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Dynamics of Quadratic Polynomial Mappings of $\mathbb{C}^2$

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Introduction

A remarkable feature of one-dimensional complex dynamics is the prominent role played by the "quadratic family" $P_c(z) = z^2 + c$. The latter has revealed an exciting source of study and inspiration for the study of general rational mappings $f: \mathbb{P}^1 \to \mathbb{P}^1$ as well as for more general dynamical systems [L]. Our purpose here is to introduce several quadratic families of polynomial self-mappings of $\mathbb{C}^2$ that we hope will be the complex two-dimensional counterpart to the celebrated quadratic family.

We partially classify quadratic polynomial endomorphisms of $\mathbb{C}^2$ (see Section 2) using some numerical invariants (dynamical degrees $\lambda_1(f), d_t(f)$ and dynamical Lojasiewicz exponent $DL_\infty(f)$), which we define in Section 1. We then use this classification to test two related questions.

Question 1. Does there exist a unique invariant probability measure of maximal entropy?

Question 2. Does there exist an algebraically stable compactification?

Simple examples show that there may be infinitely many invariant probability measures of maximal entropy when $d_t(f) = \lambda_1(f)$. When $d_t(f) > \lambda_1(f)$, it is proved in [Gu2] that the Russakovskii–Shiffman measure μ_f is the unique measure of maximal entropy. We push further the study of μ_f, when f is quadratic, by showing that it is compactly supported in $\mathbb{C}^2$ (Section 4). Moreover, every plurisubharmonic function is in $L^1(\mu_f)$ (Section 5) and the "exceptional set" is algebraic (Section 6).

When $d_t(f) < \lambda_1(f)$, one also expects the existence of a unique measure of maximal entropy (this is the case when f is a complex Hénon mapping [BLS1]). If f is algebraically stable on some smooth compactification of $\mathbb{C}^2$, then one can construct invariant currents T_+, T_- such that $f^*T_+ = \lambda_1(f)T_+$ and $f_*T_- = \lambda_1(f)T_-$ (see [Gu1]). It is usually difficult to define the invariant measure $\mu_f = T_+ \wedge T_-$. However, this can be done when f is polynomial in $\mathbb{C}^2$, since T_+ admits continuous potentials off a finite set of points. We briefly discuss Question 1 for quadratic mappings with $d_t(f) < \lambda_1(f)$ in Section 3; the answer is positive for an open set of parameters but unknown in general.

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1. Numerical Invariants

1.1. Algebraic Stability

Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial mapping. We always assume f is *dominating,* that is, the Jacobian Jf of f does not vanish identically. Let us denote by $d_t(f)$ the topological degree of f (i.e., the number of preimages of a generic point) and by $\delta_1(f)$ its algebraic degree (i.e., the degree of the preimage of a generic line in $\mathbb{C}^2$). If $f = (P, Q)$ in coordinates, then $\delta_1(f) = \max(\deg P, \deg Q)$. Clearly d_t behaves well both under iteration ($d_t(f^j) = [d_t(f)]^j$) and under conjugacy ($d_t(f) = d_t(\Phi^{-1} \circ f \circ \Phi)$). Concerning δ_1, we also have the straightforward inequality

$$\delta_1(f \circ g) \leq \delta_1(f) \cdot \delta_1(g); \tag{$*$}$$

however, equality fails in general. Nevertheless, $(*)$ shows that the sequence $(\delta_1(f^j))$ is submultiplicative, so we can define

$$\lambda_1(f) := \lim[\delta_1(f^j)]^{1/j} =: \text{first dynamical degree of } f.$$

It follows again from $(*)$ that $\lambda_1(f)$ is invariant under conjugacy.

In order to compute $\lambda_1(f)$, one needs to compute $\delta_1(f^j)$ for all $j \geq 1$. Although this can be achieved "by hand" in some simple situations, there is a subtler way of computing $\lambda_1(f)$ that, moreover, yields interesting information about the dynamics. Let $X = \mathbb{C}^2 \cup Y_\infty$ be a smooth compactification of $\mathbb{C}^2$, where Y_∞ denotes the divisor at infinity. We still denote by f the meromorphic extension of f to X and let $I_f \subset Y_\infty$ be the indeterminacy set of f, that is, the finite number of points at which f is not holomorphic.

DEFINITION 1.1. We say f is *algebraically stable* in X if, for every curve C of X and every $j \geq 1$, $f^j(C \setminus I_{f^j}) \notin I_f$, where I_{f^j} denotes the indeterminacy set of f^j.

It is known [PSch] that every smooth compactification of $\mathbb{C}^2$ is a projective algebraic surface $X = \mathbb{C}^2 \cup Y_\infty$, where the divisor at infinity $Y_\infty = C_1 \cup \cdots \cup C_s$ consists of a finite number of rational curves $C_1, \ldots, C_s$. Since we are dealing with polynomial mappings, it follows that $f(\mathbb{C}^2) \subset \mathbb{C}^2$ and that the indeterminacy set I_f is located inside Y_∞. Hence, the only curves that can be contracted to a point of indeterminacy are the C_j. The condition of algebraic stability is thus quite easy to check here.

QUESTION 1.2. Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial dominating mapping. Can one always find a smooth compactification of $\mathbb{C}^2$ on which f becomes algebraically stable?

We will see in what follows that the answer is positive when $\delta_1(f) = 2$. The answer is negative in general for rational mappings [F]. The point is that if $f : X \to X$ is algebraically stable in X, then $\lambda_1(f)$ equals the spectral radius of the linear action induced by pull-back by f on the cohomology vector space $H^{1,1}(X, \mathbb{R})$. Moreover, this is is the starting point for the construction of invariant currents (see [Gu1]).

1.2. Dynamical Lojasiewicz Exponent

A general principle is that the behavior of f at infinity governs its dynamics at bounded distance. Recall that the Lojasiewicz exponent $L_\infty(f)$ of f at infinity is defined by

$$L_\infty(f) = \sup\{\nu \in \mathbb{R} \mid \exists (C,R) > 0,\ \|m\| \geq R \Rightarrow \|f(m)\| \geq C\|m\|^\nu\}.$$

It is known that $L_\infty(f)$ is always a rational number (possibly $-\infty$) which is positive if and only if f is proper. Moreover, there are explicit formulas that yield $L_\infty(f)$ by simple computation [CK].

LEMMA 1.3. *Let* $f, g \colon \mathbb{C}^2 \to \mathbb{C}^2$ *be polynomial dominating mappings. Then*:

(i) $L_\infty(f) \leq \delta_1(f)$ *with equality if and only if* f *extends holomorphically to* $\mathbb{P}^2$;
(ii) $L_\infty(f \circ g) \leq \delta_1(f) \cdot L_\infty(g)$ *if* g *is proper*; *and*
(iii) $L_\infty(f) \cdot L_\infty(g) \leq L_\infty(f \circ g)$ *if* g *is proper.*

Proof. Let us denote by ω the Fubini–Study Kähler form on $\mathbb{P}^2$.

(i) Set $d = \delta_1(f)$. Then $f = (P, Q)$, where P, Q are polynomials such that $d = \max(\deg P, \deg Q)$, so there exists a $C_1 > 0$ such that

$$\|m\| \geq 1 \implies \|f(m)\| \leq C_1 \|m\|^d.$$

This yields $L_\infty(f) \leq d$.

Assume $\|f(m)\| \geq C\|m\|^d$ for $\|m\| \geq R$. It then follows from Taylor's lemma (see Lemma 1.5) that

$$d_t(f) = \int_{\mathbb{C}^2} f^*(\omega^2) \geq d \int_{\mathbb{C}^2} f^*\omega \wedge \omega = d^2 = \int_{\mathbb{P}^2} (\tilde{f})^*(\omega^2).$$

Hence the meromorphic extension $\tilde{f}$ of f to $\mathbb{P}^2$ has no point of indeterminacy; that is, f extends holomorphically to $\mathbb{P}^2$. Conversely, if $\tilde{f}$ is holomorphic on $\mathbb{P}^2$, then f has nondegenerate homogeneous components of degree d and so $L_\infty(f) = d$.

(ii) Set again $d = \delta_1(f)$. Then there exists a $C_1 > 0$ such that

$$\|g(m)\| \geq 1 \implies \|f \circ g(m)\| \leq C_1 \|g(m)\|^d.$$

When g is proper this reads, for every $R > 1$ large enough,

$$\|m\| \geq R \implies \|f \circ g(m)\| \leq C_1 \|g(m)\|^d.$$

The desired inequality follows.

(iii) Assume $\|f(m)\| \geq C_1\|m\|^\nu$ for $\|m\| \geq R_1$. When g is proper we may deduce $\|f \circ g(m)\| \geq C_2 \|g(m)\|^\nu$ for $\|m\| \geq R_2$. This yields $L_\infty(f \circ g) \geq \nu L_\infty(g)$, hence $L_\infty(f \circ g) \geq L_\infty(f) \cdot L_\infty(g)$. □

If f is not proper, then (ii) and (iii) of Lemma 1.1 are false, as simple examples show. The lemma shows that the sequence $(L_\infty(f^j))_{j \in \mathbb{N}}$ is supmultiplicative when f is proper.

DEFINITION 1.4. Let $f \colon \mathbb{C}^2 \to \mathbb{C}^2$ be a proper polynomial mapping. Then the *dynamical Lojasiewicz exponent* of f at infinity is

$$DL_\infty(f) := \lim [L_\infty(f^j)]^{1/j}.$$

Let us recall the following useful lemma [T].

LEMMA 1.5. *Let S be a positive closed current of bidegree $(1,1)$ on $\mathbb{P}^2$, and let u be a locally bounded plurisubharmonic function in $\mathbb{C}^2$. Assume $u(m) \geq \nu \log^+\|m\| + C$ on the support of S for some $C, \nu > 0$. Then*

$$\int_{\mathbb{C}^2} S \wedge dd^c u \geq \nu \int_{\mathbb{C}^2} S \wedge \omega.$$

The proof is an integration-by-parts argument (see [Gu1, Prop. 4.3]).

PROPOSITION 1.6. *Let $f: \mathbb{C}^2 \to \mathbb{C}^2$ be a proper polynomial mapping. Then the following statements hold.*

(i) *$L_\infty(f)$ is invariant under affine conjugacy, and $DL_\infty(f)$ is invariant under polynomial conjugacy.*
(ii) $0 < L_\infty(f) \leq DL_\infty(f) \leq \lambda_1(f) \leq \delta_1(f)$.
(iii) $L_\infty(f) \leq d_t(f)/\delta_1(f)$, *so* $DL_\infty(f) \leq d_t(f)/\lambda_1(f)$.

REMARK 1.7. All these inequalities are strict in general. Note that if $DL_\infty(f) > 1$ then infinity is an "attracting" set for f: there exist a neighborhood V of infinity in $\mathbb{C}^2$ and an $l \geq 1$ such that $\overline{f^l V} \subset V$ and $\bigcap_{j\geq 0} f^j(V) = \emptyset$. Hence every point $a \in \mathcal{B}^+(\infty) := \bigcup_{n\geq 0} f^{-n}(V)$ escapes to infinity in forward time, so the nonwandering set of f is included in the compact set $K^+ := \{p \in \mathbb{C}^2 \mid (f^n(p))_{n\geq 0}$ is bounded$\} = \mathbb{C}^2 \setminus \mathcal{B}^+(\infty)$.

Proof of Proposition 1.6. Everything follows immediately from Lemma 1.3 except for part (iii). Assume $\|f(m)\| \geq C\|m\|^\nu$ for $\|m\| \geq R$, where $C, \nu, R > 0$. It follows from two applications of Lemma 1.5 that

$$d_t(f) = \int_{\mathbb{C}^2} f^*\omega \wedge f^*\omega \geq \nu \int_{\mathbb{C}^2} f^*\omega \wedge \omega = \nu\delta_1(f).$$

Therefore $d_t(f) \geq \delta_1(f) L_\infty(f)$ and, by iteration, $d_t(f) \geq \lambda_1(f) DL_\infty(f)$. □

EXAMPLES 1.8. (1) Consider $f(z,w) = (P(w), Q(z) + R(w))$, where P, Q, R are polynomials of degree p, q, d respectively, with $d > \max(p,q)$. We obtain $\lambda_1(f) = \delta_1(f) = d$ and $d_t(f) = pq$ as well as $L_\infty(f) = DL_\infty(f) = pq/d = d_t(f)/\lambda_1(f)$.

(2) Consider $f(z,w) = (w, z^2 + aw + c)$, where $(a,b) \in \mathbb{C}^2$. Observe that the second iterate f^2 extends holomorphically to $\mathbb{P}^2$, so

$$d_t(f) = 2, \quad \lambda_1(f) = \sqrt{2} < 2 = \delta_1(f), \quad L_\infty(f) = 1 < \sqrt{2} = DL_\infty(f).$$

See [GN] for a detailed study of this mapping.

(3) Let f be a polynomial automorphism of $\mathbb{C}^2$. It is known [FrM] that f is conjugate to either an elementary automorphism or a composition of complex Hénon mappings. In the elementary case we have $d_t(f) = \lambda_1(f) = 1$ and $DL_\infty(f) = L_\infty(f) = 1/d$. In the Hénon case $d_t(f) = 1$ and $\lambda_1(f) = d$, so $DL_\infty(f) \leq 1/d$.

On the other hand, $L_\infty(f) \geq 1/d$, as follows from Lemma 1.3(ii) applied to f and f^{-1}. Therefore, $L_\infty(f) = DL_\infty(f) = 1/d$.

2. Classification of Quadratic Polynomial Mappings of $\mathbb{C}^2$

In this section we classify up to conjugacy the quadratic dominating polynomial self-mappings of $\mathbb{C}^2$ according to their dynamical degrees. For our purposes, the precise nature of the normal form is not important: the essential point will be to determine their numerical invariants and behavior at infinity. This section is devoted to a proof of the following result.

THEOREM 2.1. *Let $f\colon \mathbb{C}^2 \to \mathbb{C}^2$ be a dominating polynomial mapping with $\delta_1(f) = 2$. Then f is conjugate, by a linear affine automorphism of $\mathbb{C}^2$, to one of the following families.*

(1) $d_t(f) < \lambda_1(f)$

(1.1)
$$f(z,w) = (w + c, zw + c'),$$
where $c, c' \in \mathbb{C}$. In this case, $d_t(f) = 1$ and $\lambda_1(f) = \big(1+\sqrt{5}\big)/2$.

(1.2)
$$f(z,w) = (w + c, w[w - az] + bz + c'),$$
where $a, b, c, c' \in \mathbb{C}$ with $(a,b) \neq (0,0)$. Here $d_t(f) = 1$ and $\lambda_1(f) = 2$.

(2) $d_t(f) = \lambda_1(f)$

(2.1) $d_t(f) = \lambda_1(f) = 1$

(a) $f(z,w) = (az + c, z^2 + bw + c')$, *where $a, b, c, c' \in \mathbb{C}$ with $ab \neq 0$.*

(b) $f(z,w) = (az + c, zw + c')$, *where $a, c, c' \in \mathbb{C}$ with $a \neq 0$.*

(2.2) $d_t(f) = \lambda_1(f) = 2$

(a) $f(z,w) = (P(z), Q(z,w))$, *where* $\deg P = \deg Q = 2$ *and* $\deg_w Q = 1$.

(b) $f(z,w) = (P(z), Q(z,w))$, *where* $\deg P = 1$ *and* $\deg Q = 2 = \deg_w Q$.

(c) $f(z,w) = (w, Q(z,w))$, *where* $\deg_z Q = \deg_w Q = \deg Q = 2$.

(d) $f(z,w) = (zw + c, z[z + aw] + bz + c')$, *where $a, b, c, c' \in \mathbb{C}$.*

(3) $d_t(f) > \lambda_1(f)$

(3.1)
$$f(z,w) = (w, z^2 + aw + c),$$
where $a, c \in \mathbb{C}$. Here, $d_t(f) = 2$ and $\lambda_1(f) = \sqrt{2} = DL_\infty(f)$.

(3.2)
$$f(z,w) = (aw + c, z[z - w] + c'),$$
where $a, c, c' \in \mathbb{C}$, $a \neq 0$. In this case, $d_t(f) = 2$, $\lambda_1(f) = \big(1+\sqrt{5}\big)/2$, and $DL_\infty(f) = 1$.

(3.3)
$$f(z,w) = (az^2 + bz + c + w, z[w + \alpha z] + c'),$$
where $a, b, c, c', \alpha \in \mathbb{C}$, $a \neq 0$. Here $d_t(f) = 3$, $\lambda_1(f) = 2$, and $DL_\infty(f) > 1$.

(3.4) $$f(z,w) = (zw + c, z[z + \alpha w] + bz + c' + aw),$$

where $a, b, c, c', \alpha \in \mathbb{C}$ *with* $a \neq 0$. *Here* $d_t(f) = 3$ *and* $\lambda_1(f) = 2$, *but* $DL_\infty(f) \overset{?}{=} 1$.

(3.5) $$f(z,w) = (P(z,w) + L_1(z,w), Q(z,w) + L_2(z,w)),$$

where P, Q *are homogeneous polynomials of degree* 2 *with* $P \wedge Q = 1$ *and* L_1, L_2 *are polynomials of degree* ≤ 1. *Here we have* $d_t(f) = 4$ *and* $\lambda_1(f) = 2 = DL_\infty(f)$.

REMARK 2.2. In the sequel we shall focus on quadratic mappings with $d_t(f) \neq \lambda_1(f)$. For the remaining six families, observe that families 2.1a, 2.1b, 2.2a, and 2.2b are skew products whose dynamics are rather one-dimensional. The remaining two families 2.2c and 2.2d may display more intricate dynamical behavior. A special case of 2.2.d arises in the study of density of states of self-similar diffusion on the interval [0,1] [Sa].

Proof of Theorem 2.1. This is a case-by-case analysis.

We first decompose $f(z,w) = (P(z,w)+L_1(z,w); Q(z,w)+L_2(z,w))$, where P, Q are homogeneous polynomials of degree 2 and L_1, L_2 are polynomials of degree ≤ 1. When $P \wedge Q = 1$, f extends holomorphically to $\mathbb{P}^2$ and we obtain the family 3.5. Hence we need only consider the cases $P \equiv 0$ or $P = A\tilde{P}$ and $Q = A\tilde{Q}$ with $A, \tilde{P}, \tilde{Q}$ homogeneous of degree 1 and $\tilde{P} \wedge \tilde{Q} = 1$. Indeed, the remaining cases $Q \equiv 0$ and $P = \lambda Q$ are both conjugate to the case $P \equiv 0$ by $(z,w) \mapsto (w,z)$ and $(z,w) \mapsto (z + \lambda w, w)$, respectively.

Case 1: $P \equiv 0$. We get $f(z,w) = (\alpha z + \beta w + c, Q(z,w) + L_2(z,w))$. If $\beta = 0$ then f is a skew product, and a further case-by-case analysis yields the families 2.1a, 2.1b and 2.2b. So let us assume $\beta \neq 0$. Conjugating by $(z,w) \mapsto (z, w/\beta)$ yields $\beta = 1$. Conjugating further by $(z,w) \mapsto (z, w - \alpha z - c)$ yields $\alpha = c = 0$, hence $f(z,w) = (w, Q(z,w) + L_2(z,w))$.

Subcase A: $\deg_z Q = 0$. Since f is dominating, it follows that $\deg_z L_2 = 1$ and that $Q = Q(w)$ is a degree-2 polynomial. In this case, f is a quadratic Hénon mapping—that is, a mapping in the family 1.2 with $a = 0$ (see [FrM] for a precise normal form).

Subcase B: $\deg_z Q = 1$. Then $d_t(f) = 1$; that is, f is a birational mapping. However, its inverse is not polynomial in $\mathbb{C}^2$.

If $\deg_w Q = 2$ then we may conjugate by $(z,w) \mapsto (z, \lambda w)$ to obtain

$$f(z,w) = (w, w[w - az] + bz + b'w + c') \quad \text{with } a \neq 0.$$

Further conjugacy by a translation yields the remaining cases of the family 1.2. Observe that f is then algebraically stable in $\mathbb{P}^2$, so $\lambda_1(f) = \delta_1(f) = 2$.

If $\deg_w Q = 1$ then conjugating by $(z,w) \mapsto (\lambda z, \lambda w)$ yields

$$f(z,w) = (w, zw + bz + b'w + c').$$

We can further conjugate by a translation to derive the normal form of the family 1.1. Observe that f is then algebraically stable in $\mathbb{P}^1 \times \mathbb{P}^1$, so that $\lambda_1(f)$ is the spectral radius of the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ of the degrees of f in $\mathbb{P}^1 \times \mathbb{P}^1$. That is, $\lambda_1(f) = \left(1+\sqrt{5}\right)/2$ [FGu].

Subcase C: $\deg_z Q = 2$. Then $d_t(f) = 2$.

If $\deg_w Q = 2$, then f is algebraically stable in $\mathbb{P}^2$ and so $\lambda_1(f) = \delta_1(f) = 2$. Thus we obtain the family 2.2c.

If $\deg_w Q = 0$, then conjugating by $(z,w) \mapsto (\lambda z + c, \lambda w + c/\lambda)$ yields the family 3.1. These mappings f have the property that the second iterate f^2 is still quadratic and admits a holomorphic extension to $\mathbb{P}^2$ (i.e., f^2 belongs to the family 3.5). The assertion on the dynamical invariants easily follows.

If $\deg_w Q = 1$, then conjugating by $(z,w) \mapsto (\lambda z, \mu w)$ yields

$$f(z,w) = (aw, z[z-w] + \alpha z + \beta w + c').$$

We can further conjugate by a translation to reach the normal form of the family 3.2. Using bihomogeneous coordinates as in [Gu1], one can check that these mappings admit an algebraically stable extension to $\mathbb{P}^2$ blown up at the point $[0:1:0]$, with $\lambda_1(f) = \left(1+\sqrt{5}\right)/2$. We will check in Lemma 2.5 that $DL_\infty(f) = 1$.

Case 2: $P = A\tilde{P}$ *and* $Q = A\tilde{Q}$. Observe that f is algebraically stable in $\mathbb{P}^2$, so $\lambda_1(f) = \delta_1(f) = 2$. We can write $A(z,w) = \alpha z + \beta w$ with $(\alpha,\beta) \neq (0,0)$. Conjugating by $(z,w) \mapsto (w,z)$ if necessary, we can assume $\alpha = 1$. Further conjugacy by $(z,w) \mapsto (z-\beta, w)$ yields $\beta = 0$.

Similarly, we decompose $\tilde{P}(z,w) = az + bw$ and $\tilde{Q}(z,w) = a'z + b'w$ with $(a,b) \neq (0,0) \neq (a',b')$ and $[a:b] \neq [a':b']$ in $\mathbb{P}^1$.

Subcase A: $b = 0$. In this case, $ab' \neq 0$. Conjugating by $(z,w) \mapsto (z/b', w)$ yields $b' = 1$ and so

$$f(z,w) = (R(z) + \beta w, z[w + \alpha z] + \delta z + \varepsilon w + c'),$$

where R is a degree-2 polynomial. Either $\beta = 0$, in which case f is a skew product of type 2.2a, or we can assume $\beta = 1$ after conjugating by $(z,w) \mapsto (z, w/\beta)$. A further conjugacy by a translation yields the normal form of the family 3.3. One easily checks that $d_t(f) = 3$ in this case. The dynamical Lojasiewicz exponent at infinity will be estimated in Lemma 2.4.

Subcase B: $b \neq 0$. Conjugating by $(z,w) \mapsto (z, w/b - aw/b)$ yields $a = 0$ and $b = 1$. Thus,

$$f(z,w) = (zw + \alpha z + \beta w + c, z[a'z + b'w] + \delta z + c' + aw) \quad \text{with } a' \neq 0.$$

Further conjugacy by a translation and $(z,w) \mapsto \left(z/\sqrt{a'}, w\right)$ yields $\alpha = \beta = 0$ and $a' = 1$. If $a \neq 0$ then we have the normal form of the family 3.4. One easily checks that $d_t(f) = 3$ in this case, and the exponent $DL_\infty(f)$ will be considered in Lemma 2.6. Finally, if $a = 0$ then $d_t(f) = 2$ and f belongs to the family 2.2d. This ends the proof of the classification. □

LEMMA 2.3. *Consider* $f\colon (z,w) \in \mathbb{C}^2 \mapsto (P(z)+w, z[w+\alpha z]+c') \in \mathbb{C}^2$, *where* P *is a polynomial of degree* 2 *and* $\alpha, c' \in \mathbb{C}$. *Then* $L_\infty(f) = 1$ *and* $L_\infty(f^2) = 3/2$, *so* $DL_\infty(f) > 1$.

Proof. We leave it to the reader to check that $L_\infty(f) = 1$. Fix $(z,w) \in \mathbb{C}^2$ such that $\max(|z|,|w|) = R \gg 1$, and set $(z',w') = f(z,w)$ and $(z'',w'') = f(z',w')$. If $|z| = \max(|z|,|w|) = R$ then $|z'| \gtrsim |z|^2 = R^2$ and $|w'| \lesssim R^2$, so $|z''| \gtrsim R^4$.

We assume now that $|w| = \max(|z|,|w|) = R$. We have one of the following four possible cases.

Case 1: $C_1|w|^{1/2} \le |z| \le \varepsilon_1|w|$, where C_1 (resp. ε_1) is a fixed large (resp. small) constant. Then $|z'| \gtrsim |z|^2 \ge R^2$, $|w'| \gtrsim |zw| \gtrsim R^{3/2}$, and $|z'| \lesssim |z|^2 \le \varepsilon_1|zw| \le \varepsilon_1'|w'|$. Therefore,

$$|w'| \gtrsim |z'||w'| \gtrsim R^{5/2}.$$

Case 2: $|z| \ge \varepsilon_1|w|$. In this case, $|z'| \gtrsim |z|^2 \gtrsim R^2$ while $|w'| \lesssim |z||w| \lesssim R^2$, so $|z''| \gtrsim R^4$.

Case 3: $C_1^{-1}|w|^{1/2} \le |z| \le C_1|w|^{1/2}$. Then $|w'| \gtrsim |zw| \gtrsim R^{3/2}$ while $|z'| \lesssim R$, so $|w''| \gtrsim |z'w'| \gtrsim R^{3/2}$ if $|z'| \ge 1$. Now, if $|z'| \le 1$ then $|z''| \gtrsim |w'| \gtrsim R^{3/2}$.

Case 4: $|z| \le C_1^{-1}|w|^{1/2}$. Here $|z'| \gtrsim |w| = R$ and $|w'| \lesssim |zw| \lesssim R^{3/2}$. Therefore, $|z''| \gtrsim |z'|^2 \gtrsim R^2$.

Altogether this shows $L_\infty(f^2) \ge 3/2$. On the other hand, if $(z,w) \in \mathbb{C}^2$ is such that $P(z) + w = 0$ and $|w| = R \gg |z| \gg 1$, then $w'' = c$ and $|z''| = |P(0) + c' + z[w + \alpha z]| \lesssim |zw| \lesssim R^{3/2}$, so $L_\infty(f^2) = 3/2$. □

LEMMA 2.4. *Consider $f\colon (z,w) \in \mathbb{C}^2 \mapsto (aw + c, z[z-w] + c') \in \mathbb{C}^2$, where $a, c, c' \in \mathbb{C}$ with $a \ne 0$. Then $L_\infty(f^j) = 1$ for all j, so $DL_\infty(f) = 1$.*

Proof. It is straightforward to check that $L_\infty(f) = 1$. Observe that $f(w+a,w) = (aw + c, aw + a^2 + c')$ and hence

$$f^2(w+a,w) = (a^2w + c_2, a[c - c' - a^2]w + c_2')$$

for some constants c_2, c_2'. This shows $L_\infty(f^2) = 1$. Continuing in this fashion,

$$\begin{aligned} &f^2(w + a + \varepsilon_1/w, w) \\ &\qquad = (a^2w + c_2 + O(1/w), a[c - c' - a^2 - \varepsilon_1]w + c_2' + O(1/w)), \end{aligned}$$

and so by choosing $\varepsilon_1 = c - c' - a^2 + a$ we obtain

$$f^3(w + a + \varepsilon_1/w, w) = (a^3w + c_3 + O(1/w), a^2\alpha w + c_3' + O(1/w))$$

for some constant α that depends on the next-order term in $O(1/w)$. This shows that f^3 grows linearly on the curve $\{zw = w^2 + a + \varepsilon_1\}$ when $|w|$ is large; hence $L_\infty(f^3) = 1$. Moreover, we can choose the next-order term in $O(1/w)$ so that $\alpha = a$. We leave it to the reader to check that there exist constants ε_j, c_j, c_j' such that, for all $N \ge 2$,

$$f^N\left(w + a + \sum_{j=1}^{N-2} \frac{\varepsilon_j}{w^j}, w\right) = (a^N w + c_N + O(1/w), a^N w + c_N' + O(1/w)).$$

This yields $L_\infty(f^N) = 1$ for all N and hence $DL_\infty(f) = 1$.

Note that we have $\varepsilon_j = 0$ when $c - c' = a^2 + a$. In this case the line $L = \{z = w + a\}$ is invariant and $f|_L(z,w) = (az + c - a^2, aw + c - a^2)$. In particular: if $c = a^2$, $c' = -a$, and $a^N = 1$, then $f^N|_L = \mathrm{Id}_L$ and so L is a curve of periodic points. □

LEMMA 2.5. *Consider* $f\colon (z,w) \in \mathbb{C}^2 \mapsto (zw, z[z+\alpha w] + bz + c' + aw) \in \mathbb{C}^2$, *where* $a, b, c', \alpha \in \mathbb{C}$ *with* $a \neq 0$. *Then* $L_\infty(f^j) = 1$ *for all* j, *so* $DL_\infty(f) = 1$.

Proof. Simple estimates yield $L_\infty(f) = 1$. Observe that $f(0,w) = (0, aw + c')$ and so the line $L = (z = 0)$ is invariant and $f|_L$ is linear. This shows $L_\infty(f^j) = 1$ for all $j \geq 1$, hence $DL_\infty(f) = 1$. Note, moreover, that L is a line of periodic points when $c' = 0$ and $a^N = 1$. □

3. Birational Quadratic Mappings of $\mathbb{C}^2$

In this section we consider the families 1.1 and 1.2. Since $d_t(f) = 1$, these families admit an inverse mapping f^{-1} that is rational. There has been intensive work on these birational mappings (see references in [DF]). It is difficult in general to analyze the dynamics near the points of indeterminacy. We show that this can be done here at least for open subsets of the parameters.

3.1. Family 1.1

It is convenient to consider the meromorphic extension of $f(z,w) = (w + c, zw + c')$ to $\mathbb{P}^1 \times \mathbb{P}^1$, in bihomogeneous coordinates:

$$f\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1;$$

$$[z_0 : z_1;\ w_0 : w_1] \mapsto [w_0 : w_1 + cw_0;\ z_0w_0 : z_1w_1 + c'z_0w_0].$$

It should be understood that $\mathbb{C}^2$ coincides with the chart $(z_0 = w_0 = 1)$ and that "infinity" consists of the two lines $(z_0 = 0) = (z = \infty)$ and $(w_0 = 0) = (w = \infty)$. Observe that f has two points of indeterminacy, $m = (\infty, 0)$ and $m' = (0, \infty)$, and contracts the line $(w_0 = 0)$ to the superattractive fixed point $q_\infty = (\infty, \infty)$ while sending $(z_0 = 0)$ to the line $(w_0 = 0)$. This shows that f is algebraically stable in $\mathbb{P}^1 \times \mathbb{P}^1$ with $\lambda_1(f) = \big(1 + \sqrt{5}\big)/2$ (see [FGu] for further details). Observe also that

$$I_{f^2} = \{m, m', m''\} = I_{f^n} \quad \forall n \geq 2,$$

where $m'' = (\infty, -c)$ is sent by f to the point $m' = (0, \infty)$. The inverse mapping $f^{-1}(z,w) = \big(\frac{w-c'}{z-c}, z - c\big)$ is rational in $\mathbb{C}^2$. One easily checks that $I_{f^{-1}} = \{q_\infty, q^-\}$, where $q^- = (c, c') \in \mathbb{C}^2$, and that

$$I_{f^{-n}} = \{q_\infty,\ f^j(q^-),\ 0 \leq j \leq n-1\}.$$

It is therefore important to gain control of the orbit of q^-. Note that

$$f^{-1}(m) = m', \quad f^{-1}(m') = m'', \quad f^{-1}(m'') = m',$$

so $\{m', m''\}$ is a 2-cycle for f^{-1} to which m is strictly preperiodic (if $c \neq 0$).

LEMMA 3.1. *The 2-cycle* $\{m', m''\}$ *is* f^{-1}*-attracting if and only if* $|c| < 1$.

Proof. A simple computation shows $Df^{-2}(m')$ has eigenvalues 0 and $-c$. □

LEMMA 3.2. *Assume* $|c|, |c'| < 10^{-3}$. *Let* p_0 *denote the fixed point of* f *that is closest to* $(0,0)$. *Then* p_0 *is attracting and* q^- *belongs to the basin of* p_0.

REMARK 3.3. For such parameters, f can be considered as a small perturbation of the case $c = c' = 0$, which is the complexification of the Anosov diffeomorphism $(z,w) \mapsto (w, zw)$ on the real torus $\{|z| = |w| = 1\}$.

Proof of Lemma 3.2. Solving $f(z,w) = (z,w)$ yields two fixed points $p_0 = (\alpha, \alpha - c)$ and $p_1 = (1 + c - \alpha, 1 - \alpha)$, where α is the root of $X^2 - (1+c)X + (c + c') = 0$ with smallest modulus. The differential of f at p_0 is

$$Df(p_0) = \begin{bmatrix} 0 & \alpha - c \\ 1 & \alpha \end{bmatrix},$$

so p_0 is an attractive fixed point if $|c|, |c'|$ are small enough. Let us make a local change of coordinates to bring back p_0 to $(0,0)$. Consider

$$g(x,y) = f(\alpha + x, \alpha - c + y) - (\alpha, \alpha - c) = (y, xy + \alpha y + (\alpha - c)x).$$

In these new coordinates, $q^- = (c - \alpha, c' + c - \alpha)$; hence q^- belongs to the basin of $(0,0) = p_0$ if $|c|, |c'|$ are small enough. It is then straightforward to check that $|c|, |c'| < 10^{-3}$ is sufficient. □

LEMMA 3.4. *Assume that* $|c| < 1$ *and* $|c'| > 4/(1 - |c|)$. *Then* $q^- = (c, c')$ *belongs to the basin of the superattractive fixed point* $q_\infty = (\infty, \infty)$.

Proof. It is more comfortable to work in a local chart near q_∞. Using bihomogeneous coordinates, we work in the chart $(z_1 = w_1 = 1)$. In this chart, f defines a mapping

$$g(x,y) = \left(\frac{y}{1 + cy}, \frac{xy}{1 + c'xy} \right)$$

and q_∞ has coordinates $(0,0)$. We have

$$g^2(x,y) = \left(\frac{xy}{1 + (c + c')xy}, \frac{xy^2}{1 + cy + c'xy + (c' + cc')xy^2} \right).$$

Consider

$$\Omega := \left\{ (x,y) \in \mathbb{C}^2 \mid |y| < \frac{1}{4} \text{ and } |xy| < \frac{1}{4|c + c'|} \right\}.$$

We claim Ω is g^2-invariant and g^2 is contracting in Ω, so Ω is part of the basin of attraction of q_∞. Indeed, let $(x,y) \in \Omega$ and set $(x', y') = g^2(x,y)$. Then $|1 + (c + c')xy| > 3/4$ and so $|x'| < 4|xy|/3 < |x|/3$. Moreover,

$$|cy| < |y| < \frac{1}{4},$$

$$|c'xy| < \frac{|c'|}{4|c + c'|} < \frac{1}{3} \quad (\text{since } |c'| > 4 > 4|c|),$$

$$|c'(1 + c)xy^2| < 2|c'||xy||y| < \frac{1}{2} \cdot \frac{|c'|}{4|c + c'|} < \frac{1}{6};$$

thus $|1+cy+c'xy+(c'+cc')xy^2| > 1/4$. This yields $|y'| < 4|xy^2| < |y|/|c+c'| < |y|/3$.

Now consider $\Omega' = \Omega \cap \mathbb{C}^2$. In our original coordinates (z, w), we thus get a portion of the basin (in $\mathbb{C}^2$) of the superattractive fixed point q_∞,

$$\Omega' = \{(z, w) \in \mathbb{C}^2 \mid |w| > 1/4 \text{ and } |zw| > 4|c + c'|\}.$$

We claim that $q^- = (c, c')$ belongs to $f^{-1}(\Omega')$ under our assumptions. Indeed, $f(q^-) = (c' + c, cc' + c')$ and we know that $|cc' + c'| \geq |c'|(1 - |c|) > 4$ and $|cc' + c'||c + c'| > 4|c + c'|$. This shows that $f(q^-)$, and hence q^-, belongs to the basin of q_∞. □

3.2. *Family 1.2*

We now turn to mappings of family 1.2. When $a \neq 2$, we can further conjugate by a translation and suppose that $c = 0$. In order to simplify the exposition we will thus consider the family of three parameters.

$$f(z, w) = (w, w[w - az] + bz + c'),$$

where $a, b, c' \in \mathbb{C}$ with $(a, b) \neq (0, 0)$. We consider their meromorphic extension to $\mathbb{P}^2 = \mathbb{C}^2 \cup (t = 0)$, where $(t = 0)$ denotes the line at infinity. In homogeneous coordinates,

$$f[z : w : t] = [wt : w(w - az) + bzt + c't^2 : t^2].$$

Hence $I_f = \{m, m'\} = I_{f^n}$ for all $n \geq 1$, where $m = [1 : 0 : 0]$ and $m' = [1 : a : 0]$ and where $f((t = 0) \setminus I_f) = q_\infty := [0 : 1 : 0]$ is a superattractive fixed point for f. Thus f is algebraically stable in $\mathbb{P}^2$ and $\lambda_1(f) = \delta_1(f) = 2$. The inverse mapping f^{-1} is merely rational in $\mathbb{C}^2$ when $a \neq 0$, and $f^{-1}(z, w) = ([w - z^2 - c']/[b - az], z)$. We therefore obtain $I_{f^{-1}} = \{q_\infty, q^-\}$, where $q^- = (b/a, b^2/a^2) \in \mathbb{C}^2$ except when $a = 0$, in which case $q^- = q_\infty$ and then f is a quadratic Hénon mapping. As a result,

$$I_{f^{-n}} = \{q_\infty,\ f^j(q^-),\ 0 \leq j \leq n - 1\} \quad \forall n \geq 1.$$

Observe that $f^{-1}(m) = m' = f^{-1}(m')$.

Lemma 3.5. *The point m' is attracting for f^{-1} if and only if $|a| < 1$.*

If $|a| < 1$ and $4|a| \leq |b|$, then q^- belongs to the basin of attraction of the point q_∞.

Proof. A simple computation shows that $Df^{-1}(m')$ has eigenvalues 0 and a.

Assume $|a| < 1$ and $4|a| \leq |b|$. We work in the chart $(w = 1) \ni q_\infty$. Set $x = z/w$, $y = t/w$, and

$$g(x, y) = f[x : 1 : y] = \left(\frac{y}{1 - ax + bxy + c'y^2}, \frac{y^2}{1 - ax + bxy + c'y^2}\right).$$

We set $\Omega = \{(x, y) \in \mathbb{C}^2 \mid |x| \leq 1/4 \text{ and } |y| \leq \max(1/|b|, 1/|c'|, 1/16)\}$. Let $(x, y) \in \Omega$ and set $(x', y') = g(x, y)$. Our assumption yields

$$|1 - ax + bxy + c'y^2| > 1/4;$$

therefore,

$$|x'| \leq 4|y| \leq 1/4 \quad \text{and} \quad |y'| \leq 4|y|^2 \leq |y|/4.$$

This shows that Ω is g-invariant and that g^n uniformly converges to $q_\infty = (0,0)$ on Ω. Coming back to the canonical chart $\mathbb{C}^2 = (t = 1)$, we now have that

$$\Omega' = \Omega \cap \mathbb{C}^2 = \{(z,w) \in \mathbb{C}^2 \mid 4|z| \leq |w| \text{ and } |w| \geq \min(|b|, |4c'|, 16)\}$$

is part of the basin of attraction of the point q_∞. It remains to check that $q^- = (b/a, b^2/a^2) \in \Omega'$, but this readily follows from our assumptions $|b| \geq 4|a|$ and $|a| < 1$. □

3.3. Ergodic Properties

We mention here some basic questions about ergodic properties of these two families. Let f be one of these mappings. Since f is algebraically stable in X ($\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$), there are two well-defined Green currents T_+ and T_- such that $(f^{\pm})^* T_\pm = \lambda_1(f) T_\pm$. The current T_+ has continuous potentials in $X \setminus I_{f^2}$, so $\mu_f := T_+ \wedge T_-$ is a well-defined invariant probability measure (if T_+, T_- are properly normalized) that is mixing [FGu] and hyperbolic [BD].

When $|c| < 1$ in the family 1.1 (resp., $|a| < 1$ in the family 1.2), then μ_f has maximal entropy of $\log \lambda_1(f)$ [Gu1]. If we further assume that q^- belongs to the basin of attraction of some attractive fixed point (see Lemmas 3.2, 3.4, and 3.5), then f is a biholomorphism in a neighborhood of Supp μ_f. In this case one can copy the work of Bedford, Lyubich, and Smillie [BLS1; BLS2] on complex Hénon mappings to obtain that μ_f is the unique measure of maximal entropy and that periodic saddle points are equidistributed with respect to μ_f. It seems that the latter still holds only if we assume $|c| < 1$ (resp., $|a| < 1$). However, it would be interesting to understand the kind of bifurcation that may occur when, for example, c is fixed, $|c| < 1$, and/or $|c'|$ varies (see Lemmas 3.2 and 3.4): Can q^- belong to Supp μ_f?

Finally we raise the following question.

QUESTION. Does μ_f always have maximal entropy $= \log \lambda_1(f)$?

4. Behavior at Infinity When $d_t > \lambda_1$

Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a dominating polynomial mapping with $d_t > \lambda_1(f)$. Russakovskii and Shiffman [RSh] have proved that the sequences of probability measures $d_t^{-n}(f^n)^*\Theta$ converge toward the same limit measure μ_f. Here Θ denotes any smooth probability measure in $\mathbb{C}^2$. Our goal is to prove that μ_f has compact support in $\mathbb{C}^2$ when f is quadratic. Note that this is obvious when infinity is f-attracting and in particular when $DL_\infty(f) > 1$ (i.e., for mappings in the families 3.1, 3.3, and 3.5). For the two remaining classes, we will show that infinity is indeed attracting for an open set of parameters and that it is attracting "on the average" for remaining values of the parameters.

4.1. A Criterion of Compactness

The following proposition was inspired by a result of Douady [Do] that concerns the Newton method for solving quadratic equations in $\mathbb{C}^2$.

PROPOSITION 4.1. *Let $f: \mathbb{C}^2 \to \mathbb{C}^2$ be a proper polynomial mapping such that $d_t(f) > \lambda_1(f)$. Let g_i denote the inverse branches of f, and assume*

$$\log^+\|g_i(p)\| \le \alpha_i \log^+\|p\| + C \quad \forall p \in \mathbb{C}^2,$$

where $C, \alpha_i > 0$ and $\sum_{i=1}^{d_t(f)} \alpha_i < d_t(f)$.

Then the Russakovskii–Shiffman measure μ_f has compact support in $\mathbb{C}^2$.

Proof. Fix ρ such that $d_t^{-1}\sum \alpha_i < \rho < 1$ and $R_0 > 0$ large enough. Let ν be a probability measure in $\mathbb{C}^2$ such that

$$H_\nu(r) := \nu(\log^+\|p\| > r) \le C_0(1/r) \quad \text{for } r \ge R_0. \tag{$**$}$$

Set $\nu_n := d_t^{-n}(f^n)^*(\nu)$. We claim that

$$H_{\nu_n}(r) := \nu_n(\log^+\|p\| > r) \le \rho^n C_0(1/r) \quad \text{for } r \ge R_0.$$

This clearly implies the proposition, since every smooth probability measure ν with support in the ball of radius e^{R_0} satisfies $(**)$ and $\nu_n \to \mu_f$, so μ_f will be supported on the ball of radius e^{R_0}.

Let $h_i(r) := H_{(g_i)_*\nu}(r)$. Observe that

$$\log^+\|g_i(p)\| > r \implies \log^+\|p\| > \frac{r-C}{\alpha_i},$$

so $h_i(r) \le H_\nu((r-C)/\alpha_i)$. We may thus deduce that

$$H_{\nu_1}(r) = \frac{1}{d_t}\sum_{i=1}^{d_t} h_i(r) \le \frac{C_0}{r}\frac{1}{d_t}\sum_{i=1}^{d_t}\alpha_i\frac{r}{r-C} \le \rho\frac{C_0}{r}$$

if $r \ge R_0$ for R_0 large enough. A straighforward induction yields the claim. $\square$

REMARK 4.2. One may expect that the Russakovskii–Shiffman measure is always compactly supported in $\mathbb{C}^2$ when f is proper. Here is a heuristic argument to support this conjecture. Let α_i denote the mass of $(g_i)^*\omega$ in $\mathbb{C}^2$. Passing to an iterate we may assume $\delta_1(f) < d_t(f)$, so

$$\sum \alpha_i = \sum \int_{\mathbb{C}^2} (g_i)^*\omega \wedge \omega = \int_{\mathbb{C}^2} f_*\omega \wedge \omega = \delta_1(f) < d_t(f).$$

On the other hand, it is well known from pluripotential theory that the mass of $(g_i)^*\omega$ precisely controls the growth of $\log^+\|g_i\|$.

It should be noted that examples of polynomial mappings of $\mathbb{C}^2$ with noncompactly supported Russakovskii–Shiffman measure are given in [FGu], but these are nonproper mappings.

4.2. Family 3.2

We consider here mappings

$$f(z,w) = (aw + c, z[z-w] + c'), \quad \text{where } a \ne 0.$$

LEMMA 4.3. *If $|a| > 1$ then infinity is f-attracting.*

Proof. Assume $|a| = 1 + 2t$, $t > 0$. Set $V_\varepsilon = \{(z,w) \in \mathbb{C}^2 \mid \max(|z|,|w|) > 1/\varepsilon\}$. The lemma will follow from the existence of $\varepsilon_0 > 0$ such that

$$0 < \varepsilon < \varepsilon_0 \implies f(V_\varepsilon) \subset V_{\varepsilon/(1+t)}.$$

Fix $(z,w) \in V_\varepsilon$ and set $(z',w') = f(z,w)$. If $|w| = \max(|z|,|w|) > \varepsilon^{-1}$, then

$$|z'| = |aw + c| \geq (1+2t)|w| - |c| > \frac{1+3t/2}{\varepsilon} \quad \text{if } 0 < \varepsilon < \varepsilon_1.$$

So assume $|z| = \max(|z|,|w|) > \varepsilon^{-1}$. Either $|z - w| \geq 1 + 2t$ and so $|w'| \geq (1+2t)|z| - |c'| > (1+t)/\varepsilon$ for $0 < \varepsilon < \varepsilon_2$; or $|z - w| < 1 + 2t$, in which case $|w| > (1+t)/(1+3t/2)\varepsilon^{-1}$ yields $|z'| > (1+t)/\varepsilon$ for $0 < \varepsilon < \varepsilon_3$. We obtain the desired inclusion by choosing $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. □

We now consider the remaining cases $0 < |a| \leq 1$. Recall that $d_t(f) = 2$. Since f is proper, there are two well-defined inverse branches of f in $\mathbb{C}^2$, which we denote by g^+, g^-, ordered so that if $g^\pm(x,y) = (z^\pm, w^\pm)$ then $|z^+| \geq |z^-|$.

LEMMA 4.4. *There exists a $C > 0$ such that, for all $(x,y) \in \mathbb{C}^2$,*

$$\log^+\|g^+ \circ g^+(x,y)\| \leq \log^+\|(x,y)\| + C,$$
$$\log^+\|g^+ \circ g^-(x,y)\| \leq \tfrac{1}{2}\log^+\|(x,y)\| + C,$$
$$\log^+\|g^- \circ g^+(x,y)\| \leq \tfrac{1}{2}\log^+\|(x,y)\| + C,$$
$$\log^+\|g^- \circ g^-(x,y)\| \leq \tfrac{1}{4}\log^+\|(x,y)\| + C.$$

Therefore, μ_f has compact support in $\mathbb{C}^2$.

Proof. Fix $(x,y) \in \mathbb{C}^2$. The two preimages of (x,y) satisfy $w = (x-c)/a$ and $z^2 - (x-c)z/a + (c'-y) = 0$. From $|z^+z^-| = |c'-y|$ we get $|z^-| \leq |c'-y|^{1/2}$, hence $|z^-| \leq C_1 \max(|y|^{1/2}, 1)$. Since $|z^+ + z^-| = |x-c|/|a|$, it follows that $|z^+| \leq C_2 \max(|x|, |y|^{1/2}, 1)$. Finally, $|w^\pm| = |x-c|/|a| \leq C_3 \max(|x|, 1)$. Iterating these inequalities yields the lemma.

It follows from Proposition 4.1 that μ_f has compact support in $\mathbb{C}^2$, since here $\sum \alpha_i = 9/4 < 4 = d_t(f^2)$. □

4.3. Family 3.4

We consider here mappings of the form

$$f(z,w) = (zw + c, z[z + \alpha w] + bz + c' + aw), \quad \text{where } a \neq 0.$$

LEMMA 4.5. *If $|a| > 1$ then infinity is f-attracting.*

Proof. Define $t > 0$ by $|a| = 1 + 3t$ and fix $\lambda > 0$ small enough so that $|\alpha\lambda| < t$. For technical reasons we first conjugate f by $(z,w) \mapsto (\lambda z, w)$. Thus we will show that infinity is attracting for g, where

$$g(z,w) = (zw + c_1, z[\lambda^2 z + \alpha\lambda w] + \alpha\lambda z + c_2 + aw).$$

Set $V_\varepsilon := \{(z,w) \in \mathbb{C}^2 \mid \max(|z|,|w|) > 1/\varepsilon\}$. It is clearly sufficient to show the existence of $\varepsilon_0 > 0$ such that $g(V_\varepsilon) \subset V_{\varepsilon/(1+t)}$ for $0 < \varepsilon < \varepsilon_0$. Pick $(z,w) \in V_\varepsilon$ and set $(z',w') = g(z,w)$.

Assume first that $|z| = \max(|z|,|w|) > 1/\varepsilon$. Then $|z'| \geq |w||z| - |c_1| > (1+t)/\varepsilon$ if $|w| \geq 1+2t$ and $0 < \varepsilon < \varepsilon_1$. Now, if $|w| \leq 1+2t$ then $|w'| \geq \lambda^2|z|^2/2 > (1+t)/\varepsilon$ for $0 < \varepsilon < \varepsilon_2$, so $(z',w') \in V_{\varepsilon/(1+t)}$ in both cases.

Assume now $|w| = \max(|z|,|w|) > 1/\varepsilon$. Then $|z'| \geq (1+t)/\varepsilon$ if $|z| \geq 1+2t$ and $0 < \varepsilon < \varepsilon_3$. Now, if $|z| \leq 1+2t$, we obtain

$$|w'| \geq (|a| - |\alpha\lambda|)|w| - C \geq (1+2t)|w| - C > \frac{1+t}{\varepsilon} \quad \text{if } 0 < \varepsilon < \varepsilon_4.$$

The desired inclusion follows with $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$. □

We now consider the case $0 < |a| \leq 1$.

LEMMA 4.6. *Let f be as before. Denote by g_1, g_2, g_3 the three inverse branches of f ordered so that, if $g_i(x,y) = (z_i, w_i)$, then $|z_1| \leq |z_2| \leq |z_3|$. Then there exists a $C > 0$ such that, for all $(x,y) \in \mathbb{C}^2$:*

$$\begin{aligned}
\log^+\|g_1(x,y)\| &\leq \log^+\|(x,y)\| + C,\\
\log^+\|g_2(x,y)\| &\leq \tfrac{2}{3}\log^+\|(x,y)\| + C,\\
\log^+\|g_3(x,y)\| &\leq \tfrac{2}{3}\log^+\|(x,y)\| + C.
\end{aligned}$$

Therefore, μ_f has compact support in $\mathbb{C}^2$.

Proof. We fix $R_0 = R_0(a,b,c,c',\alpha) \gg 1$. In order to simplify notation, we will denote by $\lesssim$ an inequality $\leq$ that holds true up to a constant that depends only on the parameters a,b,c,c',α. Without loss of generality we may assume $(x,y) \in \mathbb{C}^2$ are such that $\max(|x|,|y|) > R_0$.

Let (z_i,w_i), $1 \leq i \leq 3$, be the solutions of $f(z,w) = (x,y)$ ordered so that $|z_3| \geq |z_2| \geq |z_1|$. Observe that $zw = x - c$; hence

$$z^3 + bz^2 + [\alpha(x-c) + c' - y]z + a(x-c) = 0 = (z-z_1)(z-z_2)(z-z_3).$$

From $|z_1z_2z_3| = |a(x-c)|$ it follows that

$$|z_1| \leq |a(x-c)|^{1/3} \leq |z_3|. \tag{1}$$

Assume $|x| > R_0$. Then, for R_0 chosen large enough, using $|z_1+z_2+z_3| = |b|$ yields

$$\tfrac{1}{2}|a(x-c)|^{1/3} \leq |z_2|. \tag{2}$$

Indeed, otherwise $|z_1| \leq |z_2| \leq |a(x-c)|^{1/3}/2$ yields $|z_3| \geq 4|a(x-c)|^{1/3}$ and hence $|b| = |z_1+z_2+z_3| \geq 3|a(x-c)|^{1/3}$, contradicting $|x| > R_0$. From $w_i = (x-c)/z_i$ we infer that

$$|w_3| \leq \frac{|x-c|}{|a(x-c)|^{1/3}} \lesssim \max(|x|^{2/3}, 1) \tag{3}$$

and

$$|w_2| \leq 2\frac{|x-c|}{|a(x-c)|^{1/3}} \lesssim \max(|x|^{2/3}, 1). \tag{4}$$

We now give a bound from above for $|w_1|$. Recall that $z_1 + \alpha(x-c) + bz_1 + c' + aw_1 = y$. Thus

$$|w_1| \le \frac{1}{|a|}(|y| + |\alpha(x-c)| + |c'| + |bz_1| + |z_1|^2) \lesssim \max(\|(x,y)\|, 1), \tag{5}$$

where the last inequality follows from (1). Note finally that z_3 is one of the solutions of $z^2 + bz + [c' + aw_3 - y + \alpha(x-c)] = 0$. As a result, $|z_3| \lesssim \max(|b|, |c' + aw_3 - y + \alpha(x-c)|^{1/2})$. Together with (3) this yields

$$|z_2| \le |z_3| \lesssim \max(\|(x,y)\|^{1/2}, 1). \tag{6}$$

This gives the lemma when $|x| > R_0$, so assume now that $|y| > R_0 \ge |x|$. Without loss of generality we may actually assume $|y| > R_0^2 \gg R_0 \ge |x|$. There only remains to show $|z_2| \ge \frac{1}{2}|a(x-c)|^{1/3}$. Assume the contrary; then

$$|y| \sim |\alpha(x-c) + c' - y| = |z_1 z_2 + z_1 z_3 + z_2 z_3| \lesssim |y|^{1/2+1/3}$$

by (6), a contradiction.

Using the notation of Proposition 4.1, we have $\sum \alpha_i = 7/3 < 3 = d_t(f)$ and hence μ_f has compact support in $\mathbb{C}^2$. □

5. The Russakovskii–Shiffman Measure

Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a dominating polynomial mapping such that $d_t > \lambda_1(f)$. Following [Gu2] we give in this section an elementary construction of the Russakovskii–Shiffman measure μ_f. When infinity is f-attracting, we then show that every plurisubharmonic function is in $L^1(\mu_f)$. This is stronger than the general result proved in [Gu1] that every quasi-plurisubharmonic function on $\mathbb{P}^2$ is in $L^1(\mu_f)$.

CONSTRUCTION OF μ_f. Let $a \in \mathbb{C}^2$ be a noncritical value of f and let Θ be a smooth probability measure supported near a. Then $d_t^{-1} f^*\Theta$ is again a smooth probability measure with compact support in $\mathbb{C}^2$. Thus Θ and $d_t^{-1} f^*\Theta$ are cohomologous when viewed as global smooth forms of maximal bidegree on $\mathbb{P}^2$. Hence there exists a smooth form $\mathcal{T}$ of bidegree $(1,1)$ on $\mathbb{P}^2$ such that

$$\frac{1}{d_t} f^*\Theta = \Theta + dd^c \mathcal{T}. \tag{†}$$

Adding some multiple of the Fubini–Study form ω, we can further assume that $0 \le \mathcal{T} \le C\omega$ for some constant $C > 0$. Pulling back (†) by f^n yields

$$\frac{1}{d_t^n}(f^n)^*\Theta = \Theta + dd^c \mathcal{T}_n, \quad \text{where } \mathcal{T}_n = \sum_{j=0}^{n-1} \frac{1}{d_t^j}(f^j)^*(\mathcal{T}). \tag{††}$$

The sequence $(\mathcal{T}_n)$ is an increasing sequence of positive currents of bidegree $(1,1)$ on $\mathbb{P}^2$ such that

$$0 \le \mathcal{T}_n \le C \sum_{j=0}^{n-1} \frac{1}{d_t^j}(f^j)^*\omega.$$

The latter series is convergent, since $(f^j)^*\omega$ has mass $\delta_1(f^j) \le [\lambda_1(f) + \varepsilon]^j$ for $j \ge j_\varepsilon$ and $\varepsilon > 0$ small enough that $d_t(f) > \lambda_1(f) + \varepsilon$. Therefore, $\mathcal{T}_n$ converges toward some positive current $\mathcal{T}_\infty$. This yields

$$\frac{1}{d_t^n}(f^n)^*\Theta = \Theta + dd^c\mathcal{T}_n \to \mu_f := \Theta + dd^c\mathcal{T}_\infty.$$

Observe that, if Θ' is any other smooth probability measure, then $\Theta' = \Theta + dd^cS$ for some smooth $(1,1)$-form S on $\mathbb{P}^2$, so

$$\frac{1}{d_t^n}(f^n)^*\Theta' = \frac{1}{d_t^n}(f^n)^*\Theta + dd^c\left(\frac{1}{d_t^n}(f^n)^*S\right) \to \mu_f$$

because $\|(f^n)^*S\| = \delta_1(f^n) = o(d_t^n)$. In particular $d_t^{-n}(f^n)^*\omega^2 \to \mu_f$.

REMARK 5.1. Assume that infinity is an attracting set for f in the following sense: there exists a neighborhood V of infinity in $\mathbb{C}^2$ such that $\bigcap_{j\ge1} f^j(V) = \emptyset$. In this case we get $\mathbb{C}^2 = K^+ \cup \mathcal{B}^+(\infty)$, where $K^+ = \{a \in \mathbb{C}^2 \mid (f^n(a))_{n\ge0} \text{ is bounded}\}$ is a compact subset of $\mathbb{C}^2$ and $\mathcal{B}^+(\infty)$ denotes the basin of attraction of infinity, $\mathcal{B}^+(\infty) = \bigcup_{n\ge0} f^{-n}(V)$. The measure μ_f is supported on the compact set ∂K^+ in this case. Infinity is always an attracting set for f when $DL_\infty(f) > 1$, but it may also be attracting when $DL_\infty(f) = 1$ as shown in Section 4.

An alternative construction of μ_f was given in [Gu1] under the more restrictive assumption that $DL_\infty(f) = d_t(f)/\lambda_1(f)$.

THEOREM 5.2. *Let* $f : \mathbb{C}^2 \to \mathbb{C}^2$ *be a dominating polynomial mapping such that* $d_t(f) > \lambda_1(f)$. *Assume that* (a) μ_f *has compact support and* (b) *either infinity is* f*-attracting or* $d_t(f) > \lambda_1(f)^{3/2}$. *Then every plurisubharmonic function is in* $L^1(\mu_f)$.

Proof. Let B be a ball in $\mathbb{C}^2$ containing $\operatorname{Supp}\mu_f$ and let φ be a plurisubharmonic function near $\bar{B}$. Without loss of generality, $\varphi \le 0$ on B. Let $\chi \ge 0$ be a test function in B such that $\chi \equiv 1$ near B_1 and $\operatorname{Supp}\mu_f \subset B_1 \subset\subset B$. Then

$$0 \le \int_{B_1} (-\varphi)\, d\mu_f = \int_{B_1} (-\varphi)\Theta + \int_{B_1} (-\varphi)\, dd^c(\chi\mathcal{T}_\infty).$$

Because Θ is smooth, we need only derive an upper bound on the second integral, which by Stokes's theorem reads

$$I = -\int_B \chi\mathcal{T}_\infty \wedge dd^c\varphi + \int_{B\setminus B_1} \varphi\, dd^c(\chi\mathcal{T}_\infty) = I' + I''.$$

Note that $I' \le 0$ because φ is plurisubharmonic, so we only need to obtain an upper bound on I''. Observe that $dd^c(\chi\mathcal{T}_\infty) = dd^c\chi \wedge \mathcal{T}_\infty + 2d\chi \wedge d^c\mathcal{T}_\infty + \chi dd^c\mathcal{T}_\infty$. Since $\mu_f = 0$ in $B \setminus B_1$, it follows that $\chi dd^c\mathcal{T}_\infty = -\chi\Theta$ is smooth in $B \setminus B_1$. It is therefore sufficient to get control of

$$I_1 = \int_{B\setminus B_1} \varphi\, d\chi \wedge d^c\mathcal{T}_\infty \quad \text{and} \quad I_2 = \int_{B\setminus B_1} \varphi\, dd^c\chi \wedge \mathcal{T}_\infty.$$

Since $\mathcal{T}_\infty$ is positive, we have

$$|I_2| \leq \|\chi\|_{C^2} \int_B (-\varphi)\omega \wedge \mathcal{T}_\infty \leq C_1 \sum_{j\geq 0} \int_B (-\varphi)\omega \wedge \frac{1}{d_t^j}(f^j)^*\omega.$$

Because $d_t > \lambda_1(f)$, we have $d_t^l > \delta_1(f^l)$ for l large enough. We assume for simplicity that $l = 1$ and set $d = \delta_1(f) < d_t(f)$. Now $(f^j)^*\omega = d^j\, dd^c G_j^+$ in $\mathbb{C}^2$, where G_j^+ is locally uniformly bounded in $\mathbb{C}^2$. It thus follows from Chern–Levine–Nirenberg inequalities (see [Si]) that

$$|I_2| \leq C_2 \|\varphi\|_{L^1(B_2)} \sum_{j\geq 0} \left(\frac{d}{d_t}\right)^j < +\infty,$$

where B_2 is a slightly larger ball than B.

It remains to gain control of I_1. We decompose $\mathcal{T} = \sum \mathcal{T}_{ij}\, dz_i \wedge d\bar{z}_j$ in $\mathbb{C}^2$, where the $\mathcal{T}_{ij}$ are smooth functions. By the Cauchy–Schwarz inequality,

$$\begin{aligned}
&\left| \int_{B\setminus B_1} (-\varphi)\, d\chi \wedge (f^n)^* d^c \mathcal{T} \right| \\
&\quad \leq \sum_{i,j} \left| \int_{B\setminus B_1} (-\varphi)\, d\chi \wedge (f^n)^*(d^c \mathcal{T}_{ij} \wedge dz_i \wedge d\bar{z}_j) \right| \\
&\quad \leq \sum_{i,j} \left| \int_{B\setminus B_1} (-\varphi)\, d\chi \wedge d^c\chi \wedge (f^n)^*(dz_i \wedge d\bar{z}_j) \right|^{1/2} \\
&\qquad \cdot \left| \int_{B\setminus B_1} (-\varphi)(f^n)^*(d\mathcal{T}_{ij} \wedge d^c\mathcal{T}_{ij} \wedge dz_i \wedge d\bar{z}_j) \right|^{1/2} \\
&\quad \leq C_3 \left[\int_{B\setminus B_1} (-\varphi)\omega \wedge (f^n)^*\omega \right]^{1/2} \cdot \left[\int_{B\setminus B_1} (-\varphi)(f^n)^*\omega^2 \right]^{1/2}.
\end{aligned}$$

When infinity is f-attracting, we can assume that $B \setminus B_1$ is a relatively compact subset of the basin of attraction of infinity. Therefore $\frac{1}{2}\log[1 + \|f^n\|^2] = \log\|f^n\| + u_n$, where u_n is uniformly bounded on $B \setminus B_1$. Thus $(f^n)^*(\omega^2) = (dd^c u_n)^2 + 2\, dd^c \log\|f^n\| \wedge dd^c u_n$ yields (again by Chern–Levine–Nirenberg inequalities)

$$0 \leq \int_{B\setminus B_1} (-\varphi)(f^n)^*\omega^2 \leq C_4 d^n$$

for some constant C_4 independent of n. On the other hand, $(f^n)^*\omega = d^n\, dd^c G_n^+$ with G_n^+ uniformly bounded on $B \setminus B_1$. This shows

$$\left| \int_{B\setminus B_1} (-\varphi)\, d\chi \wedge (f^n)^*(d^c \mathcal{T}) \right| \leq C_5\, d^n.$$

Therefore, $|I_1| \leq C_5 \sum_{j\geq 0} (d/d_t)^j < +\infty$.

When infinity is not f-attracting, we can still get an upper bound

$$0 \leq \int_{B\setminus B_1} (-\varphi)(f^n)^*\omega^2 \leq C_4\, d^{2n},$$

so $|I_1| \leq C_5 \sum_{j\geq 0} (d^{3/2}/d_t)^j < +\infty$ if $d_t(f) > \lambda_1(f)^{3/2}$. □

REMARK 5.3. The main ergodic properties of μ_f are established in [Gu2]. It is mixing with positive Lyapunov exponents; repelling periodic points are equidistributed with respect to μ_f; and μ_f is the unique measure of maximal entropy $h_{\mu_f}(f) = h_{\text{top}}(f) = \log d_t(f)$.

6. Algebraicity of $\mathcal{E}_f$

Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a dominating polynomial mapping such that $d_t > \lambda_1(f)$. Russakovskii and Shiffman [RSh] have demonstrated the existence of a pluripolar set $\mathcal{E}_f \subset \mathbb{C}^2$ such that

$$\frac{1}{d_t^n}(f^n)^*\varepsilon_a \to \mu_f \quad \forall a \in \mathbb{C}^2 \setminus \mathcal{E}_f.$$

Here ε_a denotes the Dirac mass at point a. Following Briend and Duval [BrDu], we show here that $\mathcal{E}_f$ is actually algebraic when f is quadratic.

We denote by $\deg_p f$ the local topological degree of f at p, that is, the number of points in $f^{-1}(q)$ that are close to p when q is close to $f(p)$. Hence $\deg_p f > 1$ if and only if p belongs to the critical set $\mathcal{C}_f$ of f. For an irreducible algebraic curve A of $\mathbb{C}^2$, we set $\deg_A f = \min_{p\in A} \deg_p f = \deg_p f$ for a generic point $p \in A$. When $A = \bigcup A_i$ is not irreducible, we set $\deg_A f = \max_i \deg_{A_i} f$.

LEMMA 6.1. *Let f, g be two proper polynomial self-mappings of $\mathbb{C}^2$. Then the following statements hold.*

(1) $\deg_p(f \circ g) = \deg_p g \cdot \deg_{g(p)} f$, *hence* $\deg_A(f \circ g) = \deg_A g \cdot \deg_{g(A)} f$.
(2) $\deg_{\mathcal{C}_{f\circ g}}(f \circ g) \le \deg_{\mathcal{C}_g} g \cdot \deg_{\mathcal{C}_f} f$.
(3) $1 \le \deg_p f \le d_t(f)$.
(4) $1 \le \deg_A f \le \deg_{\mathcal{C}_f} f \le \delta_1(f)$.
(5) *Assume $d_t(f) > \delta_1(f)$; if $\deg_{f^j(p)} f = d_t(f)$ for all $j \ge 0$, then p is periodic and the corresponding cycle is totally invariant.*

Proof. Assertion (1) is a straightforward consequence of the definition. We refer the interested reader to [GrH, Chs. 5.1 & 5.2] for further details on local topological degree. The chain rule yields $\mathcal{C}_{f\circ g} = \mathcal{C}_g \cup g^{-1}(\mathcal{C}_f)$. Therefore,

$$\begin{aligned}\deg_{\mathcal{C}_{f\circ g}}(f \circ g) &= \max(\deg_{\mathcal{C}_g}(f \circ g), \deg_{g^{-1}(\mathcal{C}_f)}(f \circ g)) \\ &= \max(\deg_{\mathcal{C}_g} g \cdot \deg_{g(\mathcal{C}_g)} f, \deg_{g^{-1}(\mathcal{C}_f)} g \cdot \deg_{\mathcal{C}_f} f) \\ &\le \deg_{\mathcal{C}_g} g \cdot \deg_{\mathcal{C}_f} f.\end{aligned}$$

Assertion (3) is clear and (4) follows easily from Bezout's theorem (see [BrDu]).

It follows from (4) that the set $E = \{p \in \mathbb{C}^2 \mid \deg_p f = d_t(f)\}$ is finite when $d_t(f) > \delta_1(f)$. So if $\deg_{f^j(p)} f = d_t(f)$ for all $j \ge 0$ then p is preperiodic to a cycle in E. To simplify we assume $f^n(p) = q$ with $q = f(q) \in E$. Now $f^{-1}(q)$ contains q with multiplicity d_t, so $f^*\varepsilon_q = d_t\varepsilon_q$ and hence q is totally invariant. This shows that $p = q$, so p is periodic and the corresponding cycle is totally invariant. □

Note in particular that $\deg_{\mathcal{C}_{f^j}} f^j$ is submultiplicative. We can therefore define the asymptotic critical degree $\mathcal{T}(f)$ of f by

$$\mathcal{T}(f) := \lim_{j\to+\infty} (\deg_{\mathcal{C}_{f^j}} f^j)^{1/j}.$$

Observe that $\mathcal{T}(f) > 1$ implies strong recurrence of the critical set, so $\mathcal{T}(f) = 1$ "generically". This motivates the following proposition, which is a weak version of a result in [BrDu] on holomorphic endomorphisms.

PROPOSITION 6.2. *Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a proper polynomial mapping, and assume that $\lambda_1(f)\mathcal{T}(f) < d_t(f)$. Then the exceptional set $\mathcal{E}_f$, if nonempty, is finite and consists of totally invariant cycles.*

Proof. Replacing f by f^l if necessary, we can assume $\delta_1(f)\deg_{\mathcal{C}_f} f < d_t(f)$. Set $E = \{p \in \mathbb{C}^2 \mid \deg_p f = d_t(f)\}$. It follows from Lemma 6.1(4) that E is a finite set. Passing to an iterate if necessary, we can further assume E is totally invariant by using Lemma 6.1(5). We claim then $\mathcal{E}_f = E$. It is sufficient to prove $\mu_{n,p}(\mathcal{C}_f) \to 0$ for all $p \notin E$, where $\mu_{n,p} = d_t^{-n}(f^n)^*\varepsilon_p$.

Set $F = \{p \in \mathbb{C}^2 \mid \deg_p f > \deg_{\mathcal{C}_f} f\}$. Then $F \setminus E$ consists of finitely many points with degree $\le d_t - 1$. Let $\rho < 1$ be close to 1 (to be chosen later) and fix $p \in \mathbb{C}^2 \setminus E$. Since E is totally invariant, $f^{-n}(p) \cap E = \emptyset$ for all n. Hence $\mu_{n,p}(F) = \mu_{n,p}(F \setminus E) \le \sharp F (d_t - 1)^n / d_t^n$. Similarly,

$$\mu_{n,p}(F \cup f^{-1}(F) \cup \cdots \cup f^{-n\rho}(F)) \le \sum_{j=0}^{n\rho} \mu_{n-j,p}(F \setminus E) \le C\left(\frac{d_t - 1}{d_t}\right)^{n(1-\rho)}.$$

Following [BrDu], we now count the number of points in $f^{-n}(p) \cap \mathcal{C}_f$. It follows from Bezout's theorem that there are no more than $\delta_1(f^n)$ points (ignoring multiplicities). Points in $f^{-n}(p) \cap \mathcal{C}_f \setminus F \cup f^{-1}(F) \cup \cdots \cup f^{-n\rho}(F)$ have multiplicity bounded from above by $(\deg_{\mathcal{C}_f} f)^{n\rho}(d_t - 1)^{n(1-\rho)}$. Therefore,

$$\begin{aligned}\mu_{n,p}(\mathcal{C}_f) &\le \mu_{n,p}(F \cup \cdots \cup f^{-n\rho}(F)) + \delta_1(f^n)\frac{(\deg_{\mathcal{C}_f} f)^{n\rho}(d_t - 1)^{n(1-\rho)}}{d_t^n} \\ &\le C\left(\frac{d_t - 1}{d_t}\right)^{n(1-\rho)} + \left(\frac{\delta_1(f)\deg_{\mathcal{C}_f}}{d_t^{\rho}}\right)^n.\end{aligned}$$

Choosing $\rho < 1$ close enough to 1 yields $\mu_{n,p}(\mathcal{C}_f) \to 0$, so $\mu_{n,p} \to \mu_f$. □

We now check that the condition $\lambda_1(f)\mathcal{T}(f) < d_t(f)$ is satisfied for quadratic families.

LEMMA 6.3.

(1) *Let f be a mapping from family 3.1. Then* $\deg_{\mathcal{C}_{f^4}} f^4 = 2$.

(2) *Let f be a mapping from family 3.2. Then* $\deg_{\mathcal{C}_{f^5}} f^5 = 2$.

(3) *Let f be a mapping from family 3.3. Then* $\deg_{\mathcal{C}_{f^2}} f^2 = 2$.

(4) *Let f be a mapping from family 3.4. Then* $\deg_{\mathcal{C}_{f^2}} f^2 = 2$.

So in all cases, $\lambda_1(f)\mathcal{T}(f) < d_t(f)$.

REMARK 6.4. Mappings from family 3.5 extend as holomorphic endomorphisms of $\mathbb{P}^2$. It follows from [BrDu] that $\mathcal{E}_f$ is algebraic in this case. The condition

$\lambda_1(f)\mathcal{T}(f) < d_t(f)$ is not necessarily satisfied, and the set $\mathcal{E}_f$ may well be infinite. In the latter case, f (or f^2) is conjugate to $(z,w) \mapsto (z^d, Q(z,w))$ and so $(z=0) \subset \mathcal{E}_f$.

Proof of Lemma 6.3. (1) Consider $f(z,w) = (w, z^2 + aw + c')$. Then $\mathcal{C}_f = (z=0)$ and $\deg_{\mathcal{C}_f} = 2 = \delta_1(f)$. One easily checks that $f(\mathcal{C}_f), f^2(\mathcal{C}_f), f^3(\mathcal{C}_f)$ and $f^{-1}(\mathcal{C}_f), f^{-2}(\mathcal{C}_f), f^{-3}(\mathcal{C}_f)$ are all different from $\mathcal{C}_f$. Hence it follows from Lemma 6.1 that $\deg_{\mathcal{C}_{f^4}} f^4 = 2$, while $\delta_1(f^4) = 4$ and $d_t(f^4) = 16$.

Observe that $E = \{p \in \mathbb{C}^2 \mid \deg_p f^2 = 4 = d_t(f^2)\}$ is empty except when $a = 0$. If $a = 0$, then $E = \{(0,0)\}$ is totally invariant only when $c' = 0$. Thus $\mathcal{E}_f = \emptyset$ except when $a = c' = 0$, in which case $\mathcal{E}_f = \{(0,0)\}$.

(2) Consider $f(z,w) = (aw + c, z[z-w] + c')$ with $a \neq 0$. The critical set is $\mathcal{C}_f = \{w = 2z\}$. By induction, we easily get that

$$f^j(\mathcal{C}_f) = \{(A_j(\zeta), B_j(\zeta)) \in \mathbb{C}^2 \mid \zeta \in \mathbb{C}^2\},$$

where A_j and B_j are polynomials of degree $\deg A_j = d_{j-1}$ and $\deg B_j = d_j$ (resp.) with $d_{j+2} = d_{j+1} + d_j$. This shows that $f^j(\mathcal{C}_f) \neq \mathcal{C}_f$ for all $j \geq 1$. Similarly,

$$f^{-j}(\mathcal{C}_f) = \{w^{d_{j-1}} z^{d_j} (z-w)^{d_j} = R_j(z,w)\},$$

where R_j is a polynomial of degree $\deg R_j < 2d_j + d_{j-1}$. As a result, $f^{-j}(\mathcal{C}_f) \neq \mathcal{C}_f$ for $j \geq 1$. In particular we have $\delta_1(f^5) \cdot \deg_{\mathcal{C}_{f^5}} f^5 = 13 \cdot 2 < 32 = d_t(f^5)$.

(3) Consider $f(z,w) = (az^2 + bz + c + w, z[w + \alpha z] + c')$ with $a \neq 0$. The critical set $\mathcal{C}_f = \{w = 2az^2 + (b - 2\alpha)z\}$ is irreducible. We obtain $f^{-1}(\mathcal{C}_f) = \{w(1-z) = (\alpha - a)z^2 - bz + c' - c\} \neq \mathcal{C}_f$ and

$$f(\mathcal{C}_f) = \{(3a\zeta^2 + (2b - 2\alpha)\zeta, 2a\zeta^3 + (b-\alpha)\zeta^2 + c') \in \mathbb{C}^2 \mid \zeta \in \mathbb{C}\} \neq \mathcal{C}_f.$$

Therefore, $\deg_{\mathcal{C}_{f^2}} f^2 = 2$ and hence $\delta_1(f^2) \cdot \deg_{\mathcal{C}_{f^2}} f^2 = 8 < 9 = d_t(f^2)$.

(4) Consider $f(z,w) = (zw + c, z[z + \alpha w] + bz + c' + aw)$ with $a \neq 0$. The critical set $\mathcal{C}_f = \{aw = 2z^2 + bz\}$ is irreducible, and straightforward computations yield $f(\mathcal{C}_f) \neq \mathcal{C}_f \neq f^{-1}(\mathcal{C}_f)$, so $\deg_{\mathcal{C}_{f^2}} f^2 = 2$. □

REMARK 6.5. It is perhaps worth mentioning that pull-backs of Dirac masses are not everywhere well-defined when f is not proper. Consider, for example, $f(z,w) = (P(z), zw^2)$, where P is a polynomial of degree $\deg P = d \geq 3$. Then $d_t(f) = 2d > d = \lambda_1(f)$. The line $(z = 0)$ is contracted to the point $a = (P(0), 0)$, so pull-backs of Dirac masses at points $f^j(a)$ by f^n are not well-defined. This shows that the orbit of point a must be included in the exceptional set $\mathcal{E}_f$; hence we cannot expect the latter to be algebraic in general.

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