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# $C^k$-Estimates for the $\bar\partial_b$-Equation on Convex Domains of Finite Type

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## 1. Introduction

Since the construction in [8] of a support function for convex domains of finite type, many results about the regularity of Cauchy–Riemann equations have been obtained on these domains. We should mention [7], in which a $\bar\partial$-solving operator for all convex domains of finite type was constructed that satisfies optimal uniform Hölder estimates. Note that this result was already obtained in [5] by using properties of the Bergman kernel. For a convex domain of finite type, Hefer [12] obtained Hölder and $L^p$-estimates depending on Catlin's multitype. In [2], a modification of the operator of [7] led to $C^k$-estimates for all $k \in \mathbb{N}$. In this work, we are interested in the regularity of tangential Cauchy–Riemann equations.

Let $D$ be a bounded convex domain in $\mathbb{C}^n$ of finite type $m$, with $bD$ its boundary. We denote by $r$ a $C^\infty$-defining convex function for $D$ such that $\operatorname{grad} r(\zeta) \neq 0$ for all $\zeta$ in a neighborhood $\mathcal{V}$ of $bD$. We use the definition of the equivalence classes and of the $\bar\partial_b$ operator given in [13] and denote by $[f]$ the class of a form $f$.

Let $C^\alpha_{0,q}(bD)$, $\alpha \geq 0$, be the set of $(0,q)$-forms of regularity $C^\alpha$ in a neighborhood of $bD$ and let $\tilde C^\alpha_{0,q}(bD)$ be the set of equivalence classes $[f]$ such that $f \in C^\alpha_{0,q}(bD)$. The tangential norm $\|[f]\|_{bD,\alpha}$ is then defined by

$$\|[f]\|_{bD,\alpha} := \inf\{\|g\|_{bD,\alpha},\ g \in C^\alpha_{0,q}(bD),\ [g] = [f]\}.$$

Now we state our main result.

Theorem 1.1. *Let $D$ be a bounded convex domain with $C^\infty$-smooth boundary of finite type $m$ in $\mathbb{C}^n$, and let $q = 1, \dots, n-1$. Then there exist two linear operators $[T_q], [\tilde T_q] \colon \tilde C^0_{0,q}(bD) \to \tilde C^0_{0,q-1}(bD)$ such that the following statements hold.*

(i) *For all $k \in \mathbb{N}$ there is a constant $c_k > 0$ such that, for all $[f] \in \tilde C^k_{0,q}(bD)$, $[T_q][f]$ and $[\tilde T_q][f]$ are in $\tilde C^{k+1/m}_{0,q-1}(bD)$ and*

$$\|[\tilde T_q][f]\|_{bD,k+1/m} + \|[T_q][f]\|_{bD,k+1/m} \leq c_k \|[f]\|_{bD,k}.$$

(ii) *For all $[f] \in \tilde C_{0,q}(bD)$ such that $\bar\partial_b[f]$ belongs to $\tilde C_{0,q+1}(bD)$ and with the additional hypothesis when $q = n-1$ that $\int_{bD} f \wedge \phi = 0$ for all $\bar\partial$-closed forms $\phi \in C^\infty_{n,0}(bD)$, we have*

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$$[f] = \bar{\partial}_b([T_q] - [\tilde{T}_q])[f] + ([T_{q+1}] - [\tilde{T}_{q+1}])\bar{\partial}_b[f]$$

(*in the case* $q = n - 1$ *we set* $[T_n] = [\tilde{T}_n] = 0$).

Theorem 1.1 for $k = 0$ was already shown for strictly pseudoconvex domains in [13], and the method we follow here is close to that used by Henkin. More precisely, when $f$ is a continuous representative of $[f] \in \tilde{C}^0_{0,q}(bD)$ we write $f$ as the *jump* on $bD$ of two $(0, q)$-forms $f_+$ and $f_-$, where $f_+$ is defined on $D$ and $f_-$ on $\mathbb{C}^n \setminus \bar{D}$. Then we use two integral formulas with the kernels $\Omega_{n,q}$ and $\tilde{\Omega}^t_{n,q}$ to represent $f_+$ and $f_-$, yielding the two operators $[T_q]$ and $[\tilde{T}_q]$.

In the strictly pseudoconvex case, Henkin first defined $\Omega_{n,q}$. Next he defined a second kernel $\tilde{\Omega}_{n,q}$ by swapping (in $\Omega_{n,q}$) the roles of $z$ and the integration variable $\zeta$. We define $\Omega_{n,q}$ as in [13] and get the kernel already used in [7]. This gives us an operator $T_q$ satisfying Hölder estimates and inducing $[T_q]$. However, we cannot define the second kernel as in Henkin [13] because the normal component in $\zeta$ of $\Omega_{n,q}$ has a bad behavior. This does not matter in [7] because the main difficulty is the control of a boundary integral and so the normal component in the integration variable does not play any role. If we define $\tilde{\Omega}_{n,q}$ by exchanging $\zeta$ and $z$ in $\Omega_{n,q}$, then the normal component in $z$ of $\tilde{\Omega}_{n,q}$ will have a bad behavior and will not disappear when integrating over the boundary. An operator $\tilde{T}_q$ defined with such a kernel and a continuous $(0, q)$-form $f$ may give a form $\tilde{T}_q f$ unbounded in a neighborhood of $bD$ and will not induce an equivalence class! However, by definition the equivalence classes do not take the normal component into account. Hence we define a suitable kernel $\tilde{\Omega}^t_{n,q}$ by keeping only the tangential component in $z$ of $\tilde{\Omega}_{n,q}$. Now $\tilde{\Omega}^t_{n,q}$ gives us an operator $\tilde{T}^t_q$ such that $\tilde{T}^t_q f$ is Hölder continuous of order $1/m$ provided $f$ is continuous. Therefore, $\tilde{T}^t_q$ induces $[\tilde{T}_q]$.

Once the two operators $[T_q]$ and $[\tilde{T}_q]$ are correctly defined, to show (i) of Theorem 1.1 we estimate the kernels and their tangential derivatives with respect to $\varepsilon$-extremal bases and, using an induction argument, we integrate by parts many times.

## 2. Definition of the Operators

We recall the definition from [8] of the support function $F$. For $\alpha \in \mathbb{R}$ we set $D_\alpha := \{z \in \mathbb{C}^n, r(z) < \alpha\}$. We fix some $\zeta$ in $\mathcal{V}$ and denote by $T^{\mathbb{C}}_\zeta bD_{r(\zeta)}$ the complex tangent space to $bD_{r(\zeta)}$ at $\zeta$ and by $\eta_\zeta$ the outer unit normal at $\zeta$ to $bD_{r(\zeta)}$. Then we choose an orthonormal basis $w'_1, \dots, w'_n$ such that $w'_1 = \eta_\zeta$. Set $r_\zeta(\omega) = r(\zeta + \omega_1 w'_1 + \cdots + \omega_n w'_n)$ and

$$F_\zeta(\omega) := 3\omega_1 + K\omega_1^2 - K' \sum_{j=2}^{m} \kappa_j M^{2^j} \sum_{\substack{|\beta|=j \\ \beta_1=0}} \frac{1}{\beta!} \frac{\partial r_\zeta}{\partial \omega^\beta}(0)\omega^\beta,$$

where $K, K', M$ are positive real numbers and

$$\kappa_j = \begin{cases} 1 & \text{if } j \equiv 0 \bmod 4, \\ -1 & \text{if } j \equiv 2 \bmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

Let $z \in \mathbb{C}^n$ be written as $z = \zeta + \omega_{1,z} w_1' + \cdots + \omega_{n,z} w_n'$. We define

$$F(\zeta, z) := F_\zeta(\omega_{1,z}, \ldots, \omega_{n,z}).$$

THEOREM 2.1. *The neighborhood $\mathcal{V}$ of $bD$ and the constants $K, K', M$ in the definition of $F$ can be chosen such that, for some positive real numbers $k', c, R$ and for any $\zeta \in \mathcal{V}$, any unit vector $v \in T_\zeta^{\mathbb{C}} bD_{r(\zeta)}$, and any $w = (w_1, w_2) \in \mathbb{C}^2$ with $|w| < R$ and $r(\zeta + w_1\eta_\zeta + w_2 v) - r(\zeta) \leq 0$, $F$ satisfies*

$$\begin{aligned} &\Re F(\zeta, \zeta + w_1\eta_\zeta + w_2 v) \\ &\quad \leq -\left|\frac{\Re w_1}{2}\right| - \frac{K}{2}(\Im w_1)^2 - \frac{K'k'}{4}\sum_{j=2}^{m}\sum_{\alpha+\beta=j}\left|\left.\frac{\partial^j r(\zeta + \lambda v)}{\partial\lambda^\alpha \partial\bar\lambda^\beta}\right|_{\lambda=0}\right| |w_2|^j \\ &\quad\quad + c(r(\zeta + w_1\eta_\zeta + w_2 v) - r(\zeta)). \end{aligned}$$

This theorem was proved in [8]. However, we may have $F(\zeta, z) = 0$ when $|\zeta - z| > R$ and so we should use the global version $S$ of [2]. The construction of $S$ does not require any ideas other than those of [16]. As in the strictly pseudoconvex case (see [16, Proof of Thm. 1.13, p. 224]), $S$ satisfies the following conditions.

(i) $S$ is of regularity $C^\infty$ in $\mathcal{V} \times U$, where $\mathcal{U}$ is a neighborhood of $\bar{D}$ and $S(\zeta, \cdot)$ is holomorphic on $\mathcal{U}$.
(ii) $S(\zeta, \zeta) = 0$ for $\zeta \in \mathcal{U} \cap \mathcal{V}$.
(iii) There exists a constant $c > 0$ such that $\Re S(\zeta, z) \leq -c|\zeta - z|^m$ for all $(\zeta, z) \in \mathcal{V} \times \mathcal{U}$ with $r(\zeta) \geq r(z)$.
(iv) On $\{(\zeta, z) \in \mathcal{V} \times \mathcal{U}, |\zeta - z| < R/2\}$, there is a $C^\infty$-function $A$ with

$$\tfrac{1}{2} \leq |A(\zeta, z)| \leq \tfrac{3}{2} \quad \text{and} \quad S = A \cdot F.$$

Moreover, $A(\zeta, z) = 1/(1 + (m' - v(\zeta, z))F(\zeta, z))$, where $m'$ is a constant and $v$ a bounded $C^\infty$ function defined on $\mathcal{V} \times \mathcal{U}$ such that all its derivatives are also bounded on $\mathcal{V} \times \mathcal{U}$.

We need a Hefer–Leray section for $S$. We choose an arbitrary unitary matrix $U$ of $\mathbb{C}^{n\times n}$ and set

$$\Sigma(\zeta, \omega) = S(\zeta, \zeta + U\omega), \tag{1}$$

$$\sigma_j(\zeta, \omega) = \int_0^1 \frac{\partial\Sigma}{\partial\omega_j}(\zeta, t\omega)\, dt, \tag{2}$$

$$Q(\zeta, z) = -\bar{U}(\sigma_1(\zeta, \bar{U}^t(z - \zeta)), \ldots, \sigma_n(\zeta, \bar{U}^t(z - \zeta))). \tag{3}$$

One can easily see that $\Sigma(\zeta, \omega) = \sum_{j=1}^n \omega_j \sigma_j(\zeta, \omega)$ and that $Q$ does not depend on $U$ and satisfies $S(\zeta, z) = \sum_{j=1}^n Q_j(\zeta, z)(\zeta_j - z_j)$.

Later on we will choose $U = U(\zeta)$ so that $\bar{U}^t\eta_\zeta = (1, 0, \ldots, 0)$. With that choice, the $\sigma_j$ will locally have the same behavior as the $Q_\zeta^j$ of [7].

Now we define the kernels and set $\eta_0(\zeta, z) = \sum_{j=1}^n \overline{\zeta_j - z_j}\, d\zeta_j$, $\eta_1(\zeta, z) = \sum_{j=1}^n Q_j(\zeta, z)\, d\zeta_j$, and

$$\eta(\zeta, \lambda, z) = (1 - \lambda)\frac{\eta_0(\zeta, z)}{|\zeta - z|^2} + \lambda\frac{\eta_1(\zeta, z)}{S(\zeta, z)}.$$

For $0 \le q \le n-1$, set

$$\Omega_{n,q} = \frac{(-1)^{q(q-1)/2}}{(2i\pi)^n} \binom{n-1}{q} \eta \wedge (\bar\partial_{\zeta,\lambda}\eta)^{n-q-1} \wedge (\bar\partial_z \eta)^q$$

and $\Omega_{n,-1} = \Omega_{n,n} = 0$. Then, for all $z \in D$ and all $f \in C_{0,q}(bD)$, we define

$$T_q f(z) = \int_{bD\times[0,1]} f(\zeta) \wedge \Omega_{n,q-1}(\zeta, \lambda, z).$$

In order to define $\tilde\Omega^t_{n,q}$ we set $\tilde S(\zeta, z) = S(z, \zeta)$ and $\tilde Q(\zeta, z) = -Q(z, \zeta)$, so that $\tilde S(\zeta, z) = \sum_{j=1}^n \tilde Q_j(\zeta, z)(\zeta_j - z_j)$ for all $(\zeta, z) \in \mathcal{U} \times \mathcal{V}$. We then set $\tilde\eta_1(\zeta, z) = \sum_{j=1}^n \tilde Q_j(\zeta, z) d\zeta_j$ and

$$\tilde\eta(\zeta, \lambda, z) = (1-\lambda)\frac{\eta_0(\zeta, z)}{|\zeta - z|^2} + \lambda \frac{\tilde\eta_1(\zeta, z)}{\tilde S(\zeta, z)}.$$

Next we define an operator $\bar\partial^t$ that removes the normal component of $\bar\partial$. For $z \in \mathcal{V}$ let $\tilde\Psi := \tilde\Psi(z)$ be a unitary matrix such that $\tilde\Psi \eta_z = (1, 0, \dots, 0)$. We set

$$\bar L^z_i := \sum_{j=1}^n \tilde\Psi_{ij} \frac{\partial}{\partial \bar z_j} \quad \text{and} \quad \bar l^z_i := \sum_{j=1}^n \overline{\tilde\Psi_{ij}} d\bar z_j.$$

Note that, for $i = 2, \dots, n$, the $L^z_i$ are tangential vectors fields and the $l^z_i$ are tangential forms. Moreover, since $\tilde\Psi$ is a unitary matrix, for all $(p, q)$-forms $f$ we have $\sum_{j=1}^n \bar L^z_j(f) \wedge \bar l^z_j = (-1)^{p+q}\bar\partial f$.

For all $(p, q)$-forms $f$ we set $\bar\partial^t f = \bar\partial f + (-1)^{p+q+1} \bar L^z_1(f) \wedge \bar l^z_1$. Note that $\bar\partial^t$ is well-defined and does not depend on $\tilde\Psi$, since

$$\bar L^z_1 = \sum_{j=1}^n \frac{1}{|\bar\partial r(z)|} \frac{\partial r}{\partial z_j}(z) \frac{\partial}{\partial \bar z_j} \quad \text{and} \quad \bar l^z_1 = \frac{1}{|\bar\partial r(z)|} \bar\partial_z r(z).$$

For $q = 1, \dots, n-1$ we define

$$\tilde\Omega^t_{n,q} = \frac{(-1)^{q(q-1)/2}}{(2i\pi)^n} \binom{n-1}{q} \tilde\eta \wedge (\bar\partial_{\zeta,\lambda}\tilde\eta)^{n-q-1} \wedge (\bar\partial^t_z \tilde\eta)^q$$

and $\tilde\Omega^t_{n,-1} = \tilde\Omega^t_{n,n} = 0$. For $z \in \mathcal{V} - D$ and $f \in C_{0,q}(bD)$ we set

$$\tilde T^t_q f(z) = \int_{bD\times[0,1]} f(\zeta) \wedge \tilde\Omega^t_{n,q-1}(\zeta, \lambda, z).$$

LEMMA 2.2. *For* $\Delta = \frac{\partial}{\partial z_j}$ *or* $\frac{\partial}{\partial \bar z_j}$, $j = 1, \dots, n$, *and for* $f \in C^0_{0,q}(bD)$, $q = 1, \dots, n-1$, *the following inequalities hold uniformly with respect to* $z$ *and* $f$:

(i) $|\Delta T_q f(z)| \lesssim \|f\|_{bD,0} |r(z)|^{1/m-1}$ *for all* $z \in D \cap \mathcal{V}$;
(ii) $|\Delta \tilde T^t_q f(z)| \lesssim \|f\|_{bD,0} |r(z)|^{1/m-1}$ *for all* $z \in \mathcal{V} - \bar D$.

We will prove Lemma 2.2 in Section 5. This lemma and Hardy–Littlewood's lemma imply that $T_q f$ and $\tilde T^t_q f$ belong to $C^{1/m}_{0,q}(bD)$ when $f$ belongs to $C^0_{0,q}(bD)$.

Moreover, for $f, g \in C^0_{0,q}(bD)$ such that $[f] = [g]$ we have $[T_q f] = [T_q g]$. Indeed, we set $\Phi_z(\zeta) = \int_{\lambda\in[0,1]} \Omega_{n,q-1}(\zeta,\lambda,z)$ for $z \in D$ and $\zeta \in \mathcal{V} \setminus \overline{D_{r(z)}}$. Since $S(\zeta, z) \neq 0$ for all $(\zeta, z) \in \mathcal{V} \times D$ with $r(\zeta) > r(z)$, it follows that $\Phi_z$ is of regularity $C^\infty$ on $\mathcal{V} \setminus \overline{D_{r(z)}}$. Therefore, by definition of the equivalent class, $T_q f(z) = T_q g(z)$ for all $z \in D$. Lemma 2.2 implies that $T_q f$ and $T_q g$ belong to $C^{1/m}_{0,q-1}(\bar D)$. Thus we have $[T_q f] = [T_q g]$. Analogously, we also have $[\tilde T^t_q f] = [\tilde T^t_q g]$. Hence we can set

$$[T_q][f] = [T_q f] \quad \text{and} \quad [\tilde T_q][f] = [\tilde T^t_q f]$$

for $[f] \in \tilde C^0_{0,q}(bD)$ such that $f$ belongs to $C^0_{0,q}(bD)$.

*Proof of Theorem 1.1(ii).* We fix $[f] \in \tilde C^0_{0,q}(bD)$ with $f \in C^0_{0,q}(bD)$, let $g \in C^0_{0,q+1}(bD)$ be a representative of $\bar\partial_b[f]$, and set

$$\iota_1 : \begin{cases} \mathbb{C}^n \times \{1\} \times \mathbb{C}^n \to \mathbb{C}^n \times [0,1] \times \mathbb{C}^n, \\ (\zeta,\lambda,z) \mapsto (\zeta,\lambda,z); \end{cases}$$

$$\iota_0 : \begin{cases} \mathbb{C}^n \times \{0\} \times \mathbb{C}^n \to \mathbb{C}^n \times [0,1] \times \mathbb{C}^n, \\ (\zeta,\lambda,z) \mapsto (\zeta,\lambda,z). \end{cases}$$

We also set $B^t_{n,q} = \iota_0^*(\tilde\Omega^t_{n,q})$, $\tilde K^t_{n,q} = \iota_1^*(\tilde\Omega^t_{n,q})$, $K_{n,q} = \iota_1^*(\Omega_{n,q})$, $B_{n,q} = \iota_0^*(\Omega_{n,q})$,

$$\tilde\Omega_{n,q} = \frac{(-1)^{q(q-1)/2}}{(2i\pi)^n} \binom{n-1}{q} \tilde\eta \wedge (\bar\partial_{\zeta,\lambda}\tilde\eta)^{n-q-1} \wedge (\bar\partial_z \tilde\eta)^q, \quad q = 1, \dots, n-1,$$

and $\tilde\Omega_{n,-1} = \tilde\Omega_{n,n} = 0$. Since $B^t_{n,q}$ is the tangential part in $z$ of the Bochner–Martinelli kernel, it follows that $|B^t_{n,q}(\zeta,z)| \lesssim 1/|\zeta - z|^{2n-1}$ for all $z, \zeta \in \mathbb{C}^n$, $\zeta \neq z$.

Next we set

$$f_+(z) = \int_{bD} f(\zeta) \wedge B_{n,q}(\zeta,z), \quad z \in D,$$

$$f_-(z) = \int_{bD} f(\zeta) \wedge B_{n,q}(\zeta,z), \quad z \in \mathbb{C}^n \setminus \bar D,$$

$$f^t_-(z) = \int_{bD} f(\zeta) \wedge B^t_{n,q}(\zeta,z), \quad z \in \mathcal{V} \setminus \bar D,$$

and first show that $f_+ = T_{q+1} g + \bar\partial_z T_q f$. Using a cutoff function, we assume that $f$ has compact support in $\mathbb{C}^n$. In order to apply the Stokes theorem we consider a sequence $(f_N)_{N\in\mathbb{N}} \subset C^\infty_{0,q}(\mathbb{C}^n)$ that converges uniformly to $f$. The Stokes theorem gives

$$\begin{aligned} &\int_{bD} f_N(\zeta) \wedge B_{n,q}(\zeta,z) - \int_{bD} f_N(\zeta) \wedge K_{n,q}(\zeta,z) \\ &\quad = \int_{bD\times[0,1]} \bar\partial_\zeta f_N(\zeta) \wedge \Omega_{n,q}(\zeta,\lambda,z) \\ &\qquad + (-1)^q \int_{bD\times[0,1]} f_N(\zeta) \wedge \bar\partial_{\zeta,\lambda}\Omega_{n,q}(\zeta,\lambda,z). \end{aligned} \tag{4}$$

On one hand, by Stokes' theorem and the definition of $\bar\partial_b[f]$ we have

$$\int_{bD\times[0,1]} g(\zeta)\wedge\Omega_{n,q}(\zeta,\lambda,z)=\lim_{N\to\infty}\int_{bD\times[0,1]}\bar\partial_\zeta f_N(\zeta)\wedge\Omega_{n,q}(\zeta,\lambda,z).$$

On the other hand, $(-1)^q\bar\partial_z\Omega_{n,q-1}=\bar\partial_{\zeta,\lambda}\Omega_{n,q}$ and so $N\to\infty$ in (4) yields

$$\begin{aligned}\int_{bD} f(\zeta)\wedge B_{n,q}(\zeta,z)-\int_{bD} f(\zeta)\wedge K_{n,q}(\zeta,z)\\ =\int_{bD\times[0,1]} g(\zeta)\wedge\Omega_{n,q}(\zeta,\lambda,z)+\int_{bD\times[0,1]} f(\zeta)\wedge\bar\partial_z\Omega_{n,q-1}(\zeta,\lambda,z).\end{aligned}$$

Both $Q$ and $S$ are holomorphic in $z$, so $\int_{bD} f(\zeta)\wedge K_{n,q}(\zeta,z)=0$ for all $q=1,\dots,n-1$ and the following equality holds on $D$:

$$f_+=T_{q+1}g+\bar\partial_z(T_qf). \tag{5}$$

Observe that $\tilde\Omega^t_{n,q}$ is the tangential part in $z$ of $\tilde\Omega_{n,q}$ and therefore $\bar\partial_{\zeta,\lambda}\tilde\Omega^t_{n,q}=(-1)^q\bar\partial^t_z\tilde\Omega^t_{n,q-1}$. Moreover, since $\tilde S$ and $\tilde Q$ are holomorphic with respect to $\zeta$, $\int_{bD} f(\zeta)\wedge\tilde K^t_{n,q}(\zeta,z)=0$ for all $z\in\mathcal V-D$ and all $q=1,\dots,n-2$. When $q=n-1$, we have $\int_{bD} f(\zeta)\wedge\tilde K^t_{n,n-1}(\zeta,z)=0$ for all $z$ in $\mathcal V-D$ because $\tilde K^t_{n,n-1}$ is a smooth $\bar\partial$-closed form of bidegree $(n,0)$ in $\zeta$. Therefore, as for (5) one can show that, on $\mathcal V-D$,

$$f^t_-=\tilde T^t_{q+1}g+\bar\partial^t_z(\tilde T^t_qf). \tag{6}$$

Now we use the jump formula, which was already used in [13] and proved in [16, Prop. IV 2.2] for the case of a function and in [11, par. 7; 3, Chaps. 19.2 & 24.1] when $q>0$. For all $\phi\in C^\infty_{n,n-q-1}(bD)$,

$$\int_{bD} f(z)\wedge\phi(z)=\lim_{\substack{\varepsilon\to0\\ \varepsilon>0}}\int_{bD}(f_+(z-\varepsilon\eta_z)-f_-(z+\varepsilon\eta_z))\wedge\phi(z). \tag{7}$$

In (7) we replace $f_-$ by $f^t_-$. There exists a form $h$ of bidegree $(n,n-q-1)$ in $\zeta$ and $(0,q-1)$ in $z$ such that $|h(\zeta,z)|\lesssim 1/|\zeta-z|^{2n-1}$ and $B^t_{n,q}(\zeta,z)-B_{n,q}(\zeta,z)=h(\zeta,z)\wedge\bar\partial_zr(z)$ for all $z,\zeta\in\mathbb C^n$, $\zeta\neq z$. So for all $\phi\in C^\infty_{n,n-q-1}(bD)$ and all $\varepsilon>0$ we have

$$\begin{aligned}\int_{bD}(f^t_-(z+\varepsilon\eta_z)-f_-(z+\varepsilon\eta_z))\wedge\phi(z)\\ =\int_{z\in bD}\left(\int_{\zeta\in bD} f(\zeta)\wedge h(\zeta,z+\varepsilon\eta_z)\right)\wedge(\bar\partial_zr(z+\varepsilon\eta_z)-\bar\partial_zr(z))\wedge\phi(z).\end{aligned}$$

Now, for $\zeta,z\in bD$, $|\zeta-(z+\varepsilon\eta_z)|\geq\varepsilon$. Therefore $|h(\zeta,z+\varepsilon\eta_z)|\lesssim\varepsilon^{-1/2}/|\zeta-z|^{2n-3/2}$ and $\left|\int_{\zeta\in bD} f(\zeta)\wedge h(\zeta,z+\varepsilon\eta_z)\right|\lesssim\|f\|_{bD,0}\varepsilon^{-1/2}$.

By the smoothness of $r$ we have $|\bar\partial r(z+\varepsilon\eta_z)-\bar\partial r(z)|\lesssim\varepsilon$ for all $z\in bD$, so $\left|\int_{bD}(f^t_-(z+\varepsilon\eta_z)-f_-(z+\varepsilon\eta_z))\wedge\phi(z)\right|\lesssim\varepsilon^{1/2}$ and then (7) becomes

$$\int_{bD} f(z)\wedge\phi(z)=\lim_{\substack{\varepsilon\to0\\ \varepsilon>0}}\int_{bD}(f_+(z-\varepsilon\eta_z)-f^t_-(z+\varepsilon\eta_z))\wedge\phi(z). \tag{8}$$

As in the strictly pseudoconvex case, (5) together with the Hölder continuity of $T_q f$ and $T_{q+1}g$ lead to

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \int_{bD} f_+(z-\varepsilon\eta_z)\wedge\phi(z) = \int_{bD} T_{q+1}g(z)\wedge\phi(z) + (-1)^q \int_{bD} T_q f(z)\wedge\bar\partial_z\phi(z), \quad (9)$$

whereas (6) together with the Hölder continuity of $\tilde T_q^t f$ and $\tilde T_{q+1}^t g$ yield

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \int_{bD} f_-^t(z+\varepsilon\eta_z)\wedge\phi(z) = \int_{bD} \tilde T_{q+1}^t g(z)\wedge\phi(z) + (-1)^q \int_{bD} \tilde T_q^t f(z)\wedge\bar\partial_z\phi(z). \quad (10)$$

We plug (9) and (10) into the jump formula (8) and obtain, by definition of equivalence classes, $[f] = ([T_{q+1}] - [\tilde T_{q+1}])\bar\partial_b[f] + \bar\partial_b([T_q]-[\tilde T_q])[f]$. □

## 3. Estimates of the Hefer Sections

In this section we estimate the Hefer sections and their derivatives, which will be needed in Section 4.

For a vector field

$$B^z = \sum_{i=1}^n a_i(z)\frac{\partial}{\partial z_i} + b_i(z)\frac{\partial}{\partial \bar z_i},$$

we set

$$B^\zeta = \sum_{i=1}^n a_i(\zeta)\frac{\partial}{\partial \zeta_i} + b_i(\zeta)\frac{\partial}{\partial \bar\zeta_i}.$$

Let us define a local basis of vector fields. We fix some point $\zeta_0 \in bD$. Since $\operatorname{grad} r(\zeta)\neq 0$ for all $\zeta\in bD$, there exist $R', c>0$ and $i$ such that

$$\left|\frac{\partial r}{\partial\zeta_i}(\zeta)\right| > c \quad \text{for all } \zeta\in B(\zeta_0,R') := \{\zeta\in\mathbb{C}^n,\ |\zeta-\zeta_0|<R'\}.$$

Moreover, there is no restriction in assuming that $i=1$ and that $R'$ and $c$ do not depend on $\zeta_0$. We set

$$Z_1^\zeta = \frac12\left(\left(\frac{\partial r}{\partial\zeta_1}(\zeta)\right)^{-1}\frac{\partial}{\partial\zeta_1} - \left(\frac{\partial r}{\partial\bar\zeta_1}(\zeta)\right)^{-1}\frac{\partial}{\partial\bar\zeta_1}\right),$$

$$Z_j^\zeta = \frac{\partial}{\partial\zeta_j} - \frac{\partial r}{\partial\zeta_j}(\zeta)\left(\frac{\partial r}{\partial\zeta_1}(\zeta)\right)^{-1}\frac{\partial}{\partial\zeta_1} \quad \text{when } j=2,\dots,n,$$

$$\bar Z_j^\zeta = \frac{\partial}{\partial\bar\zeta_j} - \frac{\partial r}{\partial\bar\zeta_j}(\zeta)\left(\frac{\partial r}{\partial\bar\zeta_1}(\zeta)\right)^{-1}\frac{\partial}{\partial\bar\zeta_1} \quad \text{when } j=2,\dots,n.$$

Here $Z_1^\zeta,\dots,Z_n^\zeta,\bar Z_2^\zeta,\dots,\bar Z_n^\zeta$ is a basis of tangential vectors fields on $B(\zeta_0,R')$. Next we set

$$V_1^\zeta := \frac{1}{|\partial r(\zeta)|}\sum_{i=1}^n \frac{\partial r}{\partial \bar\zeta_i}(\zeta)\frac{\partial}{\partial \zeta_i} - \frac{\partial r}{\partial \zeta_i}(\zeta)\frac{\partial}{\partial \bar\zeta_i}.$$

Note that $V_1^\zeta$ is also a tangential vector field, which will be useful in Section 4 thanks to the following lemma.

LEMMA 3.1. *If $R' \in ]0, R/2]$ is sufficiently small, with $R$ as given by Theorem 2.1, then for all $z, \zeta \in B(\zeta_0, R')$ we have*

$$|V_1^\zeta S(\zeta,z)| \geq 1 \quad \textit{and} \quad |V_1^\zeta \tilde{S}(\zeta,z)| \geq \frac{1}{2}.$$

*Proof.* This is obvious because $V_1^\zeta F(\zeta,z) = O(|\zeta - z|) - 3$. □

For $z \in \mathcal{V}$ near $bD$ and $\varepsilon > 0$, as in [7] we denote by $w_1^*, \dots, w_n^*$ an $\varepsilon$-extremal basis at $z$ such that $w_1^* = \eta_z$. We use $\zeta^* = (\zeta_1^*, \dots, \zeta_n^*)$ to denote the $\varepsilon$-extremal coordinates at $z$ of a point $\zeta$. We seek estimates of the Hefer coefficients and their derivatives in terms of the following complex directional level distances:

$$\tau(z,v,\varepsilon) := \sup\{\tau, r(z+\lambda v) - r(z) < \varepsilon \text{ for all } \lambda \in \mathbb{C},\ |\lambda| < \tau\}$$

(see [15]). We write $\tau_i(z,\varepsilon) = \tau(z, w_i^*, \varepsilon)$, $i = 1, \dots, n$, and set $\mathcal{P}_\varepsilon(z) := \{\zeta \in \mathbb{C}^n, |\zeta_i^*| < \tau_i(z,\varepsilon),\ i = 1, \dots, n\}$ the polydisc of McNeal centered at $z$. As in [7], for $\varepsilon_0 > 0$ sufficiently small we cover $\mathcal{P}_{\varepsilon_0}(z)$ with the polyannuli $\mathcal{P}_\varepsilon^i(z) := \mathcal{P}_{2^{-i}\varepsilon}(z) \setminus c_1\mathcal{P}_{2^{-i}\varepsilon}(z)$, where $c_1$ (given by [7, Prop. 3.1(i)]) is such that $c_1\mathcal{P}_{2^{-i}\varepsilon}(z)$ is included in $\mathcal{P}_{2^{-1}(2^{-i}\varepsilon)}(z)$ for all $z$, all $\varepsilon > 0$, and all $i \in \mathbb{N}$. This gives us the covering

$$\mathcal{P}_{\varepsilon_0}(z) \subset \mathcal{P}_{|r(z)|}(z) \cup \bigcup_{i=0}^{j_0} \mathcal{P}_{\varepsilon_0}^i(z), \tag{11}$$

where $j_0$ satisfies $2^{-j_0}\varepsilon_0 \eqsim |r(z)|$ uniformly with respect to $z$ and $\varepsilon_0$.

We assume that $\varepsilon_0$ is sufficiently small that:

(i) $\mathcal{P}_\varepsilon(z)$ is included in $B(z,R')$ for all $z \in \mathcal{V}$ and $\varepsilon \in ]0, \varepsilon_0]$, with $R'$ given by Lemma 3.1; and

(ii)
$$\left|\frac{\partial r}{\partial w_1^*}(\zeta)\right| \gtrsim 1 \quad \text{for all } \zeta \in \mathcal{P}_\varepsilon(z_0)$$
uniformly with respect to $z_0$, $\zeta$, and $\varepsilon \in ]0, \varepsilon_0]$.

Now we fix $z_0 \in \mathcal{V}$, $\varepsilon \in ]0, \varepsilon_0]$, and an $\varepsilon$-extremal basis at $z_0$. We denote by $\Phi_*$ the unitary matrix such that $\zeta^* = \Phi_*(\zeta - z_0)$. To derive our estimates we use the matrix $\Psi(\zeta)$, defined in [2], that satisfies $\Psi(\zeta)\Phi_*\eta_\zeta = (1, 0, \dots, 0)$ for all $\zeta \in B(\zeta_0, R')$.

In (1), (2), and (3) we set $U = \overline{\Psi(\zeta)\Phi_*}^t$, and we express $\Omega_{n,q}$ in the $\varepsilon$-extremal basis by setting $Q^*(\zeta,z) := \bar\Phi_* Q(\zeta,z)$. Thus we have $\eta_1(\zeta,z) = \sum_{i=1}^n Q_i^*(\zeta,z)\, d\zeta_i^*$ and

$$\bar\partial_\zeta \eta_1(\zeta,z) = \sum_{i,j=1}^n \frac{\partial Q_i^*}{\partial \bar\zeta_j^*}(\zeta,z)\,d\bar\zeta_j^* \wedge d\zeta_i^*.$$

In order to express $\tilde\Omega_{n,q}^t$ in the $\varepsilon$-extremal basis we set $\tilde Q^*(\zeta,z) := \bar\Phi_* \tilde Q(\zeta,z)$,

$$\bar V_i^z = \sum_{j=1}^n \Psi_{ij}(z)\frac{\partial}{\partial \bar z_j^*}, \quad \text{and} \quad \bar q_i^z = \sum_{j=1}^n \overline{\Psi_{ij}(z)}\,d\bar z_j^*.$$

Thus we have

$$\tilde\eta_1(\zeta,z) = \sum_{i=1}^n \tilde Q_i^*(\zeta,z)\,d\zeta_i^*,$$

$$\bar\partial_z^t \tilde\eta_1(\zeta,z) = \sum_{\substack{i=1\\ j=2}}^n \bar V_i^z(\tilde Q_i^*)(\zeta,z)\bar q_j^z \wedge d\zeta_i^*.$$

LEMMA 3.2. (i) *For $\zeta \in \mathcal{P}_\varepsilon^0(z_0)$ with $r(\zeta) \ge r(z_0)$ we have, uniformly with respect to $\zeta$, $z_0$, and $\varepsilon$,*

$$|S(\zeta,z_0)| \gtrsim \varepsilon + r(\zeta) - r(z_0).$$

(ii) *For $\zeta \in \mathbb{C}^n$ with $r(\zeta) \ge r(z_0)$ we have, uniformly with respect to $\zeta$ and $z_0$,*

$$|S(\zeta,z_0)| \gtrsim r(\zeta) - r(z_0).$$

(iii) *For $\zeta \in \mathcal{P}_\varepsilon^0(z_0)$ with $r(\zeta) \le r(z_0)$ we have, uniformly with respect to $\zeta$, $z_0$, and $\varepsilon$,*

$$|\tilde S(\zeta,z_0)| \gtrsim \varepsilon + r(z_0) - r(\zeta).$$

(iv) *For $\zeta \in \mathbb{C}^n$ with $r(\zeta) \le r(z_0)$ we have, uniformly with respect to $\zeta$ and $z_0$,*

$$|\tilde S(\zeta,z_0)| \gtrsim r(z_0) - r(\zeta).$$

*Proof.* Parts (i) and (ii) were shown in [2]. Part (iv) holds by (ii) and the definition of $\tilde S$. To show (iii), we note that if $\zeta \in \mathcal{P}_\varepsilon^0(z_0)$ is written as $\zeta = z_0 + \lambda\eta_{z_0} + \mu v$, where $v$ is a unit vector in $T_{z_0}^{\mathbb{C}} bD_{r(z_0)}$, then $|\lambda| \gtrsim c_1\varepsilon$ or $|\mu| \gtrsim c_1\tau(z_0,v,\varepsilon)$. Indeed, by [7, Prop. 3.1(iii)] we have

$$\frac{|\mu|}{\tau(z_0,v,\varepsilon)} \eqsim \sum_{i=2}^n \frac{|\zeta_i^*|}{\tau_i(z_0,\varepsilon)}.$$

Therefore, if $|\mu| \le c_1\tilde c\tau(z_0,v,\varepsilon)$ for $\tilde c$ sufficiently small (uniformly with respect to $z_0$, $\zeta$, and $\varepsilon$), then $|\zeta_i^*| < c_1\tau_i(z_0,\varepsilon)$ for $i = 2,\dots,n$. But $\zeta$ does not belong to $c_1\mathcal{P}_\varepsilon(z_0)$, so $|\zeta_1^*| = |\lambda| \ge c_1\tau_1(z_0,\varepsilon) \gtrsim c_1\varepsilon$. Now the proof of (iii) is the same as the proof of (i) in [2]. □

We define the differential operator

$$\delta_j^* := \frac{\partial}{\partial z_j^*} + \frac{\partial}{\partial \zeta_j^*}.$$

LEMMA 3.3. *For all $\zeta \in \mathcal{P}_\varepsilon(z_0)$, $i, j, k = 1,\dots,n$, we have uniformly in $\zeta$, $z_0$, and $\varepsilon$*

$$(|\tilde{Q}_i^*(\zeta,z_0)|+|Q_i^*(\zeta,z_0)|)\lesssim \frac{\varepsilon}{\tau_i(z_0,\varepsilon)},$$

$$|\delta_j^*\tilde{Q}_i^*(\zeta,z_0)|+\left|\frac{\partial\tilde{Q}_i^*}{\partial\overline{z_j^*}}(\zeta,z_0)\right|\lesssim \frac{\varepsilon}{\tau_i(z_0,\varepsilon)\tau_j'(z_0,\varepsilon)},$$

$$|\delta_j^*Q_i^*(\zeta,z_0)|+\left|\frac{\partial Q_i^*}{\partial\overline{\zeta_j^*}}(\zeta,z_0)\right|\lesssim \frac{\varepsilon}{\tau_i(z_0,\varepsilon)\tau_j'(z_0,\varepsilon)},$$

$$\left|\delta_k^*\frac{\partial Q_i^*}{\partial\overline{\zeta_j^*}}(\zeta,z_0)\right|+\left|\delta_k^*\frac{\partial\tilde{Q}_i^*}{\partial\overline{z_j^*}}(\zeta,z_0)\right|\lesssim \frac{\varepsilon}{\tau_i(z_0,\varepsilon)\tau_j'(z_0,\varepsilon)\tau_k'(z_0,\varepsilon)},$$

*where* $\tau_l'(z_0,\varepsilon)=\tau_l(z_0,\varepsilon)$ *for* $l=2,\dots,n$ *and* $\tau_1'(z_0,\varepsilon)=\varepsilon^{1/2}$.

*Proof.* The estimates for $Q_i^*$, $\partial Q_i^*/\partial\bar{\zeta}_j^*$, $\delta_j^*Q_i^*$ and $\delta_k(\partial Q_i^*/\partial\bar{\zeta}_j^*)$ have already been shown in Lemma 4.6 of [2]. The other estimates can be shown in the same way using Lemma 4.5 of [2]. □

COROLLARY 3.4. *The following inequality holds uniformly for* $i=2,\dots,n$, $j=1,\dots,n$, *and* $\zeta\in\mathcal{P}_\varepsilon(z_0)$:

$$\tau_i(z_0,\varepsilon)\tau_j(z_0,\varepsilon)|\bar{V}_i^z(\tilde{Q}_j^*)(\zeta,z_0)\bar{q}_i^{z_0}|\lesssim\varepsilon.$$

*Proof.* We have

$$\bar{V}_i^z(\tilde{Q}_j^*)(\zeta,z_0)\bar{q}_i^{z_0}=\sum_{k,l=1}^n\overline{\Psi_{il}(z_0)}\Psi_{jk}(z_0)\frac{\partial\tilde{Q}_j^*}{\partial\bar{z}_k^*}(\zeta,z_0)d\bar{z}_l^*.$$

Since $\Psi(z_0)$ is the identity matrix we have

$$V_i^z(\tilde{Q}_j^*)(\zeta,z_0)\bar{q}_i^{z_0}=\frac{\partial\tilde{Q}_j^*}{\partial\bar{z}_i^*}(\zeta,z_0)d\bar{z}_i^*,$$

and Lemma 3.3 then brings the desired estimate. □

COROLLARY 3.5. *For* $B^z=Z_1^z,\dots,Z_n^z,\bar{Z}_2^z,\dots,\bar{Z}_n^z$, $i=2,\dots,n$, $j=1,\dots,n$, *and* $\zeta\in\mathcal{P}_\varepsilon(z_0)$, *we have uniformly with respect to* $\zeta$, $z_0$, *and* $\varepsilon$

$$\tau_j(z_0,\varepsilon)(|(B^z+B^\zeta)Q_j^*(\zeta,z_0)|+|(B^z+B^\zeta)\tilde{Q}_j^*(\zeta,z_0)|)\lesssim\varepsilon^{1/2},$$

$$\left|(B^z+B^\zeta)\frac{\partial Q_j^*}{\partial\bar{\zeta}_i^*}(\zeta,z_0)\right|+|(B^z+B^\zeta)(\bar{V}_i^z\tilde{Q}_j^*(\zeta,z_0)\bar{q}_i^{z_0})|\lesssim\frac{\varepsilon^{1/2}}{\tau_j(z_0,\varepsilon)\tau_i(z_0,\varepsilon)},$$

$$|(B^z+B^\zeta)S(\zeta,z_0)|+|(B^z+B^\zeta)\tilde{S}(\zeta,z_0)|\lesssim\varepsilon^{1/2}.$$

*Proof.* We set $\delta_j:=\frac{\partial}{\partial z_j}+\frac{\partial}{\partial\zeta_j}$. For $k\neq1$ we have

$$\begin{aligned}(Z_k^z+Z_k^\zeta)Q_j^*(\zeta,z_0)&\\ &=\delta_kQ_j^*(\zeta,z_0)-\frac{\partial r}{\partial z_k}(z_0)\left(\frac{\partial r}{\partial z_1}(z_0)\right)^{-1}\delta_1Q_j^*(\zeta,z_0)\\ &\quad+\left(\left(\frac{\partial r}{\partial z_1}(z_0)\right)^{-1}\frac{\partial r}{\partial z_k}(z_0)-\left(\frac{\partial r}{\partial\zeta_1}(\zeta)\right)^{-1}\frac{\partial r}{\partial\zeta_k}(\zeta)\right)\frac{\partial Q_j^*}{\partial\zeta_1}(\zeta,z_0).\end{aligned}$$

Since $\tau_l'(z_0,\varepsilon) \gtrsim \varepsilon^{1/2}$ for all $l$, our Lemma 3.3 gives $|\delta_l^* Q_i^*(\zeta,z_0)| \lesssim \varepsilon^{1/2}/\tau_i(z_0,\varepsilon)$ and so

$$|\delta_l Q_i^*(\zeta,z_0)| \lesssim \frac{\varepsilon^{1/2}}{\tau_i(z_0,\varepsilon)}. \tag{12}$$

Lemma 3.3 from [2] gives us

$$\left|\frac{\partial r}{\partial \zeta_1^*}(\zeta) - \frac{\partial r}{\partial \zeta_1^*}(z_0)\right| \lesssim \varepsilon^{1/2},$$

and Propositions 3.1(iv) and (vii) from [7] yield

$$\left|\frac{\partial r}{\partial \zeta_l^*}(\zeta) - \frac{\partial r}{\partial \zeta_l^*}(z_0)\right| \lesssim \frac{\varepsilon}{\tau_l(z_0,\varepsilon)} \quad \text{for all } l \neq 1.$$

Since for all $l \neq 1$ we have $\tau_l(z_0,\varepsilon) \gtrsim \varepsilon^{1/2}$, this implies for all $l$ that

$$\left|\frac{\partial r}{\partial \zeta_l}(\zeta) - \frac{\partial r}{\partial \zeta_l}(z_0)\right| \lesssim \varepsilon^{1/2}. \tag{13}$$

Thus we have

$$\left|\left(\frac{\partial r}{\partial z_1}(z_0)\right)^{-1}\frac{\partial r}{\partial z_k}(z_0) - \left(\frac{\partial r}{\partial \zeta_1}(\zeta)\right)^{-1}\frac{\partial r}{\partial \zeta_k}(\zeta)\right| \lesssim \varepsilon^{1/2},$$

which together with (12) yields $|(Z_k^z + Z_k^\zeta)Q_j^*(\zeta,z_0)| \lesssim \varepsilon^{1/2}/\tau_j(z_0,\varepsilon)$ for all $k \neq 1$.

Because $Q_j^*$ is holomorphic in $z$, for all $k \neq 1$ we have

$$(\bar{Z}_k^z + \bar{Z}_k^\zeta)Q_j^*(\zeta,z_0) = \frac{\partial Q_j^*}{\partial \bar{\zeta}_k}(\zeta,z_0) - \frac{\partial r}{\partial \bar{\zeta}_k}(\zeta)\left(\frac{\partial r}{\partial \bar{\zeta}_1}(\zeta)\right)^{-1}\frac{\partial Q_j^*}{\partial \bar{\zeta}_1}(\zeta,z_0).$$

Then Lemma 3.3 implies that, for all $l$,

$$\left|\frac{\partial Q_j^*}{\partial \bar{\zeta}_l}(\zeta,z_0)\right| \lesssim \frac{\varepsilon^{1/2}}{\tau_j(z_0,\varepsilon)}, \tag{14}$$

from which we deduce $|(\bar{Z}_k^z + \bar{Z}_k^\zeta)Q_j^*(\zeta,z_0)| \lesssim \varepsilon^{1/2}/\tau_j(z_0,\varepsilon)$ for all $k \neq 1$.

Observe that

$$\begin{aligned}
&(Z_1^z + Z_1^\zeta)Q_j^*(\zeta,z_0)\\
&\quad= \frac{1}{2}\left(\frac{\partial r}{\partial z_1}(z_0)\frac{\partial r}{\partial \zeta_1}(\zeta)\right)^{-1}\left(\frac{\partial r}{\partial z_1}(z_0) - \frac{\partial r}{\partial \zeta_1}(\zeta)\right)\frac{\partial Q_j^*}{\partial \zeta_1}(\zeta,z_0)\\
&\qquad+ \frac{1}{2}\left(\frac{\partial r}{\partial z_1}(z_0)\right)^{-1}\delta_1 Q_j^*(\zeta,z_0) - \frac{1}{2}\left(\frac{\partial r}{\partial \bar{\zeta}_1}(\zeta)\right)^{-1}\frac{\partial Q_j^*}{\partial \bar{\zeta}_1}(\zeta,z_0).
\end{aligned}$$

The estimates (12), (13), and (14) then give $|(Z_1^z + Z_1^\zeta)Q_j^*(\zeta,z_0)| \lesssim \varepsilon^{1/2}/\tau_j(z_0,\varepsilon)$.

The inequalities

$$|(B^z + B^\zeta)\tilde{Q}_j^*(\zeta,z_0)| \lesssim \frac{\varepsilon^{1/2}}{\tau_j(z_0,\varepsilon)}$$

and

$$\left|(B^z + B^\zeta)\frac{\partial Q_j^*}{\partial \bar\zeta_i^*}(\zeta, z_0)\right| \lesssim \frac{\varepsilon^{1/2}}{\tau_j(z_0,\varepsilon)\tau_i(z_0,\varepsilon)}$$

can be shown using the same method together with Lemma 3.3. The inequality

$$|(B^z + B^\zeta)(\bar V_i^z(\tilde Q_j^*)(\zeta, z_0)\bar q_i^{z_0})| \lesssim \frac{\varepsilon^{1/2}}{\tau_i(z_0,\varepsilon)\tau_j(z_0,\varepsilon)}$$

requires more work. Lemma 3.3 and

$$\bar V_i^z \tilde Q_j^*(\zeta, z_0) = \frac{\partial \tilde Q_j^*}{\partial \bar z_i^*}(\zeta, z_0)$$

imply that

$$|\bar V_i^z \tilde Q_j^*(\zeta, z_0)| \lesssim \frac{\varepsilon}{\tau_i(z_0,\varepsilon)\tau_j(z_0,\varepsilon)}.$$

Therefore, $|\bar V_i^z \tilde Q_j^*(\zeta, z_0)(B^z + B^\zeta)\bar q_i^{z_0}| \lesssim \varepsilon/\tau_i(z_0,\varepsilon)\tau_j(z_0,\varepsilon)$. Next we have

$$(B^z + B^\zeta)\bar V_i^z(\tilde Q_j^*)(\zeta, z_0) = \sum_{l=1}^n B^z(\Psi_{il})(z_0)\frac{\partial \tilde Q_j^*}{\partial \bar z_l^*}(\zeta, z_0) + (B^z + B^\zeta)\frac{\partial \tilde Q_j^*}{\partial \bar z_i^*}(\zeta, z_0).$$

Lemma 3.3 implies that, for all $l$,

$$\left|\frac{\partial \tilde Q_j^*}{\partial \bar z_l^*}(\zeta, z_0)\right| \lesssim \frac{\varepsilon^{1/2}}{\tau_j(z_0,\varepsilon)}.$$

In the same way as $|(B^z + B^\zeta)Q_j^*(\zeta, z_0)| \lesssim \varepsilon^{1/2}/\tau_j(z_0,\varepsilon)$ one shows that

$$\left|(B^z + B^\zeta)\frac{\partial \tilde Q_j^*}{\partial \bar z_i}(\zeta, z_0)\right| \lesssim \frac{\varepsilon^{1/2}}{\tau_j(z_0,\varepsilon)\tau_i(z_0,\varepsilon)}$$

in order to get $|(B^z + B^\zeta)(\bar V_i^z(\tilde Q_j^*)(\zeta, z_0))\bar q_i^{z_0}| \lesssim \varepsilon^{1/2}/\tau_j(z_0,\varepsilon)\tau_i(z_0,\varepsilon)$. The last inequality of Corollary 3.5 is a consequence of the first one and of (13), because $S(\zeta, z) = \sum_{j=1}^n (\zeta_j^* - z_j^*)Q_j^*(\zeta, z)$ and $|\zeta_j^* - (z_0)_j^*| = |\zeta_j^*| \lesssim \tau_j(z_0,\varepsilon)$. Of course, the same holds for $\tilde S$. □

LEMMA 3.6. *For $z, \zeta \in B(\zeta_0, R')$ and two tangential vectors fields $B^z, \tilde B^z$ we have, uniformly with respect to $\zeta$ and $z$, that $(B^z + B^\zeta)S(\zeta, z)$, $(B^z + B^\zeta)\tilde S(\zeta, z)$, $(\tilde B^z + \tilde B^\zeta)(B^z + B^\zeta)S(\zeta, z)$, and $(\tilde B^z + \tilde B^\zeta)(B^z + B^\zeta)\tilde S(\zeta, z)$ are $O(|\zeta - z|)$.*

*Proof.* This is obvious because $S(\zeta, z) = \sum_{i=1}^n (\zeta_i - z_i)Q_i(\zeta, z)$ and $\tilde S(\zeta, z) = \sum_{i=1}^n (\zeta_i - z_i)\tilde Q_i(\zeta, z)$ and because the derivatives of $Q$ and $\tilde Q$ are uniformly bounded. □

We also have to show good estimates of $V_1^\zeta Q_i^*$ and $V_1^\zeta(\partial Q_i^*/\partial \bar\zeta_j^*)$. We set $\omega(\zeta, z) = \Psi(\zeta)(z^* - \zeta^*)$ so that $F(\zeta, z) = F_\zeta(\omega(\zeta, z))$.

LEMMA 3.7. *For $i = 1, \dots, n$, $j = 2, \dots, n$, and $\zeta \in \mathcal{P}_\varepsilon(z_0)$, the following inequalities hold uniformly with respect to $\zeta$ and $z_0$:*

$$\left|\frac{\partial \omega_j}{\partial \zeta_1^*}(\zeta, z_0)\right| \lesssim \tau_j(z_0,\varepsilon); \qquad \left|\frac{\partial^2 \omega_j}{\partial \zeta_1^* \partial \bar\zeta_i^*}(\zeta, z_0)\right| \lesssim \frac{\tau_j(z_0,\varepsilon)}{\tau_i(z_0,\varepsilon)}.$$

*Proof.* We have

$$\frac{\partial \omega_j}{\partial \zeta_1^*}(\zeta, z) = -\Psi_{j1}(\zeta) + \sum_{k=1}^{n} \frac{\partial \Psi_{jk}}{\partial \zeta_1^*}(\zeta)(z_k^* - \zeta_k^*).$$

For all $k$, $|\zeta_k^*| < \tau_k(z_0, \varepsilon)$ and so [2, Prop. 4.2] gives

$$\left|\frac{\partial \omega_j}{\partial \zeta_1^*}(\zeta, z_0)\right| \lesssim \frac{\varepsilon}{\tau_j(z_0, \varepsilon)}.$$

By [7, Prop. 3.1] we have $\tau_j(z_0, \varepsilon) \gtrsim \varepsilon^{1/2}$. Therefore, the first inequality of the lemma holds. The second is analogous. $\square$

LEMMA 3.8. *For $i, j = 1, \dots, n$ and $\zeta \in \mathcal{P}_\varepsilon(z_0)$, we have uniformly with respect to $\zeta$, $z_0$, and $\varepsilon$*

$$\tau_i(z_0, \varepsilon)\left|V_1^\zeta \frac{\partial Q_j^*}{\partial \bar\zeta_i^*}(\zeta, z_0)\right| + |V_1^\zeta Q_j^*(\zeta, z_0)| \lesssim \frac{\varepsilon^{1/2}}{\tau_j(z_0, \varepsilon)}.$$

*Proof.* Since $\tau_1(z_0, \varepsilon) \approx \varepsilon$, the inequality is obvious for $j = 1$ and so we assume $j \geq 2$. We have

$$V_1^\zeta Q_j^*(\zeta, z_0) = \sum_{k=1}^{n} \overline{\Psi_{1k}(\zeta)} \frac{\partial Q_j^*}{\partial \zeta_k^*}(\zeta, z_0) - \Psi_{1k}(\zeta) \frac{\partial Q_j^*}{\partial \bar\zeta_k^*}(\zeta, z_0),$$

and by [7, Prop. 3.1(v); 2, Prop. 4.2] it follows that $|\Psi_{1k}(\zeta)| \lesssim \varepsilon^{1/2}$ for all $k \neq 1$. Moreover, by Lemma 3.3 we have

$$\left|\frac{\partial Q_j^*}{\partial \bar\zeta_1^*}(\zeta, z_0)\right| \lesssim \frac{\varepsilon^{1/2}}{\tau_j(z_0, \varepsilon)},$$

so it suffices to show that

$$\left|\frac{\partial Q_j^*}{\partial \zeta_1^*}(\zeta, z_0)\right| \lesssim \frac{\varepsilon^{1/2}}{\tau_j(z_0, \varepsilon)}.$$

By the definition of $Q_j^*$ we have

$$\frac{\partial Q_j^*}{\partial \zeta_1^*}(\zeta, z_0) = -\sum_{k=1}^{n} \frac{\partial \Psi_{kj}}{\partial \zeta_1^*}(\zeta)\sigma_k(\zeta, \omega(\zeta, z_0)) - \Psi_{kj}(\zeta)\frac{\partial \sigma_k(\zeta, \omega(\zeta, z_0))}{\partial \zeta_1^*}.$$

Using once again Proposition 3.1(v) of [7] and Proposition 4.2 of [2], we need only estimate $\partial \sigma_j(\zeta, \omega(\zeta, z_0))/\partial \zeta_1^*$.

For $\zeta \in \mathcal{V}$ and $\omega \in \mathbb{C}^n$ we set $A_\zeta(\omega) := A(\zeta, \zeta + \overline{\Psi(\zeta)}^t \omega)$. Then

$$\sigma_j(\zeta, \omega(\zeta, z_0)) = \int_0^1 \frac{\partial A_\zeta}{\partial \omega_j}(t\omega(\zeta, z_0)) F_\zeta(t\omega(\zeta, z_0))\, dt$$
$$+ \int_0^1 A_\zeta(t\omega(\zeta, z_0)) \frac{\partial F_\zeta}{\partial \omega_j}(t\omega(\zeta, z_0))\, dt. \qquad (15)$$

Lemmas 4.3 and 4.4 of [2] together with Lemma 3.7 imply that, for $j = 2, \dots, n$ and $t \in [0, 1]$,

$$\left|\frac{\partial}{\partial\zeta_1^*}\frac{\partial F_\zeta}{\partial\omega_j}(t\omega(\zeta, z_0))\right| \lesssim \frac{\varepsilon^{1/2}}{\tau_j(z_0,\varepsilon)}. \tag{16}$$

Set $v_\zeta(\omega) = v(\zeta, \zeta + \overline{\Psi(\zeta)\Phi_*}{}^t\omega)$. We have $A_\zeta(\omega) = 1/(1+(m'-v_\zeta(\omega))F_\zeta(\omega))$. Since $v$ and its derivatives are bounded, Lemma 4.5 of [2] gives

$$\left|\frac{\partial A_\zeta}{\partial\omega_j}(t\omega(\zeta, z_0))\right| \lesssim \frac{\varepsilon}{\tau_j(z_0,\varepsilon)}. \tag{17}$$

Now we plug the estimates of [2, Lemma 4.5] and the estimates (16) and (17) into (15) to obtain

$$\left|\frac{\partial\sigma_j(\zeta,\omega(\zeta,z_0))}{\partial\zeta_1^*}\right| \lesssim \frac{\varepsilon^{1/2}}{\tau_j(z_0,\varepsilon)}$$

and finally $|V_1^\zeta Q_j^*(\zeta, z_0)| \lesssim \varepsilon^{1/2}/\tau_j(z_0,\varepsilon)$. As for the estimate of $V_1^\zeta Q_j^*(\zeta, z_0)$, in order to show

$$\left|V_1^\zeta\frac{\partial Q_j^*}{\partial\bar\zeta_i^*}(\zeta,z_0)\right| \lesssim \frac{\varepsilon^{1/2}}{\tau_i(z_0,\varepsilon)\tau_j(z_0,\varepsilon)}$$

it suffices to show that

$$\left|\frac{\partial^2\sigma_j(\zeta,\omega(\zeta,z_0))}{\partial\zeta_1^*\partial\bar\zeta_i^*}\right| \lesssim \frac{\varepsilon^{1/2}}{\tau_i(z_0,\varepsilon)\tau_j(z_0,\varepsilon)}.$$

By [2, Lemmas 4.3 & 4.4] and Lemma 3.7, for all $t \in [0, 1]$ we have

$$\left|\frac{\partial^2}{\partial\zeta_1^*\partial\bar\zeta_i^*}\left(\frac{\partial F_\zeta}{\partial\omega_j}(t\omega(\zeta,z_0))\right)\right| \lesssim \frac{\varepsilon^{1/2}}{\tau_j(z_0,\varepsilon)\tau_i(z_0,\varepsilon)}.$$

Hence (16) and [2, Lemma 4.5] yield

$$\left|\frac{\partial^2}{\partial\zeta_1^*\partial\bar\zeta_i^*}\int_0^1 A_\zeta(t\omega(\zeta,z_0))\frac{\partial F_\zeta}{\partial\omega_j}(t\omega(\zeta,z_0))\,dt\right| \lesssim \frac{\varepsilon^{1/2}}{\tau_i(z_0,\varepsilon)\tau_j(z_0,\varepsilon)}. \tag{18}$$

Next, [2, Lemma 4.5] shows that

$$\left|\frac{\partial}{\partial\bar\zeta_i^*}\left(\frac{\partial A_\zeta}{\partial\omega_j}(t\omega(\zeta,z_0))\right)\right| \lesssim \frac{\varepsilon}{\tau_i(z_0,\varepsilon)\tau_j(z_0,\varepsilon)}.$$

So again using Lemma 4.5 of [2] and the estimate (17), we have

$$\left|\frac{\partial^2}{\partial\zeta_1^*\partial\bar\zeta_i^*}\int_0^1 \frac{\partial A_\zeta}{\partial\omega_j}(t\omega(\zeta,z_0))F_\zeta(t\omega(\zeta,z_0))\,dt\right| \lesssim \frac{\varepsilon}{\tau_j(z_0,\varepsilon)\tau_i(z_0,\varepsilon)}. \tag{19}$$

Putting together (18) and (19) then yields

$$\left|\frac{\partial^2\sigma_j(\zeta,\omega(\zeta,z_0))}{\partial\zeta_1^*\partial\bar\zeta_i^*}\right| \lesssim \frac{\varepsilon^{1/2}}{\tau_j(z_0,\varepsilon)\tau_i(z_0,\varepsilon)}$$

and finally

$$\left|V_1^\zeta\frac{\partial Q_j^*}{\partial\bar\zeta_i^*}(\zeta,z_0)\right| \lesssim \frac{\varepsilon^{1/2}}{\tau_i(z_0,\varepsilon)\tau_j(z_0,\varepsilon)}. \qquad \square$$

LEMMA 3.9. *For $i = 1, \dots, n$, $l = 2, \dots, n$, and $\zeta \in \mathcal{P}_\varepsilon(z_0)$, we have uniformly with respect to $\zeta$, $z_0$, and $\varepsilon$*

$$\frac{\partial \omega_l}{\partial \zeta_1^*}(z_0, \zeta) = 0 \quad \textit{and} \quad \left|\frac{\partial^2 \omega_l}{\partial \zeta_1^* \partial \overline{z_i^*}}(z_0, \zeta)\right| \lesssim \frac{\tau_l(z_0, \varepsilon)}{\tau_i(z_0, \varepsilon)}.$$

*Proof.* By definition we have $\omega_l(z, \zeta) = \sum_{k=1}^n \Psi_{lk}(z)(\zeta_k^* - z_k^*)$, and since $\Psi(z_0)$ is the identity matrix it follows that

$$\frac{\partial \omega_l}{\partial \zeta_1^*}(z_0, \zeta) = \Psi_{l1}(z_0) = 0 \quad \text{for all } \ l \neq 1.$$

Next we have

$$\frac{\partial^2 \omega_l}{\partial \zeta_1^* \partial \overline{z_i^*}}(z_0, \zeta) = \frac{\partial \Psi_{l1}}{\partial \overline{z_i^*}}(z_0),$$

and Proposition 4.2 of [2] implies

$$\left|\frac{\partial^2 \omega_l}{\partial \zeta_1^* \partial \overline{z_i^*}}(z_0, \zeta)\right| \lesssim \frac{\tau_l(z_0, \varepsilon)}{\tau_i(z_0, \varepsilon)}.$$ □

LEMMA 3.10. *For $j = 1, \dots, n$, $k = 2, \dots, n$, and $\zeta \in \mathcal{P}_\varepsilon(z_0)$, we have uniformly with respect to $\zeta$, $z_0$, and $\varepsilon$*

$$\tau_k(z_0, \varepsilon)|V_1^\zeta(\bar{V}_k^z(\tilde{Q}_j^*)(\zeta, z_0)\bar{q}_k^{z_0})| + |V_1^\zeta \tilde{Q}_j^*(\zeta, z_0)| \lesssim \frac{\varepsilon^{1/2}}{\tau_j(z_0, \varepsilon)}.$$

*Proof.* The inequality is obvious for $j = 1$ because $\tau_1(z_0, \varepsilon) \approx \varepsilon$. Let $j > 1$. We show $|V_1^\zeta \tilde{Q}_j^*(\zeta, z_0)| \lesssim \varepsilon^{1/2} \tau_j(z_0, \varepsilon)^{-1}$.

Since for all $i \neq 1$ we have

$$V_1^\zeta \tilde{Q}_j^*(\zeta, z_0) = \sum_{i=1}^n \overline{\Psi_{1i}(\zeta)} \frac{\partial \tilde{Q}_j^*}{\partial \zeta_i^*}(\zeta, z_0)$$

and $|\Psi_{1i}(\zeta)| \lesssim \varepsilon^{1/2}$ (see [2, Prop. 4.2; 7, Prop. 3.1(v)]), we have to show that

$$\left|\frac{\partial \tilde{Q}_j^*}{\partial \zeta_1^*}(\zeta, z_0)\right| \lesssim \frac{\varepsilon^{1/2}}{\tau_j(z_0, \varepsilon)};$$

since $\Psi(z_0)$ is the identity matrix, we will actually show

$$\left|\frac{\partial \tilde{\sigma}_j(z_0, \omega(z_0, \zeta))}{\partial \zeta_1^*}\right| \lesssim \frac{\varepsilon^{1/2}}{\tau_j(z_0, \varepsilon)}.$$

We have

$$\tilde{\sigma}_j(z_0, \omega(z_0, \zeta)) = \int_0^1 \frac{\partial A_{z_0}}{\partial \omega_j}(t\omega(z_0, \zeta)) F_{z_0}(t\omega(z_0, \zeta))\, dt + \int_0^1 A_{z_0}(t\omega(z_0, \zeta)) \frac{\partial F_{z_0}}{\partial \omega_j}(t\omega(z_0, \zeta))\, dt. \tag{20}$$

According to Lemma 3.9,

$$\frac{\partial}{\partial \zeta_1^*}\frac{\partial F_{z_0}}{\partial \omega_j}(t\omega(z_0,\zeta)) = 0 \quad \text{for all } t \in [0,1]. \tag{21}$$

Using $A(z_0,\zeta) = 1/(1+(m'-v(z_0,\zeta))F(z_0,\zeta))$ and Lemma 4.5 of [2] now yields

$$\left|\frac{\partial A_{z_0}}{\partial \omega_j}(t\omega(z_0,\zeta))\right| \lesssim \frac{\varepsilon}{\tau_j(z_0,\varepsilon)} \quad \text{for all } t \in [0,1]. \tag{22}$$

Again using [2, Lemma 4.5] with (21) and (22) we obtain

$$\left|\frac{\partial \tilde{\sigma}_j(z_0,\omega(z_0,\zeta))}{\partial \zeta_1^*}\right| \lesssim \frac{\varepsilon}{\tau_j(z_0,\varepsilon)},$$

which shows that $|V_1^\zeta \tilde{Q}_j^*(\zeta,z_0)|\tau_j(z_0,\varepsilon) \lesssim \varepsilon^{1/2}$.

Now

$$V_1^\zeta(\bar{V}_k^z(\tilde{Q}_j)(\zeta,z_0)\bar{q}_k^{z_0}) = \sum_{i=1}^n \overline{\Psi_{1i}(\zeta)}\frac{\partial}{\partial \zeta_i^*}\bar{V}_k^z\tilde{Q}_j(\zeta,z_0)d\bar{z}_k$$

and, by [2, Prop. 4.2], $|\Psi_{1i}(\zeta)| \lesssim \varepsilon^{1/2}$ for all $i \neq 1$. Hence it suffices to show that

$$\left|\frac{\partial}{\partial \zeta_1^*}\bar{V}_k^z(\tilde{Q}_j^*)(\zeta,z_0)\right| \lesssim \frac{\varepsilon^{1/2}}{\tau_j(z_0,\varepsilon)\tau_k(z_0,\varepsilon)}.$$

Moreover,

$$\frac{\partial}{\partial \zeta_1^*}\bar{V}_k^z\tilde{Q}_j^*(\zeta,z_0) = \sum_{i=1}^n \frac{\partial \Psi_{ij}}{\partial \overline{z_k^*}}(z_0)\frac{\partial \tilde{\sigma}_i(z_0,\omega(z_0,\zeta))}{\partial \zeta_1^*} + \frac{\partial^2 \tilde{\sigma}_j(z,\omega(z,\zeta))}{\partial \zeta_1^* \partial \overline{z_k^*}}\bigg|_{z=z_0};$$

since (by [2, Prop. 4.3])

$$\left|\frac{\partial \Psi_{ij}}{\partial \overline{z_k^*}}(z_0)\right| \lesssim \frac{\varepsilon}{\tau_j(z_0,\varepsilon)\tau_k(z_0,\varepsilon)},$$

it suffices to estimate $\partial^2\tilde{\sigma}_j(z,\omega(z,\zeta))/\partial\zeta_1^*\partial\overline{z_k^*}|_{z=z_0}$.

We use Lemmas 4.3 and 4.4 of [2] together with Lemma 3.9 to get, for all $t \in [0,1]$,

$$\left|\frac{\partial^2}{\partial \zeta_1^* \partial \overline{z_k^*}}\left(\frac{\partial F_z}{\partial \omega_j}(t\omega(z,\zeta))\right)\bigg|_{z=z_0}\right| \lesssim \frac{\varepsilon}{\tau_k(z_0,\varepsilon)\tau_j(z_0,\varepsilon)}. \tag{23}$$

Next using the estimates of [2, Lemma 4.5], we obtain

$$\left|\frac{\partial}{\partial \bar{z}_k^*}\frac{\partial A_z(t\omega(z,\zeta))}{\partial \omega_j}\bigg|_{z=z_0}\right| \lesssim \frac{\varepsilon}{\tau_j(z_0,\varepsilon)\tau_k(z_0,\varepsilon)}$$

for all $t \in [0,1]$. With (23) and [2, Lemma 4.5] this implies that

$$\left|\frac{\partial^2 \tilde{\sigma}_j(z,\omega(z,\zeta))}{\partial \zeta_1^* \partial \overline{z_k^*}}\bigg|_{z=z_0}\right| \lesssim \frac{\varepsilon}{\tau_j(z_0,\varepsilon)\tau_k(z_0,\varepsilon)},$$

which was to be shown. □

## 4. Multiple Integrations by Parts

We fix a $C^p$-$(0,q)$-form $f$, $p \in \mathbb{N}$, with compact support in $\overline{B(\zeta_0, R'')}$, $\zeta_0 \in bD$, $R' > R'' > 0$, and denote by $\Gamma^s f$ a $C^{p-s}$-$(0,q)$-form, $p \geq s \geq 0$, with the same support as $f$ and such that $\|\Gamma^s f\|_{bD,p-s} \leq c_{s,D}\|f\|_{bD,p}$, where $c_{s,D}$ does not depend on $f$. Next, for $z \in B(\zeta_0, R') \setminus \bar D$ we set

$$\begin{aligned}\tilde I[f](j,j',k,k',l,l',s)(z)\\ &= \int_{bD} \Gamma^s f(\zeta)\\ &\quad\wedge \frac{X^{k'}(\tilde\eta_1(\zeta,z)\wedge(\bar\partial_z^t\tilde\eta_1(\zeta,z))^{k-1})\wedge\varpi_{l'}(\zeta,z)}{\tilde S^j(\zeta,z)|\zeta-z|^{2l}}\prod_{i=1}^{j'}(\tilde Z_i^z+\tilde Z_i^\zeta)\tilde S(\zeta,z),\end{aligned}$$

$$\begin{aligned}\tilde J[f](j,j',k,k',l,l',s)(z)\\ &= \int_{bD} \Gamma^s f(\zeta)\wedge \frac{X^{k'}((\bar\partial_z^t\tilde\eta_1(\zeta,z))^{k})\wedge\varpi_{l'}(\zeta,z)}{\tilde S^j(\zeta,z)|\zeta-z|^{2l}}\prod_{i=1}^{j'}(\tilde Z_i^z+\tilde Z_i^\zeta)\tilde S(\zeta,z),\end{aligned}$$

and for $z \in B(\zeta_0, R') \cap D$ we set

$$\begin{aligned}I[f](j,j',k,k',l,l',s)(z)\\ &= \int_{bD} \Gamma^s f(\zeta)\\ &\quad\wedge \frac{X^{k'}(\eta_1(\zeta,z)\wedge(\bar\partial_\zeta\eta_1(\zeta,z))^{k-1})\wedge\varpi_{l'}(\zeta,z)}{S^j(\zeta,z)|\zeta-z|^{2l}}\prod_{i=1}^{j'}(\tilde Z_i^z+\tilde Z_i^\zeta)S(\zeta,z),\end{aligned}$$

$$\begin{aligned}J[f](j,j',k,k',l,l',s)(z)\\ &= \int_{bD} \Gamma^s f(\zeta)\wedge \frac{X^{k'}((\bar\partial_\zeta\eta_1(\zeta,z))^{k})\wedge\varpi_{l'}(\zeta,z)}{S^j(\zeta,z)|\zeta-z|^{2l}}\prod_{i=1}^{j'}(\tilde Z_i^z+\tilde Z_i^\zeta)S(\zeta,z).\end{aligned}$$

Here $j, j', k, k', l, l', s \in \mathbb{N}$; $p \geq s$, $j \geq 1$, and $k \geq 1$ in $I[f]$ and $\tilde I[f]$; $X^{k'} = V_1 \dots V_{k'}$ with $V_1, \dots, V_{k'} \in \{V_1^\zeta, Z_1^z + Z_1^\zeta, \dots, Z_n^z + Z_n^\zeta, \bar Z_2^z + \bar Z_2^\zeta, \dots, \bar Z_n^z + \bar Z_n^\zeta\}$, $\tilde Z_i^z \in \{Z_1^z, \dots, Z_n^z, \bar Z_2^z, \dots, \bar Z_n^z\}$ for $i = 1, \dots, j'$; and $\varpi_{l'}$ is a form of bidegree $(n-k, n-q-1)$ in $\zeta$ in $\tilde I[f]$ and $\tilde J[f]$, with $(n-k, n-q-k)$ in $I[f]$ and $(n-k, n-q-k-1)$ in $J[f]$ and such that $|\varpi_{l'}(\zeta,z)| = O(|\zeta - z|^{l'})$.

We say that $(j, j', k, k', l, l')$ satisfies (CI) if

$$\text{(CI)}\begin{cases} 1 < j\\ 2j - j' \leq 2k - k'\\ k' \leq k\\ 2k + 2l - l' \leq 2n - 1\end{cases}\quad\text{or}\quad\begin{cases} k = j = 1\\ k' = j' = 0\\ 2l - l' \leq 2n - 3\end{cases}$$

and that it satisfies (CJ) when

$$\text{(CJ)}\begin{cases} 1 < j \\ 2j - j' \le 2k - k' \\ k' \le k \\ 2k + 2l - l' \le 2n - 2 \end{cases} \quad \text{or} \begin{cases} j = 1 \\ k = k' = j' = 0 \\ 2l - l' \le 2n - 3. \end{cases}$$

Later on we will show that $\tilde{T}_q^t f$ and $T_q f$ are (respectively) finite sums of $\tilde{I}[f](k, 0, k, 2(n-k), 1, 0)$ and $I[f](k, 0, k, 2(n-k), 1, 0)$, $k \in \{1, \dots, q-1\}$, that satisfy (CI).

**PROPOSITION 4.1.** *For $B^z \in \{Z_1^z, \dots, Z_n^z, \bar{Z}_2^z, \dots, \bar{Z}_n^z\}$ and $s < p$, the following statements hold.*

(i) *If $(j, j', k, k', l, l')$ satisfies (CI) then, for all $z \in (\mathcal{V} \cap B(\zeta_0, R')) - \bar{D}$, $B^z \tilde{I}[f](j, j', k, k', l, l', s)(z)$ is a finite sum of $\tilde{I}[f](\tilde{j}, \tilde{j}', \tilde{k}, \tilde{k}', \tilde{l}, \tilde{l}', \tilde{s})(z)$ and $\tilde{J}[f](\tilde{j}, \tilde{j}', \tilde{k}, \tilde{k}', \tilde{l}, \tilde{l}', \tilde{s})(z)$, $\tilde{s} \le s + 1$, satisfying respectively (CI) and (CJ).*

(ii) *If $(j, j', k, k', l, l')$ satisfies (CJ) then, for all $z \in (\mathcal{V} \cap B(\zeta_0, R')) - \bar{D}$, $B^z \tilde{J}[f](j, j', k, k', l, l', s)(z)$ is a finite sum of $\tilde{J}[f](\tilde{j}, \tilde{j}', \tilde{k}, \tilde{k}', \tilde{l}, \tilde{l}', \tilde{s})(z)$, $\tilde{s} \le s + 1$, satisfying (CJ).*

(iii) *If $(j, j', k, k', l, l')$ satisfies (CI) then, for all $z \in \mathcal{V} \cap B(\zeta_0, R') \cap D$, $B^z I[f](j, j', k, k', l, l', s)(z)$ is a finite sum of $I[f](\tilde{j}, \tilde{j}', \tilde{k}, \tilde{k}', \tilde{l}, \tilde{l}', \tilde{s})(z)$ and $J[f](\tilde{j}, \tilde{j}', \tilde{k}, \tilde{k}', \tilde{l}, \tilde{l}', \tilde{s})(z)$, $\tilde{s} \le s + 1$, satisfying respectively (CI) and (CJ).*

(iv) *If $(j, j', k, k', l, l')$ satisfies (CJ) then, for all $z \in \mathcal{V} \cap B(\zeta_0, R') \cap D$, $B^z J[f](j, j', k, k', l, l', s)(z)$ is a finite sum of $J[f](\tilde{j}, \tilde{j}', \tilde{k}, \tilde{k}', \tilde{l}, \tilde{l}', \tilde{s})(z)$, $\tilde{s} \le s + 1$, satisfying (CJ).*

*Proof.* We show (i). With an integration by parts we get

$$\begin{aligned} & B^z \tilde{I}[f](j, j', k, k', l, l', s)(z) \\ & \quad = \int_{bD} \Gamma^s f(\zeta) \wedge (B^z + B^\zeta) \bigg( \frac{X^{k'}(\tilde{\eta}_1(\zeta, z) \wedge (\bar{\partial}_z^t \tilde{\eta}_1(\zeta, z))^{k-1}) \wedge \varpi_{l'}(\zeta, z)}{\tilde{S}^j(\zeta, z) |\zeta - z|^{2l}} \\ & \qquad \qquad \cdot \prod_{i=1}^{j'} (\tilde{Z}_i^z + \tilde{Z}_i^\zeta) \tilde{S}(\zeta, z) \bigg) \\ & \qquad + \int_{bD} B^\zeta \Gamma^s f(\zeta) \wedge \bigg( \frac{X^{k'}(\tilde{\eta}_1(\zeta, z) \wedge (\bar{\partial}_z^t \tilde{\eta}_1(\zeta, z))^{k-1}) \wedge \varpi_{l'}(\zeta, z)}{\tilde{S}^j(\zeta, z) |\zeta - z|^{2l}} \\ & \qquad \qquad \cdot \prod_{i=1}^{j'} (\tilde{Z}_i^z + \tilde{Z}_i^\zeta) \tilde{S}(\zeta, z) \bigg) \\ & \quad = X + Y. \end{aligned}$$

Now $Y = \tilde{I}[f](j, j', k, k', l, l', s+1)(z)$, where $(j, j', k, k', l, l')$ satisfies (CI). For $X$, we should distinguish many cases depending on the values of $j, k, \dots$.

*Case 1:* $j = 1$.

$$\begin{aligned} X &= -\int_{bD} \Gamma^s f(\zeta) \wedge \frac{\tilde\eta_1(\zeta,z) \wedge \varpi_{l'}(\zeta,z)(B^z + B^\zeta)\tilde S(\zeta,z)}{\tilde S(\zeta,z)^2 |\zeta - z|^{2l}} \\ &\quad + \int_{bD} \Gamma^s f(\zeta) \wedge \frac{\tilde\eta_1(\zeta,z)}{\tilde S(\zeta,z)} \wedge (B^z + B^\zeta)\left(\frac{\varpi_{l'}(\zeta,z)}{|\zeta - z|^{2l}}\right) \\ &\quad + \int_{bD} \Gamma^s f(\zeta) \wedge \frac{(B^z + B^\zeta)\tilde\eta_1(\zeta,z) \wedge \varpi_{l'}(\zeta,z)}{\tilde S(\zeta,z)|\zeta - z|^{2l}} \\ &= X_1 + X_2 + X_3. \end{aligned}$$

Since $(1, j', k, k', l, l')$ satisfies (CI), we have $k = 1$ and $2l - l' \le 2n-3$. Moreover, $(B^z + B^\zeta)\tilde\eta_1$ is bounded, so $X_3 = \tilde J[f](1,0,0,0,l,l',s)$ and satisfies (CJ). We have

$$(B^z + B^\zeta)\left(\frac{\varpi_{l'}(\zeta,z)}{|\zeta - z|^{2l}}\right) = \frac{\varpi_{l'}(\zeta,z)}{|\zeta - z|^{2l}},$$

so $X_2 = \tilde I[f](1,0,1,0,l,l',s)(z)$ and satisfies (CI).

For $X_1$ we integrate by parts again. By Lemma 3.1 we have $|V_1^\zeta \tilde S(\zeta,z)| \gtrsim 1$ for all $(\zeta,z) \in B(\zeta_0, R')$. Therefore,

$$\begin{aligned} X_1 &= \int_{bD} \Gamma^s f(\zeta) \wedge \frac{\tilde\eta_1(\zeta,z) \wedge \varpi_{l'}(\zeta,z)}{|\zeta - z|^{2l}} \frac{(B^z + B^\zeta)\tilde S(\zeta,z)}{V_1^\zeta \tilde S(\zeta,z)} V_1^\zeta\left(\frac{1}{\tilde S(\zeta,z)}\right) \\ &= -\int_{bD} V_1^\zeta(\Gamma^s f(\zeta)) \wedge \frac{\tilde\eta_1(\zeta,z) \wedge \varpi_{l'}(\zeta,z)(B^z + B^\zeta)\tilde S(\zeta,z)}{\tilde S(\zeta,z)|\zeta - z|^{2l} V_1^\zeta \tilde S(\zeta,z)} \\ &\quad - \int_{bD} \Gamma^s f(\zeta) \wedge \frac{(B^z + B^\zeta)\tilde S(\zeta,z)}{\tilde S(\zeta,z) V_1^\zeta \tilde S(\zeta,z)} \tilde\eta_1(\zeta,z) \wedge V_1^\zeta\left(\frac{\varpi_{l'}(\zeta,z)}{|\zeta - z|^{2l}}\right) \\ &\quad - \int_{bD} \Gamma^s f(\zeta) \wedge \frac{\tilde\eta_1(\zeta,z) \wedge \varpi_{l'}(\zeta,z)}{\tilde S(\zeta,z)|\zeta - z|^{2l}} V_1^\zeta\left(\frac{(B^z + B^\zeta)\tilde S(\zeta,z)}{V_1^\zeta \tilde S(\zeta,z)}\right) \\ &\quad - \int_{bD} \Gamma^s f(\zeta) \wedge \frac{(B^z + B^\zeta)\tilde S(\zeta,z) V_1^\zeta(\tilde\eta_1(\zeta,z)) \wedge \varpi_{l'}(\zeta,z)}{\tilde S(\zeta,z)|\zeta - z|^{2l} V_1^\zeta \tilde S(\zeta,z)} \\ &= X_{11} + X_{12} + X_{13} + X_{14}. \end{aligned}$$

Since

$$\frac{(B^z + B^\zeta)\tilde S(\zeta,z)}{V_1^\zeta \tilde S(\zeta,z)} \quad \text{and} \quad V_1^\zeta\left(\frac{(B^z + B^\zeta)\tilde S(\zeta,z)}{V_1^\zeta \tilde S(\zeta,z)}\right)$$

are bounded, it follows that $X_{11} = \tilde I[f](1,0,1,0,l,l',s+1)(z)$ and $X_{13} = \tilde I[f](1,0,1,0,l,l',s)(z)$ satisfy (CI). We have

$$V_1^\zeta\left(\frac{\varpi_{l'}(\zeta,z)}{|\zeta - z|^{2l}}\right) = \frac{\varpi_{1+l'}(\zeta,z)}{|\zeta - z|^{2(l+1)}} + \frac{\varpi_{l'-1}(\zeta,z)}{|\zeta - z|^{2l}}$$

and, by Lemma 3.6, $(B^z+B^\zeta)\tilde{S}(\zeta,z)=O(|\zeta-z|)$. Therefore $X_{12}=\tilde{I}[f](1,0,1,0,l,l',s)(z)+\tilde{I}[f](1,0,1,0,l+1,l'+2,s)(z)$ satisfies (CI). Finally, $V_1^\zeta\tilde{\eta}_1$ is bounded and $2l-l'\le 2n-3$, so $X_{14}=\tilde{J}[f](1,0,0,0,l,l',s)(z)$ satisfies (CJ).

*Case 2: $j>1$.* We have

$$\begin{aligned}
X=&\int_{bD}\Gamma^s(f)(\zeta)\wedge\left(\frac{X^{k'}(\tilde{\eta}_1(\zeta,z)\wedge(\bar{\partial}_z^t\tilde{\eta}_1(\zeta,z))^{k-1})\wedge\varpi_{l'}(\zeta,z)}{\tilde{S}^j(\zeta,z)|\zeta-z|^{2l}}\right)\\
&\cdot\left((B^z+B^\zeta)\prod_{i=1}^{j'}(\tilde{Z}_i^z+\tilde{Z}_i^\zeta)\tilde{S}(\zeta,z)\right)\\
&+\int_{bD}\Gamma^s(f)(\zeta)\wedge\left(\frac{(B^z+B^\zeta)X^{k'}(\tilde{\eta}_1(\zeta,z)\wedge(\bar{\partial}_z^t\tilde{\eta}_1(\zeta,z))^{k-1})\wedge\varpi_{l'}(\zeta,z)}{\tilde{S}^j(\zeta,z)|\zeta-z|^{2l}}\right)\\
&\cdot\prod_{i=1}^{j'}(\tilde{Z}_i^z+\tilde{Z}_i^\zeta)\tilde{S}(\zeta,z)\\
&+\int_{bD}\Gamma^s(f)(\zeta)\wedge\left(\frac{X^{k'}(\tilde{\eta}_1(\zeta,z)\wedge(\bar{\partial}_z^t\tilde{\eta}_1(\zeta,z))^{k-1})\wedge\varpi_{l'}(\zeta,z)}{\tilde{S}^{j+1}(\zeta,z)|\zeta-z|^{2l}}\right)\\
&\cdot(B^z+B^\zeta)\tilde{S}(\zeta,z)\prod_{i=1}^{j'}(\tilde{Z}_i^z+\tilde{Z}_i^\zeta)\tilde{S}(\zeta,z)\\
&+\int_{bD}\Gamma^s(f)(\zeta)\wedge\left(\frac{X^{k'}(\tilde{\eta}_1(\zeta,z)\wedge(\bar{\partial}_z^t\tilde{\eta}_1(\zeta,z))^{k-1})}{\tilde{S}^j(\zeta,z)}\right)\\
&\wedge(B^z+B^\zeta)\left(\frac{\varpi_{l'}(\zeta,z)}{|\zeta-z|^{2l}}\right)\prod_{i=1}^{j'}(\tilde{Z}_i^z+\tilde{Z}_i^\zeta)\tilde{S}(\zeta,z)\\
=&\,X_1'+X_2'+X_3'+X_4'.
\end{aligned}$$

Because

$$(B^\zeta+B^z)\frac{\varpi_l(\zeta,z)}{|\zeta-z|^{2l}}=\frac{\varpi_l(\zeta,z)}{|\zeta-z|^{2l}},$$

we have that $X_4'=\tilde{I}[f](j,j',k,k',l,l',s)$ and satisfies (CI).

For $X_1'$ we integrate by parts and obtain

$$\begin{aligned}
X_1'=&-\int_{bD}\Gamma^s(f)(\zeta)\wedge\frac{X^{k'}(\tilde{\eta}_1(\zeta,z)\wedge(\bar{\partial}_z^t\tilde{\eta}_1(\zeta,z))^{k-1})\wedge\varpi_{l'}(\zeta,z)}{|\zeta-z|^{2l}}\frac{(j-1)^{-1}}{V_1^\zeta\tilde{S}(\zeta,z)}\\
&\cdot(B^z+B^\zeta)\prod_{i=1}^{j'}(\tilde{Z}_i^z+\tilde{Z}_i^\zeta)\tilde{S}(\zeta,z)V_1^\zeta\left(\frac{1}{\tilde{S}^{j-1}(\zeta,z)}\right)\\
=&\int_{bD}V_1^\zeta\frac{(j-1)^{-1}\Gamma^s(f)(\zeta)}{V_1^\zeta\tilde{S}(\zeta,z)}\wedge\frac{X^{k'}(\tilde{\eta}_1(\zeta,z)\wedge(\bar{\partial}_z^t\tilde{\eta}_1(\zeta,z))^{k-1})\wedge\varpi_{l'}(\zeta,z)}{\tilde{S}^{j-1}(\zeta,z)|\zeta-z|^{2l}}
\end{aligned}$$

$$\cdot (B^z + B^\zeta)\prod_{i=1}^{j'}(\tilde Z_i^z + \tilde Z_i^\zeta)\tilde S(\zeta,z)$$

$$+\int_{bD}\Gamma^s(f)(\zeta)\wedge\frac{X^{k'}(\tilde\eta_1(\zeta,z)\wedge(\bar\partial_z^t\tilde\eta_1(\zeta,z))^{k-1})\wedge\varpi_{l'}(\zeta,z)}{\tilde S^{j-1}(\zeta,z)|\zeta-z|^{2l}}\frac{(j-1)^{-1}}{V_1^\zeta\tilde S(\zeta,z)}$$

$$\cdot V_1^\zeta\left((B^z + B^\zeta)\prod_{i=1}^{j'}(\tilde Z_i^z + \tilde Z_i^\zeta)\tilde S(\zeta,z)\right)$$

$$+\int_{bD}\Gamma^s(f)(\zeta)\wedge\frac{V_1^\zeta X^{k'}(\tilde\eta_1(\zeta,z)\wedge(\bar\partial_z^t\tilde\eta_1(\zeta,z))^{k-1})\wedge\varpi_{l'}(\zeta,z)}{\tilde S^{j-1}(\zeta,z)|\zeta-z|^{2l}}\frac{(j-1)^{-1}}{V_1^\zeta\tilde S(\zeta,z)}$$

$$\cdot (B^z + B^\zeta)\prod_{i=1}^{j'}(\tilde Z_i^z + \tilde Z_i^\zeta)\tilde S(\zeta,z)$$

$$+\int_{bD}\Gamma^s(f)(\zeta)\wedge\frac{X^{k'}(\tilde\eta_1(\zeta,z)\wedge(\bar\partial_z^t\tilde\eta_1(\zeta,z))^{k-1})}{\tilde S^{j-1}(\zeta,z)}\wedge V_1^\zeta\frac{\varpi_{l'}(\zeta,z)}{|\zeta-z|^{2l}}$$

$$\cdot (B^z + B^\zeta)\prod_{i=1}^{j'}(\tilde Z_i^z + \tilde Z_i^\zeta)\tilde S(\zeta,z)\frac{(j-1)^{-1}}{V_1^\zeta\tilde S(\zeta,z)}$$

$$= X'_{11} + X'_{12} + X'_{13} + X'_{14}.$$

*Case 3:* $j-1=1$. Here we show that (CJ) holds. We have $2l-l'\le 2n-3$ since $k\ge 1$. Now $V_1^\zeta((j-1)^{-1}\Gamma^s(f)/V_1^\zeta\tilde S) = \Gamma^{s+1}f$, and

$$V_1^\zeta X^{k'}(\tilde\eta_1(\zeta,z)\wedge(\bar\partial_z^t\tilde\eta_1(\zeta,z))^{k-1}),\qquad V_1^\zeta\left((B^z + B^\zeta)\prod_{i=1}^{j'}(\tilde Z_i^z + \tilde Z_i^\zeta)\tilde S(\zeta,z)\right),$$
$$\left((B^z + B^\zeta)\prod_{i=1}^{j'}(\tilde Z_i^z + \tilde Z_i^\zeta)\tilde S(\zeta,z)\right),\qquad X^{k'}(\tilde\eta_1(\zeta,z)\wedge(\bar\partial_z^t\tilde\eta_1(\zeta,z))^{k-1})$$

are uniformly bounded. Therefore $X'_{11} = \tilde J[f](1,0,0,0,l,l',s+1)$, $X'_{12}$, and $X'_{13} = \tilde J[f](1,0,0,0,l,l',s)$ all satisfy (CJ). By Lemma 3.6,

$$(B^\zeta + B^z)(\tilde Z_1^\zeta + \tilde Z_1^z)\tilde S(\zeta,z) = \varpi_1(\zeta,z)\quad\text{and}\quad(\tilde Z_1^z + \tilde Z_1^\zeta)\tilde S(\zeta,z) = \varpi_1(\zeta,z).$$

Necessarily $j'>0$, so $X'_{14} = \tilde J[f](1,0,0,0,l,l',s) + \tilde J[f](1,0,0,0,l+1,l'+2,s)$ satisfies (CJ).

*Case 4:* $j-1>1$. We have that $V_1^\zeta((j-1)^{-1}\Gamma^s(f)/V_1^\zeta\tilde S) = \Gamma^{s+1}f$ and that $(B^z + B^\zeta)(\tilde Z_i^z + \tilde Z_i^\zeta)\tilde S(\zeta,z)$ is bounded for all $i=1,\dots,j'$. Consequently, $X'_{11} = \tilde I[f](j-1,j'-1,k,k',l,l',s+1)$ satisfies (CI).

The terms of

$$V_1^\zeta(\tilde Z_i^z + \tilde Z_i^\zeta)\tilde S(\zeta,z),\qquad(B^\zeta + B^z)(\tilde Z_i^z + \tilde Z_i^\zeta)\tilde S(\zeta,z),$$
$$\text{and}\quad(B^\zeta + B^z)V_1^\zeta(\tilde Z_i^z + \tilde Z_i^\zeta)\tilde S(\zeta,z)$$

are uniformly bounded for $i = 1, \dots, j'$. Hence $X'_{12}$ is a finite sum of $\tilde{I}[f](j-1, j'-1, k, k', l, l', s)$ and, if $j' > 1$, of $\tilde{I}[f](j-1, j'-2, k, k', l, l', s)$, both of which satisfy (CI).

Note that $(B^\zeta + B^z)(\tilde{Z}_i^z + \tilde{Z}_i^\zeta)\tilde{S}(\zeta, z)$ is bounded for all $i = 1, \dots, j'$. Since $(j, j', k, k', l, l')$ satisfies (CI), we have either $k > k'$ or $k = k'$. If $k > k'$ then $X'_{13} = \tilde{I}[f](j-1, j'-1, k, k'+1, l, l', s)$ satisfies (CI). But if $k = k'$ then, in $X^{k'+1}(\tilde{\eta}_1(\zeta, z) \wedge (\bar{\partial}_z^t \tilde{\eta}_1(\zeta, z))^{k-1})$, either $\tilde{\eta}_1$ or $\bar{\partial}_z^t \tilde{\eta}_1$ is differentiated at least two times. Since those derivatives are uniformly bounded, if $\tilde{\eta}_1$ is differentiated at least two times then we get a $\tilde{J}[f](j-1, j'-1, k-1, \tilde{k}', l, l', s)$, and if $\bar{\partial}_z^t \tilde{\eta}_1$ is differentiated at least two times then we get an $\tilde{I}[f](j-1, j'-1, k-1, \tilde{k}', l, l', s)$, $\tilde{k}' \le k'-1$, which satisfy (CJ) and (CI), respectively.

By Lemma 3.6, $(B^z + B^\zeta)(\tilde{Z}_i^\zeta + \tilde{Z}_i^z)\tilde{S}(\zeta, z) = O(|\zeta - z|)$ for all $i$. Therefore, $X'_{14} = \tilde{I}[f](j-1, j'-1, k, k', l, l', s) + \tilde{I}[f](j-1, j'-1, k, k', l+1, l'+2, s)$ satisfies (CI).

Both $X'_2$ and $X'_3$ can be treated as $X'_1$ using integrations by parts. Then (ii), (iii), and (iv) can be shown via the same method used to demonstrate (i). □

We now show that (CI) and (CJ) lead to estimates like those in [7, Lemma 5.5]. We denote by $\iota\colon bD \times [0,1] \times \mathbb{C}^n \to \mathbb{C}^n \times [0,1] \times \mathbb{C}^n$ the canonical injection and denote by $\iota^*$ the pullback by $\iota$.

PROPOSITION 4.2. *Fix* $\Delta = \frac{\partial}{\partial z_t}$ *or* $\frac{\partial}{\partial \bar{z}_t}$, $t = 1, \dots, n$, *and let* $\varepsilon \in ]0, \varepsilon_0]$.

(i) *For* $s \le p$, $(j, j', k, k', l, l')$ *satisfying* (CI), *and* $z_0 \in (\mathcal{V} \cap B(\zeta_0, R')) \setminus \bar{D}$, *respectively* $z_0 \in \mathcal{V} \cap D \cap B(\zeta_0, R')$,

$$\Delta\Bigg(\Gamma^s f(\zeta) \wedge \frac{X^{k'}(\tilde{\eta}_1(\zeta, z_0) \wedge (\bar{\partial}_z^t \tilde{\eta}_1(\zeta, z_0))^{k-1}) \wedge \varpi_{l'}(\zeta, z_0)}{\tilde{S}^j(\zeta, z_0)|\zeta - z_0|^{2l}} \prod_{i=1}^{j'} (\tilde{Z}_i^z + \tilde{Z}_i^\zeta)\tilde{S}(\zeta, z_0)\Bigg),$$

*respectively*

$$\iota^*\Bigg(\Delta\Bigg(\Gamma^s f(\zeta) \wedge \frac{X^{k'}(\eta_1(\zeta, z_0) \wedge (\bar{\partial}_\zeta \eta_1(\zeta, z_0))^{k-1}) \wedge \varpi_{l'}(\zeta, z_0)}{S^j(\zeta, z_0)|\zeta - z_0|^{2l}} \prod_{i=1}^{j'} (\tilde{Z}_i^z + \tilde{Z}_i^\zeta)S(\zeta, z_0)\Bigg)\Bigg),$$

*can be uniformly estimated by a sum of products of the form*

$$\|f\|_{bD, p-s} \frac{\varepsilon^{-1}}{\prod_{i=0}^{k-1} \tau_{\nu_i}(z_0, \varepsilon) \prod_{i=1}^{k-1} \tau_{\mu_i}(z_0, \varepsilon)|\zeta - z_0|^{2n-2k-1}}$$

*for* $\zeta \in bD \cap \mathcal{P}_\varepsilon^0(z_0)$ *if* $\varepsilon \ne |r(z_0)|$ *and for* $\zeta \in bD \cap \mathcal{P}_{|r(z_0)|}(z_0)$ *otherwise.*

(ii) *For* $s \le p$, $(j, j', k, k', l, l')$ *satisfying* (CJ), *and* $z_0 \in (\mathcal{V} \cap B(\zeta_0, R')) \setminus \bar{D}$, *respectively* $z_0 \in \mathcal{V} \cap D \cap B(\zeta_0, R')$,

$$\Delta\Bigg(\Gamma^s f(\zeta) \wedge \frac{X^{k'}(\bar{\partial}_z^t \tilde{\eta}_1(\zeta, z_0))^k \wedge \varpi_{l'}(\zeta, z_0)}{\tilde{S}^j(\zeta, z_0)|\zeta - z_0|^{2l}} \prod_{i=1}^{j'} (\tilde{Z}_i^z + \tilde{Z}_i^\zeta)\tilde{S}(\zeta, z_0)\Bigg),$$

*respectively*

$$\iota^*\left(\Delta\left(\Gamma^s f(\zeta)\wedge\frac{X^{k'}(\bar\partial_\zeta\tilde\eta_1(\zeta,z_0))^k\wedge\varpi_{l'}(\zeta,z_0)}{\tilde S^j(\zeta,z_0)|\zeta-z_0|^{2l}}\prod_{i=1}^{j'}(\tilde Z_i^z+\tilde Z_i^\zeta)\tilde S(\zeta,z_0)\right)\right),$$

*can be uniformly estimated by a sum of products of the form*

$$\|f\|_{bD,p-s}\frac{\varepsilon^{-1}}{\prod_{i=1}^k\tau_{\nu_i}(z_0,\varepsilon)\prod_{i=1}^k\tau_{\mu_i}(z_0,\varepsilon)|\zeta-z_0|^{2n-2k-2}}$$

*when* $j>1$ *and otherwise by*

$$\|f\|_{bD,p-s}\frac{\varepsilon^{-2}}{|\zeta-z_0|^{2n-3}}$$

*for* $\zeta\in bD\cap\mathcal{P}^0_\varepsilon(z_0)$ *if* $\varepsilon\neq|r(z_0)|$ *and for* $\zeta\in bD\cap\mathcal{P}_{|r(z_0)|}(z_0)$ *otherwise.*

*In all cases we have* $\nu_i\neq\nu_{i'}$ *and* $\mu_i\neq\mu_{i'}$ *when* $i\neq i'$ *and* $\mu_i>1$ *for all* $i$.

*Proof.* We estimate

$$I:=\iota^*\left(\Delta\left(\Gamma^s f(\zeta)\wedge\frac{X^{k'}((\bar\partial_\zeta\eta_1(\zeta,z_0))^k)\wedge\varpi_{l'}(\zeta,z_0)}{S^j(\zeta,z_0)|\zeta-z_0|^{2l}}\prod_{i=1}^{j'}(\tilde Z_i^z+\tilde Z_i^\zeta)S(\zeta,z)\right)\right).$$

We fix $\varepsilon>0$ and choose an $\varepsilon$-extremal basis $w_1^*,\dots,w_n^*$ at $z_0$, with $\zeta\in\mathcal{P}^0_\varepsilon(z_0)$ if $\varepsilon\neq|r(z_0)|$ and $\zeta\in\mathcal{P}_\varepsilon(z_0)$ otherwise. We express $I$ in the basis $w_1^*,\dots,w_n^*$. We must estimate terms such as

$$\iota^*\left(\Gamma^s(f)(\zeta)\wedge\Delta\frac{X^{k'}\left(\bigwedge_{i=1}^k\frac{\partial Q^*_{\nu_i}}{\partial\bar\zeta^*_{\mu_i}}(\zeta,z_0)d\bar\zeta^*_{\mu_i}\wedge d\zeta^*_{\nu_i}\right)\wedge\varpi_{l'}(\zeta,z)}{S^j(\zeta,z)|\zeta-z|^{2l}}\prod_{i=1}^{j'}(\tilde Z_i^z+\tilde Z_i^\zeta)S(\zeta,z)\right),\tag{24}$$

where necessarily $\mu_i\neq\mu_{i'}$ and $\nu_i\neq\nu_{i'}$ for all $i\neq i'$.

Now we use the estimate

$$\left|\frac{\partial Q_i^*}{\partial\bar\zeta_j^*}(\zeta,z_0)\right|\lesssim\frac{\varepsilon}{\tau_i(z_0,\varepsilon)\tau_j'(z_0,\varepsilon)}$$

given by Lemma 3.3. For $V_1,\dots,V_p\in\{V_1^\zeta,Z_1^z+Z_1^\zeta,\dots,Z_n^z+Z_n^\zeta,\bar Z_2^z+\bar Z_2^\zeta,\dots,\bar Z_n^z+\bar Z_n^\zeta\}$, $p\geq1$, we use the estimate

$$\left|V_1\dots V_p\left(\frac{\partial Q_i^*}{\partial\bar\zeta_j^*}(\zeta,z_0)\right)\right|\lesssim\frac{\varepsilon^{1-p/2}}{\tau_i(z_0,\varepsilon)\tau_j'(z_0,\varepsilon)}.$$

When $p=1$ this estimate was shown in Lemma 3.8 and Corollary 3.5. When $p\geq2$ we observe that $\varepsilon^{1-p/2}/\tau_i(z_0,\varepsilon)\tau_j'(z_0,\varepsilon)$ is bounded away from 0 and that

$$\left|V_1\dots V_p\left(\frac{\partial Q_i^*}{\partial\bar\zeta_j^*}(\zeta,z_0)\right)\right|\text{ is uniformly bounded w.r.t. }z_0\text{ and }\zeta.$$

Therefore the estimate holds for all $p \geq 2$. We also use the estimates

$$\left|\Delta\left(\frac{\partial Q_i^*}{\partial \bar{\zeta}_j^*}(\zeta, z_0)\right)\right| \lesssim \frac{1}{\tau_i(z_0,\varepsilon)\tau_j'(z_0,\varepsilon)}$$

and

$$\left|\Delta V_1 \dots V_p\left(\frac{\partial Q_i^*}{\partial \bar{\zeta}_j^*}(\zeta, z_0)\right)\right| \lesssim \frac{\varepsilon^{-p/2}}{\tau_i(z_0,\varepsilon)\tau_j'(z_0,\varepsilon)}, \quad p \geq 1,$$

which also hold because the LHS above is uniformly bounded and the RHS is bounded away from 0.

Thus when $\mu_i \neq 1$ for all $i$, we may use Lemmas 3.2, 3.3, and 3.8 together with Corollary 3.5 to estimate (24) as

$$\|f\|_{bD,p-s} \frac{\varepsilon^{k-j-1-(k'-j')/2}}{\prod_{i=1}^k \tau_{\nu_i}(z_0,\varepsilon)\prod_{i=1}^k \tau_{\mu_i}(z_0,\varepsilon)|\zeta - z_0|^{2l-l'}}.$$

If $\mu_{i_0} = 1$, we estimate an $(n, n-1)$-form in $\zeta$ and note that

$$\iota^*\left(\bigwedge_{i=1}^n d\zeta_i^* \bigwedge_{\substack{i=1\\ i\neq i'}}^n d\bar{\zeta}_i^*\right) = \iota^*\left(\frac{(-1)^{i'+1}\dfrac{\partial r}{\partial \overline{\zeta_{i'}^*}}(\zeta)}{\dfrac{\partial r}{\partial \overline{\zeta_1^*}}(\zeta)} \bigwedge_{i=1}^n d\zeta_i^* \wedge \bigwedge_{i=2}^n d\overline{\zeta_i^*}\right).$$

Moreover, $\varepsilon_0$ is so small that $|(\partial r/\partial \bar{\zeta}_1^*)(\zeta)| \gtrsim 1$, and by [7, Props. 3.1(vii), (iv), (v)] we have

$$\left|\frac{\partial r}{\partial \bar{\zeta}_p^*}(\zeta)\right| \lesssim \frac{\tau_1(z_0,\varepsilon)}{\tau_p(z_0,\varepsilon)}.$$

Hence there exists a $\mu_0$ with $\mu_0 \neq 1$ and $\mu_0 \neq \mu_i$ for all $i \geq 1$ such that (24) can be estimated by

$$\|f\|_{bD,p-s} \frac{\varepsilon^{k-j-1-(k'-j')/2}}{\prod_{i=1}^k \tau_{\nu_i}(z_0,\varepsilon)\prod_{i=1,\mu_i\neq 1}^k \tau_{\mu_i}(z_0,\varepsilon)|\zeta - z_0|^{2l-l'}} \frac{1}{\tau_{\mu_0}(z_0,\varepsilon)}.$$

If (CJ) is fulfilled with $j > 1$ then $k - j - (k' - j')/2 - 1 \geq -1$ and $2l - l' \leq 2n - 2k - 2$. So $I$ can be estimated by a sum of terms such as

$$\|f\|_{bD,p-s} \frac{\varepsilon^{-1}}{\prod_{i=1}^k \tau_{\nu_i}(z_0,\varepsilon)\prod_{i=1}^k \tau_{\mu_i}(z_0,\varepsilon)|\zeta - z_0|^{2n-2k-2}},$$

with $\nu_i \neq \nu_{i'}$ and $\mu_i \neq \mu_{i'}$ when $i \neq i'$ and $\mu_i > 1$ for all $i$.

If (CJ) is fulfilled with $j = 1$ then $k = k' = j' = 0$ and $2l - l' \leq 2n - 3$, so $I$ can be estimated by

$$\|f\|_{bD,p-s} \frac{\varepsilon^{-2}}{|\zeta - z_0|^{2n-3}}.$$

The other estimates of the lemma can be shown by the same method. □

COROLLARY 4.3. *For* $\Delta = \frac{\partial}{\partial z_t}$ *or* $\frac{\partial}{\partial \bar{z}_t}$, $t = 1, \dots, n$, *we have uniformly with respect to* $z$ *and* $f$

$$|\Delta L[f](j,j',k,k',l,l',s)(z)| \lesssim d(z,bD)^{1/m-1}\|f\|_{bD,s}$$

*when* (CI) *for* $L = I, \tilde I$ *and* (CJ) *for* $L = J, \tilde J$ *is fulfilled. Here* $z$ *is closed to* $bD$*, with* $z \in D$ *for* $L = I, J$ *and* $z \in \mathcal{V} \setminus \bar D$ *for* $L = \tilde I, \tilde J$.

*Proof.* Since $\Gamma^s f$ has a compact support in $\overline{B(\zeta_0, R'')}$ and since $S(\zeta, z)$ is bounded away from zero when $r(\zeta) \ge r(z)$ and when $|\zeta - z|$ is bounded away from zero, we need only consider the case $z_0 \in B(\zeta_0, R')$ and then integrate over $\overline{B(\zeta_0, R'')} \cap bD \cap \mathcal{P}_{\varepsilon_0}(z_0)$ for some $\varepsilon_0 > 0$ sufficiently small and not depending on $z_0$.

As in [7] we use the covering (11) and Proposition 4.2. When (CI) is fulfilled, for all $z_0 \in D \cap B(\zeta_0, R') \cap \mathcal{V}$ we have

$$\left|\int_{\mathcal{P}^0_{2^{-i}\varepsilon_0}(z_0)\cap bD\cap B(\zeta_0,R'')} \Delta\Bigg(\prod_{i=1}^{j'}(\tilde Z_i^z + \tilde Z_i^\zeta)S(\zeta,z)\Gamma^s f(\zeta) \wedge \frac{X^{k'}(\eta_1(\zeta,z_0)\wedge(\bar\partial_\zeta\eta_1(\zeta,z_0))^{k-1})\wedge\varpi_{l'}(\zeta,z_0)}{S^j(\zeta,z_0)|\zeta-z_0|^{2l}}\Bigg)\right| \lesssim (2^{-i}\varepsilon_0)^{1/m-1}\|f\|_{bD,s}, \quad (25)$$

$$\left|\int_{\mathcal{P}_{|r(z_0)|}(z_0)\cap bD\cap B(\zeta_0,R'')} \Delta\Bigg(\prod_{i=1}^{j'}(\tilde Z_i^z + \tilde Z_i^\zeta)S(\zeta,z)\Gamma^s f(\zeta) \wedge \frac{X^{k'}(\eta_1(\zeta,z_0)\wedge(\bar\partial_\zeta\eta_1(\zeta,z_0))^{k-1})\wedge\varpi_{l'}(\zeta,z_0)}{S^j(\zeta,z_0)|\zeta-z_0|^{2l}}\Bigg)\right| \lesssim |r(z_0)|^{1/m-1}\|f\|_{bD,s}. \quad (26)$$

Adding (25) for $i = 0, \ldots, j_0$ and (26) and then using $2^{-j_0}\varepsilon_0 \eqsim |r(z_0)|$, we get $|\Delta I[f](j,j',k,k',l,l',s)(z_0)| \lesssim d(z_0, bD)^{1/m-1}\|f\|_{bD,s}$.

The other estimates can be shown in the same way. □

## 5. Final Integral Estimates

*Proof of Lemma 2.2 and Theorem 1.1(i).* Let $[f] \in C^p_{0,q}(bD)$, $p \in \mathbb{N}$, with $f \in C^p_{0,q}(bD)$. Using the compactness of $bD$, we may assume that $f$ has a compact support in $B(\zeta_0, R'')$ for $\zeta_0 \in bD$ and $R' > R'' > 0$ not depending on $\zeta_0$. For $p$ vector fields $B_1^z, \ldots, B_p^z \in \{Z_1^z, \ldots, Z_n^z, \bar Z_2^z, \ldots, \bar Z_n^z\}$ and $\Delta = \frac{\partial}{\partial z_t}$ or $\frac{\partial}{\partial \bar z_t}$, $t = 1, \ldots, n$, we show that

$$|\Delta B_1^z \ldots B_p^z T_q f(z)| \lesssim d(z, bD)^{1/m-1}\|f\|_s$$

uniformly with respect to $z \in D$ sufficiently close to $bD$.

After integrating with respect to $\lambda \in [0,1]$, $T_q f$ can be written as a sum for $i = 0$ to $n-q-1$ of $I[f](i+1, 0, i, 0, 2(n-i-1), 1, 0)$. An induction argument using Proposition 4.1 shows that $B_1^z \ldots B_p^z T_q f$ is a finite sum of $J[f](j,j',k,k',l,l',s)$ and $I[f](j,j',k,k',l,l',s)$, $s \le p$, satisfying respectively (CJ) and (CI). Now Corollary 4.3 implies that $|\Delta B_1^z \ldots B_p^z T_q f(z)| \lesssim d(z,bD)^{1/m-1}\|f\|_{bD,p}$ uniformly with respect to $z$ and $f$. One may show analogously that $|\Delta B_1^z \dots B_p^z \tilde{T}_q^t f(z)| \lesssim d(z, bD)^{1/m-1} \|f\|_{bD,p}$ for all $z \in \mathcal{V} - \bar{D}$.

When $p = 0$, this proves Lemma 2. Theorem 1.1(i) then follows by the Hardy–Littlewood lemma. □

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