Michigan Math. J. 53 (2005)

On the Semisimplicity of Cyclotomic Temperley–Lieb Algebras

Hebing Rui, Changchang Xi, & Weihua Yu

1. Introduction

The Temperley–Lieb algebras were first introduced in [15] in order to study the single-bond transfer matrices for the Ising model and for the Potts model. Jones [9] defined a trace function on a Temperley–Lieb algebra so that he could construct the Jones polynomial of a link when the trace is nondegenerate. It is known that the trace is nondegenerate if the Temperley–Lieb algebra is semisimple. So it is an interesting question to provide a criterion for a Temperley–Lieb algebra to be semisimple. In [16, Sec. 5], there is a simple criterion for the semisimplicity of the Temperley–Lieb algebra in terms of q if the parameter is written $\delta = -(q+q^{-1})$. More explicitly, Westbury computed the determinants of Gram matrices associated to all "cell modules" via Tchebychev polynomials. This implies that a Temperley–Lieb algebra is semisimple if and only if such polynomials do not take values zero for the parameters.

As a generalization of a Temperley–Lieb algebra, the cyclotomic Temperley–Lieb algebra $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$ of type $G(m,1,n)$ was introduced in [13]. It is proved in [13] that $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$ is a cellular algebra in the sense of [3]. Thus $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$ is semisimple if and only if all of its "cell modules" are pairwise nonisomorphic irreducible. In order to determine when a cell module is irreducible, Rui and Xi computed the determinants of Gram matrices of certain cell modules [13, 8.1]. In general, it is hard to compute the determinants for all cell modules.

In this note, we shall consider the semisimplicity of cyclotomic Temperley–Lieb algebras. This is analogous to the question considered in [14] (see [2] for the case $m = 1$). Following [11], we study two functors F and G between certain categories in Section 3. Via these functors and [13, 8.1], in Section 4 we show our main result (Theorem 4.6), which states that $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$ is semisimple if and only if generalized Tchebychev polynomials do not take values zero for the parameters $\bar{\delta}_i$, $1 \le i \le m$.

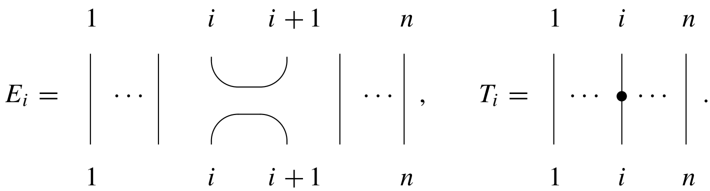
2. Cyclotomic Temperley–Lieb Algebras

In this section, we recall some of results on the cyclotomic Temperley–Lieb algebras in [13]. Throughout the paper, we fix two natural numbers m and n.

A labeled Temperley–Lieb diagram (or labeled TL diagram) D of type $G(m,1,n)$ is a Temperley–Lieb diagram with $2n$ vertices and n arcs. Each arc is labeled by

Received September 9, 2003. Revision received July 6, 2004.

an element in $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$, which will be considered as the number of dots on it. It should be noted that the arcs in a labeled TL diagram do not intersect. The following are two special labeled TL diagrams:

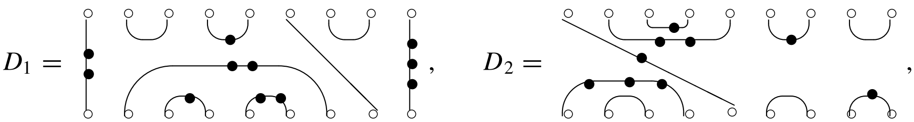


An arc in a labeled TL diagram D is said to be horizontal if its endpoints both lie in the top row or in the bottom row; otherwise, it is said to be vertical. Given a horizontal arc $\{i, j\}$ with $i < j$, we denote by i (resp. j) the left (resp. right) endpoint of the arc. For a horizontal (resp. vertical) arc, we always assume that the dots on this arc concentrate on the left endpoint (resp. the endpoint on the top row of the labeled TL diagram D).

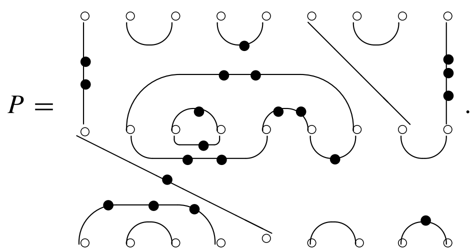
In order to define the composite of two labeled TL diagrams, we always assume that a dot in the left (resp. right) endpoint of an horizontal arc, when moved to the right (resp. left) endpoint, will be replaced by $m-1$ dots at the right (resp. left) endpoint of the arc. A dot in a vertical arc can move freely from one endpoint to another.

Suppose an arc l_1 joins another arc l_2 with a common endpoint j. A dot on the arc l_1 can move to the arc l_2. We always assume that a dot at the endpoint $j \in l_1$ can be replaced by a dot at $j \in l_2$.

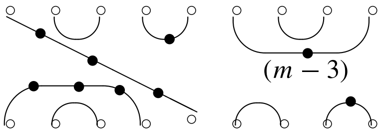
Given two labeled TL diagrams D_1 and D_2 of type $G(m, 1, n)$, we follow [13] and define a new labeled TL diagram $D_1 \circ D_2$ as follows. First, compose D_1 and D_2 in the same way as was done for the Temperley–Lieb algebra to obtain a new diagram P; second, apply the rule for the movement of dots to relabel each arc of P. We get a new labeled TL diagram, and this is defined to be $D_1 \circ D_2$. Let $n(\bar{i}, D_1, D_2)$ be the number of the relabeled closed cycles on which there are $\bar{i}$ dots. We display an example from [13] to illustrate the definition. If



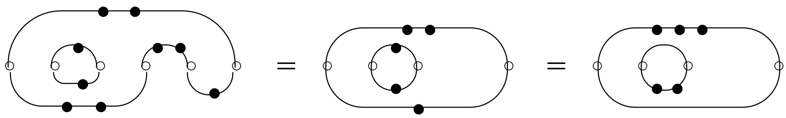
then we have a diagram



Thus the composition $D_1 \circ D_2$ of D_1 and D_2 is as follows:

$D_1 \circ D_2 =$ 

Now we relabel the closed cycles in P. By definition,



In this case, $n(\bar{i}, D_1, D_2) = 0$ if $\bar{i} \neq \bar{2}, \bar{3}$ and $n(\bar{2}, D_1, D_2) = n(\bar{3}, D_1, D_2) = 1$ under the assumption $m \geq 4$.

DEFINITION 2.1 [13, 3.3]. Let R be a commutative ring containing 1 and $\delta_0, \ldots, \delta_{m-1}$. Put $\boldsymbol{\delta} = (\delta_0, \ldots, \delta_{m-1})$. The *cyclotomic Temperley–Lieb algebra* $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$ is an associative algebra over R with a basis consisting of all labeled TL diagrams of type $G(m,1,n)$, and the multiplication is given by

$$D_1 \cdot D_2 = \prod_{i=0}^{m-1} \delta_i^{n(\bar{i}, D_1, D_2)} D_1 \circ D_2.$$

Note that if we set $\delta_0 = -(q + q^{-1})$ and $m = 1$ then we will get the usual Temperley–Lieb algebra. However, the cyclotomic Temperley–Lieb algebra of type $G(2,1,n)$ is not the same as the blob algebra considered in [12] or [6] since they have different defining relations. One can compare our generator's T_i with c_0 in [6, 5.3]. It would be interesting to know if there is an epimorphism from an extended affine Temperley–Lieb algebra [4; 5; 8] to our cyclotomic Temperley–Lieb algebra, generalizing some of results on blob algebras in [12; 1]. It was shown in [13] that $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$ can be defined by generators and relations. For the details, see [13, 2.1].

In the remaining part of this section we recall some results on the representations of $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$. First, we recall the notion of a cellular algebra in [3], which depends on the existence of a certain basis. There is also a basis-free definition of cellular algebras; for this we refer to [10].

DEFINITION 2.2 [3, 1.1]. An associative R-algebra A is called a *cellular algebra* with cell datum (I, M, C, i) if the following conditions are satisfied.

(C1) The finite set I is partially ordered. Associated with each $\lambda \in I$ is a finite set $M(\lambda)$. The algebra A has an R-basis $C^{\lambda}_{S,T}$, where (S,T) runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in I$.

(C2) The map i is an R-linear anti-automorphism of A, with $i^2 = \mathrm{id}$, that sends $C^{\lambda}_{S,T}$ to $C^{\lambda}_{T,S}$.

(C3) For each $\lambda \in I$ and $S, T \in M(\lambda)$ and each $a \in A$, the product $aC^{\lambda}_{S,T}$ can be written as

$$aC^{\lambda}_{S,T} = \sum_{U \in M(\lambda)} r_a(U,S)C^{\lambda}_{U,T} + r',$$

where r' belongs to $A^{<\lambda}$ consisting of all R-linear combinations of basis elements with upper index μ strictly smaller than λ and where the coefficients $r_a(U,S) \in R$ do not depend on T.

Assume that R is a field. For each $\lambda \in I$, one can define a cell module $\Delta(\lambda)$ and a symmetric associative bilinear form $\Phi_\lambda \colon \Delta(\lambda) \otimes_R \Delta(\lambda) \to R$ in the following way (see [3, Sec. 2]). As an R-module, $\Delta(\lambda)$ has an R-basis $\{C^{\lambda}_S \mid S \in M(\lambda)\}$, and the A-module structure is given by

$$aC^{\lambda}_{S} = \sum_{U \in M(\lambda)} r_a(U,S)C^{\lambda}_{U}. \tag{2.1}$$

The bilinear form Φ_λ is defined by

$$\Phi_\lambda(C^{\lambda}_S, C^{\lambda}_T)C^{\lambda}_{U,V} \equiv C^{\lambda}_{U,S}C^{\lambda}_{T,V} \pmod{A^{<\lambda}},$$

where U and V are arbitrary elements in $M(\lambda)$.

Let $\operatorname{rad}\Delta(\lambda) = \{c \in \Delta(\lambda) \mid \Phi_\lambda(c,c') = 0 \text{ for all } c' \in \Delta(\lambda)\}$. Then $\operatorname{rad}\Delta(\lambda)$ is an A-submodule of $\Delta(\lambda)$. Put $L(\lambda) = \Delta(\lambda)/\operatorname{rad}\Delta(\lambda)$. Then either $L(\lambda) = 0$ or $L(\lambda)$ is irreducible [3, 3.2]. We will need the following result in the next section.

LEMMA 2.3. $\operatorname{rad}\Delta(\lambda)$ *is annihilated by* $A^{\leq\lambda}$.

Proof. Let $a = C^{\mu}_{S_1,T_1} \in A^{\leq\lambda}$ and $C^{\lambda}_S \in \operatorname{rad}\Delta(\lambda)$. If $\mu < \lambda$, then $aC^{\lambda}_S = 0$ in $\Delta(\lambda)$. If $\mu = \lambda$, then we still have $aC^{\lambda}_S = 0$ because $r_a(S_1,S) = \Phi_\lambda(C^{\lambda}_{T_1}, C^{\lambda}_S)$ and $C^{\lambda}_S \in \operatorname{rad}\Delta(\lambda)$. □

From now on, we assume that R *is a splitting field of* $x^m - 1$. Then $x^m - 1 = \prod_{i=1}^{m}(x - u_i)$ for some $u_i \in R$, $1 \leq i \leq m$. Let $G_{m,n}$ be the R-subalgebra of $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$ generated by $T_1, T_2, \dots, T_n$. Then $G_{m,n}$ is a commutative algebra of dimension m^n. The cell modules over $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$ will be studied by restricting to $G_{m,n}$.

Let $\Lambda(m,n) = \{(i_1, i_2, \dots, i_n) \mid 1 \leq i_j \leq m\}$. Define $\mathbf{i} \leq \mathbf{j}$ if $i_k \geq j_k$ for all $1 \leq k \leq n$. Then $(\Lambda(m,n), \leq)$ is a poset. For any $\mathbf{i} \in \Lambda(m,n)$, we define $C^{\mathbf{i}}_{1,1} = \prod_{j=1}^{n}\prod_{l=i_j+1}^{m}(T_j - u_l)$.

LEMMA 2.4. *The set* $\{C^{\mathbf{i}}_{1,1} \mid \mathbf{i} \in \Lambda(m,n)\}$ *is a cellular basis of* $G_{m,n}$.

The cell module over $G_{m,n}$ corresponding to $\mathbf{i} \in \Lambda(m,n)$ with respect to the cellular basis just described will be denoted by $\Delta(\mathbf{i})$.

An (n,k)-*labeled parenthesis graph* is a graph consisting of n vertices $\{1, 2, \dots, n\}$ and k horizontal arcs (hence $2k \leq n$ and there are $n - 2k$ free vertices that do not belong to any arc) such that:

(1) there are at most $m - 1$ dots on each arc;
(2) there are no arcs $\{i, j\}$ and $\{q, l\}$ satisfying $i < q < j < l$; and
(3) there is no arc $\{i, j\}$ and a free vertex q such that $i < q < j$.

Condition (2) shows that the arcs in an (n,k)-labeled parenthesis graph do not intersect.

Let $P(n,k)$ be the set of all (n,k)-labeled parenthesis graphs. A labeled TL diagram D with k horizontal arcs can be determined by a triple pair (v_1, v_2, x), $x \in G_{m,n-2k}$, and $v_1, v_2 \in P(n,k)$ (see [13, Sec. 5]) and vice versa. Such a D will be denoted by $v_1 \otimes v_2 \otimes x$. In this case, we define $\mathrm{top}(D) = v_1$ and $\mathrm{bot}(D) = v_2$.

Let $\Lambda_{m,n} = \{(k,\mathbf{i}) \mid 0 \le k \le [n/2], \mathbf{i} \in \Lambda(m, n-2k)\}$. For any $(k,\mathbf{i}), (l,\mathbf{j}) \in \Lambda_{m,n}$, we say $(k,\mathbf{i}) \le (l,\mathbf{j})$ if either $k > l$ or $k = l$ and $\mathbf{i} \le \mathbf{j}$. Then $(\Lambda_{m,n}, \le)$ is a poset. For $v_1, v_2 \in P(n,k)$ and $\mathbf{i} \in \Lambda(m, n-2k)$, define $C^{(k,\mathbf{i})}_{v_1,v_2} = v_1 \otimes v_2 \otimes C^{\mathbf{i}}_{1,1}$.

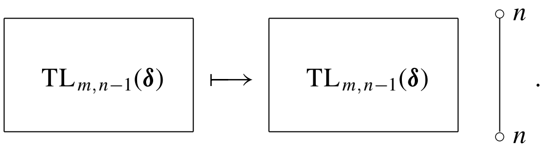
PROPOSITION 2.5 [13, 5.3]. *Let R be a splitting field of $x^m - 1$. The set $\{C^{(k,\mathbf{i})}_{v_1,v_2} \mid (k,\mathbf{i}) \in \Lambda_{n,m}, v_1, v_2 \in P(n,k)\}$ is a cellular basis of* $\mathrm{TL}_{m,n}(\delta)$.

Let $\Delta(k,\mathbf{i})$ be the cell module with respect to the cellular basis given in Proposition 2.5. Then

$$\Delta(k,\mathbf{i}) \cong V(n,k) \otimes_R v_0 \otimes_R \Delta(\mathbf{i}), \tag{2.2}$$

where $V(n,k)$ is the free R-module generated by $P(n,k)$ and v_0 is a fixed element in $P(n,k)$.

The algebra $\mathrm{TL}_{m,n-1}(\delta)$ can be considered as a subalgebra of $\mathrm{TL}_{m,n}(\delta)$ by adding the vertical arc $\{n,n\}$ to the right side of each labeled TL diagram in $\mathrm{TL}_{m,n-1}(\delta)$. This embedding can be visualized as follows:



Our next result is known as branching rule for the cell module $\Delta(k,\mathbf{i})$.

PROPOSITION 2.6 [13, 7.1]. *Suppose that* $\mathrm{ch}\, R \nmid m$. *For* $\mathbf{i} = (i_1, i_2, \ldots, i_{n-2k}) \in \Lambda(m, n-2k)$, *define* $\mathbf{i}_0 = (i_1, i_2, \ldots, i_{n-2k-1}) \in \Lambda(m, n-2k-1)$ *and* $\mathbf{i} \cup j = (i_1, i_2, \ldots, i_{n-2k}, j) \in \Lambda(m, n-2k+1)$. *Then there is a short exact sequence*

$$0 \to \Delta(k,\mathbf{i}_0) \to \Delta(k,\mathbf{i})\downarrow \to \bigoplus_{j=1}^{m} \Delta(k-1, \mathbf{i} \cup j) \to 0, \tag{2.3}$$

where we denote by $M\downarrow$ the restriction of a $\mathrm{TL}_{m,n}(\delta)$*-module M to a* $\mathrm{TL}_{m,n-1}(\delta)$*-module.*

Proof. It is proved in [13, 7.1] that

$$0 \to \Delta(k,\mathbf{i}_0) \to \Delta(k,\mathbf{i})\downarrow \to V(n-1,k-1) \otimes_R v_0 \otimes_R \Delta(\mathbf{i}) \otimes_R R\langle t_{n-2k+1}\rangle \to 0.$$

Since $\mathrm{ch}\, R \nmid m$, it follows that $R\langle t_{n-2k+1}\rangle$ is semisimple. Therefore, $R\langle t_{n-2k+1}\rangle \cong \bigoplus_{j=1}^{m} \Delta(j)$, where $\Delta(j)$ is the cell module of $R\langle t_{n-2k+1}\rangle$ with respect to the cellular basis given in Lemma 2.4 (the case $m = 1$). By direct computation, we have

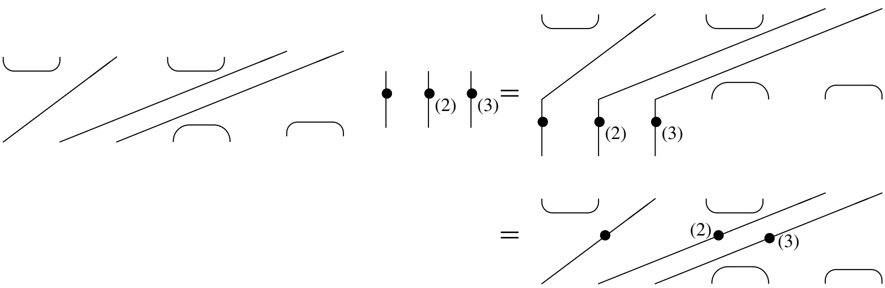
$$\Delta(\mathbf{i}) \otimes_R \Delta(j) \cong \Delta(\mathbf{i} \cup j).$$

By (2.2), we obtain (2.3). □

As $G_{m,n}$-modules, $\Delta(0,\mathbf{i}) \cong \Delta(\mathbf{i})$. Note that a cellular algebra is semisimple if and only if all of its cell modules are pairwise nonisomorphic irreducible [3]. Therefore, that $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$ is semisimple implies that all $\Delta(\mathbf{i})$ are pairwise nonisomorphic irreducible as $G_{m,n}$-modules. So, $G_{m,n}$ is semisimple, which is equivalent to the fact $\mathrm{ch}\, R \nmid m$. Moreover, $u_i \neq u_j$ for any $i \neq j$, $1 \leq i, j \leq m$.

Henceforth, we assume $\mathrm{ch}\, R \nmid m$ and $u_i = \xi^i$ for $1 \leq i \leq m$, where ξ is a primitive mth root of unity. The reason for making this assumption is that the semisimplicity of $G_{m,n}$ is necessary for $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$ to be semisimple.

For later use, we need another construction of the cell modules as follows. Let $J_{m,n}^{\geq k}$ (resp. $J_{m,n}^{>k}$) be the free R-submodule of $\mathrm{TL}_{m,n}$ generated by labeled TL diagrams with l horizontal arcs such that $l \geq k$ (resp. $l > k$). Let $I_{m,n}^{k}(\boldsymbol{\delta})$ be the submodule of $J_{m,n}^{\geq k}/J_{m,n}^{>k}$ generated by the coset of $v \otimes v_0 \otimes x$, with $v \in P(n,k)$, $x \in G_{m,n-2k}$, and $v_0 = \mathrm{top}(E_{n-2k+1} \cdots E_{n-1}) \in P(n,k)$. Then $I_{m,n}^{k}(\boldsymbol{\delta})$ is a right $G_{m,n-2k}$-module in which $x \in G_{m,n-2k}$ acts on the free vertices of $\mathrm{bot}(D)$, $D \in I_{m,n}^{k}(\boldsymbol{\delta})$. The following is an example that illustrates the action.



By the construction of cell modules, we have

$$\Delta(k,\mathbf{i}) \cong I_{m,n}^{k}(\boldsymbol{\delta}) \otimes_{G_{m,n-2k}} \Delta(\mathbf{i}). \tag{2.4}$$

Moreover, $\{v \otimes v_0 \otimes_{G_{m,n-2k}} C_{11}^{\mathbf{i}} \mid v \in P(n,k)\}$ is a free R-basis of $\Delta(k,\mathbf{i})$.

3. Restriction and Induction

In this section, we assume that there is at least one nonzero parameter, say δ_i. Otherwise $\bar{\delta}_j = 0$ for $1 \leq j \leq m$ (see (4.1) for the definition of $\bar{\delta}_j$). By [13, 8.1], $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$ is not semisimple.

LEMMA 3.1. *Suppose $\delta_i \neq 0$. Let $e = \delta_i^{-1} T_n^i E_{n-1} \in \mathrm{TL}_{m,n}(\boldsymbol{\delta})$. Then $e^2 = e$, and $e\mathrm{TL}_{m,n}(\boldsymbol{\delta})e \cong \mathrm{TL}_{m,n-2}(\boldsymbol{\delta})$.*

Proof. Each element in $e\mathrm{TL}_{m,n}(\boldsymbol{\delta})e$ is a linear combination of the labeled TL diagrams D in which $\mathrm{top}(D)$ (resp. $\mathrm{bot}(D)$) contains a horizontal arc $\{n-1, n\}$ where there are i (resp. 0) dots. Let D^0 be the labeled TL diagram obtained from D by removing the horizontal arc $\{n-1, n\}$ on $\mathrm{top}(D)$ and $\mathrm{bot}(D)$. By the definition of the product of two labeled TL diagrams (Definition 2.1), one can easily verify that the R-linear isomorphism $\phi\colon e\mathrm{TL}_{m,n}(\boldsymbol{\delta})e \to \mathrm{TL}_{m,n-2}(\boldsymbol{\delta})$ with $\phi(D) = \delta_i D^0$ is an isomorphism of R-algebras. □

Now we may use the idempotent e to define two functors F and G as follows.

DEFINITION 3.2. Let $F\colon \mathrm{TL}_{m,n}(\boldsymbol{\delta})$-mod $\to \mathrm{TL}_{m,n-2}(\boldsymbol{\delta})$-mod with $F(M) = eM$ and $G\colon \mathrm{TL}_{m,n-2}(\boldsymbol{\delta})$-mod $\to \mathrm{TL}_{m,n}(\boldsymbol{\delta})$-mod with $G(M) = \mathrm{TL}_{m,n}(\boldsymbol{\delta})e \otimes_{\mathrm{TL}_{m,n-2}(\boldsymbol{\delta})} M$.

In the following, we give a description of the image of the cell modules under the functors F and G. A similar method is also used in [11; 2; 14].

PROPOSITION 3.3. *Assume* $\mathbf{i} \in \Lambda(m, n-2k)$.

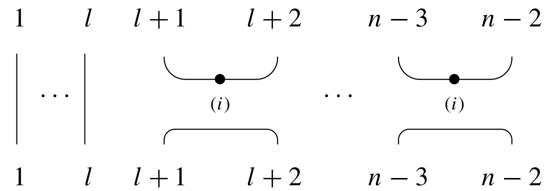
(a) *If* φ *is a nonzero* $\mathrm{TL}_{m,n-2}(\boldsymbol{\delta})$*-homomorphism, then* $G(\varphi) \neq 0$.
(b) *FG is an identity functor.*
(c) $G(\Delta(k-1,\mathbf{i})) = \Delta(k,\mathbf{i})$ *and* $G(\Delta(k-1,\mathbf{i})\!\downarrow) = \Delta(k,\mathbf{i})\!\downarrow$;
(d) $F(\Delta(k,\mathbf{i})) = \Delta(k-1,\mathbf{i})$ *and* $F(\Delta(k,\mathbf{i})\!\downarrow) = \Delta(k-1,\mathbf{i})\!\downarrow$.

Proof. (a) and (b) follow from a general result in [7, 6.2]. Part (d) follows from (c) and (b) by applying the functor F to both sides of (c).

Let $v_0 = \mathrm{top}(E_{n-2k+1}E_{n-2k+3}\cdots E_{n-1}) \in P(n,k)$. We claim that, as $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$-modules,

$$I^k_{m,n}(\boldsymbol{\delta}) \cong \mathrm{TL}_{m,n}(\boldsymbol{\delta})e \otimes_{\mathrm{TL}_{m,n-2}(\boldsymbol{\delta})} I^{k-1}_{m,n-2}(\boldsymbol{\delta}). \tag{3.1}$$

In fact, let $l = n-2k$. Then $\varepsilon = T^i_{l+1}T^i_{l+3}\cdots T^i_{n-3}E_{l+1}E_{l+3}\cdots E_{n-3} \in I^{k-1}_{m,n-2}(\boldsymbol{\delta})$; that is,

$\varepsilon =$ 

.

Suppose $D_1e \otimes D_2 \in \mathrm{TL}_{m,n}(\boldsymbol{\delta})e \otimes_{\mathrm{TL}_{m,n-2}(\boldsymbol{\delta})} I^{k-1}_{m,n-2}(\boldsymbol{\delta})$. Then $D_2 \cdot \varepsilon = \delta_i^{k-1}D_2$, $eD_2 = D_2e$, and

$$D_1e \otimes D_2 = \delta_i^{1-k}D_1e \otimes D_2\varepsilon = \delta_i^{-k}D_1D_2^0e \otimes \varepsilon,$$

where D_2^0 can be obtained from D_2 by adding two horizontal arcs $\{n-1, n\}$ to the top and bottom row of D_2. Obviously, $D_1D_2^0 \in I^k_{m,n}(\boldsymbol{\delta}_{\mathbf{i}})$. Therefore, any element in $\mathrm{TL}_{m,n}(\boldsymbol{\delta})e \otimes_{\mathrm{TL}_{m,n-2}(\boldsymbol{\delta})} I^{k-1}_{m,n-2}(\boldsymbol{\delta})$ can be expressed as a linear combination of the element $D_3e \otimes \varepsilon$ with $D_3 = D_1D_2^0$. Define the R-linear map $\alpha\colon \mathrm{TL}_{m,n}(\boldsymbol{\delta})e \otimes_{\mathrm{TL}_{m,n-2}(\boldsymbol{\delta})} I^{k-1}_{m,n-2}(\boldsymbol{\delta}) \to I^k_{m,n}(\boldsymbol{\delta})$ with $\alpha(D_3e \otimes \varepsilon) = D_3$. Then α is an epimorphism. If $D_3 = 0$, then either $0 = D_3 \in \mathrm{TL}_{m,n}(\boldsymbol{\delta})$ or $\mathrm{bot}(D_3)$ contains at least one extra arc, say $(i', i'+1)$, $i' \leq n-2k-1$, in which there are s dots. So,

$$D_3e \otimes \varepsilon = \delta_i^{-1}D_3T_{i'}^{i-s}E_{i'}T_{i'}^{s}e \otimes \varepsilon = \delta_i^{-1}D_3e \otimes T_{i'}^{i-s}E_{i'}T_{i'}^{s}\varepsilon = \delta_i^{-1}D_3e \otimes 0 = 0.$$

Therefore, α is injective. By (3.1) and (2.4),

$$\begin{aligned} G(\Delta(k-1,\mathbf{i})) &= \mathrm{TL}_{m,n}(\boldsymbol{\delta})e \otimes_{\mathrm{TL}_{m,n-2}(\boldsymbol{\delta})} \left(I^{k-1}_{m,n-2}(\boldsymbol{\delta}) \otimes_{G_{m,n-2k}} \Delta(\mathbf{i})\right) \\ &\cong \left(\mathrm{TL}_{m,n}(\boldsymbol{\delta})e \otimes_{\mathrm{TL}_{m,n-2}(\boldsymbol{\delta})} I^{k-1}_{m,n-2}(\boldsymbol{\delta})\right) \otimes_{G_{m,n-2k}} \Delta(\mathbf{i}) \\ &\cong I^{k}_{m,n}(\boldsymbol{\delta}) \otimes_{G_{m,n-2k}} \Delta(\mathbf{i}) \\ &= \Delta(k,\mathbf{i}). \end{aligned}$$

This completes the proof of the first isomorphism given in (c). The second isomorphism can be proved similarly. □

DEFINITION 3.4. For any $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$-modules M and N, define

$$\langle M, N\rangle_n = \langle M, N\rangle_{\mathrm{TL}_{m,n}(\boldsymbol{\delta})} = \dim_R \mathrm{Hom}_{\mathrm{TL}_{m,n}(\boldsymbol{\delta})}(M,N).$$

PROPOSITION 3.5. *Suppose* $\mathbf{i} \in \Lambda(m,n)$, $\mathbf{j} \in \Lambda(m,n-2k)$, *and* $k_0 \in \mathbb{N}$. *Then* $\langle \Delta(k_0,\mathbf{i}), \Delta(k+k_0,\mathbf{j})\rangle_{n+2k_0} \neq 0$ *if and only if* $\langle \Delta(0,\mathbf{i}), \Delta(k,\mathbf{j})\rangle_n \neq 0$.

Proof. "$\Leftarrow$" follows from Proposition 3.3(a) and (c) by applying G repeatedly.

"$\Rightarrow$" Suppose that $0 \neq \varphi \in \mathrm{Hom}_{\mathrm{TL}_{m,n+2k_0}(\boldsymbol{\delta})}(\Delta(k_0,\mathbf{i}), \Delta(k+k_0,\mathbf{j}))$ and $W = \varphi(\Delta(k_0,\mathbf{i}))$. Let $e = \delta_i^{-1}T^i_{n+2k_0-1}E_{n+2k_0-1}$. We claim

$$eW \neq 0. \tag{3.2}$$

Otherwise, we have $eW = 0$. Let $v_i = \mathrm{top}(E_i) = \mathrm{bot}(E_i)$. Then

$$E_1 = \delta_i^{-2}(v_1 \otimes v_{n+2k_0-1} \otimes \mathrm{id}) \cdot T^i_{n+2k_0-1}E_{n+2k_0-1}T^i_{n+2k_0-1} \cdot (v_{n+2k_0-1} \otimes v_1 \otimes \mathrm{id}).$$

Hence $E_1W = 0$, which implies $EW = 0$ with $E = E_1E_3\cdots E_{2k_0-1}$. On the other hand, let $U_0 = \mathrm{rad}\,\Delta(k_0,\mathbf{i})$. Then either $\Delta(k_0,\mathbf{i}) = U_0$ or $\Delta(k_0,\mathbf{i})/U_0$ is irreducible [3, 3.2]. Let $\mathbf{m} = (m,m,\dots,m) \in \Lambda(m,n)$. Since $E \in \mathrm{TL}^{(k_0,\mathbf{m})}_{m,n+2k_0} \subset \mathrm{TL}^{\leq(k_0,\mathbf{i})}_{m,n+2k_0}$, Lemma 2.3 shows $EU_0 = 0$. We have $W = \varphi(\Delta(k_0,\mathbf{i})) \cong \Delta(k_0,\mathbf{i})/U$. We claim $U \subset U_0$. Otherwise, $U + U_0 = \Delta(k_0,\mathbf{i})$ and hence $U/(U_0 \cap U) \cong \Delta(k_0,\mathbf{i})/U_0$ is irreducible. So, there is a composition series of $\Delta(k_0,\mathbf{i})$ such that the multiplicity of $L(k_0,\mathbf{i})$ is greater than 2, a contradiction.

Let $y = \mathrm{top}(T^i_1T^i_3\cdots T^i_{2k_0-1}E)$. Then $v = y \otimes v_0 \otimes C^{\mathbf{i}}_{1,1} \in \Delta(k_0,\mathbf{i})$ is a nonzero element, where v_0 is a fixed element in $P(n+2k_0,k_0)$. Since $\delta_i \neq 0$ we have $T^i_1T^i_3\cdots T^i_{2k_0-1}E \cdot v = (\delta_i)^{k_0}v \neq 0$, which implies $v \notin U$. Therefore, $T^i_1T^i_3\cdots T^i_{2k_0-1}E(v+U) = \delta_i^{k_0}(v+U) \not\equiv 0 \bmod U$, which contradicts the fact $eW = 0$. This completes the proof of (3.2).

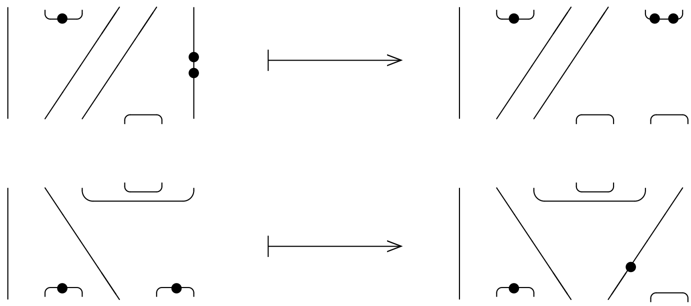
If $eW \neq 0$, then $F(\varphi) \neq 0$. Now the result follows from induction and (3.2). □

PROPOSITION 3.6. *Suppose M is a* $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$*-module. Then* $M\!\uparrow\, \cong G(M)\!\downarrow$, *where* $M\!\uparrow$ *is the induced module of a* $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$*-module M to* $\mathrm{TL}_{m,n+1}(\boldsymbol{\delta})$. *In particular, for any* $\mathbf{i} \in \Lambda(m,n-2k)$, $\Delta(k,\mathbf{i})\!\uparrow\, \cong \Delta(k+1,\mathbf{i})\!\downarrow$.

Proof. We shall define a linear map $\alpha\colon \mathrm{TL}_{m,n+1}(\boldsymbol{\delta}) \to \mathrm{TL}_{m,n+2}(\boldsymbol{\delta})e$. Suppose $x \in \mathrm{TL}_{m,n+1}(\boldsymbol{\delta})$. Add a $(n+2)$th vertex on $\mathrm{top}(x)$ and $\mathrm{bot}(x)$ to get a new labeled TL diagram D in which the following statements hold.

(1) The $(n+2)$th vertex of $\mathrm{top}(D)$ joins the vertex j if $\{j, n+1\}$ is an arc in x; here $n+1$ is the $(n+1)$th vertex in $\mathrm{bot}(x)$. Moreover, if there are s dots on the arc $\{j, n+1\}$ then there are s dots in the new arc $\{j, n+2\}$ also.
(2) $\{n+1, n+2\}$ is a horizontal arc in $\mathrm{bot}(D)$ in which there is no dot.

We give two examples to illustrate this definition.



Now, we define an R-linear map $\alpha\colon \mathrm{TL}_{m,n+1}(\boldsymbol{\delta}) \to \mathrm{TL}_{m,n+2}(\boldsymbol{\delta})e$ by $\alpha(x) = D$. Obviously, α is an R-linear isomorphism. By the definition of the product of two labeled TL diagrams in Definition 2.1, α is a $(\mathrm{TL}_{m,n+1}(\boldsymbol{\delta}), \mathrm{TL}_{m,n}(\boldsymbol{\delta}))$-bimodule isomorphism; that is,

$$\mathrm{TL}_{m,n+1}(\boldsymbol{\delta}) \cong \mathrm{TL}_{m,n+2}(\boldsymbol{\delta})e. \tag{3.3}$$

For any $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$-module M,

$$\begin{aligned} M\!\uparrow &\cong \mathrm{TL}_{m,n+1}(\boldsymbol{\delta}) \otimes_{\mathrm{TL}_{m,n}(\boldsymbol{\delta})} M \\ &\cong \mathrm{TL}_{m,n+2}(\boldsymbol{\delta})e \otimes_{\mathrm{TL}_{m,n}(\boldsymbol{\delta})} M \quad \text{(by (3.3))} \\ &\cong G(M)\!\downarrow. \end{aligned}$$

□

Corollary 3.7. *Suppose* $\mathrm{ch}\, R \nmid m$, *and assume that* $\mathbf{i} = (i_1, i_2, \ldots, i_n) \in \Lambda(m,n)$. *If* $\mathbf{j} = (i_1, i_2, \ldots, i_n, j) \in \Lambda(m, n+1)$, *then* $\langle \Delta(0,\mathbf{i})\!\uparrow, \Delta(0,\mathbf{j})\rangle_{n+1} \neq 0$.

Proof. By Proposition 3.6, $\langle \Delta(0,\mathbf{i})\!\uparrow, \Delta(0,\mathbf{j})\rangle_{n+1} = \langle \Delta(1,\mathbf{i})\!\downarrow, \Delta(0,\mathbf{j})\rangle_{n+1}$. Now Proposition 2.6 implies that $\langle \Delta(1,\mathbf{i})\!\downarrow, \Delta(0,\mathbf{j})\rangle_{n+1} \neq 0$ for all $\mathbf{j} = (i_1, i_2, \ldots, i_n, j)$, $1 \le j \le m$. □

Proposition 3.8. *Suppose* $\mathrm{ch}\, R \nmid m$ *and* $\langle \Delta(0,\mathbf{i}), \Delta(k,\mathbf{j})\rangle_n \neq 0$ *for* $\mathbf{i} \in \Lambda(m,n)$ *and* $\mathbf{j} \in \Lambda(m, n-2k)$.

(a) *If* $\mathbf{i}^0 = (i_1, i_2, \ldots, i_{n-1}) \in \Lambda(m, n-1)$, *then* $\langle \Delta(0,\mathbf{i}^0), \Delta(k,\mathbf{j})\!\downarrow\rangle_{n-1} \neq 0$.
(b) *Let* $\mathbf{j}^0 = (j_1, j_2, \ldots, j_{n-2k-1})$ *and* $\mathbf{j}^1 = (j_1, j_2, \ldots, j_{n-2k}, j_0)$, $1 \le j_0 \le m$. *Then either* $\langle \Delta(0,\mathbf{i}^0), \Delta(k,\mathbf{j}^0)\rangle_{n-1} \neq 0$ *or* $\langle \Delta(0,\mathbf{i}^0), \Delta(k-1,\mathbf{j}^1)\rangle_{n-1} \neq 0$.

Proof. Since $\mathbf{i}^0 \in \Lambda(m, n-1)$, Corollary 3.7 implies $\langle \Delta(0,\mathbf{i}^0)\!\uparrow, \Delta(0,\mathbf{i})\rangle_n \neq 0$. Since $\mathrm{ch}\, R \nmid m$, it follows that $\Delta(\mathbf{i})$ is a simple $G_{m,n}$-module, forcing $\Delta(0,\mathbf{i})$ to be an irreducible $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$-module. So, $\langle \Delta(0,\mathbf{i}^0)\!\uparrow, \Delta(k,\mathbf{j})\rangle_n \neq 0$. Using Frobenius reciprocity, we get (a).

Let $V = \Delta(k, \mathbf{j})\downarrow$. By Proposition 2.6, there is a submodule $W \subset V$ such that $W \cong \Delta(k, \mathbf{j}^0)$, where $\mathbf{j}^0 = (j_1, j_2, \dots, j_{n-2k-1})$.

Let $0 \neq S$ be the image of $\Delta(0, \mathbf{i}^0)$ in V. Since $\Delta(0, \mathbf{i}^0)$ is irreducible, $S \cong \Delta(0, \mathbf{i}^0)$. If $S \subset W$, then $\langle \Delta(0, \mathbf{i}^0), \Delta(k, \mathbf{j}^0)\rangle_{n-1} \neq 0$. If $S \not\subset W$, then $S \cap W = 0$. Thus, $(S \oplus W)/W \cong S/(W \cap S) = S$ is an irreducible submodule of V/W. By Proposition 2.6,

$$V/W \cong \bigoplus_{j=1}^{m} \Delta(k-1, \mathbf{j} \cup j).$$

Hence there exists a $\mathbf{j}^1 = (j_1, j_2, \dots, j_{n-2k}, j_0) \in \Lambda(m, n-2k+1)$ such that $(S \oplus W)/W \subset \Delta(k-1, \mathbf{j}^1)$, forcing $\langle \Delta(0, \mathbf{i}^0), \Delta(k-1, \mathbf{j}^1)\rangle_{n-1} \neq 0$. □

4. Semisimplicity of the Cyclotomic Temperley–Lieb Algebras

In this section we shall give the necessary and sufficient conditions for the semisimplicity of $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$. The key is [13, 8.1]. First, we recall some of the results in [13].

Let $u_i = \xi^i$, where ξ is a primitive mth root of unity. For any

$$\mathbf{i} = (i_1, i_2, \dots, i_{n-2}) \in \Lambda(m, n-2),$$

let

$$\Psi_{\mathbf{i}}(n,1) = \begin{pmatrix} A & B_1 & & & & \\ B_1^T & A & B_2 & & & \\ & B_2^T & A & B_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & A & B_{n-2} \\ & & & & B_{n-2}^T & A \end{pmatrix},$$

where $B_j = (b_{st})$ with $b_{st} = u_{i_j}^{s-t}$ $(1 \leq s, t \leq m)$, B_i^T stands for the transpose of B_i, and

$$A = \begin{pmatrix} \delta_0 & \delta_1 & \cdots & \delta_{m-1} \\ \delta_1 & \delta_2 & \cdots & \delta_0 \\ \vdots & \vdots & \cdots & \vdots \\ \delta_{m-1} & \delta_0 & \cdots & \delta_{m-2} \end{pmatrix}.$$

Let $p(x) = \delta_0 x^{m-1} + \delta_1 x^{m-2} + \cdots + \delta_{m-1}$. Write

$$\frac{p(x)}{x^m - 1} = \frac{\bar{\delta}_1}{x - u_1} + \frac{\bar{\delta}_2}{x - u_2} + \cdots + \frac{\bar{\delta}_m}{x - u_m}. \tag{4.1}$$

Then

$$\bar{\delta}_j = \frac{p(u_j)}{\prod_{i \neq j}(u_j - u_i)}. \tag{4.2}$$

Following [13], we partition $\mathbf{i} = (i_1, i_2, \dots, i_{n-2})$ into $(i_{1,1}, i_{1,2}, \dots, i_{1,j_1}, i_{2,1}, i_{2,2}, \dots, i_{2,j_2}, \dots, i_{r,j_r})$, with $j_1 + j_2 + \cdots + j_r = n-2$, such that (a) m divides $i_{p,q} + i_{p,q+1}$ for all p with $1 \leq q < j_p$ and (b) m does not divide $i_{p,j_p} + i_{p+1,1}$ for all $1 \leq p < r$. Let

$$P_n(x_1,\dots,x_n)=\det\begin{pmatrix} x_1 & 1 & & & \\ 1 & x_2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & x_{n-1} & 1 \\ & & & 1 & x_n \end{pmatrix}.$$

We call $P_n(x_1,x_2,\dots,x_n)$ the nth generalized Tchebychev polynomial. The following result was proved in [13, Sec. 8].

PROPOSITION 4.1. *Keep the setup. Then*

$$\det\Psi_{\mathbf{i}}(n,1)=(-1)^{m(m-1)(n-1)/2}m^{m(n-1)} \times \frac{(\bar\delta_1\bar\delta_2\cdots\bar\delta_m)^{n-1}}{\prod_{p=1}^{r}\big(\bar\delta_{m-i_{p,j_p}}\prod_{q=1}^{j_p}\bar\delta_{i_{p,q}}\big)}\prod_{p=1}^{r}P_{j_p}(\bar\delta_{i_{p,1}},\bar\delta_{i_{p,2}},\dots,\bar\delta_{i_{p,j_p}}).$$

PROPOSITION 4.2. *Suppose that* $\mathbf{i}\in\Lambda(m,n)$ *and that* $\mathbf{j}\in\Lambda(m,n-2)$. *Then, if* $\langle\Delta(0,\mathbf{i}),\Delta(1,\mathbf{j})\rangle_n\neq 0$, *it follows that* $\det\Psi_{\mathbf{j}}(n,1)=0$.

Proof. Since $\langle\Delta(0,\mathbf{i}),\Delta(1,\mathbf{j})\rangle_n\neq 0$, there is a $\varphi\in\operatorname{Hom}_{\mathrm{TL}_{m,n}(\boldsymbol{\delta})}(\Delta(0,\mathbf{i}),\Delta(1,\mathbf{j}))$ such that $\varphi(v)\neq 0$ for some $v\in\Delta(0,\mathbf{i})$. Consider an element

$$T=\sum_{i=1}^{n-1}\sum_{s=0}^{m-1}T_i^sE_iT_i^s\in\mathrm{TL}_{m,n}(\boldsymbol{\delta}).$$

We have $T\varphi(v)=\varphi(Tv)=\varphi(0)=0$. Write

$$\varphi(v)=\sum_{i=1}^{n-1}\sum_{s=0}^{m-1}a_{i,s}v_i^{(s)}\otimes v_0\otimes C_{1,1}^{\mathbf{j}},$$

where $v_i^{(s)}=\operatorname{top}(T_i^sE_i)$ and v_0 is a fixed element in $P(n,1)$. We have

$$(v_1\otimes v_1\otimes C_{1,1}^{\mathbf{j}})(v_2\otimes v_2\otimes C_{1,1}^{\mathbf{j}})\equiv v_1\otimes v_2\otimes\phi_{v_1,v_2}^{(n,1)}(t_1,t_2,\dots,t_{n-2})(C_{1,1}^{\mathbf{j}})^2$$

(mod $\mathrm{TL}_{n,m}^{<(1,\mathbf{j})}$) for some elements $\phi_{v_1,v_2}^{(n,1)}(t_1,t_2,\dots,t_{n-2})$ in $G_{m,n-2}$. By a direct computation,

$$0=T\varphi(v)=\sum_{1\le i,j\le n-1}\sum_{0\le s,t\le m-1}\phi_{v_i^{(s)},v_j^{(t)}}^{(n,1)}(u_{j_1},u_{j_2},\dots,u_{j_{n-2}})a_{j,t}v_i^{(s)}\otimes v_0\otimes C_{1,1}^{\mathbf{j}}.$$

Hence for all i,s we have

$$\sum_{1\le j\le n-1}\sum_{0\le t\le m-1}\phi_{v_i^{(s)},v_j^{(t)}}^{(n,1)}(u_{j_1},u_{j_2},\dots,u_{j_{n-2}})a_{j,t}=0.$$

Since $\varphi(v)\neq 0$, there is at least one of $a_{i,t}\neq 0$; this implies $\det\Psi_{\mathbf{j}}(n,1)=0$. □

PROPOSITION 4.3. *Suppose* R *is a splitting field of* x^m-1 *with* $\operatorname{ch}R\nmid m$. *If* $\det\Psi_{\mathbf{i}}(l,1)\neq 0$ *for all* $2\le l\le n$ *and all* $\mathbf{i}\in\Lambda(m,l-2)$, *then* $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$ *is semisimple.*

Proof. It is proved in [13] that $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$ is a cellular algebra. Note that a cellular algebra is semisimple if and only if the determinants of the Gram matrices for all cell modules are not equal to zero (see [3]). Now, suppose $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$ is not semisimple. Then there is a determinant of the Gram matrix with respect to a cell module, say $\Delta(k_1, \mathbf{i})$, that is equal to zero. Thus, we can find an irreducible module $D \subset \operatorname{rad}\Delta(k_1, \mathbf{i})$. Observe that any simple module of a cellular algebra is the simple head of a cell module. As a result, D is the simple quotient of a cell module, say $\Delta(k_2, \mathbf{j})$. Since D is a composition factor of $\Delta(k_1, \mathbf{i})$, it follows from Definition 2.2 and (2.1) that $(k_1, \mathbf{i}) \le (k_2, \mathbf{j})$. Moreover, $(k_1, \mathbf{i}) \ne (k_2, \mathbf{j})$, for otherwise $\Delta(k_1, \mathbf{i})$ would have a simple head D. So, the multiplicity of D in $\Delta(k_1, \mathbf{i})$ is at least 2, a contradiction. We have $\langle \Delta(k_2, \mathbf{j}), \Delta(k_1, \mathbf{i})\rangle_n \ne 0$. Moreover, either $k_1 > k_2$ or $k_1 = k_2$ and $\mathbf{i} < \mathbf{j}$.

Suppose $k_1 > k_2$. Using Proposition 3.5, we can assume that $\mathbf{j} \in \Lambda(m, l)$ for $l = n - 2k_2$. Let $k = k_1 - k_2$. Then $\langle \Delta(0, \mathbf{j}), \Delta(k, \mathbf{i})\rangle_l \ne 0$. Applying Proposition 3.8 repeatedly, we can assume $k = 1$. By Proposition 4.2, $\det \Psi_{\mathbf{i}}(l, 1) = 0$, a contradiction.

Suppose $k_1 = k_2$ and $\mathbf{i} < \mathbf{j}$. By Proposition 3.5, $\langle \Delta(0, \mathbf{j}), \Delta(0, \mathbf{i})\rangle_{n-2k_1} \ne 0$. This is a contradiction since $\Delta(0, \mathbf{j}) \not\cong \Delta(0, \mathbf{i})$ and since both are irreducible $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$-modules. Hence $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$ is semisimple. □

LEMMA 4.4. *Suppose* $\det \Psi_{\mathbf{i}}(n, 1) \ne 0$ *for all* $\mathbf{i} \in \Lambda(m, n-2)$ *with* $m \ge 2$. *Then* $\bar{\delta}_i \ne 0$ *for any* i, $1 \le i \le m$.

Proof. Take $\mathbf{i} = (m, m, \dots, m) \in \Lambda(m, n-2)$. Then $\mathbf{i}$ can be divided into one part with $j_1 = n-2$. By Proposition 4.1, $\bar{\delta}_i \ne 0$ $(1 \le i \le m-1)$ because they are the factors of $\det \Psi_{\mathbf{i}}(n, 1)$. Take $\mathbf{i} = (1, 1, \dots, 1) \in \Lambda(m, n-2)$. Then $\mathbf{i}$ can be divided into either one part if $m = 2$ or $n-2$ parts if $m > 2$. By Proposition 4.1, $\bar{\delta}_m \ne 0$ since it is a factor of $\det \Psi_{\mathbf{i}}(n, 1)$ in any case. □

It is proved in [13, 8.1] that $\det \Psi_{\mathbf{i}}(n, 1) \ne 0$ for all $\mathbf{i} \in \Lambda(m, n-2)$ and that $\operatorname{ch} R \nmid m$ if $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$ is semisimple. The following proposition is the inverse of this result.

PROPOSITION 4.5. *Suppose R is a splitting field of $x^m - 1$ with* $\operatorname{ch} R \nmid m$ *and* $m \ge 2$. *If* $\det \Psi_{\mathbf{i}}(n, 1) \ne 0$ *for all* $\mathbf{i} \in \Lambda(m, n-2)$, *then* $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$ *is semisimple.*

Proof. By Proposition 4.3, we need prove $\det \Psi_{\mathbf{i}}(l, 1) \ne 0$ for all $2 \le l \le n$ and $\mathbf{i} \in \Lambda(m, l-2)$ under our assumption. If $\det \Psi_{\mathbf{i}}(l, 1) = 0$ for some $l, l \ne n$ and $\mathbf{i} \in \Lambda(m, l-2)$, then $P_{j_p}(\bar{\delta}_{i_{p,1}}, \bar{\delta}_{i_{p,2}}, \dots, \bar{\delta}_{i_{p,j_p}}) = 0$ for some p, $1 \le p \le r$, by Proposition 4.1 and Lemma 4.4.

On the other hand, take $\mathbf{i}_0 = (i_1, i_2, \dots, i_{l-2}, a, a, \dots, a) \in \Lambda(m, n-2)$ with $m \nmid (i_{l-2} + a)$. By Proposition 4.1, $P_{j_p}(\bar{\delta}_{i_{p,1}}, \bar{\delta}_{i_{p,2}}, \dots, \bar{\delta}_{i_{p,j_p}})$ must be a factor of $\det \Psi_{\mathbf{i}_0}(n, 1)$ and hence $\det \Psi_{\mathbf{i}_0}(n, 1) = 0$, a contradiction. □

REMARK. The reason we assume $m \ge 2$ is that we need the fact that i_{l-2} and a cannot be in the same part. When $m = 1$, we cannot use the foregoing argument.

However, one can obtain a necessary and sufficient condition for $\mathrm{TL}_{1,n}$ to be semisimple [16, Sec. 5].

Together with [13, 8.1] and Proposition 4.5, we now have the main result of this paper as follows.

THEOREM 4.6. *Suppose $m \geq 2$. Let R be a splitting field of $x^m - 1$ that contains $1, \delta_0, \dots, \delta_{m-1}$. Then the following conditions are equivalent.*

(a) $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$ *is semisimple.*
(b) $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$ *is split semisimple.*
(c) $\mathrm{ch}\, R \nmid m$ *and* $\det \Psi_{\mathbf{i}}(n,1) \neq 0$ *for all* $\mathbf{i} \in \Lambda(m, n-2)$.
(d) *All cell modules* $\Delta(k, \mathbf{i})$ *with* $(k, \mathbf{i}) \in \Lambda_{n,m}$ *are pairwise nonisomorphic irreducible.*

Proof. Since $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$ is a cellular algebra, it follows that (a), (b), and (d) are equivalent. By Proposition 4.5 and [13, 8.1], (a) and (c) are equivalent. □

Our next corollary follows immediately from [13, 8.1] and Proposition 4.5.

COROLLARY 4.7. *Keep the setup. Then* $\mathrm{TL}_{m,n}(\boldsymbol{\delta})$ *is semisimple if and only if*

(a) $\mathrm{ch}\, R \nmid m$,
(b) $P_1(\bar{\delta}_i) = \bar{\delta}_i \neq 0, 1 \leq i \leq m$, *and*
(c) $P_l(\bar{\delta}_{i_1}, \bar{\delta}_{i_2}, \dots, \bar{\delta}_{i_l}) \neq 0, 2 \leq l \leq n$, *for any* $(i_1, i_2, \dots, i_l) \in \Lambda(m, l)$ *with* $m \mid (i_j + i_{j+1}), 1 \leq j \leq l-1$.

REMARK. Note that Theorem 4.6 is not true if $m = 1$. In this case, $\Lambda(m, n)$ contains only one element $(1, 1, \dots, 1)$ that can be partitioned into one part. Corollary 4.7 for $m = 1$ is Westbury's theorem given in [16, Sec. 5].

ACKNOWLEDGMENTS. Xi is supported in part by NSFC and a China–U.K. joint project of the Royal Society, United Kingdom. Rui is supported in part by NSFC and the foundation of Shanghai for priority academic discipline. He wishes to thank RIMS, Kyoto University, for its hospitality during his visit.

References

[1] A. Cox, J. Graham, and P. Martin, *The blob algebra in positive characteristic,* J. Algebra 266 (2003), 584–635.
[2] W. Doran, D. Wales, and P. Hanlon, *On the semisimplicity of the Brauer centralizer algebras,* J. Algebra 211 (1999), 647–685.
[3] J. Graham and G. Lehrer, *Cellular algebras,* Invent. Math. 123 (1996), 1–34.
[4] ———, *The representation theory of affine Temperley–Lieb algebras,* Enseign. Math. (2) 44 (1998), 173–218.
[5] ———, *The two-step nilpotent representations of the extended affine Hecke algebra of type A,* Compositio Math. 133 (2002), 173–197.

[6] ———, *Diagram algebras, Hecke algebras and decomposition numbers at roots of unity,* Ann. Sci. École Norm. Sup. (5) 36 (2003), 479–524.
[7] J. A. Green, *Polynomial representations of* GL_n, Lecture Notes in Math., 830, Springer-Verlag, New York, 1980.
[8] R. M. Green, *On representations of affine Temperley–Lieb algebras,* Algebras and modules, II (Geiranger, 1996), CMS Conf. Proc., 24, pp. 245–261, Amer. Math. Soc., Providence, RI, 1998.
[9] V. F. R. Jones, *Hecke algebra representations of braid groups and link polynomials,* Ann. of Math. (2) 126 (1987), 335–388.
[10] S. König and C. C. Xi, *On the structure of cellular algebras,* Algebras and modules, II (Geiranger, 1996), CMS Conf. Proc., 24, pp. 365–386, Amer. Math. Soc., Providence, RI, 1998.
[11] P. Martin, *The structure of the partition algebras,* J. Algebra 183 (1996), 319–358.
[12] P. Martin and H. Saleur, *The blob algebra and the periodic Temperley–Lieb algebra,* Lett. Math. Phys. 30 (1994), 189–206.
[13] H. Rui and C. C. Xi, *The representations of the cyclotomic Temperley–Lieb algebras,* Comment. Math. Helv. 79 (2004), 427–450.
[14] H. Rui and W. Yu, *On the semisimplicity of the cyclotomic Brauer algebras,* J. Algebra 277 (2004), 187–221.
[15] H. N. V. Temperley and E. H. Lieb, *Relations between percolation and colouring problems and other graph theoretical problems associated with regular planar lattices: Some exact results for the precolation problem,* Proc. Roy. Soc. London Ser. A 322 (1971), 251–273.
[16] B. W. Westbury, *The representation theory of the Temperley–Lieb algebras,* Math. Z. 219 (1995), 539–565.

H. Rui
Department of Mathematics
East China Normal University
Shanghai, 200062
China

hbrui@math.ecnu.edu.cn

C. C. Xi
School of Mathematical Sciences
Beijing Normal University
Beijing, 100875
China

ccxi@bnu.edu.cn

W. Yu
Department of Mathematics
East China Normal University
Shanghai, 200062
China