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Detonation Waves in a Transverse Magnetic Field

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1. Preliminaries

The equations describing the evolution of a combustible gas display a rich variety of nonlinear phenomena, including those encountered in reaction diffusion equations and in shock wave theory. In particular, there are two distinct mechanisms that can lead to the formation of combustion waves. *Deflagration* occurs when an exothermic chemical reaction is initiated in a heat-conducting gas. The subsequent diffusion of liberated heat into the surrounding medium leads to the formation of a flame that propagates into the unburned region. The fluid dynamics of the gas mixture play a negligible role in this regime. Fast combustion, or *detonation,* occurs in a dramatically different manner. Here, the process is initiated by a strong fluid dynamical shock layer that propagates into the unburned region. If the shock is sufficiently strong then the gas will be heated above its ignition temperature, causing the gas to burn in a reaction zone behind the shock. In other words, detonation waves are compressive, exothermically reacting shock waves whereas deflagration waves are expansive shock waves. For the exothermic, irreversible reactions considered here, strong deflagrations violate the second law of thermodynamics and are unphysical, and weak detonations are rare. If we permit an endothermic region then strong deflagrations and weak detonations are possible and perhaps even probable [7]. So deflagration waves will not be discussed in this paper. When the shock wave amplitude is minimum, the detonation wave is in the Chapman–Jouguet regime and plays a special role in magnetofluiddynamics, as in ordinary fluid dynamics or classical aerodynamics. This regime is realized when the shock wave occurs as a result of the energy released in a chemical reaction. The effect of this released energy appears in the conservation law of energy, which is in the related system of conservation law equations (see [2; 5; 6]).

Now we consider a one-step exothermic chemical reaction as Reactant $\to$ Product occurring in the presence of magnetic and electrical fields. The equations governing the reaction flow are

$$(\lambda_1 + 2\mu_1)\frac{du}{dx} = mu + p + \frac{1}{2}\mu(H_y^2 + H_z^2) - P,$$

$$\mu_1 \frac{dv}{dx} = mv - \mu H_x H_y - P_1,$$

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$$
\begin{aligned}
\mu_1 \frac{dw}{dx} &= mw - \mu H_x H_z - P_2, \\
\lambda \frac{dT}{dx} &= m\left(\varepsilon_i - \frac{u^2+v^2+w^2}{2}\right) - \mu u \frac{H_{y^2} + H_z^2}{2} \\
&\quad + \mu H_x(vH_y + wH_z) + uP + vP_1 + wP \\
&\quad + E_y H_z - E_z H_y - E - (h_0 - h_1)A, \\
\sigma^{-1}\frac{dH_y}{dx} &= E_z + \mu u H_y - \mu v H_x, \\
\sigma^{-1}\frac{dH_z}{dx} &= -E_y + \mu u H_z - \mu w H_x, \\
D\frac{dY}{dx} &= \frac{u}{m}A, \\
\frac{dA}{dx} &= \frac{1}{D}uA + K\phi(T)\frac{m}{u}Y,
\end{aligned}
\tag{1.1}
$$

where x is a space coordinate, $\mu > 0$ an electrical constant, (u, v, w) the velocity vector of the fluid (with u in the direction of flow), ε_i the internal energy, p the pressure, and T the temperature. The vector (H_x, H_y, H_z) is the magnetic field and (E_y, E_z) the projection of the electrical field in the (y, z)-plane, where H_x, E_y, and E_z are nonnegative constants. System (1.1) contains four dissipation coefficients; the two coefficients of viscosities λ_1 and μ_1, the coefficient of thermal conductivity λ, and the coefficient of electrical conductivity σ. These coefficients are nonnegative functions of absolute temperature T. In addition, Y is the mass fraction of the unburned gas and h_0 and h_1 are the specific enthalpies of the reactant and the product, respectively; A is an auxiliary variable. Moreover, D is the diffusion rate for the reactant and K the reaction rate coefficient, and both are positive constants. The characters m, P, P_1, P_2, and E are constants of integrations. Also, the function $\phi(T)$, known as the "reaction rate function", is defined by

$$
\phi(T) = \begin{cases} 0 & \text{for } T < T_i, \\ \phi_1(T) & \text{for } T \geq T_i, \end{cases} \tag{1.2}
$$

where $\phi_1(T)$ is a smooth positive function and T_i is the "ignition temperature" of the reaction. A typical example of $\phi_1(T)$ is the Arrhenius law: $\phi_1(T) = T^{\gamma}e^{-A/T}$ (for some positive constant γ and A). Finally, if the absolute internal energy and pressure of the unburned and burned gases are denoted by ε_0, p_0 and ε_1, p_1 (respectively), then

$$
\varepsilon_i = Y\varepsilon_0 + (1-Y)\varepsilon_1 \quad \text{and} \quad p = Yp_0 + (1-Y)p_1 \tag{1.3}
$$

are the absolute energy and pressure (respectively) of the mixture gas. For the details and derivations of these equations see [2]. Also note that in the exothermic case we have

$$
e_0 > e_1, \quad h_0 > h_1, \quad \text{and} \quad p_0 > p_1. \tag{1.4}
$$

In order to take advantage of some results from previous works, we make some simplifications and changes of variables in (1.1). Since in (1.1) the components E_y

and E_z of the electric fields are constants, one can always find a frame of reference in which $E_y = 0$. On the other hand, m is a mass quantity; in fact $m = \rho u$, where ρ is density and μ an electric quantity. One may also assume without loss of generality that $\mu = m = 1$. Then $u = V = 1/\rho$, where V is the specific volume of the fluid. Now, in (1.1) we replace X, $\lambda_1 + 2\mu_1$, u, m, P, μ_1, σ^{-1}, λ, H_x, H_y, H_z, $v - P_1$, $\omega - P_2$, ε_i, E_z, E_y, μ, D, K, Y, A with (respectively) t, μ_1, V, 1, J, μ, ν, k, δ, x_2, y_2, x_1, y_1, e, ε, 0, 1, α, β^{-1}, $1 - X$, $1 - X + Z$. Then (1.1) becomes:

$$
\begin{aligned}
\mu \dot{x}_1 &= x_1 - \delta x_2, \\
\nu \dot{x}_2 &= -\delta x_1 + V x_2 + \varepsilon, \\
\mu_1 \dot{V} &= \tfrac{1}{2}(x_2^2 + y_2^2) + V - J + p(V, T, X), \\
\kappa \dot{T} &= -\tfrac{1}{2}(x_1^2 - 2\delta x_1 x_2 + V x_2^2) - \tfrac{1}{2}(y_1^2 - 2\delta y_1 y_2 + V y_2^2) \\
&\quad - \varepsilon x_2 - \tfrac{1}{2}V^2 + JV - E + e(V, T, X) \\
&\quad - [h_0(V, T) - h_1(V, T)](1 - X + Z), \\
\mu \dot{y}_1 &= y_1 - \delta y_2, \\
\nu \dot{y}_2 &= -\delta y_1 + V y_2, \\
\alpha \dot{X} &= -V(1 - X + Z), \\
\beta V \dot{Z} &= (1 - X)\phi(T).
\end{aligned}
\tag{1.5}
$$

Note that, for $\alpha = \beta = 0$, this system reduces to (2.1) in [3] and (1.3.1) in [11].

The existence of weak and strong detonation waves has been studied by many authors. Hesaaraki [11] investigated the magnetohydrodynamic problem and proved the existence of shock waves by topological methods. Gardner [7] considered (1.5) when $\delta = \mu = \nu = \varepsilon = 0$ and $h_0(V, T) = h_1(V, T)$, and he proved the existence of weak and strong detonation waves for one-step chemical reactions. Also, Wagner [24] studied Gardner's system when h_0 and h_1 are constant and $h_0 > h_1$; he found necessary and sufficient conditions for the existence of weak and strong detonations by applying topological methods. In addition, Gasser and Szmolyan [8] considered Wagner's system with $h_0(V, T) > h_1(V, T)$. They proved the existence of weak and strong detonations and deflagrations by applying singular perturbation theory. Finally, under general assumptions of thermodynamics, Hessaraki and Razani [14; 15] proved the existence of weak, strong and CJ detonation waves for the resulting system in the paper of Gasser and Szmolyan [8].

Here, under general assumptions of thermodynamics, we prove the existence of weak and strong detonation waves in a transverse magnetic field for a one-step exothermic reaction. This means the reaction occurs in the presence of magnetic and electrical fields; that is, we consider (1.5) with $\delta = 0$.

The rest of this paper is organized as follows. In Section 2 we investigate the rest points of the reduced system of (1.5) in the case of a transverse magnetic field, and we make some observations related to the problem. In Section 3, we study the stable and unstable manifolds of the resulting system and prove the existence of weak and strong detonation waves. The uniqueness and nonuniqueness of these waves are also considered.

2. Reduction to $\delta = H_x = 0$

In this section we reduce (1.5) to a system where the reaction occurs in the presence of magnetic fields. This means that we assume $\delta = H_x = 0$, so (1.5) is reduced to the following system:

$$\begin{aligned} \mu \dot{x}_1 &= x_1, \\ \nu \dot{x}_2 &= V x_2 + \varepsilon, \\ \mu_1 \dot{V} &= \tfrac{1}{2}(x_2^2 + y_2^2) + V - J + p(V,T,X), \\ \kappa \dot{T} &= -\tfrac{1}{2}(x_1^2 + V x_2^2) - \tfrac{1}{2}(y_1^2 + V y_2^2) - \varepsilon x_2 - \tfrac{1}{2}V^2 + JV \\ &\quad - E + e(V,T,X) - [h_0(V,T) - h_1(V,T)](1 - X + Z), \\ \mu \dot{y}_1 &= y_1, \\ \nu \dot{y}_2 &= V y_2, \\ \alpha \dot{X} &= -V(1 - X + Z), \\ \beta V \dot{Z} &= (1 - X)\phi(T). \end{aligned} \tag{2.1}$$

The existence of some heteroclinic orbits between two different rest points of (2.1) proves the existence of detonation waves. At each rest point of (2.1) we must have $x_1 = y_1 = y_2 = 0$. Moreover, the subspace $x_1 = y_1 = y_2 = 0$ is an invariant subspace, and it is obvious that if an orbit of (2.1) intersects the outside part of the subspace $x_1 = y_1 = y_2 = 0$ then its ω-limit or α-limit sets couldn't intersect it. This means that our desired heteroclinic orbits must be lying in this subspace. Given this, (2.1) becomes

$$\begin{aligned} \nu \dot{x}_2 &= V x_2 + \varepsilon := G_1(u), \\ \mu_1 \dot{V} &= \tfrac{1}{2}x_2^2 + V - J + p(V,T,X) := G_2(u), \\ \kappa \dot{T} &= -\tfrac{1}{2}V x_2^2 - \varepsilon x_2 - \tfrac{1}{2}V^2 + JV - E \\ &\qquad + e(V,T,X) - q(V,T)(1 - X + Z) := G_3(u), \\ \alpha \dot{X} &= -V(1 - X + Z) := G_4(u), \\ \beta V \dot{Z} &= (1 - X)\phi(T) := G_5(u), \end{aligned} \tag{2.2}$$

where $q(V,T) = h_0(V,T) - h_1(V,T)$ and $u = (x_2, V, T, X, Z)^T$. Note that the equalities (1.3) change to

$$\begin{aligned} e(V,T,X) &= (1 - X)e_0(V,T) + X e_1(V,T), \\ p(V,T,X) &= (1 - X)p_0(V,T) + X p_1(V,T), \end{aligned} \tag{2.3}$$

where X denotes the mass fraction of the burned gas. As we mentioned before, if $\alpha = \beta = 0$ then system (2.2) reduces to (1.3.1) in [11], and if $\nu = \varepsilon = 0$ then (2.2) reduces to (2.1) in [14].

In order to obtain the rest points of (2.2), we must make some assumptions on the functions $p(V,T,X)$, $e(V,T,X)$, and $S(V,T,X)$ (the entropy of the system).

Following [1; 5; 11; 12; 13; 14], we consider a general form for thermodynamic state functions (instead of giving a specific expression) and we assume that the functions $p(V,T,X)$, $e(V,T,X)$, $S(V,T,X)$ and $h_i(V,T)$, $i=0,1$, satisfy the following hypotheses.

H_1 For $0 \le X \le 1$ and $T, V > 0$, the functions p, e, and S are positive.

H_2 For fixed $X \in [0,1]$ and $T > 0$, we have $p(V,T,X) \to +\infty$ as $V \to 0$.

H_3 For fixed $X \in [0,1]$ and given K_0 and V_0 there exists a $T_0 > 0$ such that, if $0 < V \le V_0$ and $T \ge T_0$, then $e(V,T,X) > K_0$.

H_4 If p is a function of V, T, and X, then $p_V < 0$, $p_{VV} > 0$, and $p_T > 0$.

H_5 Gibbs's law of thermodynamics is given by $de = TdS - pdV + (h_0 - h_1)dX$, where $h_i = h_i(V,T)$ for $i = 0,1$.

H_6 The following identities hold for $i = 0,1$: $e_{iV} = TS_{iV} - p_i$, $e_{iT} = TS_{iT}$, $p_{iT} = S_{iV}$, $h_i = e_i + p_{iV}$, and $S_{iT} > 0$, where $S_i = S(V,T,i)$.

These hypotheses are fairly mild and have clear thermodynamical interpretations (for more details see [17, pp. 125–132; 22, p. 516; 23; 25]). Also note that the ideal gas satisfies all the listed conditions. We will use these hypotheses directly or will take advantage of some results from previous works that are based on them.

From now on we call $\nu, \mu_1, \kappa, \alpha, \beta$ the viscosity parameters when they appear as the coefficients of derivatives in (2.2). Also, we assume that these values are positive functions of V and T.

In order to prove the existence of traveling wave solutions of (2.2), we look for an orbit of this system that is defined for all $t \in \mathbb{R}$ and then connect two different rest points of it. Hence, in the first step we need to determine the rest points of (2.2). For this, we would have

$$G_i(u) = 0, \quad 1 \le i \le 5.$$

From $G_5(u) = 0$, at a rest point we must have $X = 1$ or $T < T_i$. Also, $G_1(u) = 0$ will imply that

$$x_2 = \frac{-\varepsilon}{V}. \tag{2.4}$$

Case 1: $X = 1$. Then $G_4(u) = 0$ and $V > 0$ imply $Z = 0$. By considering this, we arrive at the following criterion for a rest point:

$$\begin{aligned} F_{11}(V,T) &= \frac{1}{2}\frac{\varepsilon^2}{V^2} + V - J + p(V,T,1) = 0, \\ F_{12}(V,T) &= \frac{1}{2}\frac{\varepsilon^2}{V} - \frac{1}{2}V^2 + JV - E + e(V,T,1) = 0. \end{aligned} \tag{2.5}$$

Case 2: $T < T_i$ *or* $\phi(T) = 0$. It follows that $G_5(u) = 0$ and hence X can take some value in $[0,1)$. If $X = m \in [0,1)$ is this value, then $G_4(u) = 0$ yields $Z = m - 1$. From $G_2(u) = G_3(u) = 0$ we have

$$\begin{aligned} F_{m1}(V,T) &= \frac{1}{2}\frac{\varepsilon^2}{V^2} + V - J + p(V,T,m) = 0, \\ F_{m2}(V,T) &= \frac{1}{2}\frac{\varepsilon^2}{V} - \frac{1}{2}V^2 + JV - E + e(V,T,m) = 0. \end{aligned} \tag{2.6}$$

By hypotheses H_4 we have $p_T > 0$, and so $\partial F_{m1}/\partial T > 0$; also, $S_T > 0$ implies $\partial F_{m2}/\partial T = e_T = TS_T > 0$ (from $de = -pdV + TdS$) for $0 \le m \le 1$. Thus the equations $F_{m1}(V,T) = 0$ and $F_{m2}(V,T) = 0$ determine the graphs of the functions $T_{m1}(V)$ and $T_{m2}(V)$, respectively, in the (V,T)-plane.

The following Lemmas 2.1–2.3 correspond to Lemmas 2.1–2.3 in [12], so their proofs are omitted. These lemmas give some information about the rest points of (2.2). See also [11, Sec. 3.2; 14, Sec. 2].

LEMMA 2.1. *For $0 \le m \le 1$, we have $dT_{m1}(V)/dV = 0$ for precisely one value of V. This is a relative maximum.*

LEMMA 2.2. *For fixed $J > 0$, there is a number $C_m \in \mathbb{R}$ such that, for $C > C_m$, the system of algebraic equations* (2.6) (*or* (2.5)) *admits no solutions. For $C = C_m$ it admits one solution and for $C < C_m$ it admits two solutions, $0 \le m \le 1$.*

Here, we assume that $F_{m1}(V,T) = 0$ and $F_{m2}(V,T) = 0$ intersect each other at two points, say (V_{mj}, T_{mj}) with $j = 0,1$, where $V_{m0} > V_{m1}$ for $0 \le m < 1$. For $m = 1$ these two rest points may coincide with each other. Using this notation, from [13, Cor. 2.1] we have the following two corollaries (see [14] for their proofs).

COROLLARY 2.1. *The curve $\{(V,T) : F_{m2}(V,T) = 0, V_{m1} < V < V_{m0}\}$ lies in the region $\{(V,T) : F_{m1}(V,T) < 0\}$ for $0 \le m \le 1$.*

COROLLARY 2.2. *For $0 \le m \le 1$, we have $S(V_{m0}, T_{m0}, m) < S(V_{m1}, T_{m1}, m)$.*

LEMMA 2.3. *The function $T_{m2}(V)$ is strictly decreasing in the interval $[V_{m1}, V_{m0}]$.*

The graphs of $F_{m1}(V,T) = 0$ and $F_{m2}(V,T) = 0$ are depicted in [13, Fig. 2.1].

THEOREM 2.1. *Let V_{mj} and T_{mj} $(0 \le m \le 1,\ j = 0,1)$ be the same as before. If $p_X = p_1 - p_0 > 0$ and $e_X = e_1 - e_0 < 0$, then for $0 \le m < n \le 1$ we have*

$$V_{m0} > V_{n0} > V_{n1} > V_{m1} \quad \textit{and} \quad T_{m0} < T_{n0} < T_{n1}.$$

Proof. This follows easily from [13, Cor. 2.1]. □

As a result of this theorem, $T_{00} < T_{10}$. Thus we may assume that the ignition temperature T_i satisfies the inequalities

$$T_{00} < T_i < T_{10}. \tag{2.7}$$

In light of these results, the rest points of (2.2) are:

$$\begin{aligned} u_{1j} &= \left(-\frac{\varepsilon}{V_{1j}}, V_{1j}, T_{1j}, 1, 0\right)^T, \quad j = 0,1; \\ u_{mj} &= \left(-\frac{\varepsilon}{V_{mj}}, V_{mj}, T_{mj}, m, m-1\right)^T, \quad 0 \le m < 1,\ T_{mj} < T_i,\ j = 0,1. \end{aligned} \tag{2.8}$$

In this work we assume that the rest points u_{10} and u_{11} exist, though they may coincide.

COROLLARY 2.3. *If the rest point u_{10} or u_{11} exists, then the rest point u_{m0} exists for some $0 \le m < 1$.*

3. Existence of Detonations

In this section, we show the existence of weak and strong detonation waves in a transverse magnetic field. The uniqueness and nonuniqueness of these waves are also considered. In order to do this we must prove the existence of some heteroclinic orbits between rest points of (2.2). The existence proof for heteroclinic orbits is carried out by some general topological arguments in ordinary differential equations. This requires that we make some observations related to the nature of stable and unstable manifolds of (2.2) at the rest points u_{10} and u_{11}. Toward this end we shall use the following lemma (which is exactly [13, Lemma 2.4]).

LEMMA 3.1. *At the rest point u_{10}, $S_T(1+p_V-\varepsilon^2/V^3)-S_V^2 > 0, 1+p_V > 0$, and $1+p_V-(\varepsilon^2/V^3) > 0$. But at the rest point u_{11}, $S_T(1+p_V-\varepsilon^2/V^3)-S_V^2 < 0$.*

The linearized system of (2.2) at the rest points u_{1j} $(j = 0, 1)$ can be written as

$$\dot{u} = M_{1j}(u - u_{1j}), \quad j = 0, 1,$$

where

$$M_{1j} = \begin{bmatrix} \dfrac{V}{\nu} & \dfrac{x_2}{\nu} & 0 & 0 & 0 \\ \dfrac{x_2}{\mu_1} & \dfrac{1+p_V}{\mu_1} & \dfrac{p_T}{\mu_1} & \dfrac{p_X}{\mu_1} & 0 \\ 0 & \dfrac{TS_V}{\kappa} & \dfrac{TS_T}{\kappa} & \dfrac{e_X+q}{\kappa} & \dfrac{-q}{\kappa} \\ 0 & 0 & 0 & \dfrac{V}{\alpha} & -\dfrac{V}{\alpha} \\ 0 & 0 & 0 & -\dfrac{\phi_1(T)}{\beta V} & 0 \end{bmatrix},$$

where $q = q(V,T) = h_0(V,T) - h_1(V,T)$, the entries of M_{1j} are considered at the rest point u_{1j}, and we use the identities $e_V = TS_V - p$, $e_T = TS_T$, and $p_T = S_V$ of hypotheses H_6. If $f(\lambda)$ is the characteristic polynomial of this matrix, then

$$f(\lambda) = \left(\lambda^2 - \lambda\frac{V}{\alpha} - \frac{\phi_1(T)}{\alpha\beta}\right)\frac{h(\lambda)}{\mu_1\nu\kappa}, \tag{3.1}$$

where

$$h(\lambda) = -[(V-\nu\lambda)[(1+p_V-\mu_1\lambda)(TS_T-\kappa\lambda)-TS_V^2]-x_2^2(TS_T-\kappa\lambda)]. \tag{3.2}$$

Let $g(\lambda) = \lambda^2 - \lambda(V/\alpha) - \phi_1(T)/\alpha\beta$. Since $V > 0$ and $\phi_1(T) > 0$, it follows that $g(\lambda)$ has one positive root and one negative root. Let $\lambda_1 < 0 < \lambda_2$ be these roots. On the other hand, by [11, Thm. 2.2.1] (due to Germain in [9] or [10]), the polynomial $h(\lambda)$ at the rest point u_{1j} $(j = 0, 1)$ has $3 - j$ positive roots and j negative roots; we denote them by $\lambda_3 \le \lambda_4 \le \lambda_5$. Thus we have proved the following theorem.

THEOREM 3.2. *Let λ_k, $1 \le k \le 5$, be the eigenvalues of the matrix M_{1j} at the rest points u_{1j}, $j = 0, 1$. Then: at the rest point u_{10} we have $\lambda_1 < 0 < \lambda_2, \lambda_3 > 0, \lambda_4 > 0$, and $\lambda_5 > 0$; but at the rest point u_{11} we have $\lambda_1 < 0, \lambda_2 > 0, \lambda_3 < 0, \lambda_4 > 0$, and $\lambda_5 > 0$.*

Our next theorem concerns the eigenvectors of these eigenvalues.

THEOREM 3.3. *Let $(y_1, y_2, y_3, y_4, y_5)^T$ be an eigenvector of the matrix M_{1j} corresponding to the negative eigenvalue λ_1.*

(i) *At u_{10}, either $y_1 < 0$, $y_2 < 0$, $y_3 > 0$, $y_4 > 0$, and $y_5 > 0$ or the reverse inequalities hold.*
(ii) *At u_{11}, either $y_1 > 0$, $y_2 > 0$, $y_4 > 0$, and $y_5 > 0$ or the reverse inequalities hold.*
(iii) *If $(z_1, z_2, z_3, z_4, z_5)^T$ is an eigenvector corresponding to the second negative eigenvalue λ_3 at the rest point u_{11}, then either $z_1 < 0$, $z_2 < 0$, $z_3 > 0$, and $z_4 = z_5 = 0$ or the reverse inequalities hold.*

Proof. (i) The eigenvalue $(y_1, y_2, y_3, y_4, y_5)^T$ must satisfy the following equations at the rest points u_{10} and u_{11}:

$$
\begin{aligned}
\left(\frac{1}{\nu}V - \lambda_1\right)y_1 - \frac{\varepsilon}{\nu V}y_2 &= 0,\\
\frac{1}{\mu_1}\left(-\frac{\varepsilon}{V}\right)y_1 + \left[\frac{1}{\mu_1}(1 + p_V) - \lambda_1\right]y_2 + \frac{1}{\mu_1}p_T y_3 + \frac{1}{\mu_1}p_X y_4 &= 0,\\
\frac{TS_V}{\kappa}y_2 + \left(\frac{TS_T}{\kappa} - \lambda_1\right)y_3 + \frac{e_X + q}{\kappa}y_4 + \frac{q}{\kappa}y_5 &= 0,\\
\left(\frac{V}{\alpha} - \lambda_1\right)y_4 - \frac{V}{\alpha}y_5 &= 0,\\
-\frac{\phi_1(T)}{\beta V}y_4 - \lambda_1 y_5 &= 0.
\end{aligned}
\tag{3.3}
$$

Since $\lambda_1 < 0$, from the first equation we obtain $\operatorname{sgn} y_1 = \operatorname{sgn} y_2$. Also

$$
\begin{vmatrix} \dfrac{V}{\alpha} - \lambda_1 & -\dfrac{V}{\alpha} \\ -\dfrac{\Phi_1(T)}{\beta V} & \lambda_1 \end{vmatrix} = 0,
$$

so the two last equations of (3.3) will imply that $\operatorname{sgn} y_4 = \operatorname{sgn} y_5$.

Now, if we obtain y_1 from the first equation and substitute in the second equation of (3.3) and then consider the third equation of (3.1), we have

$$\begin{aligned} Ay_2 + \frac{1}{\mu_1} p_T y_3 + \frac{1}{\mu_1} p_X y_4 = 0, \\ \frac{TS_V}{\kappa} y_2 + \left(\frac{TS_V}{\kappa} - \lambda_1\right) y_3 + \frac{1}{\kappa}(p_0 - p_1) V y_4 - \frac{q}{\kappa} y_5 = 0, \end{aligned} \tag{3.4}$$

where

$$A = \frac{1}{\mu_1(V - \nu\lambda_1)} \left[V\left(1 + p_V - \frac{\varepsilon^2}{V^3}\right) - \lambda_1 \nu (1 + p_V) - \mu_1 \lambda_1 (V - \nu \lambda_1) \right]$$

and $(p_0 - p_1)V = e_X + q$. Solving (3.4) for y_2 in terms of y_4 and y_5 yields

$$y_2 = \frac{(V/\nu - \lambda_1)}{h(\lambda_1)} \left(\frac{1}{\mu_1}(p_1 - p_0) \left(\frac{TS_T}{\kappa} - \lambda_1 + \frac{p_T}{\kappa}(p_1 - p_0)V \right) y_4 + \frac{q}{\kappa} \frac{1}{\mu_1} p_T y_5 \right), \tag{3.5}$$

where $h(\lambda)$ is defined in (3.2) and $h(\lambda) = (\lambda - \lambda_3)(\lambda - \lambda_4)(\lambda - \lambda_5)$. Since $\lambda_1 < 0$, it follows that $h(\lambda_1) < 0$ and thus $\operatorname{sgn} y_2 = -\operatorname{sgn} y_4 = -\operatorname{sgn} y_5$. Using this and the second equation of (3.4), we obtain

$$y_3 = \left[\frac{TS_T}{\kappa} - \lambda_1 \right]^{-1} \left[-\frac{TS_V}{\kappa} y_2 - \frac{1}{\kappa}(p_0 - p_1) V y_4 + \frac{q}{\kappa} y_5 \right]; \tag{3.6}$$

this shows that $\operatorname{sgn} y_3 = -\operatorname{sgn} y_2 = \operatorname{sgn} y_4 = \operatorname{sgn} y_5$.

(ii) Note that $\lambda_1 = -\frac{2}{\beta}\Phi_1(T)\left[V + \sqrt{V^2 + 4\frac{\alpha}{\beta}\Phi_1(T)}\,\right]^{-1}$. Since the rest points and λ_1 are independent of μ_1, κ, and ν and since $\beta^{-1} = K$ is the reaction rate coefficient, we may assume that $\mu_1 \ll \beta$, $\kappa \ll \beta$, and $\nu \ll \beta$ (see [14] and [22]). Then, by Lemma 3.1 we may assume that $h(\lambda_1) > 0$ at the rest point u_{11}. Thus (3.5) at the rest point u_{11} implies that $\operatorname{sgn} y_2 = \operatorname{sgn} y_4 = \operatorname{sgn} y_5$.

(iii) Consider (3.3) in terms of λ_3 instead of λ_1. The two last equations will imply:

$$\begin{aligned} \left(\frac{V}{\alpha} - \lambda_3\right) z_4 - \frac{V}{\alpha} z_5 = 0; \\ -\frac{\Phi_1(T)}{\beta V} z_4 - \lambda_3 z_5 = 0. \end{aligned} \tag{3.7}$$

Since λ_3 depends on μ_1, ν, and κ and is independent of α and β, it follows that

$$\begin{vmatrix} \dfrac{V}{\alpha} - \lambda_3 & -\dfrac{V}{\alpha} \\ -\dfrac{\Phi_1(T)}{\beta V} & -\lambda_3 \end{vmatrix} = 0.$$

Thus $z_4 = z_5 = 0$ is the only solution of (3.7). Hence (z_1, z_2, z_3) must be a solution of the following equations at the rest point u_{11}:

$$
\begin{aligned}
\left(\frac{1}{\mu}V-\lambda_3\right)z_1-\frac{\varepsilon}{\nu V}z_2&=0,\\
\frac{1}{\mu_1}\left(-\frac{\varepsilon}{V}\right)z_1+\left[\frac{1}{\mu_1}(1+p_V)-\lambda_3\right]z_2+\frac{1}{\mu_1}p_T z_3&=0,\\
\frac{TS_V}{\kappa}z_2+\left(\frac{TS_T}{\kappa}-\lambda_3\right)z_3&=0.
\end{aligned}
\tag{3.8}
$$

From the first and third equations of (3.8), we have $\operatorname{sgn} z_1 = \operatorname{sgn} z_2$ and $\operatorname{sgn} z_2 = -\operatorname{sgn} z_3$. □

In order to prove the existence of structure for weak and strong detonations, we define the set

$$
\begin{aligned}
D = \{u \in \mathbb{R}^5 : G_1(u) < 0,\ G_2(u) < 0,\ G_3(u) > 0,&\\
-b < 1 - X + Z < 0,\ 0 < X < 1\},&
\end{aligned}
$$

where the $G_i(u)$, $0 \le i \le 3$, are introduced in (2.2) and where b is defined as

$$
b = 1 + \frac{\alpha}{\beta} \sup_{G_1(u)\le 0,\, G_2(u)\ge 0,\, G_3(u)\ge 0} \frac{\Phi_1(T)}{V^2},
$$

so $b > 1$ is a constant. Now observe that the rest points $u_{1j} = (-\varepsilon/V_{1j}, V_{1j}, T_{1j}, 1, 0)^T$, $j = 0, 1$, are located on ∂D. Moreover, by Theorem (3.2), the stable manifold at u_{10} is 1-dimensional and at u_{11} is 2-dimensional whenever these rest points exist and are different. With regard to these two manifolds, we have the following lemma.

LEMMA 3.4. *Let D be as before. Then the stable manifold at u_{10} intersects D on a curve, and the stable manifold at u_{11} intersects D on a 2-dimensional manifold.*

Proof. As we have already shown, the linearized system of (2.2) at the rest point u_{1j} $(j = 0, 1)$ has the following form:

$$
\dot u = M_{1j}(u - u_{1j}) := (G_{1L}(u), \dots, G_{5L}(u))^T. \tag{3.9}
$$

Let $(y_1, y_2, \dots, y_5)^T$ be an eigenvector corresponding to λ_1 at the rest point u_{10}, and consider the solution

$$
u(t) = (x_2(t), V(t), T(t), X(t), Z(t))^T = (y_1, \dots, y_5)^T e^{\lambda_1 t} + u_{10}
$$

of the linearized system (3.9). Thus, for $t \in \mathbb{R}$ we have

$$
\begin{aligned}
(G_{1L}(u), \dots, G_{5L}(u))^T &= M_{10}(u - u_{10}) = M_{10}(y_1, \dots, y_5)^T e^{\lambda_1 t}\\
&= \lambda_1 (y_1, \dots, y_5)^T e^{\lambda_1 t} = (\lambda_1 y_1, \dots, \lambda_1 y_5)^T e^{\lambda_1 t}.
\end{aligned}
\tag{3.10}
$$

Since $\lambda_1 < 0$, Theorem 3.3(i) and (3.10) imply that $(G_{1L}(u), \dots, G_{5L}(u))^T \in D_\omega$, where

$$
\begin{aligned}
D_\omega = \{u \in \mathbb{R}^5 : G_{1L}(u) < 0,\ G_{2L}(u) < 0,\ G_{3L}(u) > 0,&\\
G_{4L}(u) > 0,\ G_{5L}(u) > 0\}.&
\end{aligned}
$$

This means that the stable manifold of (3.9) at the rest point u_{10}, which is the line

$$M_w = \{u \in \mathbb{R}^5 : u - u_{10} = (y_1, \ldots, y_5)^T s,\ s \in \mathbb{R}\},$$

lies in D_ω for $s > 0$ and lies in D for $s > 0$ and small. Thus, the stable manifold of (2.2) at the rest point u_{10} intersects D on a curve.

Now consider the rest point u_{11}, the negative eigenvalues λ_1 and λ_3, and the linear system (3.9) at this rest point:

$$\dot{u} = M_{11}(u - u_{11}) := (\tilde{G}_{1L}(u), \ldots, \tilde{G}_{5L}(u))^T.$$

In a similar way, it can be shown that the stable manifold of (3.9) at u_{11}, which is 2-dimensional (Theorem 3.2), has the form

$$M_s = \{u \in \mathbb{R}^5 : u - u_{11} = (y_1, y_2, y_3, y_4, y_5)^T s_1 + (z_1, z_2, z_3, 0, 0)^T s_2, \\ s_1, s_2 \in \mathbb{R}\},$$

where $(y_1, y_2, y_3, y_4, y_5)^T$ and $(z_1, z_2, z_3, 0, 0)^T$ are eigenvectors corresponding to λ_1 and λ_3, respectively. Hence, by Theorem 3.3(ii) and (iii), those points of M_s with $s_1, s_2 > 0$ and $s_1 \ll s_2$ lie in the set

$$D_s = \{u \in \mathbb{R}^5 : G_{1L}(u) < 0,\ G_{2L}(u) < 0,\ G_{iL}(u) > 0,\ 3 \le i \le 5\}.$$

Therefore, those points of M_s with $s_1, s_2 > 0$, $s_1 \ll s_2$, and $s_1 + s_2$ small lie in D. Thus the stable manifold of (2.2) at u_{11} intersects D in a 2-dimensional manifold. □

Now consider the following system of ordinary differential equations:

$$\begin{aligned} \nu \dot{x}_2 &= G_1(u), \\ \mu_1 \dot{V} &= G_2(u), \\ \kappa \dot{T} &= G_3(u), \\ \alpha \dot{X} &= G_4(u), \\ \beta V \dot{Z} &= (1 - X)\phi_1(T) := \tilde{G}_5(u), \end{aligned} \tag{3.11}$$

where the $G_k(u), 1 \le k \le 4$, are as before and where $\phi_1(T)$ is given by (1.2). Note that the system (3.11) is mathematically well-defined for all $V > 0$, $T > 0$, $Z \in \mathbb{R}$, and $X \in \mathbb{R}$; moreover, it is the same as (3.2) for $0 \le X \le 1$ and $T > T_i$. This system and Theorems 3.1 and 3.2 of [14] lead us to the proof of the existence of structure for weak and strong detonation waves.

Here, we restate [14, Thm. 3.1]. Let

$$\frac{dx}{dt} = f(x), \quad x = (x_1, x_2, \ldots, x_n)^T, \tag{3.12}$$

be an autonomous system of ordinary differential equations on $\mathbb{R}^n$.

THEOREM 3.5. *Suppose the function f in* (3.12) *is locally Lipschitz in a neighborhood of the closure of a bounded open set D that is homeomorphic to the cylinder* $\left\{x \in \mathbb{R}^n : \sum_{i=1}^{n-1} x_i^2 < 1,\ 0 < x_n < 1\right\}$, *and suppose* (3.12) *is gradient-like with respect to a function h in D. Moreover, suppose the following conditions hold.*

C_1 *The set* $\{x \in \bar{D} : h(x) = c\}$ *corresponds to the set* $\left\{x \in \mathbb{R}^n : \sum_{i=1}^{n-1} x_i^2 \le 1, x_n = c\right\}$ *for* $0 \le c \le 1$ *under the homomorphism.*

C_2 *System* (3.12) *has finitely many rest points located in the set* $\{x \in \partial D : h(x) = 1\}$. *Moreover, this system has no other rest point in* $\bar{D}$.

C_3 *The flow goes out of D on* $\{x \in \partial D : 0 < h(x) < 1\}$.

C_4 *For* $\tilde{x}$ *one of the rest points of* (3.12), *the stable manifold of this system at this rest point intersects D in a nonempty set.*

Then there is a point $p \in \{x \in \partial D : h(x) = 0\}$ *such that* $\lim_{t\to\infty} p.t = \tilde{x}$. *Moreover, if the intersection of D and the stable manifold at* $\tilde{x}$ *is* 1*-dimensional, then this point is unique. If this dimension* > 1 *then there exist infinitely many such points.*

LEMMA 3.6. *Let D be as before. If* $(\nu, \mu_1, \kappa, \alpha, \beta) > 0$ *then there is a unique orbit of* (3.11) *that lies in D and whose* ω*-limit set is* u_{10}, *and this orbit intersects the set* $\Delta = \{u \in \bar{D} : G_1(u) < 0, G_2(u) < 0, G_3(u) > 0, X = 0\}$. *Moreover, there are infinitely many orbits of* (3.11) *that lie in D and whose* ω*-limit sets are* u_{11}. *Each of these orbits intersects the set* Δ *just described. Along all of these orbits,* $-x_2(t)$, $-V(t)$, $T(t)$, $X(t)$, *and* $Z(t)$ *are increasing.*

Proof. First, note that (3.11) is gradient-like with respect to $h(u) = X$ in D and is locally Lipschitz in a neighborhood of $\bar{D}$. Next we show that (3.11)—together with D (as D), u_{10} (and similarly u_{11}) (as the rest point), and the real-valued function $h(u) = X$ (as h)—satisfy all the conditions of Theorem 3.5.

In order to see that D is homeomorphic to the cylinder and that condition C_1 holds, define

$$\begin{aligned} Q := \{u \in \mathbb{R}^4 : G_{ic}(u) \le 0 \ (1 \le i \le 2), \\ G_{3c}(u) \ge 0 \ (-b - 1 + c \le Z \le c - 1)\}, \end{aligned}$$

where $G_{ic}(u) = G_i(x_2, V, T, X, c)$. From the hypotheses we have $e_T > 0$, $e_V > 0$, $q_T > 0$, $q_V > 0$, and $e_V = TS_V - p$; and as in the proof of [13, Lemma 2.2.2], it follows that the set $Q \cap \{u : Z = Z_0\}$ is homeomorphic to the unit disk. Hence D is homeomorphic to the cylinder and condition C_1 holds. The set of rest points of (3.11) is $\{u_{10}, u_{11}\}$, and this set is located on $\partial D \cap \{u \in \mathbb{R}^5 : X = 1\}$. Therefore, condition C_2 holds also. By Lemma 3.4, the stable manifold of (3.11) at the rest points u_{10} and u_{11} intersects D in 1- and 2-dimensional manifolds, respectively. Thus, condition C_4 of Theorem 3.5 holds as well.

Finally, we show that condition C_3 is fulfilled. Note that $G_1(u) < 0$ and $V > 0$ imply that $x_2 < 0$ in D. Now, let $u_0 \in \{u \in \partial D : 0 < X < 1\}$. Then $G_i(u_0) = 0$ for one i $(1 \le i \le 3)$ or $1 - X + Z = 0$ or $1 - X + Z = -b$. Suppose $G_1(u) = 0$. Differentiating $G_1(u)$ along the orbits of (3.11) yields

$$\left.\frac{dG_1(u)}{dt}\right|_{G_1(u_0)=0} = \frac{x_2}{\nu\mu_1} G_2(u_0) + \left.\frac{V}{\nu} G_1(u_0)\right|_{G_1(u_0)=0} = \frac{x_2}{\nu\mu_1} G_2(u_0) > 0;$$

thus the flow goes out of $\bar{D}$ on $G_1(u_0) = 0$. Let $G_2(u_0) = 0$. Now differentiating $G_2(u)$ along the orbits yields

$$\left.\frac{dG_2(u)}{dt}\right|_{G_2(u_0)=0}$$
$$= \frac{x_2}{\nu}G_1(u_0) + \frac{1}{\mu_1}G_2(u_0) + p_V\frac{G_2(u_0)}{\mu_1} + \frac{p_T}{\kappa}G_3(u_0) + \frac{p_X}{\alpha}G_4(u_0) > 0;$$

thus the flow goes out of $\bar{D}$ on $G_2(u_0) = 0$. If we now differentiate $G_3(u)$ along the orbits, we obtain

$$\left.\frac{dG_3(u)}{dt}\right|_{G_3(u_0)=0}$$
$$= (-Vx_2 - \varepsilon)\frac{G_1(u_0)}{\nu} + \left(-\frac{1}{2}x_2^2 - V + J + e_V - q_V(1 - X + Z)\right)\frac{G_2(u)}{\mu_1}$$
$$+ (e_T - q_T(1 - X + Z))\frac{G_3(u)}{\kappa} + (e_X + q)\frac{G_4(u)}{\alpha} - \frac{q}{\beta V}\tilde{G}_5(u).$$

On the other hand, in $\bar{D}$ we have

$$-\frac{1}{2}x_2^2 + J - V + e_V = -\frac{1}{2}x_2^2 + J - V + TS_V - p = TS_V - G_2(u) > 0,$$

$$q_V > 0, \quad 1 - X + Z < 0, \quad q > 0, \quad (-Vx_2 - \varepsilon)\frac{G_1(u_0)}{\nu} = -\frac{G_1(u_0)^2}{\nu} < 0,$$

and

$$e_X + q = e_1 - e_0 + h_0 - h_1 = (p_0 - p_1)V < 0.$$

Therefore, $\left.\frac{dG_3(u)}{dt}\right|_{G_3(u_0)=0} < 0$. Thus the flow goes out of $\bar{D}$ on $G_3(u_0) = 0$. Finally, if we differentiate $\tilde{G}_4(u) := 1 - X + Z$ along the orbits then we obtain

$$\frac{d\tilde{G}_4(u)}{dt} = -\dot{X} + \dot{Z} = \frac{V}{\alpha}(1 - X + Z) + \frac{1 - X}{\beta V}\phi_1(T).$$

Therefore,

$$\left.\frac{d\tilde{G}_4(u)}{dt}\right|_{1-X+Z=0} = \frac{1 - X}{\beta V}\phi_1(T) > 0 \quad \text{and}$$
$$\left.\frac{d\tilde{G}_4(u)}{dt}\right|_{1-X+Z=-b} = -b\frac{V}{\alpha} + \frac{1 - X}{\beta V}\phi_1(T) < 0.$$

Thus the flow goes out of $\bar{D}$ on $\tilde{G}_4(u_0) = 0$ or $\tilde{G}_4(u_0) = -b$ and so condition C_3 of Theorem 3.5 holds, too. Hence, by that theorem there must be an orbit of (3.11) lying in D, initiating at a point on the surface $X = 0$, and running to the rest point u_{10} as $t \to +\infty$. There are likewise infinitely many points in the plane $X = 0$ such that, if an orbit of (3.11) is started at each of these points, then the ω-limit set of these orbits is the set $\{u_{11}\}$. Finally, from (3.11) and our definition of the set D it follows that, along these orbits, $X(t), Z(t), T(t)$ are increasing but $V(t)$ and $x_2(t)$ are decreasing. □

Let $\tilde{u}(t)$, $t \in [t_0, \infty)$, be one of the orbits given by Lemma 3.6, and let $\tilde{u}(t_0) \in \{u \in \bar{D}, X = 0\}$ and $\lim_{t\to\infty} \tilde{u}(t) = u_{10}$ (or u_{11}). Then we have the following lemma.

LEMMA 3.7. *Let $\tilde{u}(t)$ be as just described. If* $(\mu, \mu_1, \kappa, \alpha, \beta) > 0$ *with*

$$\beta \gg \max(\alpha, \nu, \mu_1, \kappa),$$

then the orbit $\tilde{u}(t)$ meets the hypersurface $T = T_i$ for some $\tilde{t} \in (t_0, \infty)$ with $0 < \tilde{X}(\tilde{t}) < 1$ *and* $-1 < \tilde{Z}(\tilde{t}) < 0$.

Proof. Let $(x_{2i}, V_i, T_i, X_i, Z_i)$ be the unique solution of the equations

$$G_1(u) = G_2(u) = G_3(u) = 0, \quad 1 - X + Z = 0, \quad T = T_i.$$

Then, from $T_{00} < T_i < T_{10}$, it follows that $0 < X_i < 1$, $-1 < Z_i < 0$, $V_{10} < V_i < V_{00}$, $x_{2i} = -\varepsilon/V_i$, and

$$\{u \in \bar{D} : G_1(u) = G_2(u) = G_3(u) = 0,\, 1 - X + Z = 0,\, 0 < X < X_i\} \\ \subset \{u \in \bar{D} : T \leq T_i\}.$$

Let $0 < X_0 < X_i$, $Z_0 = X_0 - 1$, $D_0 = \{u \in D : -1 < Z < Z_0, T > T_i\}$, and $\sigma = \min_{u \in D_0}[G_4(u) - G_3(u) - G_2(u) - G_1(u)]$. Then $\sigma > 0$.

Now suppose the orbit $\tilde{u}(t)$, $t \in [t_0, +\infty)$, does not meet the set $\{u \in D : T = T_i, 0 < X < 1, -1 < Z < 0\}$. Then, from $G_3(u) > 0$ in D and $\lim_{t\to\infty} \tilde{T}(t) > T_i$, we get $\tilde{T}(t) > T_i$ for all $t \in [t_0, +\infty)$.

Let t_2 and t_1 be the solutions of the equations $\tilde{Z}(t) = Z_0$ and $\tilde{Z}(t) = -1$, respectively. Since $1 - X(t) + Z(t) < 0$ in D and since $\tilde{X}(t_0) = 0$, it follows that $\tilde{Z}(t_0) < -1$. Now from $\tilde{Z}(+\infty) = 0$ and $\dot{\tilde{Z}} > 0$ in D we get $t_2 > t_1 > t_0$ and $-1 < \tilde{Z}(t) < Z_0$ for $t \in (t_1, t_2)$. Thus $\tilde{u}(t) \in D_0$ for $t \in (t_1, t_2)$.

Now, along the orbit $\tilde{u}(t)$ in D_0 we must have

$$\begin{aligned}
\frac{d}{dZ}[X + T - V - x_2] &= \left[\frac{dZ}{dt}\right]^{-1}\left[\frac{dX}{dt} + \frac{dT}{dt} - \frac{dV}{dt} - \frac{dx_2}{dt}\right] \\
&= \frac{\beta V}{(1-X)\phi_1(T)}\left[\frac{G_4(u)}{\alpha} + \frac{G_3(u)}{\kappa} - \frac{G_2(u)}{\mu_1} - \frac{G_1(u)}{\nu}\right] \\
&\geq \frac{\beta\sigma\varepsilon_1}{\max(\alpha, \kappa, \mu_1, \nu)},
\end{aligned}$$

where $1/\varepsilon_1 = \max[(1 - X)\phi_1(T)/V]$. Therefore,

$$\begin{aligned}
&\tilde{X}(t) + \tilde{T}(t) - \tilde{V}(t) - \tilde{x}_2(t)\Big|_{t_0}^{\infty} \\
&\quad \geq \int_{t_1}^{t_2}\left[\frac{G_4(\tilde{u})}{\alpha} + \frac{G_3(\tilde{u})}{\kappa} - \frac{G_2(\tilde{u})}{\mu_1} - \frac{G_1(\tilde{u})}{\nu}\right]dt \\
&\quad = \int_{-1}^{Z_0} \frac{\beta V}{(1-X)\phi_1(T)}\left[\frac{G_4(\tilde{u})}{\alpha} + \frac{G_3(\tilde{u})}{\kappa} - \frac{G_2(\tilde{u})}{\mu_1} - \frac{G_1(\tilde{u})}{\nu}\right]dZ \\
&\quad \geq \frac{\beta\sigma\varepsilon_1(Z_0 + 1)}{\max(\alpha, \kappa, \mu_1, \nu)}.
\end{aligned} \tag{3.13}$$

On the other hand, $x_2 < 0$ in D and $\tilde{x}_2(\infty) = -\varepsilon/V_{11}$. Also, from $G_2(u) < 0$ in D we get $V < J$ in D. Thus we have

$$1 + T_{11} - T(t_0) + J - V_{11} + \frac{\varepsilon}{V_{11}} \geq \frac{\beta\sigma\varepsilon(Z_0 + 1)}{\max(\alpha, \kappa, \mu_1, \nu)}.$$

But this last inequality is impossible for $\beta \gg \max(\alpha, \kappa, \mu_1, \nu)$ (this inequality makes sense; see [14; 18; 19; 20; 21; 24]). Hence the orbit $\tilde{u}(t)$ meets the set

$$\{u \in \bar{D} : T = T_i,\ 0 < X < 1,\ -1 < Z < 0\},$$

and such a $\tilde{t} > t_0$ exists. □

From now on we assume that $\beta \gg \max(\alpha, \kappa, \mu_1, \nu)$ or that the orbit $\tilde{u}(t)$ meets the hypersurface $T = T_i$ at the point $\tilde{u}_i = (\tilde{x}_{2i}, \tilde{V}_i, T_i, \tilde{X}_i, \tilde{Z}_i)$, where $0 < \tilde{X}_i < 1$ and $-1 < \tilde{Z}_i < 0$. We call the point $\tilde{u}_i$ the *ignition point*. According to Lemma 3.4, for weak detonation this point is unique, but for strong detonation there is a curve of ignition points. We are now in a position to state our Main Theorem—on the existence of structure for detonation waves—but before this we restate [14, Thm. 3.2] as follows.

THEOREM 3.8. *Suppose f in* (3.12) *is locally Lipschitz in a neighborhood of the closure of an open bounded set D that is homeomorphic to a semisphere* $\{x \in \mathbb{R}^n : |x| < 1, x_n > 0\}$, *and suppose* (3.12) *is gradient-like with respect to a real-valued function g in D. Furthermore, let the following conditions hold.*

C_1' *The set* $\{x \in \bar{D} : g(x) = 1 - c\}$ *corresponds to the set* $\{x \in \mathbb{R}^n : |x| \leq 1, x_n = c\}$ *for* $0 \leq c \leq 1$ *under the homeomorphism.*

C_2' *The set* $\{x \in \bar{D} : g(x) = 0\}$, *which consists of a single point* (*say,* $\tilde{x}$), *is a rest point of* (3.12), *and $\tilde{x}$ is the only rest point of* (3.12) *in $\bar{D}$.*

C_3' *If* $F = \{x \in \partial D : 0 < g(x) < 1\}$, *then for* $p \in \bar{F}\backslash\{\tilde{x}\}$ *we have* $p.t \notin D$ *for small positive t.*

C_4' *For* $p \in \partial D\backslash\bar{F}$, $p.t \in D$ *for* $t < 0$ *and small.*

Then, for each point $p \in \partial D\backslash\bar{F}$, *we must have* $\lim_{t\to\infty} p.t = \tilde{x}$.

THEOREM 3.9. *Suppose that* (2.2) *admits the rest points* u_{10}, u_{11}, *and* u_{m0} *for some* $0 \leq m < 1$, *and suppose that* (1.4), (2.3), *and hypotheses* H_1–H_6 *hold. Then, for given* $\nu, \mu_1, \kappa, \alpha, \beta > 0$ *with* $\beta \gg \max(\nu, \mu_1, \kappa, \alpha)$, *there is a unique orbit of* (2.2) *running from* u_{m0} *to* u_{10} *for some* $0 \leq m < 1$. *Similarly, there are infinitely many orbits of this system that run from* u_{m0} *to* u_{11} *for some* $0 \leq m < 1$.

Proof. In the region $T < T_i$ we have $\phi(T) = 0$, and from the last equation of (2.2) it follows that $Z(t)$ is constant along the orbits of this system in this region. Here we let $Z(t) = \tilde{Z}_i$, where $\tilde{Z}_i$ is the last component of $\tilde{u}_i$, the ignition point. On the hypersurface $Z = \tilde{Z}_i$, (2.2) reduces to the following 4-dimensional system of ordinary differential equations:

$$\begin{aligned}
\nu \dot{x}_2 &= Vx_2 + \varepsilon := F_1(u),\\
\mu_1 \dot{V} &= \tfrac{1}{2}x_2^2 + V - J + p(V,T,X) := F_2(u),\\
\kappa \dot{T} &= -\tfrac{1}{2}Vx_2^2 - \varepsilon x_2 - \tfrac{1}{2}V^2 + JV - E + e(V,T,X)\\
&\quad - q(V,T)(1 - X + \tilde{Z}_i) := F_3(u),\\
\alpha \dot{X} &= -V(1 - X + \tilde{Z}_i) := F_4(u),
\end{aligned} \tag{3.14}$$

where $u = (x_2, V, T, X)$. Now consider the set

$$D' = \{u \in \mathbb{R}^4 : F_1(u) < 0,\ F_2(u) < 0,\ F_3(u) > 0,\ 1 - X + \tilde{Z}_i < 0,\\ T - V - x_2 + X < T_i - \tilde{V}_i - \tilde{x}_{2i} + \tilde{X}_i\}.$$

Note that the ignition point $\tilde{u}_i$ belongs to $\partial D'$. By using Theorem 3.8, we will show that any orbit of (3.14) initiating at a point on $\partial D' \cap \{u \in \mathbb{R}^4 : T - V - x_2 + X = T_i - \tilde{V}_i - \tilde{x}_{2i} + X_i\}$ approaches the unique rest point of (3.14) given by (2.8) and located in the region $T < T_i$ as t tends to $-\infty$. We denote this rest point by $(\bar{x}_{2i}, \bar{V}_i, \bar{T}_i, \bar{X}_i)$. In order to do this, we show that the set D', the unique rest point, and the function

$$g(u) = \frac{\bar{x}_{2i} - x_2 + \bar{V}_i - V + T - \bar{T}_i + X - \bar{X}_i}{\bar{x}_{2i} - \tilde{x}_{2i} + \bar{V}_i - \tilde{V}_i + \tilde{T}_i - \bar{T}_i + \tilde{X}_i - \bar{X}_i}$$

satisfy all the conditions of Theorem 3.8.

By our definition of D', it follows that D' is homeomorphic to a semisphere, that the set $\{u \in D' : g(u) = c\}$ is homeomorphic to a disk for $0 < c < 1$, and that $(\bar{x}_{2i}, \bar{V}_i, \bar{T}_i, \bar{X}_i)$ is the only rest point of (3.14) in $\bar{D}'$. Moreover, (3.14) is gradient-like with respect to $g(u)$ in D'. Hence, conditions C_1' and C_2' of Theorem 3.8 are fulfilled.

To see that condition C_3' is satisfied, we differentiate $F_i(u)$, $1 \le i \le 3$, and $g(u)$ along the orbits of (3.14) on $\partial D'$:

$$\begin{aligned}
\left.\frac{dF_1(u)}{dt}\right|_{F_1(u)=0} &= x_2 \frac{F_2(u)}{\mu_1} > 0 \quad (\text{since } x_2 < 0 \text{ in } D'),\\
\left.\frac{dF_2(u)}{dt}\right|_{F_2(u)=0} &= x_2 \frac{F_1(u)}{\nu} + p_T \frac{F_3(u)}{\kappa} > 0 + p_X \frac{F_4(u)}{\alpha} > 0,\\
\left.\frac{dF_3}{dt}\right|_{F_3(u)=0} &= \left(-\frac{1}{2}x_2^2 - V + J + e_V - q_V(1 - X + \tilde{Z}_i)\right)\frac{F_2(u)}{\mu_1}\\
&\quad - (-Vx_2 - \varepsilon)\frac{F_1(u)}{\nu} + (e_X + q)\frac{F_4(u)}{\alpha}\\
&= (TS_V - F_2(u) - q_V(1 - X + \tilde{Z}_i))\frac{F_2(u)}{\mu_1}\\
&\quad - \frac{F_1(u)^2}{\nu} + (p_0 - p_1)V\frac{F_4}{\alpha} < 0,\\
\left.\frac{dg(u)}{dt}\right|_{g(u)=1} &= a\left(-\frac{F_1(u)}{\nu} - \frac{F_2(u)}{\mu_1} + \frac{F_3(u)}{\kappa} + \frac{F_4(u)}{\alpha}\right) > 0,
\end{aligned}$$

where $a = (\bar{x}_{2i} - \tilde{x}_{2i} + \bar{V}_i - \tilde{V}_i + \tilde{T}_i - \bar{T}_i + \tilde{X}_i - \bar{X}_i)^{-1} > 0$. Thus the flow goes out of D' on $\bigcup_{i=1}^{3}\{u : F_i(u) = 0\} \cup \{u : g(u) = 1\}$. Moreover, $\partial D' \cap \{u : F_4(u) = 0\}$ is invariant with respect to (3.14). Hence condition C_3' is satisfied, too. Therefore, by Theorem 3.8, each orbit of (3.14) initiating at a point on $\partial D' \cap \{u : T - x_2 - V + X = \tilde{T}_i - \tilde{x}_{2i} - \tilde{V}_i + \tilde{X}_i\}$ lies in $\bar{D}'$ for $t < 0$ and goes to $(\bar{x}_{2i}, \bar{V}_i, \bar{T}_i, \bar{X}_i)$ as $t \to -\infty$. Observe that, along these orbits, $-x_2(t)$, $-V(t)$, $T(t)$ are increasing and $X(t)$ is nondecreasing.

Now consider the ignition point $\tilde{u}_i$ and the unique orbit of (3.14), say

$$\tilde{\tilde{u}}(t) = (\tilde{\tilde{x}}_2(t), \tilde{\tilde{V}}(t), \tilde{\tilde{T}}(t), \tilde{\tilde{X}}(t), \tilde{Z}_i), \quad -\infty < t < t_0,$$

with $\tilde{\tilde{u}}(t_0) = (\tilde{x}_{2i}, \tilde{V}_i, \tilde{T}_i, \tilde{X}_i, \tilde{Z}_i)$ and $\lim_{t\to-\infty} \tilde{\tilde{u}}(t) = (\bar{x}_{2i}, \bar{V}_i, \bar{T}_i, \bar{X}_i, \bar{Z}_i)$, which exists by the foregoing argument. Along this orbit, $-x_2(t)$, $-V(t)$, $T(t)$, $X(t)$ are increasing and $Z(t)$ is constant. This orbit lies in D, the domain used in the proof of Lemma 3.6. Now define

$$u(t) = \begin{cases} \tilde{u}(t) & \text{for } t \geq t_0, \\ \tilde{\tilde{u}}(t) & \text{for } t \leq t_0. \end{cases}$$

Then $u(t)$ is a complete orbit of (2.2) lying in D, and it runs from u_{m0} to either u_{10} or u_{11} for some $0 \leq m < 1$. This completes the proof. □

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