Michigan Math. J. 52 (2004)

Projective Geometry of Freudenthal's Varieties of Certain Type

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Dedicated to Professor Hiroshi Asano on the occasion of his 70th birthday

0. Introduction

H. Freudenthal constructed, in a series of his papers (see [10] and its references), the exceptional Lie algebras of type E_8, E_7, E_6, and F_4 by defining various projective varieties. The purpose of our work is to study projective geometry for his varieties of certain type, which are called *varieties of planes* in the symplectic geometry of Freudenthal (see [10, 4.11] and [23, 2.3]).

Let $\mathfrak{g}$ be a graded, simple, finite-dimensional Lie algebra over the complex number field $\mathbb{C}$ with grades between -2 and 2, $\dim \mathfrak{g}_2 = 1$, and $\mathfrak{g}_1 \neq 0$, namely, a *graded Lie algebra of contact type*: $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ (see Section 1). We set

$$\mathcal{V} := \{x \in \mathfrak{g}_1 \setminus \{0\} \mid (\operatorname{ad} x)^2 \mathfrak{g}_{-2} = 0\},$$

and define an algebraic set V in $\mathbb{P}(\mathfrak{g}_1)$ to be the projectivization of $\mathcal{V}$:

$$V := \pi(\mathcal{V}),$$

where $\pi : \mathfrak{g}_1 \setminus \{0\} \to \mathbb{P}(\mathfrak{g}_1)$ is the natural projection. Then we call $V \subseteq \mathbb{P}(\mathfrak{g}_1)$ (with the reduced structure) the *Freudenthal variety* associated to the graded Lie algebra $\mathfrak{g}$ of contact type, which is a natural generalization of Freudenthal's varieties mentioned previously. Note that V is not necessarily connected in this general setting. Moreover, we here consider the projectivization of a closed set $\{x \in \mathfrak{g}_1 \mid (\operatorname{ad} x)^{k+1} \mathfrak{g}_{-2} = 0\}$ and denote it by V_k; we have

$$\emptyset = V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 \subseteq V_4 = \mathbb{P},$$

where we set $\mathbb{P} := \mathbb{P}(\mathfrak{g}_1)$ for short. Clearly, V_3 is a quartic hypersurface, V_2 is an intersection of cubics, and $V_1 = V$ is an intersection of quadrics (with a few exceptions).

In the literature, several results have been known about the structure of $\mathfrak{g}_1$ as a $\mathfrak{g}_0$-space—case by case for each exceptional Lie algebra of types E_8, E_7, E_6, and F_4—from the viewpoint of the invariant theory of prehomogeneous vector spaces (see [13; 15; 19; 22]). By virtue of those results, it can be shown, for example, that the stratification of $\mathbb{P}$ given by the differences of the V_k exactly corresponds to the

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Received August 7, 2002. Revision received May 15, 2004.

Both of the authors were supported in part by Grant-in-Aid for Scientific Research (C), no. 10640046, Japan Society for the Promotion of Science. The first author was supported in part by Grant for Special Research Projects, Individual Research, no. 99A-187, Waseda University.

orbit decomposition of the $\mathfrak{g}_0$-space $\mathfrak{g}_1$ for those exceptional Lie algebras, and also that Freudenthal varieties V associated to the algebras of type E_8, E_7, E_6, and F_4 are projectively equivalent to (respectively) the 27-dimensional E_7-variety arising from the 56-dimensional irreducible representation, the orthogonal Grassmann variety of isotropic 6-planes in $\mathbb{C}^{12}$ (namely, the 15-dimensional spinor variety), the Grassmann variety of 3-planes in $\mathbb{C}^6$, and the symplectic Grassmann variety of isotropic 3-planes in $\mathbb{C}^6$—with $\dim \mathbb{P} = 55$, 31, 19, and 13, respectively (see Appendix); for those homogeneous projective varieties, we refer to [12, Sec. 23.3].

In this article we study the Freudenthal varieties V with the filtration $\{V_k\}$ of the ambient space $\mathbb{P}$ from the viewpoint of projective geometry, not individually but systematically in terms of abstract Lie algebras, without depending on the classification of simple Lie algebras or on the known results for each case of types E_8, E_7, E_6, and F_4.

Before stating the main result, we note that the Lie bracket $\mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_2 \simeq \mathbb{C}$ defines a nondegenerate skew-symmetric form on $\mathfrak{g}_1$, so that this form allows us to identify $\mathfrak{g}_1$ with its dual space (and hence $\mathbb{P}$ with its dual space), and $\mathfrak{g}_1$ is even-dimensional. Moreover, the quartic form on $\mathfrak{g}_1$ defining V_3 has a differential that, via the symplectic form, defines a vector field on $\mathfrak{g}_1$, and this vector field defines a 1-dimensional distribution on $\mathbb{P}$ away from the singular locus of V_3 (see Proposition A1). We denote by L_P the (closure of the) integral curve of this distribution passing through $P \in \mathbb{P} \setminus \operatorname{Sing} V_3$. On the other hand, we have a rational map $\gamma \colon \mathbb{P} \dashrightarrow \mathbb{P}$ defined by $x \mapsto (\operatorname{ad} x)^3 \mathfrak{g}_{-2}$ with base locus V_2, which turns out to be a Cremona transformation of $\mathbb{P}$. It is deduced that $\gamma^{-1}(V) = V_3 \setminus V_2$, $\gamma^{-1}(\mathbb{P} \setminus V_3) = \mathbb{P} \setminus V_3$, $\gamma^2 = 1$ on $\mathbb{P} \setminus V_3$, and γ is explicitly given by the partial differentials of q (see Proposition A2). Note that our γ is a special case of the Cremona transformations in [7, Thm. 2.8(ii)].

Our main results are summarized as follows (see Theorems A, B, C, D, and E as well as Corollaries A2, B1, B3, and C).

THEOREM. *Assume that V is irreducible. Then the following statements hold.*

(1) *V is a Legendrian subvariety of $\mathbb{P}$ (i.e., the projectivization of a Lagrangian subvariety of $\mathfrak{g}_1$) with $\dim V = n - 1$, spans $\mathbb{P}$, and is an orbit of the group of inner automorphisms of $\mathfrak{g}$ with Lie algebra $\mathfrak{g}_0$, and hence smooth, where $\dim \mathfrak{g}_1 = 2n$. In particular, the projective dual V^* of V is equal to the union of tangents to V via the symplectic form.*

(2) *V_2 is the singular locus of V_3; for any $P \in \mathbb{P} \setminus V_2$, L_P is the line in $\mathbb{P}$ joining P and $\gamma(P)$.*
   (a) *If $P \in \mathbb{P} \setminus V_3$, then L_P is a unique secant line of V passing through P, there is no tangent line to V passing through P, $L_P \cap V$ consists of harmonic conjugates with respect to P and $\gamma(P)$, and $L_P \setminus V \subseteq \mathbb{P} \setminus V_3$. Moreover, γ preserves L_P, and the automorphism of L_P induced from γ leaves each point in $L_P \cap V$ invariant and also permutes P and $\gamma(P)$.*
   (b) *If $P \in V_3 \setminus V_2$, then there is no secant line of V passing through P, L_P is a unique tangent line to V passing through P, $L_P \cap V = \gamma(P)$, and $L_P \setminus V \subseteq V_3 \setminus V_2$. Moreover, L_P is contracted by γ to the contact point*

$\gamma(P)$, and conversely the fibre of γ on $Q \in V$ consists of the points $P \in V_3 \setminus V_2$ such that $Q \in L_P$, or (equivalently) P lies on some tangent to V at Q.

In particular, V is a variety with one apparent double point, and V_3 is the union of tangents to V.

(3) *For any $P \in V_2 \setminus V$, the family of secants of V passing through P is of dimension at least 1, and all of those secants are isotropic with respect to the symplectic form. In particular, $V_2 \setminus V$ is covered by isotropic secants of V.*

(4) *For any $Q, R \in V$, the secant line joining Q and R is isotropic if and only if the tangents to V at Q and at R are disjoint.*

(5) *For any $P \in V_3 \setminus V_2$ and $Q \in V$, if the secant line joining Q and the contact point $\gamma(P)$ of L_P is not isotropic, then there is a twisted cubic curve contained in V to which L_P and L_R are tangent at $\gamma(P)$ and at Q (respectively), where R is a point on some tangent to V at Q away from V_2, determined by P and Q.*

(6) *If $V_2 \neq V$, then V is ruled—that is, covered by lines contained in V.*

(7) *For any $P \in V$, the double projection from P gives a birational map from V onto $\mathbb{P}^{n-1}$, and by the inverse map V is written as the closure of the image of a cubic Veronese embedding of a certain affine space $\mathbb{A}^{n-1}$ under some projection to $\mathbb{P}$.*

We show also that the three conditions $V = \emptyset$, $V_3 = \mathbb{P}$, and $V_2 = \mathbb{P}$ are equivalent to each other (Corollary A1) and that, if V is neither empty nor irreducible, then $\mathfrak{g}_1$ decomposes naturally into two irreducible $\mathfrak{g}_0$-submodules of dimension n and V is the (disjoint) union of the projectivizations of those summands (Corollary B2).

The contents of this paper are organized as follows. In Section 1 we give some preliminaries on graded Lie algebras of contact type, and we define a certain symmetric product $\mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$ and a ternary product in $\mathfrak{g}_1$ induced from it in order to avoid a raging flood of Lie brackets. In the literature, several authors (see [8; 9; 20; 28]) have introduced various ternary products in $\mathfrak{g}_1$ in the process of establishing their theories of triple systems. However, those ternary products themselves are essentially the same, as is easily seen. In this paper we use the one first introduced by Yamaguti and Asano [27], since their product seems suitable for our computation. In Section 2 we prove some basic results on $\mathfrak{g}_1$ with those products and the symplectic form.

In Section 3 we discuss the line field with the Cremona transformation, study the relationship between V and the family of lines L_P, and investigate the case $V = \emptyset$. In Section 4 we establish the homogeneity of V and show that $\mathcal{V}$ is a Lagrangian subvariety of $\mathfrak{g}_1$ with respect to the symplectic form; the homogeneity of V is deduced from that of $\mathbb{P} \setminus V_3$ (see Claim in the proof of Theorem B and Remark B1). Using those results, we study the irreducibility and the ruledness of V. In Sections 5 and 6 we discuss isotropic secants of V and double projections of V, respectively. We also give a geometric meaning of the ternary product we use.

In Section 7, we study the intersection of V and a certain linear subspace of dimension 3, which turns out to be the twisted cubic curve already mentioned. In the study of the ruledness and isotropic secants of V (as well as the twisted cubic curves in V), a rational map $\Phi_P\colon \mathbb{P} \dashrightarrow \mathbb{P}$ defined by $x \mapsto (\operatorname{ad} x) \circ (\operatorname{ad} a)^2 \mathfrak{g}_{-2}$ plays a key role, where $P = \pi(a)$ is a point in $V_2 \setminus V$ or in $V_3 \setminus V_2$ (see (the proof of) Corollary B3 as well as Propositions C and E and Theorem E). For instance, the point R in part (5) of the summary Theorem is written explicitly as $\Phi_P(Q)$. We give in the Appendix a classification of Freudenthal varieties for the convenience of the reader: the classification follows from known results and some direct computations.

Finally we should mention that S. Mukai announced a theorem [21, (5.8)] on cubic Veronese varieties without proofs. Our work was originated by looking for proofs of the corresponding statements for Freudenthal varieties (Corollaries A2, B1, and C and Theorem D). In fact, we see from his list [21, (5.10)] of cubic Veronese varieties (and the list in our Appendix) that the notion of our Freudenthal varieties coincides with that of his cubic Veronese varieties. Our result gives a partial explanation for this coincidence (see Theorem D).

1. Preliminaries

For a finite-dimensional, simple Lie algebra $\mathfrak{g}$ of rank ≥ 2, a graded decomposition of contact type is obtained as follows. Take a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a basis Δ of the root system R with respect to $\mathfrak{h}$, and fix an order on R defined by Δ. Denote by ρ the highest root of $\mathfrak{g}$, let E_+ and E_- be highest and lowest weight vectors (respectively), and set $H := [E_+, E_-]$. By multiplying suitable scalars, one may assume that (E_+, H, E_-) form an $\mathfrak{sl}_2$-triple; that is, those vectors have the following standard relations:

$$[H, E_+] = 2E_+, \quad [H, E_-] = -2E_-, \quad [E_+, E_-] = H.$$

Then, the eigenspace decomposition of $\mathfrak{g}$ with respect to $\operatorname{ad} H$ gives $\mathfrak{g}$ a graded decomposition of contact type. In other words, if we set $\mathfrak{g}_\lambda := \{x \in \mathfrak{g} \mid [H, x] = \lambda x\}$ for $\lambda \in \mathbb{C}$, then it follows that $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, $\dim \mathfrak{g}_2 = 1$, and $\mathfrak{g}_1 \neq 0$; in fact, $\mathfrak{g}_1 = 0$ if and only if $\mathfrak{g} = \mathfrak{sl}_2$. In terms of root spaces of $\mathfrak{g}$, we have

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+ \setminus (R_\rho \cup \{\rho\})} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}),$$

$$\mathfrak{g}_{\pm 1} = \bigoplus_{\alpha \in R_\rho} \mathfrak{g}_{\pm\alpha}, \qquad \mathfrak{g}_{\pm 2} = \mathfrak{g}_{\pm\rho} = \mathbb{C}E_\pm,$$

where R^+ is the set of positive roots and $R_\rho := \{\alpha \in R^+ \mid \rho - \alpha \in R\}$. Indeed, let $\mathfrak{s}_\rho$ be the subalgebra of $\mathfrak{g}$ spanned by E_+, H, and E_- that is isomorphic to $\mathfrak{sl}_2$. Then the irreducible decomposition of $\mathfrak{g}$ as an $\mathfrak{sl}_2$-module gives the decomposition just displayed (see [26] for full details). Conversely, for a graded decomposition $\mathfrak{g} = \sum \mathfrak{g}_i$ of contact type, taking suitable bases E_+ for $\mathfrak{g}_2$ and E_- for $\mathfrak{g}_{-2}$ with $H := [E_+, E_-]$, one may assume that (E_+, H, E_-) form an $\mathfrak{sl}_2$-triple, as before. Then, we see that E_+ and E_- are some highest and lowest weight vectors (respectively) and that each $\mathfrak{g}_i$ is recovered as an $(\operatorname{ad} H)$-eigenspace. Therefore, the graded decompositions of contact type are unique up to automorphism of $\mathfrak{g}$, so the Freudenthal variety V is essentially unique and determined by $\mathfrak{g}$ itself (see Appendix).

Now we define a symmetric product $\times: \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$ by the formula

$$-2a \times b = [b, [a, E_-]] + [a, [b, E_-]],$$

which induces a symmetric map $L: \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathrm{Hom}(\mathfrak{g}_1, \mathfrak{g}_1)$ and a ternary product $[\cdot,\cdot,\cdot]: \mathfrak{g}_1 \times \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_1$ by

$$[a, b, c] = L(a, b)c = [a \times b, c].$$

Note that the adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is faithful because $\mathfrak{g}$ is simple (see [26, Lemma 3.2(1)]); we may assume $\mathfrak{g}_0 \subseteq \mathrm{Hom}(\mathfrak{g}_1, \mathfrak{g}_1)$, so that we identify $L(a, b)$ with $a \times b$. We think of $\mathfrak{g}_1$ as an $\mathfrak{g}_0$-module via the adjoint action. For example, we often write Dx instead of $(\mathrm{ad}\, D)x$ and $[D, x]$ for $D \in \mathfrak{g}_0$ and $x \in \mathfrak{g}_1$. As the skew-symmetric form $\langle\cdot,\cdot\rangle: \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathbb{C}$ and the quartic form on $\mathfrak{g}_1$ defining V_3 (mentioned in Introduction), we use the ones determined by

$$2\langle a, b\rangle E_+ = [a, b], \qquad 2q(x)E_+ = (\mathrm{ad}\, x)^4 E_-.$$

Note that the skew-symmetric form $\langle\cdot,\cdot\rangle$ is *nondegenerate* because $\mathfrak{g}$ is simple (see [26, Lemma 3.2(2)]).

With the notation used so far, it follows that

$$\begin{aligned} V = V_1 &= \pi(\{x \in \mathfrak{g}_1 \setminus \{0\} \mid x \times x = 0\}), \\ V_2 &= \pi(\{x \in \mathfrak{g}_1 \setminus \{0\} \mid [xxx] = 0\}), \\ V_3 &= \pi(\{x \in \mathfrak{g}_1 \setminus \{0\} \mid \langle x, [xxx]\rangle = 0\}), \end{aligned}$$

and $q(x) = \langle x, [xxx]\rangle$. Note that $V_0 = \emptyset$ since $[[x, E_-]E_+] = x$ for any $x \in \mathfrak{g}_1$; indeed, it follows from the Jacobi identity that $[[x, E_-]E_+] = [[x, E_+], E_-] + [x[E_-, E_+]] = [x, -H] = x$ since $[x, E_+] \in \mathfrak{g}_3 = 0$. On the other hand, it follows from Lemma 1 that $V \neq \mathbb{P}$.

LEMMA 1. *Let $\mathfrak{g}_{00}$ be the subalgebra of $\mathfrak{g}_0$ defined by*

$$\mathfrak{g}_{00} := \mathrm{Ker}(\mathrm{ad}\, E_+|\mathfrak{g}_0) = \mathrm{Ker}(\mathrm{ad}\, E_-|\mathfrak{g}_0).$$

Then we have $\mathfrak{g}_0 = \mathfrak{g}_{00} \oplus \mathbb{C}H$, and $\mathfrak{g}_{00}$ is linearly spanned by the elements in $\mathfrak{g}_0$ of the form $a \times b$ with $a, b \in \mathfrak{g}_1$. In particular, $\mathfrak{g}_{00} \neq 0$, and $x \times x \neq 0$ for some $x \in \mathfrak{g}_1$.

Proof. First of all, we have $DE_- = 0$ if and only if $DE_+ = 0$ for any $D \in \mathfrak{g}_0$. Indeed, we have $DE_\pm \in [\mathfrak{g}_0, \mathfrak{g}_{\pm 2}] \subseteq \mathfrak{g}_{\pm 2} = \mathbb{C}E_\pm$, so that $DE_\pm = \lambda_\pm E_\pm$ for some $\lambda_\pm \in \mathbb{C}$. Then

$$0 = DH = D[E_+, E_-] = [DE_+, E_-] + [E_+, DE_-] = \lambda_+ H + \lambda_- H,$$

so that $\lambda_+ + \lambda_- = 0$; hence the claim follows. The decomposition follows from the short exact sequence

$$0 \longrightarrow \mathfrak{g}_{00} \longrightarrow \mathfrak{g}_0 \xrightarrow{\operatorname{ad} E_+|\mathfrak{g}_0} \mathfrak{g}_2 \longrightarrow 0,$$

which splits by $\operatorname{ad} E_-|\mathfrak{g}_2$. We next have $a \times b \in \mathfrak{g}_{00}$ for any $a, b \in \mathfrak{g}_1$. Indeed, it follows from the Jacobi identity that

$$\begin{aligned} -2[a \times b, E_-] &= [[b[a, E_-]]E_-] + [[a[b, E_-]]E_-] \\ &= ([[b, E_-][a, E_-]] + [b[[a, E_-]E_-]]) \\ &\quad + ([[a, E_-][b, E_-]] + [a[[b, E_-]E_-]]) \\ &= [b[[a, E_-]E_-]] + [a[[b, E_-]E_-]], \end{aligned}$$

which is equal to zero since we have $[[a, E_-]E_-], [[b, E_-]E_-] \in \mathfrak{g}_{-3} = 0$.

Now, it follows from the definition of $\langle \cdot, \cdot \rangle$ that $2\langle a, b\rangle H = [[a, b]E_-]$, so that

$$2(a \times b + \langle a, b\rangle H) = -[b[a, E_-]] - [a[b, E_-]] + [[a, b]E_-] = -2[b[a, E_-]]$$

by the Jacobi identity; that is, we obtain

$$[[a, E_-]b] = a \times b + \langle a, b\rangle H. \tag{1.1}$$

On the other hand, since $\mathfrak{g}$ is simple (see [26, Lemma 3.1]), we have

$$[[\mathfrak{g}_1, E_-], \mathfrak{g}_1] = \mathfrak{g}_0 = \mathfrak{g}_{00} \oplus \mathbb{C}H.$$

Thus $\mathfrak{g}_{00}$ is spanned by the products $a \times b$.

Finally, if $x \times x = 0$ for any $x \in \mathfrak{g}$, then $0 = (a + b) \times (a + b) = 2a \times b$. That is, $a \times b = 0$ for any $a, b \in \mathfrak{g}_1$, so that $\mathfrak{g}_{00} = 0$ by our previous reasoning. If $\mathfrak{g}_{00} = 0$ then $\mathfrak{h} = \mathfrak{g}_0 = \mathbb{C}H$, hence $\operatorname{rk} \mathfrak{g} = 1$. Therefore, it follows from our assumption $\operatorname{rk} \mathfrak{g} \geq 2$ that $\mathfrak{g}_{00} \neq 0$, and that $x \times x \neq 0$ for some $x \in \mathfrak{g}_1$. □

LEMMA 2 (Asano [3]). *For any $a, b, c \in \mathfrak{g}_1$ and $D \in \mathfrak{g}_{00}$, we have*:

(1) $\langle Da, b\rangle + \langle a, Db\rangle = 0$;
(2) $D(a \times b) = Da \times b + a \times Db$;
(3) $D[abc] = [(Da)bc] + [a(Db)c] + [ab(Dc)]$.

Proof. (1) From the definition of $\mathfrak{g}_{00}$ and the Jacobi identity, we have

$$0 = 2\langle a, b\rangle DE_+ = D[a, b] = [Da, b] + [a, Db].$$

(2) Similarly, we obtain

$$\begin{aligned} -2D(a \times b) &= D[b[a, E_-]] + D[a[b, E_-]] \\ &= [Db, [a, E_-]] + [b, D[a, E_-]] + [Da, [b, E_-]] + [a, D[b, E_-]] \\ &= [Db, [a, E_-]] + [b[Da, E_-]] + [Da, [b, E_-]] + [a[Db, E_-]] \\ &= -2Da \times b - 2a \times Db. \end{aligned}$$

(3) It follows from (2) that

$$(D(a \times b))c = [Da \times b, c] + [a \times Db, c] = [(Da)bc] + [a(Db)c].$$

On the other hand, it follows from the Jacobi identity that

$$(D(a \times b))c = [Dc, a \times b] + D([a \times b, c]) = -[ab(Dc)] + D[abc].$$

Combining these formulas, we obtain the result. □

We denote by G_{00} the group of inner automorphisms of $\mathfrak{g}$ with Lie algebra $\mathfrak{g}_{00}$. Then Lemma 2 tells us that the symplectic form $\langle\cdot,\cdot\rangle$, the symmetric product $\times$, and the ternary product $[\cdot,\cdot,\cdot]$ are equivariant with respect to the action of G_{00}, so that each V_i is stable under the action of G_{00} (i.e., a union of some orbits of G_{00}). We should mention that the proofs here of (2) and (3) in Lemma 2 are due to the referee and are much simpler than the proofs in [3].

LEMMA 3 (Asano [3]). *We have* $[abc]-[acb]=\langle a,c\rangle b-\langle a,b\rangle c+2\langle b,c\rangle a$ *for any* $a,b,c\in\mathfrak{g}_1$.

Proof. Using (1.1) and the Jacobi identity yields

$$\begin{aligned}([abc]+\langle a,b\rangle c)-([acb]+\langle a,c\rangle b) &= [[[a,E_-]b]c]-[[[a,E_-]c]b]\\ &= [[a,E_-][b,c]]\\ &= 2\langle b,c\rangle[[a,E_-]E_+]=2\langle b,c\rangle a. \qquad \square\end{aligned}$$

2. Basic Results

PROPOSITION 1. *If* $x\in\mathcal{V}$, *then the following statements hold.*

(1) $[axx]=3\langle a,x\rangle x$ *for any* $a\in\mathfrak{g}_1$; *in particular, if* $a\times x=0$ *then* $\langle a,x\rangle=0$ (see [2]).
(2) $\mathbb{C}x\subseteq\mathfrak{g}_{00}x$.

Proof. (1) Because $x\in\mathcal{V}$, it follows from Lemma 3 that

$$[axx]=[xax]=[xxa]+\langle x,x\rangle a-\langle x,a\rangle x+2\langle a,x\rangle x=3\langle a,x\rangle x.$$

(2) This follows from (1) since $\langle\cdot,\cdot\rangle$ is not identically zero. □

PROPOSITION 2. *We have* $\langle[abc],d\rangle=\langle[cda],b\rangle$ *for any* $a,b,c,d\in\mathfrak{g}_1$.

Proof. It follows from Lemma 3 that

$$\langle[abc],d\rangle=\langle[acb],d\rangle+\langle a,c\rangle\langle b,d\rangle-\langle a,b\rangle\langle c,d\rangle+2\langle b,c\rangle\langle a,d\rangle,$$
$$\langle[cda],b\rangle=\langle[cad],b\rangle+\langle c,a\rangle\langle d,b\rangle-\langle c,d\rangle\langle a,b\rangle+2\langle d,a\rangle\langle c,b\rangle,$$

so that $\langle[abc],d\rangle-\langle[cda],b\rangle=\langle[acb],d\rangle-\langle[acd],b\rangle=0$ by Lemma 2(1). □

PROPOSITION 3. *If* $x\in\mathcal{V}$ *and* $D,E\in\mathfrak{g}_{00}$, *then*:

(1) $Dx\times x=0$ [2];
(2) $\langle Dx,x\rangle=0$;
(3) $\langle Dx,Ex\rangle=0$;
(4) $[(Dx)(Ex)x]=0$.

Proof. (1) It follows from $x\in\mathcal{V}$ and Lemma 2(2) that

$$0=D(x\times x)=Dx\times x+x\times Dx=2Dx\times x.$$

(2) This follows from (1) and Proposition 1(1).

(3) It follows from Proposition 2 and (1) that $\langle [abx], [cdx] \rangle = \langle [x[cdx]a], b \rangle = 0$ for any $a, b, c, d \in \mathfrak{g}_1$, so that $\langle Dx, Ex \rangle = 0$ by Lemma 1.

(4) It follows from Lemma 3 that

$$[(Dx)(Ex)x] = [(Dx)x(Ex)] + \langle Dx, x \rangle Ex - \langle Dx, Ex \rangle x + 2\langle Ex, x \rangle Dx,$$

which is equal to zero. Indeed, we have $Dx \times x = 0$ by (1), $\langle Dx, x \rangle = \langle Ex, x \rangle = 0$ by (2), and $\langle Dx, Ex \rangle = 0$ by (3). □

PROPOSITION 4. *For any $a \in \mathfrak{g}_1$, we have*:

(1) $[aaa] \times a = 0$;
(2) $[aa[aaa]] = 3q(a)a$;
(3) $[aaa] \times [aaa] = -3q(a)a \times a$;
(4) $[[aaa][aaa][aaa]] = -9q(a)^2 a$;
(5) $q([aaa]) = 9q(a)^3$.

Proof. (1) It follows from Lemma 2(2) that

$$0 = [a \times a, a \times a] = [aaa] \times a + a \times [aaa] = 2[aaa] \times a.$$

(2) It follows from Lemma 3 and (1) that

$$[aa[aaa]] = [a[aaa]a] + 3\langle a, [aaa] \rangle a = 3q(a)a.$$

(3) It follows from Lemma 2(2) that

$$[a \times a, [aaa] \times a] = [aa[aaa]] \times a + [aaa] \times [aaa].$$

Using (1) and (2), we obtain $0 = 3q(a)a \times a + [aaa] \times [aaa]$.

(4) It follows from (3) and (2) that

$$[[aaa][aaa][aaa]] = -3q(a)[aa[aaa]] = -9q(a)^2 a.$$

(5) This follows from (4). □

PROPOSITION 5. *If $b = a + x$ with $a \in \mathfrak{g}_1$ and $x \in \mathcal{V}$, then*:

(1) $b \times b = a \times a + 2a \times x$;
(2) $[bbb] = [aaa] + 3[aax] + 6\langle x, a \rangle (a - x)$;
(3) $q(b) = q(a) + 4\langle x, [aaa] \rangle + 12\langle x, a \rangle^2$.

Proof. (1) This is clear from $x \in \mathcal{V}$.

(2) It follows from (1) that $[bbb] = [aaa] + [aax] + 2[axa] + 2[axx]$. Since $[axx] = 3\langle a, x \rangle x$ by Proposition 1(1), by Lemma 3 we have

$$\begin{aligned}[bbb] = [aaa] + [aax] + 2([aax] + \langle a, a \rangle x - \langle a, x \rangle a + 2\langle x, a \rangle a) \\ + 2(3\langle a, x \rangle x) \\ = [aaa] + 3[aax] + 6\langle x, a \rangle a + 6\langle a, x \rangle x.\end{aligned}$$

(3) It follows from (2) that

$$\begin{aligned} q(b) &= \langle a,[aaa]\rangle + \langle x,[aaa]\rangle + 3\langle a,[aax]\rangle + 3\langle x,[aax]\rangle \\ &\quad + 6\langle x,a\rangle\langle a+x,a-x\rangle \\ &= q(a) + 4\langle x,[aaa]\rangle + 12\langle x,a\rangle^2, \end{aligned}$$

since we have $\langle a,[aax]\rangle = -\langle [aaa],x\rangle$ by Lemma 2(1) and $\langle x,[aax]\rangle = 0$ by Proposition 3(2). □

PROPOSITION 6. *For any* $a \in \mathfrak{g}_1$, *we have*:

(1) $3[aa[aab]] = 8\langle b,[aaa]\rangle a + 8\langle a,b\rangle[aaa] + \langle a,[aaa]\rangle b$ *for any* $b \in \mathfrak{g}_1$;
(2) *if* $q(a) \neq 0$, *then the linear map* $L(a,a)$ *has full rank.*

Proof. (1) It follows from Lemma 3 that

$$[aa[aab]] = [aa[aba]] + 3\langle a,b\rangle[aaa]. \tag{2.1}$$

It also follows from Lemma 3 that $[aa[aba]] = [a[aba]a] + 3\langle a,[aba]\rangle a$. Since $\langle a,[aba]\rangle = \langle b,[aaa]\rangle$ by Proposition 2, we have

$$[aa[aba]] = [a[aba]a] + 3\langle b,[aaa]\rangle a. \tag{2.2}$$

On the other hand, it follows from Lemma 2(3) that $[ab[aaa]] = 2[a[aba]a] + [aa[aba]]$, and by Lemma 3 and Proposition 4(1) we have

$$[ab[aaa]] = \langle a,[aaa]\rangle b - \langle a,b\rangle[aaa] + 2\langle b,[aaa]\rangle a.$$

Therefore,

$$2[a[aba]a] + [aa[aba]] = \langle a,[aaa]\rangle b - \langle a,b\rangle[aaa] + 2\langle b,[aaa]\rangle a. \tag{2.3}$$

Thus, it follows from (2.2) and (2.3) that

$$3[aa[aba]] = \langle a,[aaa]\rangle b - \langle a,b\rangle[aaa] + 8\langle b,[aaa]\rangle a.$$

Combining this formula with (2.1), we obtain the required result.

(2) (Asano [5]) Note that $\mathfrak{g}_1 = \mathbb{C}a \oplus \mathbb{C}[aaa] \oplus (\mathbb{C}a \oplus \mathbb{C}[aaa])^\perp$ if $q(a) \neq 0$. Indeed, it follows from $\langle a,[aaa]\rangle \neq 0$ that

$$\mathbb{C}a \cap \mathbb{C}[aaa] = (\mathbb{C}a + \mathbb{C}[aaa]) \cap (\mathbb{C}a + \mathbb{C}[aaa])^\perp = \{0\}$$

and from the nondegeneracy of $\langle\cdot,\cdot\rangle$ that

$$\dim(\mathbb{C}a + \mathbb{C}[aaa]) + \dim(\mathbb{C}a + \mathbb{C}[aaa])^\perp = \dim \mathfrak{g}_1.$$

Now, it is clear from (1) that

$$L(a,a)^2|_{\mathbb{C}a+\mathbb{C}[aaa]} = 3q(a)1_{\mathbb{C}a+\mathbb{C}[aaa]},$$

$$L(a,a)^2|_{(\mathbb{C}a+\mathbb{C}[aaa])^\perp} = \frac{q(a)}{3}1_{(\mathbb{C}a+\mathbb{C}[aaa])^\perp}.$$

Therefore $L(a,a)^2$ has full rank if $q(a) \neq 0$, hence $L(a,a)$ does also. □

PROPOSITION 7. *For any $a \in \mathfrak{g}_1$ and $x \in \mathcal{V}$, we have*:

(1) $[aaa] \times x + 3[aax] \times a + 6\langle x, a\rangle a \times a = 0$;

(2) $3[aax] \times [aax] + 8\langle x, [aaa]\rangle a \times x - 8\langle x, a\rangle [aaa] \times x = 0$. *In particular, if* $[aaa] = 0$, *then* $[aax] \times [aax] = 0$ *and, moreover,* $\mathbb{C}x + \mathbb{C}[aax] \subseteq \mathcal{V} \cup \{0\}$.

Proof. (1) If $b = a + x$, then

$$\begin{aligned} 0 &= [bbb] \times b \quad (\because \text{Proposition 4(1)}) \\ &= ([aaa] + 3[aax] + 6\langle x, a\rangle (a - x)) \times (a + x) \quad (\because \text{Proposition 5(2)}) \\ &= [aaa] \times x + 3[aax] \times a + 6\langle x, a\rangle a \times a \quad (\because \text{Propositions 4(1), 3(1)}), \end{aligned}$$

where in the last equality we have also used that $x \in \mathcal{V}$.

(2) It follows from Proposition 3(1) and Lemma 2(2) that

$$[aa[aax]] \times x + [aax] \times [aax] = 0.$$

On the other hand, it follows from Proposition 6(1) that

$$3[aa[aax]] \times x = 8\langle x, [aaa]\rangle a \times x - 8\langle x, a\rangle [aaa] \times x.$$

Thus we have the desired result. If $[aaa] = 0$ then $[aax] \times [aax] = 0$, and it follows from Proposition 3(1) that $[aax] \times x = 0$, so the latter assertion follows as well. □

3. A Line Field and a Cremona Transformation

PROPOSITION A1.

(1) *The quartic form q on $\mathfrak{g}_1$ has a differential at $a \in \mathfrak{g}_1$ as follows*:

$$dq(a)\colon t_a\mathfrak{g}_1 \to \mathbb{C}; \quad b \mapsto 4\langle b, [aaa]\rangle,$$

where $t_a\mathfrak{g}_1$ is the Zariski tangent space to $\mathfrak{g}_1$ at a, which is naturally identified with $\mathfrak{g}_1$.

(2) *In particular, the singular locus of V_3 is equal to V_2.*

(3) *The vector field on $\mathfrak{g}_1$ corresponding to dq via the symplectic form $\langle\cdot,\cdot\rangle$ induces a 1-dimensional distribution $\mathcal{D}$ on $\mathbb{P}$ away from* $\operatorname{Sing} V_3 = V_2$ *that is given by*

$$\mathcal{D}\colon \pi(a) \mapsto (\mathbb{C}a + \mathbb{C}[aaa])/\mathbb{C}a,$$

where $\pi(a) \in \mathbb{P} \setminus V_2$ and we naturally identify the Zariski tangent space $t_{\pi a}\mathbb{P}$ with the quotient space $\mathfrak{g}_1/\mathbb{C}a$.

Proof. (1) By Lemma 3,

$$[baa] = [aba] = [aab] - 3\langle a, b\rangle a. \tag{3.1}$$

Since $\langle a, [aab]\rangle = \langle b, [aaa]\rangle$ by Lemma 2(1), it follows that

$$\langle a, [baa]\rangle = \langle a, [aba]\rangle = \langle a, [aab]\rangle = \langle b, [aaa]\rangle. \tag{3.2}$$

Therefore, for $\lambda \in \mathbb{C}$ and $b \in \mathfrak{g}_1$,

$$\begin{aligned}\frac{1}{\lambda}\{q(a+\lambda b) - q(a)\} &= \langle b, [aaa]\rangle + \langle a, [baa]\rangle + \langle a, [aba]\rangle + \langle a, [aab]\rangle \\ &\quad + \lambda(\cdots) \\ &= 4\langle b, [aaa]\rangle + \lambda(\cdots) \\ &\to 4\langle b, [aaa]\rangle, \quad \lambda \to 0.\end{aligned}$$

(2) The hypersurface V_3 defined by $q = 0$ is singular at $\pi(a)$ if and only if $dq(a) = 0$, which is equivalent to $\langle b, [aaa]\rangle = 0$ for any $b \in \mathfrak{g}_1$ by (1). That is, $[aaa] = 0$ because $\langle\cdot,\cdot\rangle$ is nondegenerate.

(3) This is an immediate consequence of (1); indeed, the differential dq corresponds via $\langle\cdot,\cdot\rangle$ to a vector field on $\mathfrak{g}_1$ given by $a \mapsto [aaa]$ up to scalar multiple. □

PROPOSITION A2. *Let*

$$\gamma : \mathbb{P} \dashrightarrow \mathbb{P}$$

be a rational map induced from the cubic, $a \mapsto [aaa]$. Then:

(1) $\gamma^{-1}(V) = V_3 \setminus V_2$;
(2) $\gamma^{-1}(\mathbb{P} \setminus V_3) = \mathbb{P} \setminus V_3$;
(3) $\gamma^2 = 1$ *on* $\mathbb{P} \setminus V_3$, *hence* γ *gives an automorphism of* $\mathbb{P} \setminus V_3$;
(4) γ *is explicitly given by the partial differentials of* q.

In particular, γ *is a Cremona transformation of* $\mathbb{P}(\mathfrak{g}_1)$ *with order* 2 *if* $V_2 \neq \mathbb{P}$.

Proof. The assertions (1), (2), and (3) follow (respectively) from part (3), (5), and (4) of Proposition 4. Assertion (4) follows from Proposition A(1). □

A *secant line* of V is by definition a line in $\mathbb{P}$ that passes through at least two distinct points of V yet is not contained in V. We note that, for a line L in $\mathbb{P}$, if the scheme-theoretic intersection $L \cap V$ has length more than 2 then $L \subseteq V$; indeed, V is an intersection of quadric hypersurfaces.

THEOREM A. *Let L_P be the closure of the integral curve of $\mathcal{D}$ through $P \in \mathbb{P} \setminus V_2$, where $\mathcal{D}$ is the* 1*-dimensional distribution on $\mathbb{P} \setminus V_2$ induced from the quartic form q. Then the following statements hold.*

(1) *For any $P \in \mathbb{P} \setminus V_2$, L_P is the line in $\mathbb{P}$ joining P and $\gamma(P)$.*
(2) *If $P \in \mathbb{P} \setminus V_3$, then*:
    (a) *L_P is a secant line of V, and $L_P \cap V$ consists of harmonic conjugates with respect to P and $\gamma(P)$*;
    (b) $L_P \setminus V \subseteq \mathbb{P} \setminus V_3$;
    (c) *L_P is a unique secant line of V passing through P*;
    (d) *there is no tangent line to V passing through P*;
    (e) *$\gamma(L_P \setminus V) = L_P \setminus V$, and the automorphism of L_P induced from γ leaves each point in $L_P \cap V$ invariant and also permutes P and $\gamma(P)$.*
(3) *If $P \in V_3 \setminus V_2$, then*:
    (a) *L_P is a tangent line to V, and $L_P \cap V = \{\gamma(P)\}$*;
    (b) $L_P \setminus V \subseteq V_3 \setminus V_2$;

(c) *there is no secant line of V passing through P;*
(d) *L_P is a unique tangent line to V passing through P;*
(e) *$\gamma(L_P \setminus V) = \gamma(P)$, and $\gamma^{-1}(Q) = \{P \in V_3 \setminus V_2 \mid Q \in L_P\} = T_Q V \setminus V_2$ for any $Q \in V$, where $T_Q V$ is the embedded tangent space to V at Q.*

Proof. (1) If $b = \lambda a + \mu[aaa]$ with $a \in \mathfrak{g}_1$ and $\lambda, \mu \in \mathbb{C}$, then it follows from part (1) and (3) of Proposition 4 that

$$b \times b = (\lambda^2 - 3\mu^2 q(a)) a \times a, \tag{3.3}$$

and, moreover, from Proposition 4(2) that

$$[bbb] = (\lambda^2 - 3\mu^2 q(a))(3\mu q(a) a + \lambda[aaa]). \tag{3.4}$$

Therefore, if $b \in \mathbb{C}a + \mathbb{C}[aaa]$ then $[bbb] \in \mathbb{C}a + \mathbb{C}[aaa]$. On the other hand, if $[aaa] \neq 0$ then $\dim(\mathbb{C}a + \mathbb{C}[aaa]) = 2$. Indeed, if $[aaa] = \lambda a$ for some $\lambda \in \mathbb{C}$, then $(\lambda a) \times a = [aaa] \times a = 0$ by Proposition 4(1), so that $\lambda = 0$ or $a \times a = 0$; in any case, $[aaa] = 0$. Thus, if $P = \pi(a) \in \mathbb{P} \setminus V_2$, then for any $\pi(b) \in \mathbb{P}(\mathbb{C}a + \mathbb{C}[aaa]) \setminus V_2$ we have

$$\mathbb{C}a + \mathbb{C}[aaa] = \mathbb{C}b + \mathbb{C}[bbb]$$

(which is of dimension 2) and also

$$\begin{aligned} t_{\pi(b)}\mathbb{P}(\mathbb{C}a + \mathbb{C}[aaa]) &= (\mathbb{C}a + \mathbb{C}[aaa])/\mathbb{C}b \\ &= (\mathbb{C}b + \mathbb{C}[bbb])/\mathbb{C}b = \mathcal{D}(\pi(b)) \end{aligned}$$

in $t_{\pi(b)}\mathbb{P} = \mathfrak{g}_1/\mathbb{C}b$. Therefore, $L_P = \mathbb{P}(\mathbb{C}a + \mathbb{C}[aaa]) \simeq \mathbb{P}^1$.

(2a & 3a) We see from (1) that if $P = \pi(a) \in \mathbb{P} \setminus V_2$ then one may take $(\lambda : \mu)$ as homogeneous coordinates of $\pi(b)$ in $L_{\pi(a)}$. It follows from (3.3) that the scheme-theoretic intersection $L_P \cap V$ is a closed subscheme of L_P defined by $\lambda^2 - 3\mu^2 q(a) = 0$, so that $L_P \cap V = \{(\pm\sqrt{3q(a)} : 1)\}$. The results follow from this observation. In fact, $P \notin V_3$ if and only if $L_P \cap V$ consists of two distinct points (i.e., iff L_P is a secant line of V); otherwise, $L_P \cap V$ has intersection multiplicity 2 at $\gamma(P)$, so that L_P is tangent to V at $\gamma(P)$. For $P \notin V_3$, with respect to the homogeneous coordinates given by $(\lambda' : \mu') := (\lambda : \sqrt{3q(a)}\mu)$ we have $P = (1 : 0)$, $\gamma(P) = (0 : 1)$, and $L_P \cap V = \{(1 : \pm 1)\}$, so that the cross-ratio of those points is -1.

(2b & 3b) Assume that $Q = \pi(b) \in L_P \setminus V$ with $P = \pi(a) \notin V_2$; as a result, $\lambda^2 - 3\mu^2 q(a) \neq 0$ by (3.3), where $b = \lambda a + \mu[aaa]$. Since

$$q(b) = (\lambda^2 - 3\mu^2 q(a))^2 q(a)$$

by (3.4), we see that $P \notin V_3$ if and only if $Q \notin V_3$, so that (2b) follows. Moreover, if $P \in V_3 \setminus V_2$ then $Q \in V_3 \setminus V_2$ by (3.4), so (3b) follows.

(2c & 3c) For $P = \pi(a) \in \mathbb{P} \setminus V_2$, if $a = x + y$ for some $x, y \in \mathcal{V}$, then it follows from Proposition 5(2) that $0 \neq [aaa] = -6\langle x, y\rangle(x - y)$ and from Proposition 5(3) that $q(a) = 12\langle x, y\rangle^2 \neq 0$. In particular, $P \notin V_3$, and this proves (3c). Combining these formulas, we obtain

$$x - y = \pm \frac{1}{\sqrt{3q(a)}}[aaa].$$

Therefore, $\{x, y\}$ is uniquely determined by a, and so a secant line of V passing through $P = \pi(a) \in \mathbb{P} \setminus V_3$ is unique.

(2d & 3d) It suffices to show that, if L is a tangent line to V at Q passing through $P \in \mathbb{P} \setminus V_2$, then $L = L_P$. Indeed, this implies $P \in V_3$ by the preceding proof of (2a & 3a).

Here we note that, for $Q = \pi(x) \in V$,

$$T_Q V = \mathbb{P}(t_x \mathcal{V}),$$

where $T_Q V$ is the embedded tangent space to V at Q, $t_x \mathcal{V}$ is the Zariski tangent space to $\mathcal{V}$ at x, and we identify $t_x \mathfrak{g}_1$ with $\mathfrak{g}_1$. Indeed, $T_Q V$ is equal to the projectivization of $T_x \mathcal{V}$ and $T_x \mathcal{V}$ is the translation $x + t_x \mathcal{V}$ of $t_x \mathcal{V}$ by adding x, which is equal to $t_x \mathcal{V}$ since $\mathcal{V} \cup \{0\}$ is a cone with vertex the origin of $\mathfrak{g}_1$ (here $T_x \mathcal{V}$ is the embedded tangent space to $\mathcal{V}$ at x).

Since $P = \pi(a) \in L \subseteq T_Q V$, it follows that $a \in t_x \mathcal{V}$; hence there is a curve $\xi : \Delta \to \mathcal{V}$ such that $\xi(0) = x$ and

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \{\xi(\lambda) - \xi(0)\} = a,$$

where $\Delta := \{\lambda \in \mathbb{C} \mid |\lambda| < \varepsilon\}$ with a sufficiently small $\varepsilon > 0$. For $\lambda \in \Delta \setminus \{0\}$, set

$$P(\lambda) := \pi(\xi(\lambda) - \xi(0)), \quad Q(\lambda) := \pi(\xi(\lambda)), \quad L(\lambda) := \mathbb{P}(\mathbb{C}\xi(\lambda) + \mathbb{C}\xi(0)).$$

Then $\lim_{\lambda \to 0} P(\lambda) = P$, $\lim_{\lambda \to 0} Q(\lambda) = Q$, and $\lim_{\lambda \to 0} L(\lambda) = L$. Since $P \notin V_2$, we may assume (taking ε smaller if necessary) that $P(\lambda) \notin V_2$ for any $\lambda \in \Delta \setminus \{0\}$. Moreover, since $P(\lambda)$ lies on a secant line $L(\lambda)$ of V, by (3c) we have $P(\lambda) \notin V_3$. It then follows from (2c) that $L(\lambda) = L_{P(\lambda)}$ for any $\lambda \in \Delta \setminus \{0\}$, so we obtain $L = L_P$ by taking $\lambda \to 0$.

(2e) By part (1) and (3) of Proposition 4 together with (3.4),

$$[bbb] \times [bbb] = -3q(a)(\lambda^2 - 3\mu^2 q(a))^3 a \times a$$

for $b = \lambda a + \mu[aaa]$; therefore, $\gamma(L_P \setminus V) \subseteq L_P \setminus V$. On the other hand, it follows from (2b) that $L_P \setminus V = L_P \setminus V_3$. Thus we have $L_P \setminus V = \gamma^2(L_P \setminus V) \subseteq \gamma(L_P \setminus V) \subseteq L_P \setminus V$ by Proposition A2(3), so that $\gamma(L_P \setminus V) = L_P \setminus V$. Furthermore, with respect to the homogeneous coordinates $(\lambda' : \mu')$ in the proof of (2a & 3a), it follows from (3.4) that the automorphism of L_P induced from γ is given by $(\lambda' : \mu') \mapsto (\mu' : \lambda')$. Hence the result follows.

(3e) We already know that $\gamma^{-1}(V) = V_3 \setminus V_2$ by Proposition A2(1), and it follows from (3.4) that $\gamma(L_P \setminus V) = \gamma(P)$, where we note that $L_P \setminus V = L_P \setminus V_2$. Now, if $P \in \gamma^{-1}(Q)$ then, according to (3a), L_P is tangent to V at $\gamma(P) = Q$ and so $P \in L_P \subseteq T_Q V$. Conversely, if $P \in T_Q V \setminus V_2$ then, by the uniqueness (3d), Q coincides with the contact point of the unique tangent L_P, which by (3a) is $\gamma(P)$; if $Q \in L_P$ with $P \in V_3 \setminus V_2$, then $Q \in L_P \cap V = \{\gamma(P)\}$, again by (3a). In either case, it follows that $Q = \gamma(P)$. □

COROLLARY A1. *The three conditions* $V = \emptyset$, $V_3 = \mathbb{P}$, *and* $V_2 = \mathbb{P}$ *are equivalent.*

Proof. It follows from part (2a) of Theorem A that if $V = \emptyset$ then $\mathbb{P} \setminus V_3 = \emptyset$, that is, $V_3 = \mathbb{P}$. It suffices to show that $V_3 = \mathbb{P}$ implies $V_2 = \mathbb{P}$ and that $V_2 = \mathbb{P}$ implies $V = \emptyset$. The following argument is due to Asano [4].

Assume that $V_3 = \mathbb{P}$, that is, $q \equiv 0$. Then, for any $a, b \in \mathfrak{g}_1$ and $\lambda, \mu \in \mathbb{C}$,

$$\begin{aligned} 0 &= q(\lambda a + \mu b) \\ &= \lambda^4 q(a) + \lambda^3 \mu(\langle b, [aaa]\rangle + \langle a, [baa]\rangle + \langle a, [aba]\rangle + \langle a, [aab]\rangle) \\ &\quad + \lambda^2\mu^2(\cdots) + \cdots + \mu^4 q(b). \end{aligned}$$

Since this holds for arbitrary $\lambda, \mu \in \mathbb{C}$, it follows that

$$\langle b, [aaa]\rangle + \langle a, [baa]\rangle + \langle a, [aba]\rangle + \langle a, [aab]\rangle = 0.$$

Then, by (3.2), $4\langle b, [aaa]\rangle = 0$ for any $a, b \in \mathfrak{g}_1$, so that $[aaa] = 0$ for any $a \in \mathfrak{g}_1$ since $\langle \cdot, \cdot \rangle$ is nondegenerate; that is, $V_2 = \mathbb{P}$.

Next assume that $V_2 = \mathbb{P}$. A similar argument to the foregoing shows that $[baa] + [aba] + [aab] = 0$. Using (3.1), we have

$$3[aab] = 6\langle a, b\rangle a.$$

Therefore, if $a \times a = 0$ then $\langle a, b\rangle a = 0$ for any $b \in \mathfrak{g}_1$, so $a = 0$ since $\langle \cdot, \cdot \rangle$ is nondegenerate. Thus, we have $\mathcal{V} = \emptyset$, that is, $V = \emptyset$. This completes the proof of the equivalence. □

REMARK A. It can be shown that $V = \emptyset$ if and only if the Lie algebra $\mathfrak{g}$ is of type C (see Appendix). In fact, using a theorem of Asano [29, Thm. 1.6; 4], one can show that if $q \equiv 0$ then $\mathfrak{g} \simeq \mathfrak{sp}_{2n+2}$, where $\dim \mathfrak{g}_1 = 2n$. The converse is checked by an explicit computation.

Recall that a projective variety $V \subseteq \mathbb{P}$ is called a *variety with one apparent double point* if, for a general point $P \in \mathbb{P}$, there exists a unique secant line of V passing through P (see [24, IX]).

COROLLARY A2. *If* $V \neq \emptyset$, *then* V *is a variety with one apparent double point. In particular,* V *is nondegenerate in* $\mathbb{P}$.

Proof. Set $U := \mathbb{P} \setminus V_3$. It follows from Corollary A1 that, if $V \neq \emptyset$, then $U \neq \emptyset$ and hence U is dense in $\mathbb{P}$. According to Theorem A(2), U is covered by unique secants L_P $(P \in U)$ of V. This completes the proof. □

4. The Homogeneity

THEOREM B. *Let* G_{00} *be the group of inner automorphisms of* $\mathfrak{g}$ *with Lie algebra* $\mathfrak{g}_{00}$, *where* $\mathfrak{g}_{00}$ *is the subalgebra of* $\mathfrak{g}_0$ *defined by* $\mathfrak{g}_{00} := \operatorname{Ker}(\operatorname{ad} E_{\pm}|\mathfrak{g}_0)$. *Then:*

(1) *G_{00} acts transitively on each irreducible component of $\mathcal{V}$. In particular, we have $t_x\mathcal{V} = \mathfrak{g}_{00}x$ for any $x \in \mathcal{V}$, where $t_x\mathcal{V}$ is the Zariski tangent space to $\mathcal{V}$ at x.*

(2) *$\mathfrak{g}_{00}x = (\mathfrak{g}_{00}x)^{\perp}$ with $2 \dim \mathfrak{g}_{00}x = \dim \mathfrak{g}_1$ for any $x \in \mathcal{V}$, and $\mathfrak{g}_1 = \mathfrak{g}_{00}x \oplus \mathfrak{g}_{00}y$ for any $x, y \in \mathcal{V}$ with $\langle x, y\rangle \neq 0$.*

Proof. (1) Let G_0 be the group of inner automorphisms of $\mathfrak{g}$ with Lie algebra $\mathfrak{g}_0$, and set $\mathcal{U} := \{s \in \mathfrak{g}_1 \mid q(s) \neq 0\}$. We first prove the following claim.

Claim. *G_0 acts transitively on $\mathcal{U}$.*

Proof. We may assume that $\mathcal{U} \neq \emptyset$. Since $\mathcal{U}$ is irreducible and stable under the action of G_0, it suffices to show that

$$\mathfrak{g}_0 s = \mathfrak{g}_1$$

for any $s \in \mathcal{U}$. Take an arbitrary $a \in \mathfrak{g}_1$. By Proposition 6(2), there exists a $b \in \mathfrak{g}_1$ such that $a = L(s,s)b$. On the other hand, it follows from Lemma 3 that $L(s,s)b = [sbs] + 3\langle s, b\rangle s = (s \times b + 3\langle s, b\rangle H)s \in \mathfrak{g}_0 s$. Thus, we have $a \in \mathfrak{g}_0 s$. □

Now, it follows from $\mathfrak{g}_0 = \mathfrak{g}_{00} \oplus \mathbb{C}H$ (Lemma 1) that $G_0 = G_{00} \cdot \mathbb{C}^{\times}$. Taking account of Proposition 1(2), we see that it suffices to show that G_0 acts transitively on each of the irreducible components of $\mathcal{V}$.

Take an arbitrary $x \in \mathcal{V}$. There exists a $y \in \mathcal{V}$ such that $\langle x, y\rangle \neq 0$, since V is nondegenerate (Corollary A2). It suffices to show that there exists a $g \in G_0$ such that $gx = x'$ for any x' in a Zariski open neighborhood $\mathcal{V} \setminus y^{\perp}$ of x in $\mathcal{V}$; indeed, this implies the required transitivity because any two nonempty, Zariski open subsets of an irreducible space have a nonempty intersection. Since $x + y \in \mathcal{U}$ and $x' + y \in \mathcal{U}$ by Proposition 5(3), it follows from our Claim that there exists a $g \in G_0$ such that $x' + y = g(x + y) = gx + gy$. Here we have $gx, gy \in \mathcal{V}$, since $\mathcal{V}$ is G_0-stable. It thus follows from Theorem A(2) that

$$\{gx, gy\} = \{x', y\}.$$

If $gx = x'$, then there is nothing to prove; otherwise, we have $gx = y$ and $gy = x'$, so that $g^2 x = x'$. This completes the proof of part (1).

(2) We note that

$$\dim \mathfrak{g}_{00}x \leq \tfrac{1}{2} \dim \mathfrak{g}_1 \tag{4.1}$$

holds for any $x \in \mathcal{V}$. Indeed, it follows from Proposition 3(3) that $\mathfrak{g}_{00}x \subseteq (\mathfrak{g}_{00}x)^{\perp}$ and from the nondegeneracy of $\langle \cdot, \cdot \rangle$ that $\dim(\mathfrak{g}_{00}x)^{\perp} = \dim \mathfrak{g}_1 - \dim \mathfrak{g}_{00}x$.

Now, we show that

$$\mathfrak{g}_1 = \mathfrak{g}_{00}x + \mathfrak{g}_{00}y \tag{4.2}$$

for any $x, y \in \mathcal{V}$ with $\langle x, y\rangle \neq 0$. Take an arbitrary $a \in \mathfrak{g}_1$. We set $s := x + y$; it follows from Proposition 5(3) that $s \in \mathcal{U}$, as before. According to Proposition 6(2), there exists a $b \in \mathfrak{g}_1$ such that $a = L(s,s)b$. On the other hand, since $x, y \in \mathcal{V}$, it follows from Lemma 3 that

$$\begin{aligned} L(s,s)b &= [xyb] + [yxb] \\ &= [xby] + \langle x,b\rangle y - \langle x,y\rangle b + 2\langle y,b\rangle x \\ &\quad + [ybx] + \langle y,b\rangle x - \langle y,x\rangle b + 2\langle x,b\rangle y \\ &= ([byx] + 3\langle y,b\rangle x) + ([bxy] + 3\langle x,b\rangle y), \end{aligned}$$

which is contained in $\mathfrak{g}_{00}x + \mathfrak{g}_{00}y$ by Proposition 1(2). Thus we have

$$a \in \mathfrak{g}_{00}x + \mathfrak{g}_{00}y.$$

Combining (4.1) and (4.2), we obtain that $\mathfrak{g}_{00}x \cap \mathfrak{g}_{00}y = 0$ for any $y \in \mathcal{V}$ with $\langle x,y\rangle \neq 0$. We note that, for any $x \in \mathcal{V}$, there exists $y \in \mathcal{V}$ such that $\langle x,y\rangle \neq 0$ since V is nondegenerate (Corollary A2). Hence the equality holds in (4.1), and the required results follow. □

REMARK B1. Since we have $G_0 = G_{00} \cdot \mathbb{C}^\times$, it follows from the Claim that G_{00} acts transitively on $\mathbb{P} \setminus V_3 = \pi(\mathcal{U})$. More precisely, we have $\mathfrak{g}_{00}s = [sss]^\perp$ for any $s \in \mathcal{U}$. Indeed, for any $s \in \mathfrak{g}_1$, it follows from Lemma 1, Proposition 2, and Proposition 4(1) that $\mathfrak{g}_{00}s \subseteq [sss]^\perp$ by $\langle [abs],[sss]\rangle = \langle [s[sss]a],b\rangle = 0$, and $\mathfrak{g}_{00}s + \mathbb{C}s = \mathfrak{g}_0 s$. If $q(s) \neq 0$, then $\mathfrak{g}_0 s = \mathfrak{g}_1$ by the Claim and $[sss]^\perp$ has codimension 1 in $\mathfrak{g}_1$; hence the assertion follows.

REMARK B2. One can deduce Theorem B(1) from the linear section theorem [17, Thm. B] by using a generalization of a theorem of Richardson [27, Lemma, p. 469], as well as from Theorem B(2) by using the finiteness theorem for the number of nilpotent orbits [27, Prop. 2, p. 469]. Note that both of those proofs depend essentially on the argument in [27, Lemma, p. 469]. On the other hand, by using Theorem A(2), one can deduce (2) from (1) in Theorem B as follows.

Proof of (1) $\Rightarrow$ (2) *in Theorem B.* Similarly to the foregoing proof of part (2), it suffices to show (4.2) for any $x, y \in \mathcal{V}$ with $\langle x,y\rangle \neq 0$. Take an arbitrary $a \in \mathfrak{g}_1$, and consider a line in $\mathfrak{g}_1$ such that

$$\sigma : \mathbb{C} \to \mathfrak{g}_1; \quad \lambda \mapsto (x+y) + \lambda a.$$

Since $\sigma(0) = x + y \in \mathcal{U}$ by Proposition 5(3), for a sufficiently small $\varepsilon > 0$ we have $\sigma(\lambda) \in \mathcal{U}$ for any $\lambda \in \Delta$, where $\Delta := \{\lambda \in \mathbb{C} \mid |\lambda| < \varepsilon\}$. Then it follows from Theorem A(2) that there exist (analytic) curves $\xi, \eta \colon \Delta \to \mathcal{V}$ such that $\xi(0) = x$, $\eta(0) = y$, and

$$\sigma(\lambda) = \xi(\lambda) + \eta(\lambda)$$

for any $\lambda \in \Delta$. Then

$$a = \frac{d}{d\lambda}\bigg|_{\lambda=0} \sigma(\lambda) = \frac{d}{d\lambda}\bigg|_{\lambda=0} \xi(\lambda) + \frac{d}{d\lambda}\bigg|_{\lambda=0} \eta(\lambda) \in t_x\mathcal{V} + t_y\mathcal{V}.$$

According to (1) we have $t_x\mathcal{V} = \mathfrak{g}_{00}x$ and $t_y\mathcal{V} = \mathfrak{g}_{00}y$, so $a \in \mathfrak{g}_{00}x + \mathfrak{g}_{00}y$. □

Recall that the *tangent variety* of V, denoted by $\operatorname{Tan} V$, is the union of embedded tangent spaces to V, and that the *projective dual* of V, denoted by V^*, is the set of hyperplanes tangent to V (see e.g. [11, Sec. 3]).

COROLLARY B1. *Assume that $V \neq \emptyset$. Then*:

(1) *G_{00} acts transitively on each irreducible component of V; and V is smooth and equidimensional of dimension $n-1$, where* $\dim \mathfrak{g}_1 = 2n$.
(2) *Denote by L^* the set of hyperplanes containing a linear subspace $L \subseteq \mathbb{P}$. Then $(T_Q V)^* = T_Q V$ for any $Q \in V$ and so*

$$\operatorname{Tan} V = V^*,$$

where we identify $\mathbb{P}$ with its dual space $\mathbb{P}^\vee := \mathbb{P}(\mathfrak{g}_1^*)$ via the symplectic form $\langle \cdot, \cdot \rangle$.

Proof. (1) This is an immediate consequence of Theorem B. Note that $\dim V = n-1$ does not follow directly from Corollary A2, since V is not necessarily irreducible.

(2) We have $V^* = \bigcup_{P \in V} (T_P V)^*$, and it follows from Theorem B that

$$T_Q V = \mathbb{P}(t_x \mathcal{V}) = \mathbb{P}(\mathfrak{g}_{00} x) = \mathbb{P}((\mathfrak{g}_{00} x)^\perp) = \mathbb{P}(\operatorname{Ker}(\mathfrak{g}_1^* \twoheadrightarrow (\mathfrak{g}_{00} x)^*)) = (T_Q V)^*$$

for any $Q = \pi(x) \in V$, so that $V^* = \operatorname{Tan} V$. $\square$

COROLLARY B2. *If V is neither empty nor irreducible, then there exist irreducible $\mathfrak{g}_{00}$-modules $\mathfrak{s}_1$ and $\mathfrak{s}_2$ of dimension n such that $\mathfrak{g}_1 = \mathfrak{s}_1 \oplus \mathfrak{s}_2$, and we have*

$$V = \mathbb{P}(\mathfrak{s}_1) \sqcup \mathbb{P}(\mathfrak{s}_2),$$

where $\dim \mathfrak{g}_1 = 2n$.

Proof. Let $\{Z_i\}_{1 \leq i \leq k}$ be the set of irreducible components of V with $k \geq 2$, and let $\mathfrak{s}_i$ be the linear subspace of $\mathfrak{g}_1$ spanned by $\pi^{-1} Z_i$. It follows from Corollary B1(1) that each Z_i is an orbit of G_{00}, so that $\mathfrak{s}_i$ is G_{00}-stable. Moreover, by virtue of an argument in Zak [30, pp. 49–50], we see that $\mathfrak{s}_i$ is an irreducible G_{00}-module. Since $\mathbb{P}(\mathfrak{s}_i)$ has a unique closed orbit of G_{00} (see e.g. [26, Ch. 1, Sec. 4.6.1, Lemma]), we see that $\mathfrak{s}_i \cap \mathfrak{s}_j = 0$ if $i \neq j$. Taking into account the nondegeneracy of V (Corollary A2), we obtain

$$\mathfrak{g}_1 = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_k.$$

On the other hand, we have $\dim \mathfrak{s}_i \geq \dim \pi^{-1} Z_i = n$ by Theorem B(2). Hence it follows that $\mathfrak{g}_1 = \mathfrak{s}_1 \oplus \mathfrak{s}_2$ with $k = 2$ and $Z_i = \mathbb{P}(\mathfrak{s}_i)$. $\square$

REMARK B3. It is known that V is irreducible unless $\mathfrak{g}$ is of type A or C (see Appendix). In fact, if $\mathfrak{g} = \mathfrak{so}_m$ then V is a Segre embedding of $\mathbb{P}^1 \times Q$ in $\mathbb{P}^{2m-9}$, where Q is a quadric hypersurface in $\mathbb{P}^{m-5}$; if $\mathfrak{g}$ is of type G_2, then V is a cubic Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$ (for other exceptional Lie algebras $\mathfrak{g}$, see the Introduction). Conversely, it follows from a direct computation that we are in the case just described if $\mathfrak{g} = \mathfrak{sl}_{n+2}$ with $n \geq 1$.

COROLLARY B3. *If $V \neq \emptyset$ and $V_2 \neq V$, then V is ruled—that is, covered by lines contained in V.*

Proof. We may assume that V is irreducible, for otherwise the claim is obvious from Corollary B2. For $P = \pi(u) \in \mathbb{P}$, let

$$\Phi_P : \mathbb{P} \dashrightarrow \mathbb{P}$$

be a rational map induced from $L(u,u)$ and set $B_P := \mathbb{P}(\operatorname{Ker} L(u,u))$, the base locus of Φ_P. It suffices to show that

$$V \not\subseteq B_P \cup \Phi_P(V \setminus B_P)$$

for any $P = \pi(u) \in V_2 \setminus V$. Indeed, if $Q \in V \setminus (B_P \cup \Phi_P(V \setminus B_P))$ then $Q \neq \Phi_P(Q)$, and it follows from Proposition 7(2) that the line joining Q and $\Phi_P(Q)$ is contained in V. Taking account of the homogeneity of V (Corollary B1(1)), we see that this holds for any $Q \in V$.

Now, since $\langle [uua], u\rangle = -\langle a, [uuu]\rangle = 0$ for any $a \in \mathfrak{g}_1$ by Lemma 2(1), if we set $P^\perp := \mathbb{P}(u^\perp)$ then

$$\Phi_P(V \setminus B_P) \subseteq \mathbb{P}(L(u,u)\mathfrak{g}_1) \subseteq P^\perp,$$

so that $B_P \cup \Phi_P(V \setminus B_P) \subseteq B_P \cup P^\perp$. On the other hand, we have $V \not\subseteq B_P \cup P^\perp$ since V is irreducible and nondegenerate (Corollary A2). Thus the claim follows. □

REMARK B4. It can be shown that $V = V_2$ if and only if $\mathfrak{g}$ is of type G_2.

5. Isotropic Secants

PROPOSITION C. *For $P = \pi(u) \in \mathbb{P}$, let $\Phi_P : \mathbb{P} \dashrightarrow \mathbb{P}$ be a rational map induced from $L(u,u)$ with base locus $B_P = \mathbb{P}(\operatorname{Ker} L(u,u))$. If V is irreducible and $P \in V_2 \setminus V$, then* $\dim \Phi_P(V \setminus B_P) \geq 1$; *hence* $\dim \Phi_P(\mathbb{P} \setminus B_P) \geq 1$ *and* $\operatorname{codim} B_P \geq 2$.

Proof. We have $B_P \neq \mathbb{P}$ since $P \notin V$. Suppose $\dim \Phi(V \setminus B_P) = 0$, and set

$$Q := \Phi_P(V \setminus B_P) \in \mathbb{P}.$$

Since $\Phi_P(R) = Q$ for any $R \in V \setminus B_P$, by Proposition 7(2) V contains the line joining Q and R for any $R \in V \setminus B_P$ with $Q \neq R$. Since V is irreducible, V must be a cone with vertex Q. Since V is smooth by Corollary B1(1), it follows that V is a linear variety, so that $V = \mathbb{P}$ by its nondegeneracy (Corollary A2). Yet by Lemma 1 $V \neq \mathbb{P}$, a contradiction, so $\dim \Phi_P(V \setminus B_P) > 0$. □

REMARK C1. The irreducibility condition for V is essential in Proposition C. In fact, there is an example of u satisfying this assumption such that $\operatorname{rk} L(u,u) = 1$ when $\mathfrak{g} = \mathfrak{sl}_m$, where V is not irreducible (see Remark B3).

REMARK C2. It follows easily from Proposition 6 that $\dim \Phi_P(\mathbb{P} \setminus B_P) \geq 1$ if $P \notin V_2$ and $\operatorname{codim} \Phi_P(\mathbb{P} \setminus B_P) \geq 1$ if $P \in V_3$, though we do not use these facts in this paper.

Recall that the *secant locus* Σ_P as well as the *tangent locus* Θ_P of V with respect to a given point $P \in \mathbb{P}$ are defined by

$$\Sigma_P^\circ := \{Q \in V \mid \exists R \in V \setminus \{Q\},\ P \in Q * R\}, \quad \Sigma_P := \overline{\Sigma_P^\circ},$$
$$\Theta_P := \{Q \in V \mid P \in T_Q V\},$$

where we denote by $Q * R$ the line in $\mathbb{P}$ joining Q and R and by $T_Q V$ the embedded tangent space to V at Q in $\mathbb{P}$ (see e.g. [11]).

THEOREM C. *Assume that V is irreducible. Then*:

(1) *For any $x, y \in \mathcal{V}$, $\langle x, y\rangle = 0$ if and only if $\mathfrak{g}_{00}x \cap \mathfrak{g}_{00}y \neq 0$. In particular, a secant line joining $Q, R \in V$ is isotropic with respect to the symplectic form if and only if $T_Q V \cap T_R V \neq \emptyset$.*
(2) *$V_2 \setminus V$ is covered by isotropic secants of V. More precisely, for any $u \in \mathfrak{g}_1$, we have that $[uuu] = 0$ and $u \times u \neq 0$ if and only if $u = x + y$ for some $x, y \in \mathcal{V}$ such that $\langle x, y\rangle = 0$ and $x \times y \neq 0$.*
(3) *If $P \in V_2 \setminus V$, then*

$$\Phi_P(V \setminus B_P) \subseteq \Sigma_P, \qquad \Phi_P(V \cap P^\perp \setminus B_P) \subseteq \Theta_P,$$

where $\Phi_P : \mathbb{P} \dashrightarrow \mathbb{P}$ is the rational map induced from $L(u,u)$ with base locus $B_P = \mathbb{P}(\mathrm{Ker}\, L(u,u))$ and $P^\perp = \mathbb{P}(u^\perp)$ with $P = \pi(u)$.
(4) *We have* $\dim \Sigma_P \geq 1$ *for any $P \in V_2 \setminus V$.*

Proof. (1) We show that if $\langle x, y\rangle = 0$ for some $x, y \in \mathcal{V}$, then $\mathfrak{g}_{00}x \cap \mathfrak{g}_{00}y \neq 0$; the converse has already been proved in Theorem B(2).

We set $u := x + y$, $P := \pi(u)$, and $L := \mathbb{P}(\mathbb{C}x + \mathbb{C}y)$, the line joining $\pi(x)$ and $\pi(y)$. If $P \in V$ then $x \times y = 0$, so that $L \subseteq V$. Therefore, $L \subseteq T_Q V \cap T_R V$ with $Q = \pi(x)$ and $R = \pi(y)$; that is, $0 \neq \mathbb{C}x + \mathbb{C}y \subseteq \mathfrak{g}_{00}x \cap \mathfrak{g}_{00}y$, as required. Thus we may assume that $P \notin V$ and $x \times y \neq 0$.

Now assume $\langle x, y\rangle = 0$. It follows from Proposition 5(2) that $P \in V_2$ and from Lemma 3 and Proposition 1(2) that

$$\mathfrak{g}_{00}y \ni [axy] + \langle a, x\rangle y = [ayx] + \langle a, y\rangle x \in \mathfrak{g}_{00}x$$

for any $a \in \mathfrak{g}_1$. Suppose that $\mathfrak{g}_{00}x \cap \mathfrak{g}_{00}y = 0$. It follows that $[axy] + \langle a, x\rangle y = [ayx] + \langle a, y\rangle x = 0$. Therefore, by Lemma 3,

$$\begin{aligned} L(u,u)a = 2[xya] &= 2([axy] + \langle x, a\rangle y - \langle x, y\rangle a + 2\langle y, a\rangle x) \\ &= 4\langle x, a\rangle y + 4\langle y, a\rangle x \in \mathbb{C}x + \mathbb{C}y, \end{aligned}$$

so that $L(u,u)(\mathfrak{g}_1) \subseteq \mathbb{C}x + \mathbb{C}y$, hence $\Phi_P(V \setminus B_P) \subseteq \Phi_P(\mathbb{P} \setminus B_P) \subseteq L$. Moreover, $\Phi_P(V \setminus B_P)$ is dense in L; indeed, we have $\dim \Phi_P(V \setminus B_P) \geq 1$ by Proposition C, where we note that $P \in V_2 \setminus V$. On the other hand, it follows from Proposition 7(2) that $\Phi_P(V \setminus B_P) \subseteq V$. Hence $P \in L \subseteq V$, and this is a contradiction. Thus we have $\mathfrak{g}_{00}x \cap \mathfrak{g}_{00}y \neq 0$.

(2) The "if" part follows from part (1) and (2) of Proposition 5. For the "only if", let $u \in \mathfrak{g}_1$ satisfy $[uuu] = 0$ and $u \times u \neq 0$. There exists a $v \in \mathcal{V}$ such that $\langle u, v\rangle \neq 0$, since V is nondegenerate (Corollary A2). If $z = \lambda u + \mu[uuv]$ with $\lambda, \mu \in \mathbb{C}$, then it follows from Proposition 7 that

$$z \times z = \lambda^2 u \times u + 2\lambda\mu[uuv] \times u + \mu^2[uuv] \times [uuv]$$
$$= \lambda(\lambda + 4\mu\langle u, v\rangle)u \times u.$$

Since $\lambda(\lambda + 4\mu\langle u, v\rangle) = 0$ for $(\lambda, \mu) = (1, -1/4\langle u, v\rangle), (0, 1/4\langle u, v\rangle)$, if we set

$$\{x, y\} := \left\{u - \frac{1}{4\langle u, v\rangle}[uuv], \frac{1}{4\langle u, v\rangle}[uuv]\right\}$$

then $x, y \in \mathcal{V}$ with $u = x + y$. Note that, by Proposition 7(1), $[uuv] \neq 0$ because $u \times u \neq 0$ and $\langle u, v\rangle \neq 0$.

(3) We see from the proof of part (2) that Φ_P is defined on $V \setminus P^{\perp}$ and that

$$\Phi_P(V \setminus P^{\perp}) \subseteq \Sigma_P^{\circ}$$

for any $P = \pi(u) \in V_2 \setminus V$. Since V is irreducible, we obtain $\Phi_P(V \setminus B_P) \subseteq \Sigma_P$ as required.

Next, for the tangent locus it suffices to show that, for any $v \in \mathcal{V}$, if $\langle u, v\rangle = 0$ and $[uuv] \neq 0$ then $u \in \mathfrak{g}_{00}[uuv]$. For any $a, b \in \mathfrak{g}_1$ and $v \in \mathcal{V}$, it follows from Proposition 2 and Proposition 7(1) that

$$\langle [ab[uuv]], u\rangle = \langle [[uuv]ua], b\rangle = 2\langle u, v\rangle\langle [uua], b\rangle = 0.$$

Therefore, $\langle \mathfrak{g}_{00}[uuv], u\rangle = 0$ by Lemma 1 and hence $u \in \mathfrak{g}_{00}[uuv]$ by Theorem B(2).

(4) The assertion follows directly from Proposition C and (3). □

REMARK C3. The irreducibility condition for V is essential in the proof of part (1); in fact, it is easily seen that the conclusion does not hold when $\mathfrak{g} = \mathfrak{sl}_m$.

COROLLARY C. *If V is irreducible, then $V_3 = \operatorname{Tan} V$.*

Proof. It follows from Theorem A(2d) that $V_3 \supseteq \operatorname{Tan} V$ and from Theorem A(3a) and Theorem C(3) that $V_3 \subseteq \operatorname{Tan} V$. □

6. Double Projections

PROPOSITION D. *For any $x, y \in \mathcal{V}$, let $\Psi_{xy} : \mathfrak{g}_1 \to \mathfrak{g}_1$ be a linear map defined by*

$$\Psi_{xy}(a) := [axy] + \langle a, x\rangle y.$$

(1) *If $\langle x, y\rangle \neq 0$, then* $\operatorname{Ker} \Psi_{xy} = \mathfrak{g}_{00}x$ *and* $\Psi_{xy}(\mathfrak{g}_1) = \mathfrak{g}_{00}y$. *In particular, a rational map $\Psi_{PQ} : \mathbb{P} \dashrightarrow \mathbb{P}$ induced from Ψ_{xy} is a double projection from P with image $T_Q V$ (i.e., a projection with center $T_P V$ onto $T_Q V$) and so defines a morphism*

$$\Psi_{PQ} : \mathbb{P} \setminus T_P V \to T_Q V,$$

where $T_P V$ is the embedded tangent space to V at P with $P = \pi(x)$ and $Q = \pi(y)$.

(2) *Moreover, for any $R \in V$, the four points R, $[PQR]$, $\Psi_{PR}(Q)$, and $\Psi_{QR}(P)$ are collinear, and $[PQR]$ is the harmonic conjugate of R with respect to $\Psi_{PR}(Q)$*

and $\Psi_{QR}(P)$, where we set $[PQR] := \pi([xyz])$ with $R = \pi(z)$. In particular, this holds for general $P, Q, R \in V$ and yields a geometric interpretation of our ternary product.

Proof. (1) It follows from part (1) and (2) of Proposition 3 that $\operatorname{Ker}\Psi_{xy} \supseteq \mathfrak{g}_{00}x$ and from Proposition 1(2) that $\Psi_{xy}(\mathfrak{g}_1) \subseteq \mathfrak{g}_{00}y$. On the other hand, we have the formula

$$\Psi_{xy} - \Psi_{yx} = 2\langle x, y\rangle 1_{\mathfrak{g}_1},$$

which is nothing but Lemma 3. If $\langle x, y\rangle \neq 0$ then we have the direct sum decomposition $\mathfrak{g}_1 = \mathfrak{g}_{00}y \oplus \mathfrak{g}_{00}x$ (Theorem B(2)). We thus see that Ψ_{xy} and Ψ_{yx} are surjective onto $\mathfrak{g}_{00}y$ and onto $\mathfrak{g}_{00}x$ (respectively) and hence the assertion follows.

(2) We note that $[xyz] \neq 0$. Indeed, since $q(x+y) = 12\langle x, y\rangle^2 \neq 0$ by Proposition 5(3), it follows from Proposition 6(2) that $2L(x, y) = L(x+y, x+y)$ is injective. Now the assertion follows directly from the definition of the maps Ψ_{PQ}. In fact, we have

$$\begin{aligned}\Psi_{xz}(y) &= [yxz] + \langle y, x\rangle z = [xyz] - \langle x, y\rangle z,\\ \Psi_{yz}(x) &= [xyz] + \langle x, y\rangle z,\end{aligned}$$

so that $\Psi_{PR}(Q)$ and $\Psi_{QR}(P)$ lie on the line joining R and $[PQR]$, and those four points form a harmonic quadruple. □

REMARK D1. In terms of the Lie bracket, by (1.1) we have $\Psi_{ab}(c) = [b[a[c, E_-]]]$.

THEOREM D. *For any $P, Q \in V$, if the secant line joining P and Q is not isotropic (i.e., if $T_P V \cap T_Q V = \emptyset$) then the following statements hold.*

(1) $V \setminus P^\perp = (\Psi_{PQ}|_{V\setminus T_P V})^{-1}(T_Q V \setminus P^\perp)$.

(2) *The double projection Ψ_{PQ} gives an isomorphism $V \setminus P^\perp \to T_Q V \setminus P^\perp$. In fact, a rational map $\Gamma_{QP} : T_Q V \dashrightarrow V$ induced from a map $\Gamma_{yx} : \mathfrak{g}_{00}y \to \mathcal{V} \cup \{0\}$ defined by*

$$\Gamma_{yx}(t) := \langle x, [ttt]\rangle x + 3\langle x, t\rangle[ttx] + 12\langle x, t\rangle^2 t$$

gives the inverse of $\Psi_{PQ}|_{V\setminus P^\perp}$, where $P = \pi(x)$ and $Q = \pi(y)$.

(3) *The base locus of Γ_{QP} is $T_Q V \cap P^\perp \cap V_2$.*

In particular: if V is irreducible, then Ψ_{PQ} gives a birational map from V to $T_Q V$, and V is the closure of the image of a composition of a cubic Veronese embedding of the affine space $T_Q V \setminus P^\perp$ with some projection to $\mathbb{P}$.

Proof. (1) By Proposition 2 and Proposition 1(1), for any $x, y \in \mathcal{V}$ with $\langle x, y\rangle \neq 0$ we have $\langle \Psi_{xy}(a), x\rangle = 2\langle a, x\rangle\langle x, y\rangle$, so that $a \in x^\perp$ if and only if $\Psi_{xy}(a) \in x^\perp$. Therefore,

$$\Psi_{xy}^{-1}(\mathfrak{g}_{00}y \setminus x^\perp) = \mathfrak{g}_1 \setminus x^\perp;$$

hence $\Psi_{PQ}^{-1}(T_Q V \setminus P^\perp) = \mathbb{P} \setminus P^\perp$. Observe here that $T_P V \subseteq P^\perp$ by Proposition 3(2) and that, by Theorem C(1), $\langle x, y\rangle \neq 0$ if and only if $T_P V \cap T_Q V = \emptyset$ with $P = \pi(x)$ and $Q = \pi(y)$. Restricting Ψ_{PQ} to $V \setminus T_P V$, we obtain the result.

(2) We first show that Ψ_{PQ} is unramified on $V \setminus P^{\perp}$. For any $z \in \mathcal{V}$ with $\langle x, z\rangle \neq 0$ we have $\mathfrak{g}_{00}x \cap \mathfrak{g}_{00}z = 0$, so it follows from Proposition D(1) that Ψ_{xy} induces an isomorphism $\mathfrak{g}_{00}z \to \mathfrak{g}_{00}y$. Thus Ψ_{PQ} is unramified at $\pi(z)$.

We next show that $\Psi_{PQ}|_{V\setminus P^{\perp}}$ is injective. Assume that $\Psi_{xy}(a) = \Psi_{xy}(b)$ with $a, b \in \mathcal{V} \setminus x^{\perp}$; it then follows from Proposition D(1) that

$$a - b \in \operatorname{Ker} \Psi_{xy} = \mathfrak{g}_{00}x.$$

Therefore, by (1), (2), and (4) of Proposition 3, $\langle a - b, x\rangle = 0$, $(a - b) \times x = 0$, and

$$[(a - b)(a - b)x] = 0.$$

In particular, we see that $L(a - b, a - b)$ is not injective (since $x \neq 0$) and so $q(a - b) = 0$ by Proposition 6(2); hence $\langle a, b\rangle = 0$ by Proposition 5(3). On the other hand, since $a, b \in \mathcal{V}$ we have $[(a - b)(a - b)x] = -2[abx]$, and

$$\begin{aligned}[abx] &= [axb] + \langle a, x\rangle b - \langle a, b\rangle x + 2\langle b, x\rangle a \quad (\because \text{Lemma 3})\\ &= [bxb] + \langle a, x\rangle b + 2\langle b, x\rangle a \quad (\because a \times x = b \times x,\ \langle a, b\rangle = 0)\\ &= 3\langle x, b\rangle b + \langle a, x\rangle b + 2\langle b, x\rangle a \quad (\because \text{Proposition 1(1)})\\ &= 2\langle a, x\rangle(a - b) \quad (\because \langle a, x\rangle = \langle b, x\rangle).\end{aligned}$$

Therefore, since $\langle a, x\rangle \neq 0$ we have $a = b$ as required.

We next show that $\Gamma_{yx}(t) \in \mathcal{V} \cup \{0\}$ for any $t \in \mathfrak{g}_{00}y$. It follows from Proposition 3(1) that

$$\begin{aligned}\Gamma_{yx}(t) \times \Gamma_{yx}(t) &= 24\langle x, t\rangle^2\langle x, [ttt]\rangle x \times t + 9\langle x, t\rangle^2[ttx] \times [ttx]\\ &\quad + 72\langle x, t\rangle^3 t \times [ttx] + 144\langle x, t\rangle^4 t \times t\\ &= 3\langle x, t\rangle^2(3[ttx] \times [ttx] + 8\langle x, [ttt]\rangle x \times t)\\ &\quad + 24\langle x, t\rangle^3(3t \times [ttx] + 6\langle x, t\rangle t \times t),\end{aligned}$$

which is equal to zero by Proposition 7. Thus we have $\Gamma_{yx}(t) \times \Gamma_{yx}(t) = 0$.

Since $\langle x, [ttt]\rangle x + 3\langle x, t\rangle[ttx] \in \mathfrak{g}_{00}x = \operatorname{Ker} \Psi_{xy}$ and since, by Proposition D(1), $12\langle x, t\rangle^2 t \in \mathfrak{g}_{00}y = \operatorname{Ker} \Psi_{yx}$, it follows from Lemma 3 that

$$\Psi_{xy} \circ \Gamma_{yx}(t) = (\Psi_{yx} + 2\langle x, y\rangle 1_{\mathfrak{g}_1})(12\langle x, t\rangle^2 t) = 24\langle x, y\rangle\langle x, t\rangle^2 t$$

for any $t \in \mathfrak{g}_{00}y$, so that

$$\Psi_{PQ} \circ \Gamma_{QP} = 1_{T_Q V \setminus P^{\perp}}.$$

Therefore, $\Psi_{PQ}|_{V\setminus P^{\perp}}$ is surjective onto $T_Q V \setminus P^{\perp}$ and hence an isomorphism, and Γ_{QP} on $T_Q V \setminus P^{\perp}$ gives the inverse of Ψ_{PQ}.

(3) If $\Gamma_{yx}(t) = 0$ with $t \in \mathfrak{g}_{00}y \setminus \{0\}$, then

$$\langle x, [ttt]\rangle x + 3\langle x, t\rangle[ttx] = 12\langle x, t\rangle^2 t = 0$$

by the direct sum decomposition $\mathfrak{g}_1 = \mathfrak{g}_{00}x \oplus \mathfrak{g}_{00}x$, so that $\langle x, t\rangle = 0$ because $t \neq 0$ and $\langle x, [ttt]\rangle = 0$ because $x \neq 0$. If $[ttt] \neq 0$ then $\pi([ttt]) = \pi(y)$ by Theorem A(3), so $\langle x, [ttt]\rangle$ would not be zero (by our assumption $\langle x, y\rangle \neq 0$), and this is a contradiction. Thus we have $[ttt] = 0$. □

REMARK D2. The morphism $\Psi_{PQ}\colon V \setminus T_PV \to T_QV$ is not necessarily surjective. In fact, for any $P \in V$, if $\mathfrak{g}$ is of type G_2 then: $P^\perp$ is the osculating plane to the twisted cubic $V \subseteq \mathbb{P}^3$ at P; $V \cap P^\perp = \{P\}$; and $\Psi_{PQ}(V \setminus T_PV) = T_QV \setminus P^\perp$ for any $Q \in V$ with $P \neq Q$.

REMARK D3. We have proved in the foregoing that $\Psi_{xy}\colon \mathcal{V} \setminus x^\perp \to \mathfrak{g}_{00}y \setminus x^\perp$ is an isomorphism.

Added in proof. We give here another expression of the inverse map of the double projection Ψ_{PQ} in Section 6 (see Theorem D). We first note that there is an isomorphism of affine spaces,

$$\iota\colon \mathfrak{g}_{00}y \cap x^\perp \to T_QV \setminus P^\perp$$

defined by $\iota(a) := \pi(a+y)$. Indeed, the inverse is given by

$$\iota^{-1}(\pi(t)) := \frac{\langle x, y\rangle}{\langle x, t\rangle}t - y$$

for $\pi(t) \in T_QV \setminus P^\perp$, where $T_QV = \mathbb{P}(\mathfrak{g}_{00}y)$ and $P^\perp = \mathbb{P}(x^\perp)$. Now let $\rho\colon \mathfrak{g}_{00}y \cap x^\perp \to V$ be the composition of ι with the rational map $\Gamma_{QP}\colon T_QV \dashrightarrow V$ in Theorem D(2). Then ρ is the inverse of Ψ_{PQ} via ι, and it follows from part (1) and (4) of Proposition 3 that

$$\rho(a) = \pi\left(\frac{\langle x, [aaa]\rangle}{12\langle x, y\rangle^2}x + \frac{1}{4\langle x, y\rangle}[aax] + a + y\right).$$

In particular, the Freudenthal variety V is equal to the closure of the image of the affine space $\mathfrak{g}_{00}y \cap x^\perp$ under the cubic Veronese embedding ρ.

7. Twisted Cubic Curves

PROPOSITION E. *For any $P \in V_3 \setminus V_2$ and $Q \in V$, if the secant line joining Q and the contact point $\gamma(P)$ of L_P is not isotropic, then*:

(1) $Q \in L_{\Phi_P(Q)}$ *and* $\Phi_P^3(Q) = \gamma(P) \in L_P = L_{\Phi_P^2(Q)}$ *with both* $\Phi_P(Q), \Phi_P^2(Q) \in V_3 \setminus V_2$; *and*

(2) $L_P \cap L_{\Phi_P(Q)} = \emptyset$, *so* Q, $\Phi_P(Q)$, $\Phi_P^2(Q)$, *and* $\Phi_P^3(Q)$ *are linearly independent in* $\mathbb{P}$.

Proof. (1) Take $t \in \mathfrak{g}_1$ and $x \in \mathcal{V}$ such that $\pi(t) = P$ and $\pi(x) = Q$, and set $D := L(t,t) \in \mathfrak{g}_{00}$; we have $\Phi_P^k(Q) = \pi(D^kx)$, and $\langle x, [ttt]\rangle \neq 0$ by our assumption that the secant line joining $\gamma(P) = \pi([ttt])$ and Q is not isotropic. According to Theorem A(3), the unique tangent line L_P to V passing through P has a unique contact point $\gamma(P)$. On the other hand, since $q(t) = 0$, it follows from Proposition 6(1) and Proposition 4(2) that

$$3D^2x = -8\langle x, t\rangle[ttt] + 8\langle x, [ttt]\rangle t,$$

$$3D^3x = 8\langle x, [ttt]\rangle[ttt],$$

$$D^4x = 0.$$

Therefore, $\Phi_P^3(Q) = \pi([ttt]) = \gamma(P)$ and $\Phi_P^2(Q) \in L_P$, so $\Phi_P^2(Q) \in T_{\gamma(P)}V$. Here we note that $D^3x \neq 0$, since $3\langle x, D^3x\rangle = 8\langle x, [ttt]\rangle^2 \neq 0$.

Now, similarly to the proof of Proposition 3(1), it follows from Lemma 2(2) that

$$0 = D(Dx \times x) = D^2x \times x + Dx \times Dx,$$

so that $[(Dx)(Dx)(Dx)] = -[x(D^2x)(Dx)]$. By Lemma 3, part (1) and (2) of Proposition 3, and Lemma 2(1),

$$\begin{aligned}[x(D^2x)(Dx)] &= -\langle x, D^2x\rangle Dx + 2\langle D^2x, Dx\rangle x \\ &= \langle Dx, Dx\rangle Dx - 2\langle D^3x, x\rangle x = -2\langle D^3x, x\rangle x \neq 0.\end{aligned}$$

Hence $[(Dx)(Dx)(Dx)] \neq 0$ and $\Phi_P(Q) \in T_QV \setminus V_2$. It then follows from Theorem A(3) that $Q \in L_{\Phi_P(Q)}$. Moreover, it follows that

$$9D^2x \times D^2x = 64\langle x, [ttt]\rangle^2 t \times t \neq 0,$$

so $\Phi_P^2(Q) \notin V$. Thus, by Theorem A(3) we have $\Phi_P^2(Q) \in L_P \setminus V \subseteq T_{\gamma(P)}V \setminus V_2$ and hence $\gamma(P) \in L_{\Phi_P^2(Q)} = L_P$.

(2) Since $L_P \subseteq T_{\gamma(P)}V$, $L_{\Phi_P(Q)} \subseteq T_QV$, and $T_{\gamma(P)}V \cap T_QV = \emptyset$ (because $\langle x, [ttt]\rangle \neq 0$), it follows that $L_P \cap L_{\Phi_P(Q)} = \emptyset$. The linear independence of the four points is thus a consequence of part (1). □

THEOREM E. *For any $P \in V_3 \setminus V_2$ and $Q \in V$ such that the secant line joining Q and the contact point $\gamma(P)$ of L_P is not isotropic* (*i.e.*, $T_QV \cap T_{\gamma(P)}V = \emptyset$), *let $\mathbb{P}_{PQ}$ be the linear subspace of dimension* 3 *in $\mathbb{P}$ spanned by Q, $\Phi_P(Q)$, $\Phi_P^2(Q)$* (*or, equivalently, P*), *and $\Phi_P^3(Q) = \gamma(P)$—that is, spanned by L_P and $L_{\Phi_P(Q)}$, the unique tangent lines to V passing through P and $\Phi_P(Q)$.*

(1) *The intersection $V \cap \mathbb{P}_{PQ}$ is a twisted cubic curve in $\mathbb{P}_{PQ} \simeq \mathbb{P}^3$ given explicitly by the image of L_P under the cubic map $\Gamma_{\gamma(P)Q}$*:

$$V \cap \mathbb{P}_{PQ} = \Gamma_{\gamma(P)Q}(L_P).$$

(2) *The twisted cubic curve in this $\mathbb{P}_{PQ}$ has the following properties*:
  (a) *L_P and $L_{\Phi_P(Q)}$ are the tangent lines at $\gamma(P)$ and Q* (*respectively*); *and*
  (b) *$\gamma(P)^\perp \cap \mathbb{P}_{PQ}$ and $Q^\perp \cap \mathbb{P}_{PQ}$ are* (*respectively*) *the osculating planes at $\gamma(P)$ and Q, which are spanned by L_P and $\Phi_P(Q)$ and by $L_{\Phi_P(Q)}$ and $\Phi_P^2(Q)$, respectively.*

Proof. Take $t \in \mathfrak{g}_1$ and $x \in \mathcal{V}$ such that $\pi(t) = P$ and $\pi(x) = Q$, and set $D := L(t,t) \in \mathfrak{g}_{00}$ as before. Since $D^4x = 0$ (see the proof of Proposition E), using Lemma 2(2) repeatedly yields from $D^k(x \times x) = 0$ that

$$\begin{aligned} D^3x \times x + 3D^2x \times Dx &= 0, \\ 4D^3x \times Dx + 3D^2x \times D^2x &= 0, \\ D^3x \times D^2x &= 0, \\ D^3x \times D^3x &= 0. \end{aligned}$$

If $z = \lambda_0 x + \lambda_1 Dx + \lambda_2 D^2 x + \lambda_3 D^3 x$ with $\lambda_i \in \mathbb{C}$, then it follows from the foregoing equalities that

$$\begin{aligned} z \times z = (\lambda_1^2 - 2\lambda_0\lambda_2)&Dx \times Dx \\ &+ 2(\lambda_1\lambda_2 - 3\lambda_0\lambda_3)Dx \times D^2x \\ &+ \tfrac{1}{2}(2\lambda_2^2 - 3\lambda_1\lambda_3)D^2x \times D^2x. \end{aligned}$$

Here $Dx \times Dx$, $Dx \times D^2x$, and $D^2x \times D^2x$ are linearly independent. Indeed, let $\lambda Dx \times Dx + \mu Dx \times D^2x + \nu D^2x \times D^2x = 0$ for some $\lambda, \mu, \nu \in \mathbb{C}$. Then, applying D^2 to this formula, we obtain $\lambda = 0$ because (a) it follows from our assumption $\langle x, [ttt]\rangle \neq 0$ that

$$D^2(Dx \times Dx) = 2D(Dx \times D^2x) = \tfrac{1}{2} D^2x \times D^2x \neq 0$$

and (b) other terms are killed by D^2. Therefore, $\mu Dx \times D^2x + \nu D^2x \times D^2x = 0$. Again applying D to this formula, we similarly obtain $\mu = 0$ and hence $\nu = 0$. Thus, it follows that the intersection $V \cap \mathbb{P}_{PQ}$ has defining equations

$$\lambda_1^2 - 2\lambda_0\lambda_2 = \lambda_1\lambda_2 - 3\lambda_0\lambda_3 = 2\lambda_2^2 - 3\lambda_1\lambda_3 = 0,$$

which is a twisted cubic curve in $\mathbb{P}_{PQ}$ with the required properties, as is easily shown.

We next demonstrate that $V \cap \mathbb{P}_{PQ} = \Gamma_{\gamma(P)Q}(L_P)$. If $u = \lambda D^2x + \mu D^3x$ with $\lambda, \mu \in \mathbb{C}$, then

$$\Gamma_{(D^3x)x}(u) = \tfrac{4}{3}\langle x, D^3x\rangle^2(2\lambda^3 x + 6\lambda^2\mu Dx + 9\lambda\mu^2 D^2x + 9\mu^3 D^3x)$$

with $\pi(D^3x) = \Phi_P^3(Q) = \gamma(P)$. Indeed, it follows that $\langle x, u\rangle = \mu\langle x, D^3x\rangle$ and

$$u \times u = \lambda^2 D^2x \times D^2x = \frac{64}{9}\langle x, [ttt]\rangle^2 t \times t = \frac{8}{3}\lambda^2\langle x, D^3x\rangle D,$$

so $3[uux] = 8\lambda^2\langle x, D^3x\rangle Dx$, $3[uuu] = 8\lambda^3\langle x, D^3x\rangle D^3x$, and $3\langle x, [uuu]\rangle = 8\lambda^3\langle x, D^3x\rangle^2$. Since $\Gamma_{\gamma(P)Q}(R) \in V$ for any $R = \pi(u) \in \mathbb{P}(\mathbb{C}D^2x + \mathbb{C}D^3x) = L_P$ by Theorem D, we have

$$\Gamma_{\gamma(P)Q}(L_P) \subseteq V \cap \mathbb{P}_{PQ},$$

where $\mathbb{P}_{PQ} = \mathbb{P}(\mathbb{C}x + \mathbb{C}Dx + \mathbb{C}D^2x + \mathbb{C}D^3x)$. Since both $\Gamma_{\gamma(P)Q}(L_P)$ and $V \cap \mathbb{P}_{PQ}$ are irreducible and closed subsets of $\mathbb{P}$ and of the same dimension, it follows that $\Gamma_{\gamma(P)Q}(L_P) = V \cap \mathbb{P}_{PQ}$. □

REMARK E1. The morphism $\Gamma_{\gamma(P)Q} \colon L_P \to \mathbb{P}_{PQ}$ is given by

$$(\lambda : \mu) \mapsto (2\lambda^3 : 6\lambda^2\mu : 9\lambda\mu^2 : 9\mu^3)$$

in terms of homogeneous coordinates with respect to the basis $\{D^2x, D^3x\}$ for L_P and $\{x, Dx, D^2x, D^3x\}$ for $\mathbb{P}_{PQ}$.

REMARK E2. Set $E := L(Dx, Dx)$ and $F := [D, E]$ with $D := L(t, t)$ as before, and denote by $\mathfrak{g}_{00PQ}$ the subalgebra of $\mathfrak{g}_{00}$ generated by D, E, and F. Then it follows that

Table A1 Adjoint Varieties and Freudenthal Varieties

$\mathfrak{g}$	$X \subseteq \mathbb{P}(\mathfrak{g})$	$\mathfrak{g}_{00}$	$V \subseteq \mathbb{P}(\mathfrak{g}_1)$
$\mathfrak{sl}_m$	$(\mathbb{P}^{m-1} \times \mathbb{P}^{m-1}) \cap (1) \subseteq \mathbb{P}^{m^2-2}$	$\mathfrak{gl}_1 \oplus \mathfrak{sl}_{m-2}$	$\mathbb{P}^{m-3} \sqcup \mathbb{P}^{m-3} \subseteq \mathbb{P}^{2m-5}$
$\mathfrak{so}_m$	$\mathbb{G}_{\text{orthog.}}(2, m) \subseteq \mathbb{P}^{\binom{m}{2}-1}$	$\mathfrak{sl}_2 \oplus \mathfrak{so}_{m-4}$	$\mathbb{P}^1 \times Q^{m-6} \subseteq \mathbb{P}^{2m-9}$
$\mathfrak{sp}_{2m}$	$v_2\mathbb{P}^{2m-1} \subseteq \mathbb{P}^{\binom{2m+1}{2}-1}$	$\mathfrak{sp}_{2m-2}$	$\emptyset \subseteq \mathbb{P}^{2m-3}$
$\mathfrak{e}_6$	$E_6(\omega_2)^{21} \subseteq \mathbb{P}^{77}$	$\mathfrak{sl}_6$	$\mathbb{G}(3,6) \subseteq \mathbb{P}^{19}$
$\mathfrak{e}_7$	$E_7(\omega_1)^{33} \subseteq \mathbb{P}^{132}$	$\mathfrak{so}_{12}$	$S_5 = \mathbb{G}_{\text{orthog.}}(6, 12) \subseteq \mathbb{P}^{2^5-1}$
$\mathfrak{e}_8$	$E_8(\omega_8)^{57} \subseteq \mathbb{P}^{247}$	$\mathfrak{e}_7$	$E_7(\omega_6) \subseteq \mathbb{P}^{55}$
$\mathfrak{f}_4$	$F_4(\omega_1)^{15} \subseteq \mathbb{P}^{51}$	$\mathfrak{sp}_6$	$\mathbb{G}_{\text{sympl.}}(3,6) \subseteq \mathbb{P}^{13}$
$\mathfrak{g}_2$	$G_2(\omega_2)^5 \subseteq \mathbb{P}^{13}$	$\mathfrak{sl}_2$	$v_3\mathbb{P}^1 \subseteq \mathbb{P}^3$

Notation: We denote by "$\cap$ (1)" the cutting by a general hyperplane and by v_d the Veronese embedding of degree d. We denote by $\mathbb{G}(r, m)$ a Grassmann variety of r-planes in $\mathbb{C}^m$, and by $\mathbb{G}_{\text{orthog.}}(r, m)$ and $\mathbb{G}_{\text{symp.}}(r, m)$ an orthogonal and a symplectic (respectively) Grassmann variety of isotropic r-planes in $\mathbb{C}^m$. A simple exceptional Lie algebra of Dynkin type G is denoted by $\mathfrak{g}$ (as in [12]); a simple algebraic group of type G is denoted simply by G; and for a dominant integral weight ω of G, the minimal closed orbit of G in $\mathbb{P}(V_\omega)$ is denoted by $G(\omega)$, where V_ω is the irreducible representation space of G with highest weight ω. For example, $\mathfrak{g}_2$ in the list is the simple Lie algebra of type G_2, and $G_2(\omega_2)$ is the minimal closed orbit of an algebraic group of type G_2 in $\mathbb{P}(V_{\omega_2})$, where ω_2 is the second fundamental dominant weight in the standard notation of Bourbaki [6, Ch. 4–6].

$$[F, D] = \tfrac{4}{3}\langle D^3x, x\rangle D, \qquad [F, E] = -\tfrac{4}{3}\langle D^3x, x\rangle E,$$

so that $\mathfrak{g}_{00PQ}$ is isomorphic to the Lie algebra $\mathfrak{sl}_2$. Denoting by $\mathfrak{g}_{1PQ}$ the subspace of $\mathfrak{g}_1$ spanned by x, Dx, D^2x, and D^3x, we see that $\mathfrak{g}_{1PQ}$ is an irreducible $\mathfrak{g}_{00PQ}$-module of dimension 4 with

$$F(D^kx) = (2k-3)\tfrac{2}{3}\langle D^3x, x\rangle D^kx,$$

and the twisted cubic curve $V \cap \mathbb{P}_{PQ} = \Gamma_{\gamma(P)Q}(L_P)$ is a unique closed orbit in $\mathbb{P}_{PQ} = \mathbb{P}(\mathfrak{g}_{1PQ})$ under the natural action of the group of inner automorphisms of $\mathfrak{g}_{00}$ with Lie algebra $\mathfrak{g}_{00PQ}$.

Thus, for any $P \in V_3 \setminus V_2$ and $Q \in V$ with $T_{\gamma(P)}V \cap T_QV = \emptyset$, a subalgebra $\mathfrak{g}_{00PQ}$ of $\mathfrak{g}_{00}$ isomorphic to $\mathfrak{sl}_2$ and an irreducible $\mathfrak{g}_{00PQ}$-submodule $\mathfrak{g}_{1PQ}$ of $\mathfrak{g}_1$ with dimension 4 are associated to P and Q. If $\mathfrak{g}$ is of type G_2, then $\mathfrak{g}_{00PQ}$ and $\mathfrak{g}_{1PQ}$ are (respectively) equal to $\mathfrak{g}_{00}$ and $\mathfrak{g}_1$ themselves.

Appendix: A Classification of Freudenthal Varieties

In Table A1 we give a classification of Freudenthal varieties V in terms of the root data of $\mathfrak{g}$. It would be interesting to compare V with the adjoint variety associated to $\mathfrak{g}$, since those varieties are closely related to each other. In fact, for a simple graded Lie algebra $\mathfrak{g} = \sum \mathfrak{g}_i$ of contact type, denote by V the Freudenthal variety associated to $\mathfrak{g}$ (as before) and denote by X the orbit of the inner automorphism group of $\mathfrak{g}$ through $\pi(E_+)$ in $\mathbb{P}(\mathfrak{g})$, which is the minimal closed orbit in $\mathbb{P}(\mathfrak{g})$, called the *adjoint variety* associated to $\mathfrak{g}$ (see [16]). Then, according to [17, Thm. B], we have $V = X \cap \mathbb{P}(\mathfrak{g}_1)$.

ACKNOWLEDGMENTS. Both of the authors would like to thank Prof. Hiroshi Asano and Prof. Soji Kaneyuki for their invaluable advice and stimulating discussion. In particular, the proof of Theorem B(1) was inspired by Prof. Asano [5], where the discussion with him is essential. Both authors would also like to thank the referee for invaluable suggestions and advice; for instance, the proofs of part (2) and (3) of Lemma 2 are due to the referee, and the presentation featuring the notion of a line field was also suggested by the referee. Proposition D(2) gives an answer to a question raised by Prof. Laurent Manivel. The first author would like to thank Prof. Eduardo Esteves and Prof. Arnaldo Garcia for their invaluable comments. Most of this work was completed while the first author stayed in IMPA (Instituto de Matemática Pura e Aplicada), and the first draft of this article appeared in IMPA Preprint Server as Preprint Series A2001/29 [18]. He wishes to thank the people at IMPA, especially Prof. Esteves and Prof. Garcia for their warm hospitality. Muito obrigado!

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