ. of Maryland, College Park, 1970), Lecture Notes in Math., 185, pp. 125–137, Springer-Verlag, Berlin, 1971.
[21] J. C. Sikorav, *Some properties of holomorphic curves in almost complex manifolds,* Holomorphic curves in symplectic geometry (M. Audin, J. Lafontaine, eds.) Progr. Math., 117, pp. 165–189, Birkhäuser, Basel, 1994.
[22] B. Wong, *Characterization of the unit ball in $\mathbb{C}^n$ by its automorphism group,* Invent. Math. 41 (1977), 253–257.

Department of Mathematics
Pohang University of Science and Technology
Pohang 790-784
Korea

nyawoo@postech.ac.kr