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On the Gehring–Hayman Property, the Privalov–Riesz Theorems, and Doubling Measures

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1. Introduction and Main Results

The motivation for the results in this paper is threefold. First, in [BKR] the authors observed that a number of relevant results on conformal mapping rely on only two properties of the derivative $|f'|$ of a conformal map f of the unit disk in the complex plane: the *Harnack property* (H) and the so-called *volume growth property* (VG).

Let $\rho\colon \mathbb{R}^{N+1}_+ \to (0,+\infty)$ be a continuous function (a *metric density*) in the upper half-space. We say that ρ satisfies Harnack's property (H) with constant C if, for each $z = (a,t) \in \mathbb{R}^{N+1}_+$,

$$C^{-1} \le \frac{\rho(w)}{\rho(q)} \le C$$

whenever $w, q \in B\big(z, \frac{1}{2}t\big)$. (See the end of this section for notation.)

Associated to a metric density ρ in $\mathbb{R}^{N+1}_+$, we define the ρ-length of a curve Γ in $\mathbb{R}^{N+1}_+$ as

$$\operatorname{length}_\rho(\Gamma) = \int_\Gamma \rho(z)\,|dz|$$

and the ρ-distance

$$d_\rho(w,q) = \inf_\Gamma \operatorname{length}_\rho(\Gamma)$$

for $w, q \in \mathbb{R}^{N+1}_+$, where the infimum is taken over all curves in $\mathbb{R}^{N+1}_+$ joining w, q. Then d_ρ is a distance in $\mathbb{R}^{N+1}_+$. If $\rho(x,t) = 1/t$, then d_ρ is the *hyperbolic distance* in $\mathbb{R}^{N+1}_+$. We recall that the hyperbolic geodesics are exactly the vertical lines and the circles ending orthogonally at the boundary.

If $z \in \mathbb{R}^{N+1}_+$ and $r > 0$ then $B_\rho(z,r)$ denotes the open ball of center z and radius r in the distance d_ρ. We say that ρ satisfies the *volume growth* condition (VG) with constant C if

$$\mu_\rho(B_\rho(z,r)) \le C r^{N+1}$$

for all $z \in \mathbb{R}^{N+1}_+$ and $r > 0$, where μ_ρ is the volume measure associated to ρ.

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From the distortion theorem for conformal mappings [P], it follows that if f is conformal in $\mathbb{R}^2_+$ then $\rho = |f'|$ satisfies (H) with respect to some absolute constant. On the other hand, it is elementary that $|f'|$ also satisfies (VG) for some absolute constant and $N = 1$. Following [ABH; BHR; BK; BKR], those metric densities ρ in $\mathbb{R}^{N+1}_+$ satisfying (H) and (VG) are called *conformal metrics.* In [BKR] it was shown that most of the results on the boundary behavior of conformal maps still hold for conformal metrics (the Gehring–Hayman theorem and Beurling's radial limit theorem among them).

We say that a metric density ρ in $\mathbb{R}^{N+1}_+$ satisfies the *Gehring–Hayman property* (GH) with constant C if, for each $w, q \in \mathbb{R}^{N+1}_+$ and for any curve $\Gamma \subset \mathbb{R}^{N+1}_+$ with endpoints w, q,

$$\int_\Gamma \rho \geq C \int_{\Gamma_h} \rho,$$

where Γ_h is the hyperbolic geodesic joining w and q. Then we have the following theorem.

THEOREM A [BKR, Thm. 3.1].

$$(\mathrm{H}) + (\mathrm{VG}) \implies (\mathrm{GH});$$

that is, conformal metrics satisfy (GH).

When $\rho = |f'|$ for some conformal mapping f, we recover the classical Gehring–Hayman theorem [GHa]. In [BaBu] can be found further geometrical implications of the Gehring–Hayman property in more general settings.

A second motivation comes from the Privalov–Riesz problems for quasiconformal mappings or, more generally, for conformal metrics. We recall that a classical theorem of Privalov (see [P, Chap. 6]) states that if f is analytic in the upper half-plane $\mathbb{R}^2_+$ then

$$m_1\Big\{x \in \mathbb{R} : \lim_{z \to x} f(z) = 0\Big\} = 0,$$

where the limit must be understood in the angular sense and m_1 is 1-dimensional Lebesgue measure. Observe that no growth restriction on f is needed here.

A close companion to the Privalov theorem is the Riesz theorem (see [P, Thm. 6.8]): suppose now that $F : \mathbb{R}^2_+ \to \Omega$ is conformal, where $\Omega \subset \mathbb{R}^2$ is a Jordan domain. Then F' belongs to the Hardy space H^1 if and only if $\mathcal{H}^1(\partial\Omega) < \infty$, and in this case the restriction of F to $\mathbb{R} \cup \{\infty\}$, which is a homeomorphism into $\partial\Omega$ by [P, Thm. 2.3], is absolutely continuous in both directions, in the sense that

$$m_1(E) = 0 \iff \mathcal{H}^1(F(E)) = 0$$

for $E \subset \mathbb{R}$. Here $\mathcal{H}^1$ denotes 1-dimensional Hausdorff measure in the plane. The implication to the right (*direct absolute continuity*) follows from the fact that $F' \in H^1$, while the implication to the left (*inverse absolute continuity*) requires Privalov's theorem. Observe also that, since the vertical asymptotic behavior of F'

is comparable (up to multiplicative constants) to the angular behavior (see [P, Chap. 1]), it follows from Privalov that

$$m_1\Big\{x \in \mathbb{R} : \lim_{t\to 0} F'(x,t) = 0\Big\} = 0.$$

Now, let $F\colon \mathbb{R}^{N+1}_+ \to \mathbb{R}^{N+1}$ be quasiconformal with $N \geq 2$. Then the limit

$$\lim_{t\to 0} F(x,t) = f(x)$$

exists for $x \in \mathbb{R}^N$ outside an exceptional set of $(N+1)$-capacity 0—in particular, of Hausdorff dimension 0 [V, Thm. 15.1]. The *averaged derivative* of F is defined by

$$a_F(x,t) = \left(\fint_{B((x,t),t/2)} J_F \, dm_{N+1}\right)^{1/(N+1)},$$

where J_F is the Jacobian determinant of F, which exists almost everywhere and is strictly positive in $\mathbb{R}^{N+1}_+$. An equivalent version of a_F was first introduced in [AG]. Since quasiconformal maps do not, in general, have pointwise derivatives, it turns out that a_F is a satisfactory quasiconformal substitute of the modulus of the derivative of a conformal mapping. One of the links between quasiconformal theory and the general theory of metrics developed in [BKR] is that a_F is a conformal metric if F is quasiconformal [BKR, Ex. 2.4).

On the other hand, it has been shown in [ABH, Thm. 1.1] that the boundary function f has distributional derivatives provided that F belongs to the so called *Riesz class,* meaning that a_F satisfies an appropriate Hardy-type condition. Therefore, it is natural to ask (see [ABH, Prob. 1.3]) whether

$$m_N\{x \in \mathbb{R}^N : Df(x) = 0\} = 0, \tag{1.1}$$

which can be seen as a quasiconformal analogue of Privalov's theorem.

In view of the connection between mappings and metric densities, it is also asked (in [ABH, Prob. 4.2]) whether a conformal metric ρ in $\mathbb{R}^{N+1}_+$ in the appropriate Riesz class verifies Privalov's theorem in the sense that

$$m_N\Big\{x \in \mathbb{R}^N : \lim_{t\to 0} \rho(x,t) = 0\Big\} = 0. \tag{1.2}$$

It turns out that an affirmative answer to (1.2) would yield an affirmative answer to (1.1) (see [ABH]). It is also remarked there that standard sawtooth arguments show that the general case would follow from the case where F or ρ is assumed to satisfy a Riesz condition. This fits with the spirit of Privalov's theorem in the classical case, as mentioned previously.

Regarding the problem of absolute continuity, it is proved in [BK] that F belongs to the Riesz class if $\mathcal{H}^N(\partial\Omega) < \infty$, where $\Omega = F(\mathbb{R}^{N+1}_+)$. Examples in [H] show that the direct absolute continuity does not hold in general, even when $\partial\Omega$ satisfies assumptions stronger than $\mathcal{H}^N(\partial\Omega) < \infty$. On the other hand, by [BK, Lemma 6.2] it follows that a positive answer to (1.2) would solve the problem of inverse absolute continuity. We refer to [ABH] for some partial results and further comments.

The third motivation is that an interesting link between doubling measures on $\mathbb{R}^N$, metrics in $\mathbb{R}^{N+1}_+$, and the Gehring–Hayman property has been shown in [DS] and [S1]. We say that a positive measure μ on $\mathbb{R}^N$ is *doubling* if

$$\mu(2B) \le C\mu(B)$$

for each ball B and some $C > 0$, where $2B$ is the ball concentric to B and twice its radius. Now, if μ is a doubling measure on $\mathbb{R}^N$, define a metric density ρ_μ in $\mathbb{R}^{N+1}_+$ as

$$\rho_\mu(x,t) = \left(\frac{\mu(B(x,t))}{t^N}\right)^{1/N}. \tag{1.3}$$

Then ρ_μ satisfies (H) (with constant depending only on N and the doubling constant). Motivated by questions of bi-Lipschitz embeddings (see also [S2]), David and Semmes [DS; S1] observed that the celebrated Gehring lemma [G] on the self-improving of regularity for quasiconformal mappings could be adapted to more general situations. From their program it follows in particular that the regularity of μ is closely related to certain geometric properties of ρ_μ. Namely, the following theorem is a restatement, in slightly different terms, of Proposition 3.4 in [S1].

THEOREM B [S1, Prop. 3.4]. *Let μ be a doubling measure on $\mathbb{R}^N$, $N \ge 2$, and let ρ_μ be defined by* (1.3). *Assume that ρ_μ satisfies* (GH). *Then μ is absolutely continuous with respect to N-dimensional Lebesgue measure, and its density is a strong A_∞-weight.*

Note that even though the hypothesis in [S1, Prop. 3.4] in terms of ρ_μ looks weaker than (GH), it turns out to be equivalent; this is shown by [S1, Lemma 3.3], Proposition 4.2, and a standard argument with Whitney balls. Theorem B states that, when ρ is the special lift associated to a doubling measure μ on $\mathbb{R}^N$ given by (1.3), then the assumption that ρ satisfies (GH) (it automatically satisfies (H) because of the doubling property) implies that ρ has a nice boundary behavior; in particular,

$$m_N\Big\{x \in \mathbb{R}^N : \lim_{t\to 0} \rho(x,t) = 0\Big\} = 0,$$

so ρ_μ satisfies Privalov's theorem.

For example, the metric densities $\rho_1(x,t) = t$ and $\rho_2(x,t) = 1/t$ both satisfy (H) in $\mathbb{R}^{N+1}_+$. Observe that ρ_2 trivially satisfies (GH) but ρ_1 does not. In fact, no metric in $\mathbb{R}^{N+1}_+$ that is continuous in $\mathbb{R}^N$ and vanishing on some rectifiable curve in $\mathbb{R}^N$ can satisfy (GH) (see [BKR, Secs. 2.8 & 2.9] for more examples). Furthermore, it is easy to check that ρ_2 does not satisfy (VG).

Note that the Gehring–Hayman property can be seen as a sort of subharmonicity property of the metric density that seems to prevent it from being small at a large part of the boundary. By Theorem B and all the preceding comments, one is tempted to believe that the Privalov problem might be related to (GH), independently of the conformality of the metric. For instance, we could even ask whether

(H) and (GH) together imply (1.2) for a general metric density ρ in $\mathbb{R}_+^{N+1}$. The examples in Theorems 1 and 2, the main results of the paper, show that this is not the case, although it turns out that the metrics used there are not conformal in general (see the corollary to follow). This seems to reinforce the idea that the Privalov problem should be very much related to conformality.

But Theorems 1 and 2 say also a bit more: suppose that we restrict our attention to metrics obtained as special liftings of doubling measures on $\mathbb{R}^N$. Then we can ask how stable Theorem B is to perturbations in the definition of ρ_μ. More precisely, for a doubling measure μ on $\mathbb{R}^N$ and a certain nondecreasing function $\psi\colon [0,+\infty)$, suppose we define

$$\rho(x,t) = \left(\frac{\mu(B(x,t))}{t^N\psi(t)}\right)^{1/N}. \tag{1.4}$$

Observe that the choice $\psi = 1$ gives ρ_μ. From Theorem B it follows that, in order to destroy the absolute continuity of μ in (1.4), some restriction on the size of ψ will be needed. Theorem 2 shows that $\psi(0) = 0$ is essentially the only such restriction, whereas Theorem 1 states that, for weight functions of the form $\psi(t) = t^\varepsilon$, the situation can also change dramatically as soon as $\varepsilon > 0$. All of this shows that Theorem B is quite sensitive to perturbations in the definition of ρ_μ in (1.3). We do not know whether a quasiconformal mapping in the upper half-space can be constructed in such a way that the trace of its Jacobian on the boundary exhibits similar properties to that of the measures in Theorems 1 and 2.

Theorem 1. *Let $\varepsilon > 0$ and define $\psi_\varepsilon(t) = t^\varepsilon$ if $0 < t < 1$ and $\psi_\varepsilon(t) = 1$ if $t \geq 1$. Then, for each $\eta > 0$, there is a singular doubling measure μ on $\mathbb{R}^N$* (*with doubling constant depending only on N, ε, and η*) *such that, if we define*

$$\rho(x,t) = \left(\frac{\mu(B(x,t))}{t^N\psi_\varepsilon(t)}\right)^{1/N},$$

then (a)

$$\lim_{t\to 0}\frac{\rho(x,t)}{t^\eta} = 0$$

for m_N-a.e. $x \in \mathbb{R}^N$ and (b) *ρ satisfies* (H) *and* (GH) *with constants depending only on N, ε, and η.*

It is a well-known fact that, given a doubling measure μ on $\mathbb{R}^N$, there is a positive number d (depending only on N and the doubling constant) such that $\mu(E) > 0$ implies $\dim E \geq d$, where dim denotes Hausdorff dimension. This remark applied to the set $E = \{x : D\mu(x) = \infty\}$, together with Theorem 1, gives the following corollary (see [BKR]).

Corollary. *If in Theorem 1 we choose $\varepsilon = N$ and let η be any absolute positive constant, then*

$$\dim\left\{x \in \mathbb{R}^N : \int_0^1 \rho(x,t)\,dt = \infty\right\} \geq d(N) > 0.$$

In particular, ρ cannot be a conformal metric.

THEOREM 2. *Let* $\psi_0 : [0,1] \to [0,1]$ *be continuous and nondecreasing, with* $\psi_0(0) = 0$ *and* $\psi_0(1) = 1$. *Then there is a singular doubling measure* μ *on* $\mathbb{R}^N$ (*with doubling constant depending only on* N) *and a continuous, nondecreasing function* $\psi : [0,\infty) \to [0,\infty)$ *with* $\psi(0) = 0$ *and* $\psi \geq \psi_0$ *in* $[0,1]$ *such that, if*

$$\rho(x,t) = \left(\frac{\mu(B(x,t))}{t^N \psi(t)}\right)^{1/N},$$

then ρ *satisfies* (H) *and* (GH) (*with constants depending only on* N) *and*

$$\lim_{t\to 0} \rho(x,t) = 0$$

for m_N*-a.e.* $x \in \mathbb{R}^N$.

The structure of the paper is as follows. In Section 2 we construct the class of doubling measures that will be used later, and Section 3 describes some symmetry properties of those measures. Section 4 is devoted to the analysis of the Gehring–Hayman property for metrics of the form (1.4), and in Section 5 the boundary behavior of these metrics is studied with the help of martingale techniques.

NOTATION. Whenever $x \in \mathbb{R}^n$ and $r > 0$, $B(x,r)$ denotes the Euclidean ball of center x and radius r; $\mathbb{R}^{N+1}_+$ denotes the upper half-space $\{(x,t) : x \in \mathbb{R}^N, t > 0\}$.

Given $p \in \mathbb{N}$, for each $k \in \mathbb{N}$ we define $\mathcal{F}_{k,p}$, the family of *p-adic cubes of generation k* in $\mathbb{R}^N$, by

$$\mathcal{F}_{k,p} = \left\{\prod_{j=1}^{N} [(m_j - 1)p^{-k}, m_j p^{-k}) : m_j \in \mathbb{Z},\ 1 \leq j \leq N\right\};$$

numbers of the form $\{mp^{-k}, m \in \mathbb{Z}\}$ will be referred as the *p-adic numbers of generation k*. Given $Q \in \mathcal{F}_{k-1,p}$, there is a natural decomposition of Q into p^N cubes in $\mathcal{F}_{k,p}$, which will be called the *descendents* of Q.

For any cube Q, we use $l(Q)$ to denote the side-length of Q. A curve γ in $\mathbb{R}^n$ joining two points $x, y \in \mathbb{R}^n$ is any continuous mapping $\gamma : [a,b] \to \mathbb{R}^n$ such that $\gamma(a) = x$ and $\gamma(b) = y$.

Finally, C might denote successive positive constants during the same proof.

2. A Certain Family of Doubling Measures

Fix an odd integer $p \geq 3$ and a positive number δ such that $0 < \delta \leq p^{-N}$. Let $(\delta_k)_{k=1}^{\infty}$ be a sequence of nonnegative numbers such that $\delta \leq \delta_k \leq p^{-N}$ for all $k \in \mathbb{N}$, and set

$$a_k = \frac{1 - \delta_k}{p^N - 1}$$

(observe that $a_k \geq \delta_k$). We will construct a positive measure μ in $\mathbb{R}^N$ by specifying it on each $\mathcal{F}_{k,p}$, the family of p-adic cubes of generation k.

Set $\mu(Q) = 1$ for each $Q \in \mathcal{F}_{0,p}$ and assume that μ has already been defined on $\mathcal{F}_{j,p}$ with $j \leq k-1$. If $Q_{k-1} \in \mathcal{F}_{k-1,p}$ then Q_{k-1} has p^N p-adic descendents of generation k and, since p is odd, one of them contains the center of Q_{k-1}. Let us call this centered cube Q_k^*. Then, if $Q_k \subset Q_{k-1}$ and $Q_k \in \mathcal{F}_{k,p}$, we define

$$\mu(Q_k) = \begin{cases} a_k \mu(Q_{k-1}) & \text{if } Q_k \neq Q_k^*, \\ \delta_k \mu(Q_{k-1}) & \text{if } Q_k = Q_k^*. \end{cases}$$

Hence the measure has been also defined on $\mathcal{F}_{k,p}$. Since $\prod_{k=n}^{\infty} a_k = 0$ for all $n \in \mathbb{N}$, it can be checked [Sh, p. 152] that μ defines a Borel positive measure in $\mathbb{R}^N$. This type of asymmetric weight construction is well known.

Let us first show that μ is doubling. The main idea here is that μ has been constructed in such a way that the distortion between any two adjacent p-adic cubes of the same generation is controlled by some fixed constant, even if the cubes are not p-adic "siblings". The next two properties of μ will be needed in the sequel.

(i) If $Q_k, Q_k' \in \mathcal{F}_{k,p}$ are adjacent (not necessarily siblings), then

$$\left(\max\left\{\frac{a_1}{\delta_1}, \dots, \frac{a_k}{\delta_k}\right\}\right)^{-1} \leq \frac{\mu(Q_k)}{\mu(Q_k')} \leq \max\left\{\frac{a_1}{\delta_1}, \dots, \frac{a_k}{\delta_k}\right\}. \tag{2.1}$$

(ii) If $Q_k \subset Q_{k-1}$ with $Q_k \in \mathcal{F}_{k,p}$ and $Q_{k-1} \in \mathcal{F}_{k-1,p}$, then

$$\frac{\mu(Q_{k-1})}{\mu(Q_k)} \leq \frac{1}{\delta_k}. \tag{2.2}$$

Proposition 2.1. *μ is a doubling measure; that is, there exists a $C > 0$ (depending only on N, p, and δ) such that*

$$\mu(B(x, 2t)) \leq C\mu(B(x, t))$$

for each $x \in \mathbb{R}^N$ and $t > 0$.

Proof. By periodicity, it is enough to check the doubling property for small t, so we assume that $2t \leq 1$. Let k_1 be the largest integer such that $2t \leq p^{-k_1}$, and let k_2 be the smallest integer such that $\sqrt{N}p^{-k_2} \leq t$. Now, let $Q_{k_i}(x)$ be the p-adic cube of the generation k_i containing x, $i = 1, 2$. Then it is easy to check that:

(i) $k_1 < k_2$, and $k_2 - k_1$ depends only on N and p;
(ii) $Q_{k_2}(x) \subset B(x, t)$;
(iii) $B(x, 2t)$ is contained in a union of 3^N p-adic cubes of generation k_1, with $Q_{k_1}(x)$ being one of the cubes and the others being adjacent to it but not necessarily siblings.

From property (i) we deduce that if Q is any p-adic cube of generation k_1 that is adjacent to $Q_{k_1}(x)$, then

$$\mu(Q) \leq \max\left\{\frac{a_1}{\delta_1}, \dots, \frac{a_{k_1}}{\delta_{k_1}}\right\}\mu(Q_{k_1}(x)) \leq C_0 \mu(Q_{k_1}(x)),$$

where C_0 depends only on N, p, δ. Therefore,

$$\frac{\mu(B(x, 2t))}{\mu(B(x, t))} \leq 3^N C_0 \frac{\mu(Q_{k_1}(x))}{\mu(Q_{k_2}(x))}.$$

Yet by iterating (2.2) we obtain $\mu(Q_{k_1}(x)) \leq \mu(Q_{k_2}) \prod_{j=k_1+1}^{k_2} \delta_j^{-1}$, so

$$\frac{\mu(B(x,2t))}{\mu(B(x,t))} \leq 3^N C_0^{k_2-k_1+1}. \qquad \square$$

3. Some Symmetry Properties of μ

Suppose that μ has been constructed as in Section 2. The following symmetry properties will play an important role in our subsequent analysis of the Gehring–Hayman property.

LEMMA 3.1. *For $m \in \mathbb{N}$ and $i \in \{1, \dots, N\}$, let c be a p-adic number of generation m and let H be the hyperplane $\{x_i = c\}$ in $\mathbb{R}^N$. Let $x, \bar{x} \in \mathbb{R}^N$ be symmetric with respect to H. Suppose that $lp^{-m} \leq \operatorname{dist}(x, H) < (l+1)p^{-m}$ for some $l \in \mathbb{N}$. Then, for each $t > 0$ we have*

$$C^{-1} \leq \frac{\mu(B(x,t))}{\mu(B(\bar{x},t))} \leq C,$$

where $C = C(N, p, \delta, l) > 0$.

Proof. If $t \geq p^{-m}$ then the result follows from the doubling condition. If $t \leq p^{-m}$ (say, $t \asymp p^{-k}$ with $k \geq m$) then it is enough to observe, from the symmetry of the construction of μ, that: if $Q_k \subset Q_m(x)$, $\overline{Q_k} \subset Q_m(\bar{x})$, $Q_k, \overline{Q_k} \in \mathcal{F}_{k,p}$, and Q_k and $\overline{Q_k}$ are symmetric with respect to H, then

$$\frac{\mu(Q_k)}{\mu(\overline{Q_k})} = \frac{\mu(Q_m(x))}{\mu(Q_m(\bar{x})};$$

and from the doubling condition,

$$C^{-1} \leq \frac{\mu(Q_m(x))}{\mu(Q_m(\bar{x}))} \leq C$$

where $C = C(N, p, \delta, l) > 0$. $\square$

For the following property, we need to introduce a bit more notation. Let $Q = J_1 \times \dots \times J_N \in \mathcal{F}_{m,p}$ and, for each $i = 1, 2, \dots, N$, let c_i be the middle point of J_i. Fix $k \geq m$ and set $I_i^* = [c_i - 1/2p^k, c_i + 1/2p^k)$. Then, since p is odd, $I_i^* \in \mathcal{F}_{k,p}$ for each $1 \leq i \leq N$. Now, let $R = I_1 \times \dots \times I_N \in \mathcal{F}_{k,p}$ with $R \subset Q$. For any i, let

$$R^{*i} = I_1^* \times \dots \times I_{i-1}^* \times I_i \times I_{i+1}^* \times \dots \times I_N^*$$

be the cube obtained when R is projected onto the centered k-adic bar $I_1^* \times \dots \times I_{i-1}^* \times J_i \times I_{i+1}^* \times \dots \times I_N^*$ parallel to the x_i direction.

PROPOSITION 3.2. *Let $k \geq m$, $R \in \mathcal{F}_{k,p}$, $Q \in \mathcal{F}_{m,p}$, and $R \subset Q$. For each direction i $(1 \leq i \leq N)$, if R^{*i} is the projected cube of R along the direction i (as before) then*

$$\mu(R) \geq \mu(R^{*i}).$$

Proof. Use the construction of μ, the inequalities $a_k \geq \delta_k$, and the fact that p is odd. □

Proposition 3.2 expresses the fact that, for any p-adic N-dimensional cube, μ gives less mass around the center of such cube.

Let $Q \in \mathcal{F}_{m,p}$ and $k \geq m$. A finite collection of cubes $\mathcal{C} = \{Q^1, \ldots, Q^n\}$ is a *p-adic k-chain crossing Q* if:

(i) $Q^i \subset Q$ and $Q^i \in \mathcal{F}_{k,p}$, $i = 1, \ldots, n$;
(ii) Q^i and Q^{i+1} have a common face, $i = 1, 2, \ldots, n-1$; and
(iii) ∂Q^1 and ∂Q^n intersect two parallel faces of Q.

If we remove the condition that all Q^i belong to the same generation, we will refer to $\mathcal{C}$ just as a *p-adic chain crossing Q*. Suppose now that $Q = J_1 \times \cdots \times J_N$, that c_i is the middle point of J_i, and that I_i^* is as in the remarks preceding Proposition 3.2 ($i = 1, \ldots, N$). Let $\{I_1, \ldots, I_{p^{k-m}}\}$ be the decomposition of J_1 into p-adic intervals of generation k, in increasing order. Then we define the *centered p-adic k-chain crossing Q* along the x_1 direction as $\mathcal{C}_1^* = \{Q^1, \ldots, Q^{p^{k-m}}\}$, where $Q^j = I_j \times I_2^* \times \cdots \times I_N^*$. The *centered k-chain* crossing Q along the x_i direction, $\mathcal{C}_i^*$, is defined analogously.

4. Construction of ρ and the Gehring–Hayman Property

We now describe the type of metric densities to be used in Theorems 1 and 2.

Suppose that $p, \delta, (\delta_k), \mu$ are as in Section 3. We will consider metric densities of the form

$$\rho(x,t) = \left(\frac{\mu(B(x,t))}{t^N \psi(t)}\right)^{1/N}, \tag{4.1}$$

where $\psi\colon [0,\infty) \to [0,\infty)$ is continuous and nondecreasing and verifies

$$\psi(2t) \leq M\psi(t), \tag{4.2}$$

$$\psi(1) = 1, \tag{4.3}$$

$$\left(\frac{\psi(p^{-k})}{\psi(p^{-(k-1)})}\right)^{1/N} \leq (p-1)\left(\frac{1-\delta_k}{p^N - 1}\right)^{1/N} + \delta_k^{1/N} \tag{4.4}$$

for each $k \in \mathbb{N}$ and $t > 0$, where $M > 0$.

Standard arguments about doubling measures show that ρ is continuous. On the other hand, (4.2) and (4.4) impose restrictions on the growth of ψ near zero. Inequality (4.2) directly implies that ρ verifies (H), whereas (4.4) is a technical restriction that will be needed to prove (GH). Because of the periodicity of the construction of μ, (4.3) implies that ρ is bounded above on $\{(x,t) : t \geq 1\}$. Note also that, if $N = 1$, then (4.2)–(4.4) are automatically satisfied with $\psi = 1$.

Suppose that μ is as in Section 3, ψ verifies (4.2)–(4.4), and ρ is defined by (4.1). The following theorem is the main result of this section.

THEOREM 4.1. *Let* $\psi: [0,\infty) \to [0,\infty)$ *be continuous and nondecreasing, and let* ψ *satisfy* (4.2)–(4.4). *If* ρ *has been defined by* (4.1), *then* ρ *satisfies* (H) *and* (GH) *with constants depending only on* N, p, δ, *and* M.

Our next proposition will state that, in the presence of (H), the Gehring–Hayman property is actually equivalent to an apparently weaker condition (WGH). Let ρ be a metric in $\mathbb{R}^{N+1}_+$ satisfying (H). Following [S1], for $a, b \in \mathbb{R}^N$ we let $B_{a,b}$ be the smallest ball in $\mathbb{R}^N$ containing a and b. Also, set $z_{a,b} = (c, |a-b|)$, where c is the middle point on the segment $[a,b]$. We say that ρ satisfies the *weak Gehring–Hayman property* (WGH) with constant C if, for each $w = (a, t_0)$ and $q = (b, t_1) \in \mathbb{R}^{N+1}_+$ with $a \neq b$ and for any curve $\Gamma \subset \mathbb{R}^{N+1}_+$ with endpoints w, q such that $\Gamma \subset 4B_{a,b} \times (0, 2|a-b|]$, we have

$$\int_\Gamma \rho \geq C\rho(z_{a,b})|a-b|.$$

It is clear, from Harnack's property and the geometry of hyperbolic geodesics, that (H) + (GH) $\Rightarrow$ (WGH). Proposition 4.2 gives the converse. Its proof uses well-known standard arguments (see [BKR]). We include it here for completeness.

PROPOSITION 4.2. *If* ρ *is a metric in* $\mathbb{R}^{N+1}_+$ *satisfying* (H) *and* (WGH), *then* ρ *satisfies* (GH) *with constant depending on the* (H) *and* (WGH) *constants.*

Proof. Let Γ be a curve in $\mathbb{R}^{N+1}_+$ joining (a, t_0) and (b, t_1). We can assume that $a = 0$ and $t_0 \leq t_1$. We suppose first that $t_1 \leq |b|$.

Let L_1 be the vertical segment joining $(0, t_0)$ and $(0, |b|)$, L_2 the horizontal segment joining $(0, |b|)$ and $(b, |b|)$, and L_3 the vertical segment joining $(b, |b|)$ and (b, t_1). It is easy to check that it is enough to prove

$$\int_\Gamma \rho \geq C \int_{L_1 \cup L_2 \cup L_3} \rho.$$

Let k be an integer such that $2^k t_0 \leq |b| < 2^{k+1} t_0$. Then, for $j \leq k$, set

$$A_j = B((0,0), t_0 2^{j+1}) \setminus B((0,0), t_0 2^j).$$

Observe that Γ must cross each annulus A_j for $j \leq k$. Take a subcurve $\Gamma_j \subset \Gamma \cap A_j$ that crosses A_j, and let I_j be the vertical segment joining the points $(0, 2^j t_0)$ and $(0, 2^{j+1} t_0)$. We claim that

$$\int_{\Gamma_j} \rho \geq C_1 \int_{I_j} \rho.$$

To prove the claim, let a_j, b_j be the endpoints of Γ_j, with $|a_j| = 2^j t_0$ and $|b_j| = 2^{j+1} t_0$. We distinguish two cases as follows.

Case 1: $|b_j| \leq \frac{3}{2} 2^j t_0$. Here, $b_j \in B_j = \left\{z = (x,r) : |z| = 2^{j+1} t_0,\ |x| \leq \frac{3}{2} 2^j t_0\right\}$, so we can select a subcurve α_j of Γ_j connecting $A_j \cap \{(x,r) : r = 2^j t_0\}$ and B_j with length at least $C_0 2^j t_0$ for some absolute constant C_0. Then, by (H),

$$\int_{\Gamma_j} \rho \geq \int_{\alpha_j} \rho \geq C_1 \int_{I_j} \rho.$$

Case 2: $|b_j| \geq \frac{3}{2}2^j t_0$. In this case, by (WGH) we have

$$\int_{\Gamma_j} \rho \geq \rho(z_{a_j,b_j})|a_j - b_j| \geq C \int_{I_j} \rho.$$

The claim is thus proved in both cases, and adding up in j yields

$$\int_{\Gamma} \rho \geq C \int_{L_1} \rho.$$

Exactly in the same way we get $\int_\Gamma \rho \geq C \int_{L_3} \rho$. Finally, by hypothesis we have $\int_\Gamma \rho \geq C\rho(z_{0,b})|b| \geq C \int_{L_2} \rho$ and so

$$\int_{\Gamma} \rho \geq C \int_{L_1 \cup L_2 \cup L_3} \rho$$

as desired.

If we now suppose $|b| \leq t_1$ then the result follows in the same way. Here, it is enough to show that

$$\int_{\Gamma} \rho \geq C \int_{L_1 \cup L_2} \rho.$$

Choose k such that $2^k t_0 \leq t_1 < 2^{k+1} t_0$. Now Γ must cross the annuli $A_j = B((0,0), 2^{j+1}t_0) \backslash B((0,0), 2^j t_0)$, and an argument similar to the foregoing can be repeated to yield $\int_\Gamma \rho \geq C \int_{L_1} \rho$. Also, Γ must cross $B(B, |b|/2)$ and so, by Harnack, $\int_\Gamma \rho \geq C\rho(b, t_1)|b| \geq C \int_{L_2} \rho$. Therefore,

$$\int_{\Gamma} \rho \geq C \int_{L_1 \cup L_2} \rho$$

and the proposition follows. □

Let $w = (a, t_0)$ and $q = (b, t_1) \in \mathbb{R}^{N+1}_+$ with $a \neq b$, and let $\Gamma(s) = (\gamma(s), t(s))$, $0 \leq s \leq L$, be a curve in $\mathbb{R}^{N+1}_+$ joining w and q such that $\Gamma \subset 4B_{a,b} \times (0, 2|a-b|]$. Here s denotes arc-length of Γ and we can assume that $L = \text{length}(\Gamma) < \infty$ (else the (GH)-inequality would be trivial). By Proposition 4.2, it is enough to prove

$$\int_{\Gamma} \rho \geq C\rho(z_{a,b})|a - b|, \tag{4.5}$$

where $C = C(N, p, \delta, M) > 0$.

We start with a number of reductions. The first one says that $|a - b|$ can be assumed to be small.

LEMMA 4.3. *Suppose that* (4.5) *holds whenever* $|a - b| \leq 1$. *Then it also holds when* $|a - b| \geq 1$ (*possibly with a different constant*).

Proof. With the same notation as before, suppose that $|a - b| \geq 1$ and for $m \in \mathbb{N}$ choose $s_0 = 0 < s_1 < \cdots < s_m = L$ such that, if $a_j = \gamma(s_j)$, then $1/2 \leq |a_j - a_{j-1}| \leq 1$ for $1 \leq j \leq m$. Set $\Gamma_j = \Gamma[s_{j-1}, s_j]$. Then, by hypothesis we have

$$\int_\Gamma \rho \geq \sum_{j=1}^{m} \int_{\Gamma_j} \rho \geq \sum_{j=1}^{m} \rho(z_{a_j,a_{j-1}})|a_j - a_{j-1}|.$$

Thus, from the doubling condition, (4.3), and the fact that $m \geq 2|a-b|$, we obtain

$$\int_\Gamma \rho \geq C\rho(z_{a,b})|a-b|. \qquad \square$$

The next proposition is a direct consequence of the reflection property of μ (Lemma 3.1). We use the following notation: If H is a hyperplane in $\mathbb{R}^N$, if S_+ and S_- are the half-spaces into which $\mathbb{R}^N$ is divided by H, and if $x \in \mathbb{R}^N$, then $\bar{x}$ denotes the symmetric point of x with respect to H and

$$\hat{x} = \begin{cases} x & \text{if } x \in S_+, \\ \bar{x} & \text{if } x \in S_-. \end{cases}$$

PROPOSITION 4.4. *Let c be a p-adic number of generation m, and let H be the hyperplane $\{x_i = c\}$ in $\mathbb{R}^N$ for some i, $1 \leq i \leq N$. Let $\Gamma(s) = (\gamma(s), t(s))$, $0 \leq s \leq L$, be a curve in $\mathbb{R}^{N+1}_+$ such that $\gamma \subset \{x \in \mathbb{R}^N : \operatorname{dist}(x, H) \leq lp^{-m}\}$ for some $l \in \mathbb{N}$. Define $\hat{\Gamma}(s) = (\hat{\gamma}(s), t(s))$ for $0 \leq s \leq L$. Then,*

$$C^{-1} \int_{\hat{\Gamma}} \rho \leq \int_\Gamma \rho \leq C \int_{\hat{\Gamma}} \rho,$$

where $C = C(N, p, \delta, l) > 0$.

Next we will use Proposition 4.4 to see that we can restrict to the case where the projection curve γ crosses two parallel faces of some p-adic cube. If $a \in \mathbb{R}^N$ and $m \in N$, then each of the $3^N - 1$ cubes in $\mathcal{F}_{m,p}$ surrounding a will be called a *neighbor p-adic cube of a of generation m.*

LEMMA 4.5. *Let $w = (a, t_0)$, $q = (b, t_1)$, and $\Gamma(s) = (\gamma(s), t(s))$, $0 \leq s \leq L$, be as before, with $0 < |a-b| \leq 1$. Pick $m \in \mathbb{N}$ such that $2\sqrt{N}p^{-m} \leq |a-b| < 2\sqrt{N}p^{-(m-1)}$. Then there exist a $Q \in \mathcal{F}_{m,p}$, a neighbor cube of a, and a curve $\hat{\Gamma}(s) = (\hat{\gamma}(s), \hat{t}(s))$ such that $\hat{\gamma} \subset Q$ crosses two parallel faces of Q and*

$$\int_\Gamma \rho \geq C \int_{\hat{\Gamma}} \rho$$

for some $C = C(N, p, \delta) > 0$.

Proof. Consider the grid $\mathcal{F}_{m,p}$ formed by all p-adic cubes of generation m. Let $A(a)$ be the union of the $3^N - 1$ neighbor cubes of a in $\mathcal{F}_{m,p}$. From the choice of m, it follows that γ must eventually leave $A(a)$. Denote by γ_1 the part of γ between a and the point where γ leaves $A(a)$ for the first time. Then $\gamma_1 \subset A(a) \cup Q_m(a)$. Whenever γ_1 crosses from a cube $Q \in \mathcal{F}_{m,p}$ to an adjacent cube $\hat{Q} = Q + p^{-m}v$, where $v = \pm e_i$ for some $1 \leq i \leq N$ (with $\{e_1, \ldots, e_N\}$ the canonical basis in $\mathbb{R}^N$), we will associate the vector v to that crossing. In this way, the p-adic m-chain of cubes obtained by the successive cubes in $\mathcal{F}_{m,p}$ crossed by γ_1 can be identified with a usual random walk $\sum v_j$ in $\mathbb{Z}^N$ consisting of unit steps along any of the coordinate directions. We will use the following elementary fact.

CLAIM. *Let $(k_1, \ldots, k_N) \in \mathbb{Z}^N$ be such that $|k_i| \le 2$ for all $i = 1, \ldots, N$ and $|k_j| = 2$ for some j. Then, from any given usual walk in $\mathbb{Z}^N$ joining $(0, \ldots, 0)$, and $(k_1, \ldots, k_N)$, we can extract a subpath $v_1 + \cdots + v_n$ with $n \le N + 1$ such that $v_1 = v_n$ and $\{v_1, \ldots, v_{n-1}\}$ are orthogonal.*

Let $Q_1, \ldots, Q_{n+1}$ be the p-adic m-chain associated to the walk $v_1 + \cdots + v_n$ of the claim and denote by γ_2 the corresponding subcurve of γ_1. The next step consists of modifying γ_2 by successive reflections on the coordinate hyperplanes in such a way that the corresponding walk becomes simpler. More precisely, suppose that we reflect γ_2 on the hyperplane normal to v_2 containing the "exit face" of Q_2. Then the reflected curve is such that its associated walk no longer contains the vectors $\pm v_2$. Continuing in this way (and after at most $N-1$ reflections) yields a curve $\hat{\gamma}$ such that its associated walk is $v_1 + v_1$ and must therefore cross some cube $Q \in \mathcal{F}_{m,p}$. It is clear from the construction that Q is a neighbor cube of a.

Now, Proposition 4.4 shows that, after each reflection, the integral $\int_\Gamma \rho$ changes up to a multiplicative factor depending only on N, p, δ. This proves Lemma 4.5. □

The following lemma is elementary.

LEMMA 4.6. *Let $Q \in \mathcal{F}_{k-1,p}$ and $\{Q^{1*}, \ldots, Q^{p*}\}$ be a p-adic centered k-chain crossing Q along any of the coordinate directions. Then, if ψ satisfies* (4.4),

$$\sum_{i=1}^{p} \left(\frac{\mu(Q^{i*})}{\psi(p^{-k})} \right)^{1/N} \ge \left(\frac{\mu(Q)}{\psi(p^{-(k-1)})} \right)^{1/N}.$$

Proof. From the construction of μ and (4.4),

$$\begin{aligned} \sum_{i=1}^{p} (\mu(Q_i^*))^{1/N} &= \left[(p-1) \left(\frac{1-\delta_k}{p^N - 1} \right)^{1/N} + \delta_k^{1/N} \right] (\mu(Q))^{1/N} \\ &\ge \left(\frac{\psi(p^{-k})}{\psi(p^{-(k-1)})} \right)^{1/N} (\mu(Q))^{1/N}. \end{aligned}$$ □

Now we are ready for the proof of the general case.

Proof of Theorem 4.1. By Proposition 4.2 we need only check (WGH), and by Lemmas 4.3 and 4.5 we can assume that $\Gamma(s) = (\gamma(s), t(s))$, $0 \le s \le L$, is an arc-length parameterization of a curve in $\mathbb{R}_+^{N+1}$ joining $w = (a, t_0)$ and $q = (b, t_1)$ in such a way that a and b lie on two parallel faces of some $Q \in \mathcal{F}_{m,p}$ and $\gamma \subset Q$. Then it is enough to prove

$$\int_\Gamma \rho = \int_0^L \left(\frac{\mu(\gamma(s), t(s))}{(t(s))^N \psi(t(s))} \right)^{1/N} ds \ge C \left(\frac{\mu(Q)}{\psi(p^{-m})} \right)^{1/N}, \tag{4.6}$$

where $C = C(N, p, \delta, M) > 0$.

We start by choosing a convenient covering of Γ by Whitney balls. Pick $s_1 = 0 < s_2 < \cdots < s_n \le L$ and $c = c(p) > 0$ sufficiently small that, if $z_j = \Gamma(s_j) = (a_j, t_j)$ and $\Gamma_j = \Gamma[s_j, s_{j+1}]$, then:

(i) $\Gamma_j \subset B(z_j, ct_j)$;
(ii) $|z_j - z_{j+1}| = ct_j$ if $1 \le j \le n-1$ and $|z_n - q| \le ct_n$; and
(iii) if $k_j \in \mathbb{N}$ is such that $p^{-k_j} \le t_j < p^{-(k_j-1)}$, then $|k_{j+i} - k_j| \le 1$ whenever $1 \le i \le p$ and $j + i \le n$.

Set $Q^j = Q_{k_j}(a_j)$, $1 \le j \le n$. Then $\mathcal{C} = \{Q^1, \dots, Q^n\}$ will be a p-adic chain crossing Q along a certain coordinate direction and, from (i) and (ii),

$$\int_\Gamma \rho \ge C \sum_{j=1}^n \left(\frac{\mu(Q^j)}{\psi(l(Q^j))} \right)^{1/N}.$$

Now, for each j, let Q^{j*} be the corresponding centered cube in Q along that fixed direction and define $\mathcal{C}^* = \{Q^{1*}, \dots, Q^{n*}\}$, which is also a p-adic chain crossing Q. From the projection property (Proposition 3.2) we have $\mu(Q^j) \ge \mu(Q^{j*})$, so

$$\int_\Gamma \rho \ge C \sum_{j=1}^n \left(\frac{\mu(Q^{j*})}{\psi(l(Q^{j*}))} \right)^{1/N}.$$

By eliminating superfluous cubes, we can assume that no Q^{i*} is contained in some other Q^{j*} for $i \ne j$. After these reductions it is easy to check, using (iii), that $\mathcal{C}^*$ has the following property: *if $R \in \mathcal{F}_{k-1,p}$ has some p-adic descendent in $\mathcal{C}^* \cap \mathcal{F}_{k,p}$, then $\mathcal{C}^*$ must contain the whole p-adic centered k-chain crossing R along the given direction.*

Now Lemma 4.6 allows us to "raise generations" in the following sense: whenever $R^* \in \mathcal{F}_{k-1,p}$ contains a whole p-adic centered k-chain $\{R^{1*}, \dots, R^{p*}\}$, we can replace $\{R^{1*}, \dots, R^{p*}\}$ by R^*. Proceeding in this way enables us to successively lift the largest generation of $\mathcal{C}^*$, and we finally obtain

$$\int_\Gamma \rho \ge C \left(\frac{\mu(Q)}{\psi(p^{-m})} \right)^{1/N} \ge C\rho(z_{a,b})|a-b|,$$

which proves the theorem. □

5. Basic Facts about Martingales and Application to the a.e. Behavior of μ

We will exhibit examples of functions ψ such that the corresponding ρ given by (4.1) has a bad boundary behavior; in particular, it does not verify Privalov's theorem. For that, we will need to take into account the asymptotic a.e. behavior of μ.

As shown in [GoN; LN], martingales are a useful tool for investigating the growth behavior of a given measure in terms of its multiplicative properties. In order to make the exposition as self-contained as possible, we first collect the basic facts about martingales that will be used later.

Let $Q_0 = [0,1)^N$ and $p \in \mathbb{N}$, $p \geq 2$. For any $Q_{k-1} \in \mathcal{F}_{k-1,p}$, let $\{Q_k^i\}_{i=1}^{p^N}$ be its decomposition into disjoint p-adic cubes of generation k. If $x \in Q_0$, we recall that $Q_k(x)$ denotes the only p-adic cube in $\mathcal{F}_{k,p}$ containing x. Suppose that ν is a probability measure in Q_0. We say that a real-valued sequence of functions $\{S_n\}$ in Q_0 is a *p-adic ν-martingale* if:

(i) each S_n is constant on any $Q_n \in \mathcal{F}_{n,p}$; and

(ii) for each $n \in \mathbb{N}$ and any $Q_{n-1} \in \mathcal{F}_{n-1,p}$,

$$\int_{Q_{n-1}} S_n \, d\nu = S_{n-1}|_{Q_{n-1}} \nu(Q_{n-1}),$$

that is,

$$S_{n-1}|_{Q_{n-1}} \nu(Q_{n-1}) = \sum_{i=1}^{p^N} S_n|_{Q_n^i} \nu(Q_n^i),$$

where $\{Q_n^i\}_{i=1}^{p^N}$ are the p-adic descendents of Q_{n-1}.

The differences $X_k = S_k - S_{k-1}$ are called the *increments* of the martingale. Another relevant concept associated to a martingale is the *quadratic characteristic.* If $\{S_n\}$ is a p-adic ν-martingale, we define its quadratic characteristic $\langle S \rangle_n$ as

$$\langle S \rangle_n(x) = \sum_{k=1}^{n} \sum_{i=1}^{p^N} \frac{\nu(Q_k^i(x))}{\nu(Q_{k-1}(x))} (X_k^i(x))^2,$$

where $X_k^i(x) = X_k|_{Q_k^i(x)}$. Note that $\langle S \rangle_n$ is constant on any p-adic cube of generation $n-1$ (we say that $\langle S \rangle_n$ is a *predictable* sequence) and is nondecreasing. We set $\langle S \rangle_\infty = \lim_n \langle S \rangle_n$. It turns out that the quadratic characteristic determines the asymptotic behavior of a martingale (in the same way that the area function in harmonic analysis), as the following theorem shows.

THEOREM. *Let $\{S_n\}$ be a p-adic ν-martingale in Q_0. Then the following statements hold.*

(i) $\{x \in Q_0 : \langle S \rangle_\infty(x) < \infty\} \subset \{x \in Q_0 : \exists \lim_n S_n < \infty\}$ *ν-a.e.*

(ii) (Law of the Iterated Logarithm) *Assume that $\{S_n\}$ has uniformly bounded increments; that is, assume $|X_k| \leq C$ for each k. Then*

$$\limsup_n \frac{|S_n(x)|}{\sqrt{2\langle S \rangle_n(x) \log\log\langle S \rangle_n(x)}} = 1$$

ν-a.e. on the set $\{x \in Q_0 : \langle S \rangle_\infty(x) = \infty\}$.

Part (i) is due to Doob, and part (ii) is one of the versions of the law of the iterated logarithm (LIL) as appears, for instance, in [St].

Now suppose that μ is a probability measure and take $\nu = m_N$ to be N-dimensional Lebesgue measure in Q_0. The following representation ([GoN, Lemma 2.1]; cf. [LN, Prop. 2.1]) explains the connections between measures and martingales.

PROPOSITION 5.1. *If μ is a probability measure on Q_0, then there exist a p-adic m_N-martingale $\{S_n\}$ and a nondecreasing predictable sequence $\{A_n\}$ such that*

$$\mu(Q_n) = \exp\{S_n - A_n\}.$$

Here, up to additive constants, $\{A_n\}$ and $\{S_n\}$ can be chosen to be

$$A_n(x) = p^{-N} \sum_{k=1}^{n} \sum_{i=1}^{p^N} \log \frac{\mu(Q_{k-1}(x))}{\mu(Q_k^i)}$$

and

$$S_n(x) = p^{-N} \sum_{k=1}^{n} \sum_{i=1}^{p^N} \log \frac{\mu(Q_k(x))}{\mu(Q_k^i)},$$

where (as usual) $\{Q_k^i\}$ is the p-adic decomposition of $Q_{k-1}(x)$.

The quadratic characteristic of the martingale $\{S_n\}$ in this proposition is given by

$$\langle S\rangle_n(x) = p^{-2N} \sum_{k=1}^{n} \sum_{i=1}^{p^N} \left(\sum_{j=1}^{p^N} \log \frac{\mu(Q_k^i)}{\mu(Q_k^j)} \right)^2;$$

as before, $\{Q_k\}$ denotes the p-adic "tower" of cubes containing x.

Now let $0 < \delta \le \delta_k \le p^{-N}$. We will apply martingale techniques to describe the asymptotic behavior of the measure μ constructed in Section 3. Let $\{S_n\}$ be as in Proposition 5.1. Observe that, since μ is doubling, $\{S_n\}$ has uniformly bounded increments. Furthermore, a simple computation shows:

$$A_n = nN \log p + \sum_{k=1}^{n} \left(\frac{p^N - 1}{p^N} \log \frac{p^N - 1}{p^N(1-\delta_k)} + \frac{1}{p^N} \log \frac{1}{p^N \delta_k} \right),$$

$$\langle S\rangle_n = p^{-2N} \sum_{k=1}^{n} \log^2 \left(\frac{\delta_k^{-1} - 1}{p^N - 1} \right).$$

Hereafter, we will denote

$$\alpha(\delta) = \frac{p^N - 1}{p^N} \log \frac{p^N - 1}{p^N(1-\delta)} + \frac{1}{p^N} \log \frac{1}{p^N \delta}, \tag{5.1}$$

$$\beta(\delta) = (p-1) \left(\frac{1-\delta}{p^N - 1} \right)^{1/N} + \delta^{1/N}. \tag{5.2}$$

Then α is decreasing in $(0, p^{-N})$ with $\alpha(0) = +\infty$ and $\alpha(p^{-N}) = 0$, and β is increasing in $(0, p^{-N})$ with $\beta(0) = (p-1)(p^N - 1)^{-1/N} < 1 = \beta(1)$. Let us consider two special choices of (δ_k).

(1) Set $\delta_k = \delta$ for each k, where $0 < \delta \le p^{-N}$. Then:

$$A_n = nN \log p + n\alpha(\delta),$$

$$\langle S\rangle_n = p^{-2N} n \log^2 \left(\frac{\delta^{-1} - 1}{p^N - 1} \right).$$

(2) Fix $E \subset \mathbb{N}$ (infinite), and choose $\delta_k = \delta$ if $k \in E$ and $\delta_k = p^{-N}$ otherwise. Then

$$A_n = nN \log p + \nu_n \alpha(\delta),$$

$$\langle S\rangle_n = p^{-2N} \nu_n \log^2\left(\frac{\delta^{-1} - 1}{p^N - 1}\right),$$

where $\nu_n = \#(E \cap [1, n])$.

We now apply the law of the iterated logarithm to the measures μ constructed in Section 3 from these particular choices. Since $\langle S\rangle_\infty = \infty$ pointwise in both cases, the LIL and the values of A_n and $\langle S\rangle_n$ allow us to deduce the following corollary.

COROLLARY 5.2. *There is* $C = C(N, p, \delta) > 0$ *such that*:

(i) *if* μ *is as in case* (1), *then*

$$\lim_n \frac{\mu(Q_n(x))}{p^{-n(N+\alpha(\delta)(\log p)^{-1})} \exp\{C\sqrt{n \log\log n}\}} = 0$$

for m_N*-a.e.* x;

(ii) *if* μ *is as in case* (2), *then*

$$\lim_n \frac{\mu(Q_n(x))}{p^{-nN} \exp\{C\sqrt{\nu_n \log\log \nu_n} - \alpha(\delta)\nu_n\}} = 0$$

for m_N*-a.e.* x.

In particular, μ *is singular in both cases.*

6. Proofs of Theorems 1 and 2

Proof of Theorem 1. Let $\varepsilon > 0$ and $\eta > 0$. Pick an odd integer p such that

$$\frac{\log \frac{p^N - 1}{(p-1)^N}}{\log p} < \varepsilon. \tag{6.1}$$

For this p, let us consider the functions $\alpha(\delta)$ and $\beta(\delta)$ given by (5.1) and (5.2), respectively. Then, since $\alpha(0) = \infty$, we can choose $0 < \delta \le p^{-N}$ such that

$$\alpha(\delta)(\log p)^{-1} > \varepsilon + \eta N.$$

Associated to p and to $\delta_k = \delta$, let μ be the doubling measure corresponding to case (1). Observe that (6.1) and the fact that $\beta \ge \beta(0) = (p-1)(p^N - 1)^{-1/N}$ together yield

$$\left(\frac{\psi_\varepsilon(p^{-k})}{\psi_\varepsilon(p^{-(k-1)})}\right)^{1/N} = p^{-\varepsilon/N} \le \beta(\delta),$$

so ψ_ε verifies condition (4.4). Since conditions (4.2) and (4.3) trivially hold, it follows from Theorem 4.1 that the metric

$$\rho(x, t) = \left(\frac{\mu(B(x, t))}{t^N \psi_\varepsilon(t)}\right)^{1/N}$$

verifies (H) and (GH) with constants depending only on N, ε, η. On the other hand, by Corollary 5.2, the choice of δ, and the doubling property, we have

$$\lim_{t\to 0} \frac{\mu(B(x, t))}{t^{N+\varepsilon+\eta N}} = 0$$

for m_N-a.e. $x \in \mathbb{R}^N$. Therefore,

$$\lim_{t\to 0} \frac{\rho(x,t)}{t^\eta} = 0$$

m_N-a.e. $x \in \mathbb{R}^N$. This finishes the proof of Theorem 1. $\square$

Proof of Theorem 2. Fix any odd integer p, and let $\psi_0\colon [0,1] \to [0,1]$ be as in the statement of the theorem. Choose $0 < \delta \le p^{-N}$ such that

$$\log \frac{1}{\beta(\delta)} < \alpha(\delta).$$

Let

$$M_n = \frac{\log(1/\psi_0(p^{-n}))}{\log(1/\beta(\delta))},$$

and let $E = \{k \in \mathbb{N} : [M_k] > [M_{k-1}]\}$ and $\nu_n = \#(E \cap [1,n])$ (here $[x]$ denotes integer part of x). Then E is infinite and $\nu_n \le M_n$ for all n. Set $\nu_0 = 0$ for convenience. Now we define

$$\psi(p^{-n}) = (\beta(\delta))^{\nu_n}.$$

Observe that $\psi(p^{-n}) \ge \psi_0(p^{-n})$ for $n = 0, 1, \dots$. Extend ψ to $[0,1]$ in such a way that ψ is nondecreasing and $\psi \ge \psi_0$. Note that, whatever the extension of ψ, it will satisfy a doubling condition of the type $\psi(pt) \le (\beta(\delta))^{-2}\psi(t)$ if $0 \le t \le p^{-1}$, so $\psi(2t) \le C\psi(t)$ where $C = C(N,p) > 0$. This shows that ψ satisfies (4.2). Since $\psi(1) = 1$, (4.3) also holds. Finally, if $\delta_k = \delta$ when $k \in E$ and $\delta_k = p^{-N}$ otherwise, then

$$\frac{\psi(p^{-k})}{\psi(p^{-(k-1)})} = (\beta(\delta))^{\nu_k - \nu_{k-1}} \le \beta(\delta_k)$$

for all k, so ψ satisfies condition (4.4), too; by Theorem 4.1, the metric

$$\rho(x,t) = \left(\frac{\mu(B(x,t))}{t^N\psi(t)}\right)^{1/N}$$

satisfies (H) and also (GH) with constants depending only on N.

Now, by Corollary 5.2, the definition of ψ, and the choice of δ, we have

$$\lim_n \frac{\mu(Q_n(x))}{p^{-nN}\psi(p^{-n})} \le \lim_n \exp\left\{C\sqrt{\nu_n \log\log \nu_n} - \left(\alpha(\delta) - \log\frac{1}{\beta(\delta)}\right)\nu_n\right\} = 0$$

m_N-a.e. x, and therefore

$$\lim_{t\to 0} \rho(x,t) = 0$$

m_N-a.e. x. $\square$

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