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$\mathbf{C}_+$-Actions on Contractible Threefolds

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1. Introduction

The aim of this paper is to generalize the theorem of Miyanishi [M1] stating that, for any nontrivial algebraic $\mathbf{C}_+$-action on $\mathbf{C}^3$, the algebraic quotient $\mathbf{C}^3/\!/\mathbf{C}_+$ is isomorphic to $\mathbf{C}^2$. Our main result is that, for a nontrivial algebraic $\mathbf{C}_+$-action on a smooth contractible affine algebraic threefold X, the algebraic quotient $X/\!/\mathbf{C}_+$ is isomorphic to a smooth contractible affine surface S. Since all such surfaces are rational [GS], we deduce that X is rational as well. Furthermore, if the action is free, then we conclude that X is isomorphic to $S \times \mathbf{C}$ and that the action is induced by translation on the second factor by virtue of [K3], where this result was proved under the additional assumption that S is smooth. Another consequence of our main result is that, when X admits a dominant morphism from a threefold of form $C \times \mathbf{C}^2$, the quotient S is isomorphic to $\mathbf{C}^2$. We also give an independent proof of the latter fact that (unlike our main result) does not use the difficult theorem of Taubes [T] about the absence of simply connected homology cobordisms between certain homology spheres. In fact, the rationality of X can also be proved without this theorem; however, this would require another difficult theorem that all logarithmic $\mathbf{Q}$-homology planes are rational [PS; GPS; GP]. In conclusion, we derive the following criterion: If there is a free algebraic $\mathbf{C}_+$-action on a smooth contractible affine algebraic threefold X that admits a dominant morphism from $C \times \mathbf{C}^2$, then X is isomorphic to $\mathbf{C}^3$.

2. The Main Result

Let $\rho\colon X \to S$ be the quotient morphism of a nontrivial algebraic $\mathbf{C}_+$-action on a smooth contractible affine algebraic threefold X. By Fujita's result, X is factorial (see e.g. [K1]). Some other properties of $\rho\colon X \to S$ proved in [K3, Lemma 2.1, Prop. 3.2, Rem. 3.3] are summarized in the following lemma.

Lemma 2.1.

(1) *The surface S is affine and factorial, and $\rho^{-1}(s)$ is a nonempty curve for every $s \in S$.*

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(2) *There is a curve* Γ *in* S *such that* $\breve{S} = S \setminus \Gamma$ *is smooth and* $\rho^{-1}(\breve{S})$ *is naturally isomorphic to* $\breve{S} \times \mathbf{C}$, *so that the projection onto the first factor corresponds to* ρ.

LEMMA 2.2. *In the foregoing notation, let* S^* *be the smooth part of the quotient* $S = X//\mathbf{C}_+$. *Then the groups* $\pi_1(S^*)$ *and* $H_2(S^*)$ *are trivial.*

Proof. The set F of singular points of S is finite because S is factorial. According to Lemma 2.1, $L = \rho^{-1}(F)$ is a curve and hence $\pi_1(X \setminus L) = \pi_2(X \setminus L) = 0$.

Let γ be a loop in $S^* = S \setminus F$. After a small homotopy if necessary, we may assume that $\gamma \subset \breve{S}$ with $\breve{S}$ as in Lemma 2.1. Since $\rho^{-1}(\breve{S}) = \breve{S} \times \mathbf{C}$, the loop γ lifts to a loop $\gamma' \subset X \setminus L$. The loop γ' is homotopic to zero in $X \setminus L$, so γ is homotopic to zero in S^*. This shows that $\pi_1(S^*) = 0$.

Now, by the Hurewicz theorem, $H_2(S^*)$ is isomorphic to the second homotopy group of S^*. An element of $\pi_2(S^*)$ can be viewed as a continuous map Υ from the 2-sphere S^2 to S^*. Without loss of generality, one may assume that its image meets Γ at a finite number of general points and that $\{\zeta_1, \dots, \zeta_n\} = \Upsilon^{-1}(\Gamma)$ is finite. Let $\mathcal{S}_i$ be the germ of S at $\Upsilon(\zeta_i)$. According to [K3, Lemma 4.1], there is a germ $\mathcal{P}_i \subset X$ of a surface such that $\mathcal{S}_i$ is a homeomorphic image of $\mathcal{P}_i$ under ρ. Consider small discs Δ_i in S^2 centered at ζ_i. Put

$$\Upsilon_i = (\rho|_{\mathcal{P}_i})^{-1} \circ \Upsilon|_{\bar{\Delta}_i} \quad \text{and} \quad S_0^2 = S^2 \setminus \bigsqcup_{i=1}^{n} \Delta_i;$$

then $\Upsilon(S_0^2) \subset \breve{S}$. By the Tietze extension theorem, there is a continuous map $\Upsilon_0 \colon S_0^2 \to \breve{X} \simeq \breve{S} \times \mathbf{C}$ such that $\rho \circ \Upsilon_0 = \Upsilon|_{S_0^2}$ and $\Upsilon_0|_{\partial\Delta_i} = \Upsilon_i|_{\partial\Delta_i}$ for every $i = 1, \dots, n$. Hence Υ_0 and the Υ_i together define a continuous map $\Upsilon' \colon S^2 \to X \setminus L$ such that $\rho \circ \Upsilon' = \Upsilon$. Since $\pi_2(X \setminus L) = 0$, we see that $\pi_2(S^*)$ and hence $H_2(S^*)$ are trivial. $\square$

Let $s_1, \dots, s_k$ be the singular points of S. For each $i = 1, \dots, k$, there exists a neighborhood U_i of s_i in S such that U_i is an open cone over a closed connected oriented 3-manifold $\Sigma_i = \partial \bar{U}_i$. If $S \hookrightarrow \mathbf{C}^n$ is a closed embedding, one can find a closed ball $B \subset \mathbf{C}^n$ of sufficiently large radius such that, if $U_0 = S \setminus B$, then $S \setminus U_0$ is a deformation retract of S. Hence $S_0 := S \setminus \left(\bigsqcup_{i=0}^{k} U_i\right)$ is a deformation retract of S^*; in particular, $\pi_1(S_0) = H_2(S_0) = 0$. Let $\Sigma_0 = \partial \bar{U}_0$ and $\Sigma = \partial S_0$, so that $\Sigma = \bigsqcup_{i=0}^{k} \Sigma_i$.

LEMMA 2.3. *Let* Σ *be as before. Then* $H_1(\Sigma) = H_2(\Sigma) = 0$; *that is, each of the* $\Sigma_0, \dots, \Sigma_k$ *is a homology sphere. Moreover, the* 3*-cycles* $\Sigma_1, \dots, \Sigma_k$ *form a free basis of* $H_3(S_0) = \mathbf{Z}^k$.

Proof. Since $H_1(S_0) = H_2(S_0) = 0$ by Lemma 2.2, the exact homology sequence

$$\cdots \to H_3(S_0, \Sigma) \to H_2(\Sigma) \to H_2(S_0) \\ \to H_2(S_0, \Sigma) \to H_1(\Sigma) \to H_1(S_0)$$

implies that $H_1(\Sigma) = H_2(S_0, \Sigma)$. By Lefschetz duality, $H_2(S_0, \Sigma) = H^2(S_0)$. The latter group vanishes because $H^2(S_0) = \mathrm{Hom}(H_2(S_0), \mathbf{Z}) = 0$; see Lemma 2.2. By Poincaré duality, $H_2(\Sigma) = H_1(\Sigma) = 0$. Since $H_3(S_0, \Sigma) = H^1(S_0) = 0$ and $H_4(S_0, \Sigma) = H^0(S_0) = \mathbf{Z}$, it follows by extending the homology sequence to the left that $0 \to \mathbf{Z} \to H_3(\Sigma) \to H_3(S_0) \to 0$. This yields the last claim. $\square$

The following lemma is a special case of Satz 2.8 in [B].

LEMMA 2.4. *Let $\mathcal{S}$ be the germ of a normal surface at a point s, and let $\mathcal{P}$ be the germ of a smooth surface at a point p. Let $\psi : \mathcal{P} \to \mathcal{S}$ be a finite morphism such that $\psi^{-1}(s) = p$. Then s is at worst a quotient singularity.*

PROPOSITION 2.5.

(1) *For every nontrivial algebraic $\mathbf{C}_+$-action on a smooth contractible affine algebraic threefold X, the quotient $S = X//\mathbf{C}_+$ has at worst quotient singularities of type $x^2 + y^3 + z^5 = 0$.*
(2) *S is contractible.*
(3) *If the Kodaira logarithmic dimension $\bar{\kappa}(S^*)$ of S^* is 1 then S is smooth, and if $\bar{\kappa}(S^*) = -\infty$ then $S \simeq \mathbf{C}^2$.*

Proof. We know from Lemma 2.1 that $\rho \colon X \to S$ is surjective and that the fibers of ρ are curves. Therefore, we can choose a germ $\mathcal{P}$ of a smooth surface at a smooth point p of $\rho^{-1}(s)$ (where $s \in S$) that is transversal to the curve $\rho^{-1}(s)$. The restriction of ρ to $\mathcal{P}$ yields a finite morphism $\psi : \mathcal{P} \to \mathcal{S}$, where $\mathcal{S}$ is the germ of S at s. By Lemma 2.4, s is at most a quotient singularity; in particular, its local fundamental group is finite. On the other hand, by Lemma 2.3, the local first homology group at s is trivial. Therefore, the local fundamental group is perfect. The only quotient singularity whose fundamental group is perfect is E_8—that is, it is of the type $x^2 + y^3 + z^5 = 0$ (see [B]).

To prove the second statement, note that $\pi_1(S) = 0$ because $\pi_1(S^*) = 0$ by Lemma 2.2. The statement will follow from the Whitehead and Hurewicz theorems as soon as we show that $H_2(S) = 0$ (since we already know that $H_i(S) = 0$ for $i \geq 3$; see [N]). Let U_i, Σ_i, and S_0 be as defined just before Lemma 2.3, $U^0 = \bigsqcup_{i=1}^k U_i$, and $\Sigma^0 = \bigsqcup_{i=1}^k \Sigma_i$. Then $S \setminus U_0 = S_0 \cup \bar{U}^0$ and $\Sigma^0 = S_0 \cap \bar{U}^0$. Recall that each U_i is contractible for $i \geq 1$; in particular, $H_2(\bar{U}^0) = 0$. Then $H_2(S_0) = 0$ by Lemma 2.2 and $H_1(\Sigma^0) = 0$ by Lemma 2.3. The Mayer–Vietoris sequence now implies that $H_2(S \setminus U_0) = 0$ and therefore $H_2(S) = 0$, since $S \setminus U_0$ is a deformation retract of S.

If $\bar{\kappa}(S^*) = 1$ then any singularity of S must be cyclic quotient [GM]; hence S is smooth by virtue of (1). For $\bar{\kappa}(S^*) = -\infty$, the only logarithmic contractible surfaces with at worst E_8-type singularities are $\mathbf{C}^2$ or the surface $x^2 + y^3 + z^5 = 0$ in $\mathbf{C}^3$ (see [MSu, Thm. 2.7]). The second possibility should be eliminated because $\pi_1(S^*) \neq 0$, contrary to Lemma 2.2. This implies (3). $\square$

COROLLARY 2.6. *Every smooth contractible affine algebraic threefold with a nontrivial algebraic $\mathbf{C}_+$-action is rational.*

Proof. According to Proposition 2.5, surface S is contractible logarithmic (i.e., it has at worst quotient singularities) and hence rational by [GPS; PS; GP]. Therefore, $S \times \mathbf{C}$ is rational, and so is X by virtue of Lemma 2.1(2). □

THEOREM 2.7. *For every nontrivial algebraic $\mathbf{C}_+$-action on a smooth contractible affine algebraic threefold X, the quotient $S = X//\mathbf{C}_+$ is a smooth contractible affine surface.*

Proof. Let S_0, Σ, and Σ_i be as defined before Lemma 2.3. Assume first that S has only one singular point. Then the boundary Σ of S_0 consists of two components. One component is Σ_1, which (according to Proposition 2.5) is the link of singularity at 0 of $x^2 + y^3 + z^5 = 0$. The manifold Σ_1 is also known as the Poincaré homology sphere. The other component is Σ_0, which is also a homology sphere by Lemma 2.3. Lemmas 2.2 and 2.3 imply that $\pi_1(S_0) = 0$ and that the embeddings $\Sigma_0 \hookrightarrow S_0$ and $\Sigma_1 \hookrightarrow S_0$ induce isomorphisms in homology. Thus S_0 is a simply connected homology cobordism between Σ_1 and Σ_0. But this contradicts the Taubes theorem ([T]; see also [FSt, Thm. 5.2]), which states that the Poincaré homology sphere cannot be homology cobordant to any homology sphere via a simply connected homology cobordism.

To complete the proof, it is enough to consider the case of two singular points; the general case will follow by a similar argument. If S has two singular points, then Σ is a disjoint union of Σ_0, Σ_1, and Σ_2. Let us join a point $x_0 \in \Sigma_0$ with a point $x_2 \in \Sigma_2$ by a path γ in S_0. Let V_2 and V_1 be tubular neighborhoods of γ in S_0 (i.e., each V_i is homeomorphic to $\gamma \times B_i$ where B_i is a three-dimensional ball, and V_i meets Σ_j, $j = 0, 2$, along the ball $x_j \times B_i$) such that int $V_2 \supset V_1$. Put $S_1 = S_0 \setminus V_1$. Then the boundary of S_1 consists of two components, Σ_1 and Σ', where Σ' is a connected sum of Σ_0 and Σ_2 (and hence is a homology sphere). Note that $\pi_1(S_1) = \pi_1(S_0 \setminus \gamma) = 0$ by the dimension argument. In order to show that we have a homology cobordism between Σ_1 and Σ' and thus get a contradiction with the Taubes theorem, we need only show that $H_2(S_1) = 0$ and that the 3-cycle Σ_1 generates $H_3(S_1) = \mathbf{Z}$.

The Mayer–Vietoris sequence of $S_0 = V_2 \cup S_1$ implies that $H_2(S_1)$ is the image of $H_2(V_2 \setminus V_1)$ under the natural embedding. Note that $x_2 \times (B_2 \setminus B_1)$ is a deformation retract of $V_2 \setminus V_1$. Therefore, every element of $H_2(S_1)$ can be represented by a 2-cycle in $x_2 \times (B_2 \setminus B_1) \subset \Sigma_2 \setminus (x_2 \times B_1)$. Since Σ_2 is a homology sphere, we conclude that $H_2(\Sigma_2 \setminus B_2) = 0$ and hence $H_2(S_1) = 0$. Because $H_3(V_2 \setminus V_1) = H_3(V_2) = 0$ and $H_2(V_2 \setminus V_1) = \mathbf{Z}$, applying once again the Mayer–Vietoris sequence yields the exact sequence $0 \to H_3(S_1) \to H_3(S_0) \to \mathbf{Z} \to 0$. Since $\{\Sigma_1, \Sigma_2\}$ is a free basis of $H_3(S_0)$ according to Lemma 2.3, we see that $H_3(S_1)$ is freely generated by Σ_1.

This leaves us with just one possibility: that S has no singular points and hence is smooth. That it is contractible was already proved in Proposition 2.5(2). □

COROLLARY 2.8. *Let X be a smooth contractible affine algebraic threefold with a nontrivial algebraic $\mathbf{C}_+$-action on it.*

(1) *If the action is free, then X is isomorphic to $S \times \mathbf{C}$ and the action is induced by a translation on the second factor.*

(2) *If X admits a dominant morphism from a threefold of form $C \times \mathbf{C}^2$, then the algebraic quotient $S = X//\mathbf{C}_+$ is isomorphic to $\mathbf{C}^2$.*
(3) *If the assumptions of both* (1) *and* (2) *hold, then X is isomorphic to $\mathbf{C}^3$.*

Proof. The first statement was proved in [K3, Thm. 5.4(ii)] under the additional assumption that $S = X//\mathbf{C}_+$ is smooth. Theorem 2.7 removes this assumption and proves (1) in full generality. In the second statement, we have a dominant morphism $C \times \mathbf{C} \to S$. Since the Kodaira logarithmic dimension $\bar{\kappa}(C \times \mathbf{C})$ equals $-\infty$, we conclude that $\bar{\kappa}(S) = -\infty$. Because S is also smooth and contractible, it is isomorphic to $\mathbf{C}^2$ (see e.g. [M2]). The third statement is an obvious consequence of (1) and (2). □

Two $\mathbf{C}_+$-actions on a variety are said to be *equivalent* if they have the same general orbits (i.e., if the associated locally nilpotent derivations have the same kernel). In particular, nonequivalent actions generate different quotient morphisms. Corollary 2.8(3) implies the following result.

Corollary 2.9. *Suppose that a smooth contractible affine algebraic threefold X admits two nonequivalent nontrivial algebraic $\mathbf{C}_+$-actions. Then $X//\mathbf{C}_+ = \mathbf{C}^2$ for any nontrivial algebraic $\mathbf{C}_+$-action. Furthermore, X is isomorphic to $\mathbf{C}^3$ if it admits a free $\mathbf{C}_+$-action.*

It is worth mentioning that Corollary 2.6 also follows from Theorem 2.7 and [GS].

3. The Case When $S \simeq \mathbf{C}^2$

The aim of this section is to give an independent proof of Corollary 2.8(2) (and hence of Corollary 2.8(3)) that does not use the Taubes theorem.

Let X be the complement to an effective divisor D of simple normal crossing type in a projective algebraic manifold $\bar{X}$. Consider the sheaf $\Omega^k(\bar{X}, D)$ of logarithmic k-forms on $\bar{X}$ along D (that is, each section of this sheaf over an open subset $U \subset \bar{X}$ is a holomorphic k-form on $U \cap X$ that has at most simple poles at general points of $U \cap D$). Let r be the rank of $\Omega^k(\bar{X}, D)$ (i.e., $r = C_{n,k}$, where $n = \dim \bar{X}$ and $C_{n,k}$ is the number of combinations), $S^m\Omega^k(\bar{X}, D)$ its symmetric m-power, and $\Gamma(\bar{X}, S^m\Omega^k(\bar{X}, D))$ the space of holomorphic sections of $S^m\Omega^k(\bar{X}, D)$ over $\bar{X}$. We say that the Kodaira–Iitaka–Sakai logarithmic k-dimension $\bar{\kappa}_k(X)$ of X is $-\infty$ if no symmetric power of $\Omega^k(\bar{X}, D)$ has a nontrivial global section; otherwise, we put

$$\bar{\kappa}_k(X) = \limsup_{m \to +\infty} \frac{\log \dim \Gamma(\bar{X}, S^m\Omega^k(\bar{X}, D))}{\log m} - r + 1.$$

This definition does not depend on the choice of simple normal crossing completion $\bar{X}$ of X [I; K2]. One can easily see that $\bar{\kappa}_k(X) = -\infty$ if $k > \dim X$, and $\bar{\kappa}_k(X)$ is the usual Kodaira logarithmic dimension in the case when $k = \dim X$.

Lemma 3.1 [I; K2, Prop. 4.2]. *Let $\bar{X}_1$ and $\bar{X}_2$ be complete complex algebraic manifolds, and let D_1 and D_2 be divisors of SNC-type in $\bar{X}_1$ and $\bar{X}_2$, respectively.*

Suppose that $\bar f\colon \bar X_1 \to \bar X_2$ is a morphism and that $\bar f$ is an extension of a dominant morphism $f\colon X_1 \to X_2$, where $X_i = \bar X_i - D_i$. Then $\bar f$ generates a natural homomorphism $f^*\colon S^m\Omega^k(\bar X_2, D_2) \to S^m\Omega^k(\bar X_1, D_1)$.

The word "natural" here means that we treat $\Gamma(\bar X_i, S^m\Omega^k(\bar X_i, D_i))$ as the subspace of $\Gamma(\bar X, \Omega^k(\bar X_i, D_i)^{\otimes m})$ invariant under the natural action of the symmetric group $S(m)$ and that f^* is generated by the induced mapping of k-forms. In particular, f^* sends nonzero sections of $S^m\Omega^k(\bar X_2, D_2)$ to nonzero sections of $S^m\Omega^k(\bar X_1, D_1)$. We thus have the following result.

COROLLARY 3.2. *Let $f\colon X_1 \to X_2$ be a dominant morphism of algebraic varieties and let $n_i = \dim X_i$. Then $\bar\kappa_k(X_1) + C_{n_1,k} \geq \bar\kappa_k(X_2) + C_{n_2,k}$. In particular, if $\bar\kappa_k(X_1) = -\infty$ then $\bar\kappa_k(X_2) = -\infty$.*

Let H be a hyperplane in $\mathbf{P}^s$, that is, $\mathbf{C}^s = \mathbf{P}^s \setminus H$. Then $X' = \bar X \times \mathbf{P}^s$ is a completion of $X \times \mathbf{C}^s$ and $D' = X' \setminus (X \times \mathbf{C}^s)$ is of simple normal crossing type. Using the fact that any sheaf of the form

$$\Omega^1(\mathbf{P}^s, H)^{\otimes m_1} \otimes \cdots \otimes \Omega^s(\mathbf{P}^s, H)^{\otimes m_s}$$

has no global nonzero sections over $\mathbf{P}^s$, one can show that

$$\Gamma(\bar X, \Omega^k(\bar X, D)^{\otimes m}) = \Gamma(X', \Omega^k(X', D')^{\otimes m}),$$

which implies the following.

LEMMA 3.3. *Let $Y = X \times \mathbf{C}^s$ and $n = \dim X$. Then*

$$\bar\kappa_k(Y) = \bar\kappa_k(X) + C_{n,k} - C_{n+s,k} \quad \textit{for any } k \geq 0.$$

In particular, $\bar\kappa_k(Y) = -\infty$ when $k > n$.

Applying the theorem about removing singularities of holomorphic functions in codimension 2, we obtain the following result.

LEMMA 3.4. *Let Z be a subvariety of codimension at least 2 in an algebraic manifold X. Then $\bar\kappa_k(X) = \bar\kappa_k(X \setminus Z)$ for every k.*

THEOREM 3.5. *Let X be a smooth contractible affine algebraic threefold such that $\bar\kappa_2(X) = -\infty$. Then, for every nontrivial algebraic $\mathbf{C}_+$-action on X, the algebraic quotient $S = X//\mathbf{C}_+$ is isomorphic to $\mathbf{C}^2$.*

Proof. Let F be the set of singular points of S. According to Lemma 2.1, $L = \rho^{-1}(F)$ is a curve. Therefore, $\bar\kappa_2(X \setminus L) = -\infty$; see Lemma 3.4. By Corollary 3.2, $\bar\kappa_2(S^*) = -\infty$, and the statement follows from Proposition 2.5(3). □

Now Lemma 3.3 implies Corollary 2.8(2).

REMARK. Consider an n-dimensional smooth contractible affine algebraic variety X and an algebraic action of a unipotent group U on X. Suppose that U has dimension $n-2$ (i.e., U is isomorphic to $\mathbf{C}^{n-2}$ as an affine algebraic variety) and there are only finitely many orbits nonisomorphic to $\mathbf{C}^{n-2}$. It was mentioned in [K3, Rem. 5.4] that the morphism $X \to S = X//U$ is surjective. Because surjectivity of the quotient morphism is the only crucial argument in the proof of Proposition 2.5, we can extend some of our results to this action of U. That is, $X//U$ is a smooth contractible surface that is isomorphic to $\mathbf{C}^2$ in the case when X admits a dominant morphism from an n-fold of form $C \times \mathbf{C}^{n-1}$.

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