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SAGBI Bases and Degeneration of Spherical Varieties to Toric Varieties

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Introduction

Let $X \subset \mathbb{P}(V)$ be a (normal) complex projective G-variety, where G is a (reductive) classical group and V is a complex finite-dimensional G-module. Suppose X is *spherical*—that is, a Borel subgroup has a dense orbit. Generalizing the case of toric varieties, one can associate an integral convex polytope $\Delta(X)$ to X such that the Hilbert polynomial $h(t)$ of X is the Ehrhardt polynomial of $\Delta(X)$, that is, $h(t) =$ number of integral points in $t\Delta(X)$. The polytope $\Delta(X)$ is the polytope fibred over the moment polytope of X with the Gelfand–Cetlin polytopes as fibres. This polytope was defined by Okounkov [O1] based on results of Brion. Following Okounkov, we call this polytope the *Newton polytope* of X.

In this paper, for $G = \mathrm{SP}(2n, \mathbb{C})$ we show that X can be deformed (degenerated), by a flat deformation, to the toric variety corresponding to the polytope $\Delta(X)$ (Corollary 5.5). This is a consequence of the main result of our paper: the homogeneous coordinate ring of a *horospherical* variety has a SAGBI basis (Theorem 5.1). A spherical variety is horospherical if the stabilizer of a point in the dense G-orbit contains a maximal unipotent subgroup. Flag varieties and Grassmanians are examples of horospherical varieties. It is known that any spherical variety can be deformed, by a flat deformation, to a horospherical variety such that the moment polytopes of the two varieties are the same (see [P; ABr, Sec. 2.2; Kn, Satz 2.3]).

More precisely, we prove that if $X \subset \mathbb{P}(V)$ is a projective horospherical G-variety ($G = \mathrm{SP}(2n, \mathbb{C})$), then the homogeneous coordinate ring R of X can be embedded in a Laurent polynomial algebra and has a SAGBI basis with respect to a natural term order. (SAGBI stands for subalgebra analogue of Gröbner basis for ideals.) Moreover, we show that the semi-group of initial monomials is the semi-group of integral points in the cone over the polytope $\Delta(X)$. A finite collection $f_1, \dots, f_r$ of elements of R is a SAGBI basis, with respect to a term order, if the semi-group of initial monomials is generated by the initial monomials of the f_i and if, moreover, every element of R can be represented as a polynomial in the f_i, in a finite number of steps, by means of a simple classical algorithm called the *subduction algorithm*.

Degenerations of flag and Schubert varieties to toric varieties have been studied by Gonciulea and Lakshmibai [GoL] and Caldero [Ca]. More recently, Kogan and

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Miller [KoM] have shown the existence of a SAGBI basis for the coordinate ring of the flag variety of $\mathrm{GL}(n,\mathbb{C})$. More precisely, they prove that, for any dominant weight λ in the interior of the Weyl chamber: (a) the homogeneous coordinate ring of the flag variety $\mathrm{GL}(n)/B$ embedded in $\mathbb{P}(V_\lambda)$ (i.e. the generalized Plücker embedding) has a SAGBI basis; and (b) $\mathrm{GL}(n)/B$ can be degenerated to the toric variety corresponding to the Gelfand–Cetlin polytope of λ. The main results of this paper (Theorem 5.1 and Corollary 5.5) imply a similar result for the flag varieties G/P of $G = \mathrm{SP}(2n,\mathbb{C})$.

I was informed by A. Mustata that the degenerations of spherical varieties to toric varieties can be useful in understanding the mirrors of hypersurfaces in spherical varieties. In [Gi] and [BCKV], Givental and others use the Gonciulea–Lakshmibai toric degeneration to give a mirror construction for the hypersurfaces in a (partial) flag variety.

A key step in our proof is a result of Okounkov on the representation theory of $\mathrm{SP}(2n,\mathbb{C})$. Let V_λ denote the irreducible G-module with highest weight λ, where $G = \mathrm{SP}(2n,\mathbb{C})$. It is well known that one can view V_λ as a subspace of $\mathbb{C}[G]$ and, after restriction to U, as a subspace of $\mathbb{C}[U]$, where U is the standard maximal unipotent subgroup of G. In [O2] Okounkov proves that, with respect to a natural term order on $\mathbb{C}[U]$, the set of highest terms of elements of V_λ can be identified with the Gelfand–Cetlin polytope Δ_λ (Theorem 4.2). As Okounkov informed the author, by using methods similar to those used for $\mathrm{SP}(2n,\mathbb{C})$, one can prove his result for other classical groups. But so far he has not published the proofs for other classical groups (e.g., $\mathrm{GL}(n,\mathbb{C})$ and the orthogonal group). Our results here, as well as their proofs, go verbatim for other classical groups—provided that Okounkov's result is shown to hold for them.

In Section 1 we discuss SAGBI bases. Section 2 deals with some facts about homogeneous coordinate rings of spherical varieties, and we give a description of the homogeneous coordinate ring of a horospherical variety. In Section 3, we define the Gelfand–Cetlin polytopes and the polytope $\Delta(X)$. Section 4 discusses the result of Okounkov on the initial monomials of elements of an irreducible G-module and Gelfand–Cetlin polytopes for $G = \mathrm{SP}(2n,\mathbb{C})$. Finally, in Section 5 we state and prove our main results.

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1. SAGBI Bases

In this section we define the notion of a SAGBI basis for a subalgebra of the Laurent polynomials. SAGBI bases play an important role when one deals with subalgebras of the polynomial or Laurent polynomial algebras. Their theory is more complicated than the theory of Gröbner bases. In particular, not every subalgebra has a SAGBI basis with respect to a given term order. It is an unsolved problem to determine, for a given term order, which subalgebras have a SAGBI basis.

Let $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ denote the algebra of Laurent polynomials in n variables. Let $\prec$ be a term order on $\mathbb{Z}^n$, that is, a total order compatible with addition. An important example is the lexicographic order. We will identify a monomial with its exponent and hence regard it as an element of $\mathbb{Z}^n$. Obviously, the multiplication of monomials corresponds to the addition of their exponents in $\mathbb{Z}^n$. The initial monomial (with respect to $\prec$) of a polynomial f is denoted by $\mathrm{in}(f)$. If R is a subalgebra of $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, we denote by $\mathrm{in}(R)$ the semi-group of initial monomials in R, that is, $\{\mathrm{in}(f) \mid 0 \neq f \in R\}$.

First consider the case where R is a subalgebra of $\mathbb{C}[x_1, \ldots, x_n]$. In this case, one usually assumes that $\prec$ satisfies the extra condition

$$\mathbf{a} \succ (0, \ldots, 0) \quad \forall \mathbf{a} \ \ (0 \neq \mathbf{a} \in \mathbb{N}^n).$$

Definition 1.1. Let R be a subalgebra of $\mathbb{C}[x_1, \ldots, x_n]$. A finite collection of polynomials $\{f_1, \ldots, f_r\} \subset R$ is a SAGBI basis for R if $\{\mathrm{in}(f_1), \ldots, \mathrm{in}(f_r)\}$ generates the semi-group $\mathrm{in}(R)$.

When R has a SAGBI basis, there is a simple classical algorithm (due to Kapur–Madlener and Robbiano–Sweedler [RoS]) for expressing elements of R in terms of the f_i as follows. Write $\mathrm{in}(f) = d_1 \,\mathrm{in}(f_1) + \cdots + d_r \,\mathrm{in}(f_r)$ for some $d_1, \ldots, d_r \in \mathbb{N}$. Dividing the leading coefficient of f by the leading coefficient of $f_1^{d_1} \cdots f_r^{d_r}$, we obtain a c such that the leading term of f is the same as the leading term of $c f_1^{d_1} \cdots f_r^{d_r}$. Set $g = f - c f_1^{d_1} \cdots f_r^{d_r}$. If $g = 0$ we are done; otherwise, we replace f by g and proceed inductively. Since g has a smaller leading exponent than f and since $\mathbb{N}^n$ is well-ordered with respect to $\prec$, it follows that this process will terminate, resulting in an expression for f as a polynomial in the f_i. This is referred to as the *subduction algorithm*. See [RoS] for a detailed discussion of SAGBI bases for subalgebras of $\mathbb{C}[x_1, \ldots, x_n]$.

In general, if R is a subalgebra of $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ then, since $\mathbb{Z}^n$ is not well-ordered, there is no guarantee that this algorithm terminates. Following [R, p. 2], we define the SAGBI basis as follows.

Definition 1.2. Let R be a subalgebra of $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. A finite collection of polynomials $\{f_1, \ldots, f_r\}$ is a SAGBI basis for R if:

(a) the $\mathrm{in}(f_i)$ generate $\mathrm{in}(R)$ as a semi-group; and
(b) the subduction algorithm just described terminates for every $f \in R$, no matter what choices are made for $d_1, \ldots, d_r$ in the course of the algorithm.

The algebra R is said to have a SAGBI basis if it has a SAGBI basis for some choice of a term order.

SAGBI bases are closely related with the toric degeneration of varieties. Namely, if a subalgebra R has a SAGBI basis then $\mathrm{Spec}(R)$ can be deformed, by a flat deformation over $\mathbb{C}$, to a (possibly nonnormal) affine toric variety. The deformation corresponds to the degeneration of the elements of R to their initial monomials (see the paragraph before Corollary 5.3 and [E, Thm. 15.17]).

2. Homogeneous Coordinate Ring of Spherical and Horospherical Varieties

Let V be a finite-dimensional G-module and $X \subset \mathbb{P}(V)$ a projective spherical G-variety; that is, X is normal and a Borel subgroup $B \subset G$ has a dense orbit in X. Let $R = \mathbb{C}[X]$ denote the homogeneous coordinate ring of X. This algebra is graded by the degree of polynomials,

$$R = \bigoplus_{k=0}^{\infty} R_k.$$

We decompose the spaces R_k into irreducible G-modules,

$$R_k = \bigoplus_{\lambda} m_{k,\lambda} V_\lambda,$$

where V_λ is the irreducible G-module with the highest weight λ and $m_{k,\lambda}$ is its multiplicity (here $m_{k,\lambda} V_\lambda$ denotes the direct sum of $m_{k,\lambda}$ copies of V_λ). Since X is spherical, its spectrum is multiplicity free (i.e., $m_{k,\lambda} \in \{0, 1\}$). Let $\Phi(X)$ denote the *moment polytope* of X, that is, the intersection of the image of the moment map with the positive Weyl chamber for the choice of B. (The symplectic structure on X is induced from the projective space and X is regarded as a Hamiltonian K-space, where K is a maximal compact subgroup of G.) Also, denote by Λ the weight lattice of G. The following theorem, due to Brion, determines which weights λ occur in the decomposition of R_k with multiplicity 1.

THEOREM 2.1 [Br1, Sec. 3; Br2]. *There is a sublattice Λ' of Λ such that $\Phi(X) \subset \Lambda'_{\mathbb{R}}$, the vector space spanned by Λ', and we have*

$$R_k = \bigoplus_{\lambda \in k\Phi(X) \cap \Lambda'} V_\lambda.$$

The rank of the sublattice Λ' is called the *rank* of the spherical variety X.

REMARK 2.2. It follows from Theorem 2.1 that one can recover the moment polytope $\Phi(X)$ from the multiplicities of the irreducible G-modules appearing in R_k. More precisely, we have

$$\Phi(X) = \text{closure of } \bigcup_{k=0}^{\infty} \left\{ \frac{\mu}{k} \mid V_\mu \text{ appears in the decomposition of } R_k \right\}.$$

One can show that the ring multiplication in R sends $V_\lambda \times V_\mu$ to $V_{\lambda+\mu} \oplus \bigoplus_\nu V_\nu$, where $\nu = \lambda + \mu - \xi$ and ξ is some nonnegative combination of the simple roots. Now, suppose that all the stabilizer subgroups of the points of X contain a maximal unipotent subgroup. From a theorem of Popov [P, Thm. 2.3] it can be shown that the ring multiplication sends $V_\lambda \times V_\mu$ to $V_{\lambda+\mu}$ and that this map coincides with a Cartan multiplication. (For the definition of Cartan multiplication, see [FH, p. 429].)

DEFINITION 2.3. A spherical G-variety X such that the stabilizer of a point in the dense G-orbit contains a maximal unipotent subgroup is called a *horospherical* variety.

It can be shown that, if X is horospherical, then all the stabilizer subgroups contain a maximal unipotent subgroup. Examples of horospherical varieties are toric varieties, flag varieties, and Grassmanians.

Now assume X is horospherical, and fix a point x in the dense G-orbit of X. Choose highest-weight vectors f_λ in each simple submodule V_λ of R by the condition that $f_\lambda(x) = 1$. Then the product of these highest-weight vectors is again such a vector; that is, $f_\lambda f_\mu = f_{\lambda+\mu}$. Hence, for any two λ and μ appearing in the decomposition of R, one can uniquely define Cartan multiplication. We can then give the following description for the homogeneous coordinate ring of X.

THEOREM 2.4. *We have the following isomorphism of graded algebras*:

$$R \cong \bigoplus_{k=0}^{\infty} \bigoplus_{\lambda \in k\Phi(X) \cap \Lambda'} V_\lambda,$$

where the multiplication in the right-hand side is defined as follows. Let $R_d = \bigoplus_\lambda V_\lambda$ *and* $R_e = \bigoplus_\mu V_\mu$ *be the decompositions of two graded pieces of* R. *Then the multiplication* $R_d \times R_e \to R_{d+e}$ *is given by the Cartan multiplication* $V_\lambda \times V_\mu \to V_{\lambda+\mu}$, *which is defined uniquely by our foregoing choice of the highest-weight vectors* f_λ *and* f_μ.

3. Newton Polytope of a Spherical Variety

Let G be a (reductive) classical group over $\mathbb{C}$. In this section we briefly explain, following [O1], the definition of the Newton polytope of a spherical G-variety X. We start by recalling the Gelfand–Cetlin polytopes.

To each dominant weight λ of G there corresponds a Gelfand–Cetlin (or briefly GC) polytope Δ_λ. The convex polytope Δ_λ has the property that the number of integral points in Δ_λ is equal to the dimension of the irreducible G-module V_λ. The dimension of the GC polytope is equal to the complex dimension of the maximal unipotent subgroup U of G, that is, $\frac{1}{2}(\dim(G) - \operatorname{rank}(G))$. For $G = \mathrm{GL}(n, \mathbb{C})$, the construction of this polytope is due to Gelfand and Cetlin (see [GCe]). Definitions of GC polytopes can be found in [BeZ, Sec. 4]. We shall now recall the definitions of GC polytopes for $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{SP}(2n, \mathbb{C})$.

DEFINITION 3.1 (GC Polytopes for $\mathrm{GL}(n, \mathbb{C})$). Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ be a decreasing sequence of integers representing a dominant weight in $\mathrm{GL}(n, \mathbb{C})$. The GC polytope Δ_λ is the set of all real numbers $x_1, x_2, \dots, x_{n-1}, y_1, \dots, y_{n-2}, \dots, z$ such that the following inequalities hold:

$$
\begin{array}{ccccccccccccc}
\lambda_1 & & \lambda_2 & & \lambda_3 & \cdots & \lambda_{n-2} & & & & \lambda_{n-1} & & \lambda_n \\
& x_{1,1} & & x_{2,1} & & \cdots & & & x_{n-2,1} & & & x_{n-1,1} & \\
& & x_{1,2} & & x_{2,2} & \cdots & x_{n-3,2} & & & & x_{n-2,2} & & \\
& & & \cdots & & \cdots & & & \cdots & & & & \\
& & & & \cdots & & \cdots & & & & & & \\
& & & & & x_{1,n-1}, & & & & & & &
\end{array}
$$

where the structure

$$
\begin{array}{ccc}
a & & b \\
& c &
\end{array}
$$

indicates that $a \geq c \geq b$.

DEFINITION 3.2 (GC Polytopes for $\mathrm{SP}(2n, \mathbb{C})$). Let B be the Borel subgroup of upper triangular matrices in $\mathrm{SP}(2n, \mathbb{C})$, and let

$$T = \{(t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}) \mid t_i \in \mathbb{C}^* \ \forall i = 1, \ldots, n\}$$

be the maximal torus. Every dominant weight is then represented by a decreasing sequence of positive integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$. The GC polytope Δ_λ is the set of all real numbers $x_1, \ldots, x_n, y_1, \ldots, y_{n-1}, \ldots, z, w$ such that the following inequalities hold:

$$
\begin{array}{cccccccc}
\lambda_1 & & \lambda_2 & & \cdots & \lambda_n & & 0 \\
& x_{1,1} & & x_{2,1} & \cdots & & x_{n,1} & \\
& & y_{1,1} & & \cdots & y_{n-1,1} & & 0 \\
& & & \cdots & & & \cdots & \\
& & & & \cdots & & & \\
& & & & & y_{1,1} & & 0 \\
& & & & & & x_{1,1}. &
\end{array}
$$

If the components of the weight λ are real, we still can define Δ_λ by the foregoing inequalities. Hence we can extend the definition of Δ_λ to all the real λ.

LEMMA 3.3. *The assignment $\lambda \mapsto \Delta_\lambda$ is linear; that is, $\Delta_{c\lambda} = c\Delta_\lambda$ for any positive c and $\Delta_{\lambda+\mu} = \Delta_\lambda + \Delta_\mu$, where the addition in the right-hand side is the Minkowski sum of convex polytopes.*

Proof. The proof is immediate from the definition in each of the three cases of classical groups. □

Now, let $X \subset \mathbb{P}(V)$ be a (smooth) projective spherical G-variety and $\Phi(X)$ its moment polytope. As before, let Λ denote the weight lattice and $\Lambda_\mathbb{R}$ the real vector space spanned by Λ.

DEFINITION 3.4 (Newton Polytope of a Spherical Variety). Define the set

$$\Delta(X) \subset \Lambda_\mathbb{R} \oplus \mathbb{R}^{\dim U} = \mathbb{R}^{\dim B}$$

by

$$\Delta(X) = \bigcup_{\lambda \in \Phi(X)} (\lambda, \Delta_\lambda).$$

From Lemma 3.3, it follows that $\Delta(X)$ is a convex polytope.

Remark 3.5. In [O1], as a corollary to a theorem of Brion it is shown that the polytope $\Delta(X)$ has the property

$$\dim R_k = \#\{k\Delta(X) \cap \Lambda'\},$$

where Λ' is the sublattice of Λ in Theorem 2.1. This means that the Hilbert polynomial of the variety X coincides with the Ehrhardt polynomial of the polytope $\Delta(X)$. Note that, since the Hilbert polynomial of a toric variety corresponding to a polytope Δ is the Ehrhardt polynomial of Δ and since the Hilbert polynomial is invariant under a flat deformation, the fact just stated agrees with the main result of the paper; that is, X can be deformed to the toric variety of the polytope $\Delta(X)$ (Corollary 5.5).

4. Initial Monomials of Elements of an Irreducible G-Module and Gelfand–Cetlin Polytopes

Let λ be a dominant weight and V_λ the corresponding irreducible G-module, where $G = \mathrm{SP}(2n, \mathbb{C})$. The purpose of this section is to explain the result of Okounkov [O2] regarding the initial monomials of the elements of V_λ. This result will be needed in the proof of our main theorem.

First, we explain how one can identify V_λ with a subspace of a polynomial algebra—that is, the coordinate ring of the standard maximal unipotent subgroup. Let T be the standard maximal torus of diagonal matrices in G, B_+ the Borel subgroup of upper triangular matrices, and U_+ the maximal unipotent subgroup of B_+. Denote by B_- and U_- the opposite subgroups of B_+ and U_+, respectively. Fix a B_--eigenvector ξ_λ in $(V_\lambda)^*$. It is well known that the mapping from V_λ to $\mathbb{C}[G]$, defined by

$$v \mapsto f_v,$$

$$f_v(g) = \xi_\lambda(g^{-1} \cdot v),$$

maps the G-module V_λ isomorphically to the subspace

$$\{f \in \mathbb{C}[G] \mid f(gb) = (-\lambda)(b) f(g) \ \forall b \in B_-\}, \tag{1}$$

where $-\lambda$ is regarded as a character of B_-. We identify V_λ with its image in $\mathbb{C}[G]$.

Consider the Bruhat decomposition

$$G = \bigcup_{w \in W} B_+ w B_-,$$

where W is the Weyl group. We have $G/B_- = \bigcup_{w\in W} B_+wB_-/B_-$, and the big Bruhat cell $\mathcal{U}$ in G/B_- is B_+B_-. Since $B_+ \cap B_- = T$ and $B_+ = U_+T$, the cell $\mathcal{U}$ can be identified with U_+ via $u \mapsto uB_-$. From (1) and the fact that $\mathcal{U}$ is dense in

G/B_- it follows that every element of $V_\lambda \subset \mathbb{C}[G]$ is uniquely determined by its restriction to U_+. We can thus consider V_λ as a subspace of $\mathbb{C}[U_+]$. Note that U_+ is isomorphic, as a variety, to the affine space of dimension $\frac{1}{2}(\dim(G)-\operatorname{rank}(G))$. For each dominant weight λ, choose a highest-weight vector $v_\lambda \in V_\lambda$ such that $\xi_\lambda(v_\lambda) = 1$.

PROPOSITION 4.1. *The following diagram is commutative*:

$$\begin{tikzcd}
V_\lambda \times V_\mu \arrow[r] \arrow[d, hook] & V_{\lambda+\mu} \arrow[d, hook] \\
\mathbb{C}[G] \times \mathbb{C}[G] \arrow[r] \arrow[d] & \mathbb{C}[G] \arrow[d] \\
\mathbb{C}[U_+] \times \mathbb{C}[U_+] \arrow[r] & \mathbb{C}[U_+],
\end{tikzcd}$$

where the map in the first row is the Cartan multiplication (*defined uniquely by our previous choice of* v_λ *and* v_μ) *and the maps in the second and third rows are the usual product of functions.*

Proof. From (1) it follows that each f_v defines a function on G/U_- and hence that each V_λ can be identified with a subspace of $\mathbb{C}[G/U_-]$. Now the commutativity of the top part of the diagram follows from a theorem of Popov ([P, Thm. 2.3]; see also the paragraph after Remark 2.2). The commutativity of the bottom part of the diagram is trivial. $\square$

Following [O2], we now explain how one can interpret the GC polytope Δ_λ as the convex hull of the set of initial monomials of the elements of V_λ regarded as polynomials in $\mathbb{C}[U_+]$. Choose a basis $e_1, \dots, e_{2n}$ of $\mathbb{C}^{2n}$ in which the matrix of the symplectic form is

$$\begin{bmatrix} & & & & & 1 \\ & \mathbf{0} & & & \cdots & \\ & & & 1 & & \\ & & -1 & & & \\ & \cdots & & & \mathbf{0} & \\ -1 & & & & & \end{bmatrix}.$$

Let x_{ij} be the matrix elements in this basis. We use $x_{11}, \dots, x_{nn}$ as coordinates in T and use the dual coordinates

$$g^\lambda = x_{11}^{\lambda_1} \cdots x_{nn}^{\lambda_n}, \quad g \in T,\ \lambda \in \Lambda,$$

for weights. The weights

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$$

are dominant for B_+.

We use x_{ij} ($i < j, i + j \leq 2n + 1$) as coordinates in U_+ and also as coordinates in the big Bruhat cell $\mathcal{U}$. Consider the following lexicographic ordering on $\mathbb{C}[U_+]$:

$$\prod x_{ij}^{p_{ij}} \succ \prod x_{ij}^{q_{ij}}$$

if $p_{1,2n} < q_{1,2n}$, or if $p_{1,2n} = q_{1,2n}$ and $p_{1,2n-1} < q_{1,2n-1}$, and so on. Observe that, in particular,

$$x_{1,2n} \prec x_{1,2n-1} \prec \cdots \prec x_{12} \prec x_{2,2n-1} \prec \cdots \prec x_{23} \prec \cdots \prec x_{n,n+1}, \tag{2}$$

which is exactly the reverse of the ordering of positive roots induced by the standard lexicographic order in $\mathbb{R}^n$. For a dominant weight λ and a monomial

$$\prod x_{ij}^{p_{ij}},$$

put

$$\begin{aligned} \eta_i &= \lambda_i - p_{1,2n-i+1}, & i &= 1, \dots, n, \\ \theta_i &= \eta_{i+1} + p_{1,i+1}, & i &= 1, \dots, n-1, \\ \eta_i' &= \theta_i - p_{2,2n-i}, & i &= 1, \dots, n-1, \\ \theta_i' &= \eta_{i+1}' + p_{2,i+1}, & i &= 1, \dots, n-2. \end{aligned} \tag{3}$$

THEOREM 4.2 [O2, Thm. 2]. *View V_λ as a subspace of $\mathbb{C}[U_+]$. Then, with the above grading on $\mathbb{C}[U_+]$, the monomial*

$$\prod x_{ij}^{p_{ij}}$$

is an initial monomial of a polynomial in V_λ if and only if the numbers $\eta_1, \dots, \eta_n$, $\theta_1, \dots, \theta_{n-1}, \eta_1', \dots, \eta_{n-1}', \dots$ belong to the GC polytope Δ_λ.

Let us denote the vector $(\eta, \theta, \eta', \theta', \dots) \in \mathbb{R}^{\dim U}$ by (q_{ij}), where $i < j$ and $i + j \leq 2n + 1$. The change of variables $p_{ij} \mapsto q_{ij}$ in (3) can be written in matrix form as

$$(q_{ij}) = A(p_{ij}) + B\lambda, \tag{4}$$

where A is a constant upper triangular matrix with $0, 1, -1$ as entries and $1, -1$ on the diagonal and where B is the matrix of the linear transformation

$$\lambda = (\lambda_1, \dots, \lambda_n) \mapsto (\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_2, \lambda_3, \dots, \lambda_n, \dots, \lambda_n) \in \mathbb{R}^{\dim(U)}.$$

Note that $\det(A) = \pm 1$ and hence the inverse of A also has integer entries. By (4) we can therefore write

$$(p_{ij}) = A^{-1}((q_{ij}) - B\lambda).$$

Theorem 4.2 can be stated as follows: The monomial

$$\prod x_{ij}^{p_{ij}}$$

is an initial monomial of an element of V_λ if and only if $(p_{ij}) \in A^{-1}(\Delta_\lambda - B\lambda)$.

DEFINITION 4.3. We denote the polytope $A^{-1}(\Delta_\lambda - B\lambda)$ by Δ_λ'.

One has $\Delta_\lambda = A\Delta'_\lambda + B\lambda$, and hence the two polytopes can be transformed to each other by integral translations and integral transformations. Thus Δ_λ and Δ'_λ are integrally equivalent. The following lemma is immediate from the definition.

LEMMA 4.4. *The map $\lambda \mapsto \Delta'_\lambda$ is linear. That is, $\Delta'_{c\lambda} = c\Delta'_\lambda$ for a positive c and $\Delta'_{\lambda+\mu} = \Delta'_\lambda + \Delta'_\mu$, where the addition in the right-hand side is the Minkowski sum.*

DEFINITION 4.5. As in the definition of $\Delta(X)$, for a spherical variety X we define $\Delta'(X) \subset \Lambda_{\mathbb{R}} \oplus \mathbb{R}^{\dim U} = \mathbb{R}^{\dim B}$ by

$$\Delta'(X) = \bigcup_{\lambda \in \Phi(X)} (\lambda, \Delta'_\lambda).$$

From Lemma 4.4, it follows that $\Delta'(X)$ is a convex polytope.

REMARK 4.6. The map $(\lambda, x) \mapsto (\lambda, A^{-1}(x - B\lambda))$ is an integral transformation that maps $\Delta(X)$ to $\Delta'(X)$. The inverse of this transformation is $(\lambda, x) \mapsto (\lambda, Ax + B\lambda)$, which is also integral. So the polytopes $\Delta'(X)$ and $\Delta(X)$ can be transformed to each other by integral transformations and hence are integrally equivalent.

5. Main Theorem

In this section we prove the main results of the paper.

THEOREM 5.1. *Let V be a finite-dimensional G-module and $X \subset \mathbb{P}(V)$ a projective horospherical G-variety, where $G = \mathrm{SP}(2n, \mathbb{C})$. Then the following statements hold.*

- (i) *The homogeneous coordinate ring R of X can be embedded into the Laurent polynomial algebra $\mathbb{C}[x_1, \dots, x_d, y_1^{\pm 1}, \dots, y_r^{\pm 1}, t]$, where $d = \frac{1}{2}(\dim(G) - \operatorname{rank}(G))$ and $r = \operatorname{rank}(X)$.*
- (ii) *R has a SAGBI basis with respect to a natural term order. Moreover, the semi-group of initial monomials $S = \mathrm{in}(R) \subset \mathbb{Z}^{d+r+1}$ coincides with the semi-group of integral points in the cone over the polytope $\Delta'(X)$ (see Definitions 4.3 and 4.5); that is,*

$$S = \mathbb{Z}^{d+r+1} \cap \bigcup_{k=0}^{\infty} (k\Delta'(X), k).$$

Proof. We identify $\mathbb{C}[U_+]$ with the polynomial algebra $\mathbb{C}[x_1, \dots, x_d]$ equipped with the term order $\prec$ in Theorem 4.2. For each λ, let ϕ_λ denote the embedding $V_\lambda \hookrightarrow \mathbb{C}[x_1, \dots, x_d]$. Let Λ' be the sublattice of the weight lattice in Theorem 2.1, and let $C \cong (\mathbb{C}^*)^r$ be a torus whose lattice of characters is Λ'. Let $y_1, \dots, y_r$ be a choice of coordinates in C whereby $\mathbb{C}[C] = \mathbb{C}[y_1^{\pm 1}, \dots, y_r^{\pm 1}]$. For $\lambda = (\lambda_1, \dots, \lambda_r) \in \Lambda'$ and $y = (y_1, \dots, y_r) \in C$, define $y^\lambda = y_1^{\lambda_1} y_2^{\lambda_2} \cdots y_r^{\lambda_r}$. Having the algebra isomorphism in Theorem 2.4 in mind, define the function

$$\Psi\colon R = \bigoplus_{k=0}^{\infty} \bigoplus_{\lambda \in k\Phi(X)\cap\Lambda'} V_\lambda \to \mathbb{C}[x_1,\dots,x_d,y_1^{\pm 1},\dots,y_r^{\pm 1},t]$$

by

$$\Psi(f) = t^k y^\lambda \phi_\lambda(f) \quad \forall f \in V_\lambda,\ \lambda \in k\Phi(X)\cap\Lambda',$$

where t is an extra free variable. Then we have the following lemma.

LEMMA 5.2. *Ψ is an injective homomorphism of algebras.*

Proof. Since the ϕ_λ are additive homomorphisms, it follows that Ψ is also additive. The multiplicativity of Ψ follows from Proposition 4.1. The homomorphism Ψ is one-to-one because the ϕ_λ are one-to-one. □

Now, R can be thought of as a subalgebra of $\mathbb{C}[x_1,\dots,x_d,y_1^{\pm 1},\dots,y_r^{\pm 1},t]$. Extend the term order $\prec$ to $\mathbb{C}[x_1,\dots,x_d,y_1^{\pm 1},\dots,y_r^{\pm 1},t]$ by lexicographic order such that $t \succ y_r \succ \cdots \succ y_1 \succ x_i$, $i = 1,\dots,d$. Let $S = \mathrm{in}(R) \subset \mathbb{Z}^{d+r+1}$. By Theorem 4.2 we have

$$S = \mathbb{Z}^{d+r+1} \cap \bigcup_{k=0}^{\infty} \bigcup_{\lambda\in k\Phi(X)\cap\Lambda'} (\Delta'_\lambda, \lambda, k);$$

that is, S is the semi-group of integral points in the cone over the polytope $\Delta'(X)$. This cone is a (strictly) convex rational polyhedral cone and hence S is finitely generated (Gordon's lemma). Also, from the definition of $\prec$ and S, there are only finitely many points in S that are smaller than a given point in S. This means that the subduction algorithm terminates after a finite number of steps. Thus R has a SAGBI basis and the proof of the theorem is finished. □

Suppose R is an arbitrary subalgebra of a Laurent polynomial algebra. It is standard that the polynomials in R can be continuously deformed to their initial monomials. More precisely, one can show that there is a flat family of algebras $\pi : \mathcal{R} \to \mathbb{C}$ such that $\pi^{-1}(t) \cong R$ for all $t \neq 0$ and $\pi^{-1}(0) = \mathbb{C}[\mathrm{in}(R)]$, the semi-group algebra of $\mathrm{in}(R)$ [E, Thm. 15.17]. If the semi-group $\mathrm{in}(R)$ is finitely generated, then $\mathbb{C}[\mathrm{in}(R)]$ is the coordinate ring of an affine (possibly nonnormal) toric variety. Geometrically speaking, this means that $\mathrm{Spec}(R)$ can be deformed, by a flat deformation, to this affine toric variety.

COROLLARY 5.3. *Let $G = \mathrm{SP}(2n,\mathbb{C})$. Any projective horospherical G-variety $X \subset \mathbb{P}(V)$ can be deformed, by a flat deformation, to the toric variety corresponding to the polytope $\Delta(X)$. That is, there exists a flat family of varieties $\pi : \mathcal{X} \to \mathbb{C}$ such that $\pi^{-1}(t) \cong X$ for all $t \neq 0$ and $\pi^{-1}(0)$ is the toric variety of the polytope $\Delta(X)$.*

Proof. Let R be the homogeneous coordinate ring of X. From [E, Thm. 15.17, p. 343] we know that $\mathrm{Spec}(R)$ can be deformed, by a flat deformation, to the affine toric variety whose coordinate ring is the semi-group algebra $\mathbb{C}[S]$. Since $\Delta'(X)$ and $\Delta(X)$ can be transformed to each other by integral transformations

(Remark 4.6), it follows that the semi-group S is isomorphic to S_0, the semi-group of integral points in the cone over $\Delta(X)$. So $\mathrm{Spec}(R)$ can be deformed to the toric variety $\mathrm{Spec}(\mathbb{C}[S_0])$. It is well known that the projectivization of this affine toric variety is the toric variety corresponding to the polytope $\Delta(X)$ (see [St, p. 36]). This finishes the proof. □

Now let $X \subset \mathbb{P}(V)$ be a projective spherical G-variety. By a general result of Popov applied to the spherical varieties, one can deform X, by a flat deformation, to a horospherical variety X_0. This can be stated more precisely as follows.

THEOREM 5.4 (see [P; ABr, Sec. 2.2; Kn, Satz 2.3]). *Let G be a reductive group and Y an affine spherical G-variety. There exists a flat family of affine G-varieties $\pi : \mathcal{Y} \to \mathbb{C}$ such that*

1. *the $Y_t = \pi^{-1}(t)$ are isomorphic to Y as G-varieties for $t \neq 0$,*
2. *$Y_0 = \pi^{-1}(0)$ is horospherical, and*
3. *$\mathbb{C}[Y]$ and $\mathbb{C}[Y_0]$ are isomorphic as graded G-modules; in particular, the multiplicities of the irreducible representations V_λ appearing in the graded pieces $\mathbb{C}[Y]_d$ and $\mathbb{C}[Y_0]_d$ are the same for any $d \geq 0$.*

If $X \subset \mathbb{P}(V)$ is a projective spherical variety, let Y in Theorem 5.4 be the cone over X in V. We obtain that X can be degenerated to a projective horospherical variety X_0, where X_0 is the projectivization of Y_0 in the theorem. Since the multiplicities of the irreducible G-modules appearing in the homogeneous coordinate rings of X and X_0 are the same, we know that the moment polytopes of X and X_0 are also the same (see Remark 2.2). It is then immediate from the definition that $\Delta(X) = \Delta(X_0)$.

COROLLARY 5.5. *Let $G = \mathrm{SP}(2n, \mathbb{C})$. Any projective spherical G-variety $X \subset \mathbb{P}(V)$ can be deformed, by a flat deformation, to the toric variety corresponding to the polytope $\Delta(X)$. That is, there exists a flat family of varieties $\pi : \mathcal{X} \to \mathbb{C}$ such that $\pi^{-1}(t) \cong X$ for all $t \neq 0$ and $\pi^{-1}(0)$ is the toric variety of the polytope $\Delta(X)$.*

Proof. By the foregoing comment, X can be deformed to a horospherical variety X_0 and $\Delta(X) = \Delta(X_0)$. The corollary now follows from Corollary 5.3. □

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