Michigan Math. J. 52 (2004)

Blow-up of Positive Solutions for a Family of Nonlinear Parabolic Equations in General Domain in $\mathbb{R}^N$

Mahmoud Hesaaraki & Abbas Moameni

1. Introduction

We consider the following nonlinear parabolic problem,

$$u_t - \triangle u = F(u, \triangledown u), \quad x \in \Omega, \ t > 0 \tag{1.1}$$

$$u(t,x) = 0, \qquad x \in \partial\Omega, \tag{1.2}$$

$$u(0,x) = u_0(x), \qquad x \in \Omega, \tag{1.3}$$

where Ω is a bounded (or unbounded) and sufficiently regular (say, uniformly regular of class C^2) open domain in $\mathbb{R}^N$, $F \in C^1(\mathbb{R} \times \mathbb{R}^N)$, and u_0 satisfies the compatibility condition (i.e., $u_0(x) = 0$ on $\partial\Omega$). It is well known that the problem (1.1)–(1.3) admits a unique classical solution u, of maximal existence time $T^* \in (0, \infty]$, when Ω is a bounded domain and $u_0(x) \in C^1(\bar{\Omega})$ [14, Thm. 10, p. 206]. Moreover, if $T^* < \infty$, then u blows up in finite time in C^1 norm; that is,

$$\limsup_{t \to T^*} \sup_{x \in \bar{\Omega}} |u(x,t)| + |\triangledown u(x,t)| = +\infty.$$

It is also known that if $F(u, \triangledown u) = b|\triangledown u|^p + au^q$ ($p > 1$, $q > 1$, and $a, b \in \mathbb{R}$) then (1.1)–(1.3) admits a unique, maximal-in-time solution $u \in C([0, T^*); W_0^{1,s}(\Omega))$ for all sufficiently regular initial data. For example, $u_0 \in W_0^{1,s}(\Omega)$ with $s \ge N \max(p,q)$ when Ω is an unbounded domain. Moreover, if $T^* < \infty$ then $\lim_{t \to T^*} \|u(t)\|_{W_0^{1,\infty}(\Omega)} = \infty$.

The foregoing two regularity assumptions on u_0 will be maintained throughout the paper.

The equation

$$u_t - \triangle u = |\triangledown u|^p, \quad t > 0, \ x \in \Omega, \ p > 1, \tag{1.4}$$

serves as a typical model case in the theory of parabolic partial differential equations. In fact, it is the simplest example of parabolic PDE with a nonlinearity depending on the first-order spatial derivatives of u, and it can be considered as an analogue of the extensively studied equation with zero-order nonlinearity, $u_t - \triangle u = u|u|^{p-1}$. This equation was studied by many authors in the past few years [3; 4; 5; 6; 7; 8; 16; 17; 21; 22; 29].

Received March 3, 2003. Revision received September 10, 2003.

Assume that the initial data is given by $u_0 = \lambda\psi$, where $\psi \not\equiv 0$ is a positive function on Ω. The authors in [1] and [29] proved that the problem (1.2)–(1.4) for $p > 2$ cannot admit a global solution whenever λ is sufficiently large and Ω is a bounded domain.

The first aim of the present paper is the study of the problem (1.2)–(1.4) for unbounded domains. In fact, for $p > 2$ and $u_0 = \lambda\psi$ ($\psi \not\equiv 0$ with λ sufficiently large), we shall prove that the finite-time blow-up occurs whenever Poincaré's inequality is valid in $W_0^{1,1}(\Omega)$; that is:

$$\|v\|_{L^1(\Omega)} \le C(\Omega)\|\nabla v\|_{L^1(\Omega)} \quad \forall v \in W_0^{1,1}(\Omega),$$

and there exist some positive functions $\phi \in W_0^{1,\infty}(\Omega)$ such that, for $\delta = \frac{1}{p-1}$ and $\delta = \frac{1}{pN}$,

$$\int_\Omega \frac{1}{\phi(x)^\delta}\,dx < \infty.$$

REMARK 1.1. Here we should mention that the foregoing conditions hold for every bounded domain. In fact, if ϕ is the first eigenfunction of the $-\triangle$ operator in $H_0^1(\Omega)$, then $\phi \in W^{2,2}(\Omega) \cap W_0^{1,\infty}(\Omega)$ and we can choose ϕ such that $\phi > 0$ in Ω. Furthermore, the author in [29, Lemma 5.1] proved that

$$\int_\Omega \phi^{-\alpha}(x)\,dx = C(\alpha,\Omega) < \infty \quad \forall \alpha \in (0,1).$$

Moreover, these conditions may hold for some unbounded domains. For example, if we take

$$\Omega = \{(x,y) \mid |x| \le 1,\ |y| \le 3x^4 - 8x^2 + 6\} \cup \{(x,y) \mid |x| \ge 1,\ |y| \le 1/x^4\} \subseteq \mathbb{R}^2,$$

and choose

$$\phi(x,y) = \begin{cases} (1/x^4 - |y|)e^{(1/x^4-|y|)}, & |x| \ge 1, \\ 3x^4 - 8x^2 + 6 - |y|, & |x| \le 1, \end{cases}$$

whenever $p = 3$, then the conditions hold for this unbounded domain.

In the case of $F(u,\nabla u) = b|\nabla u|^p + au^q$ ($q > 1$, $p > 1$), several authors have studied the existence of nonglobal positive solutions and blow-up under certain assumptions on p, q, N, a, b, and Ω (see e.g. [2; 10; 11; 12; 13; 17; 18; 23; 24; 25; 26; 27; 28; 31]). The author in [29] and [28] recently introduced some open problems about boundedness of global solutions and blow-up in finite time for these types of equations. The second aim of this article is to consider these problems. In fact, we prove the following statements.

(i) If $a > 0$ and $b > 0$, we show that blow-up in finite time occurs for large values of initial data in every regular domain in $\mathbb{R}^N$.

(ii) If $b > 0$, $a < 0$, $|a| \ll 1$, and $\Omega = (-1,1)$ is an interval in $\mathbb{R}^1$, we show that either blow-up occurs or global solutions are unbounded for large values of initial data.

(iii) If $a > 0$, $b < 0$, and Ω is bounded, we show that the solutions decay exponentially whenever the initial data is small.

In Section 2 we establish the blow-up for the equation $u_t - \triangle u = F(\nabla u)$. In Section 3, we study blow-up in finite time and global solutions for the equation $u_t - \triangle u = b|\nabla u|^p + au^q$.

2. Blow-up for the Equation $u_t - \Delta u = F(\nabla u)$

In this section we consider the problem

$$\begin{aligned} u_t - \triangle u &= F(\nabla u), && x \in \Omega,\ t > 0, \\ (x,0) &= u_0(x), && x \in \Omega, \\ u(x,t) &= 0, && x \in \partial\Omega,\ t > 0, \end{aligned} \tag{2.5}$$

where Ω is a (possibly unbounded) domain in $\mathbb{R}^N$. At first we assume that $F(\nabla u) = |\nabla u|^p$. In [29] it was shown that the nonnegative solutions of (1.2)–(1.4) for $p > 2$, blow up in finite time for large values of initial data whenever Ω is a bounded domain in $\mathbb{R}^N$. We recall this theorem here.

THEOREM 2.1. *Let Ω be a bounded domain with smooth boundary (say C^2) and let $\varphi_1 > 0$ be the first eigenfunction of the $-\triangle$ in $H_0^1(\Omega)$. Assume that $p > 2$ and consider the problem* (1.2)–(1.4). *There exists $K_0 = K_0(\Omega, p) > 0$ such that, if $\int_\Omega u_0(x)\varphi_1(x)\,dx > K_0$, then gradient blow-up occurs.*

We now extend this theorem for regular domains that are possibly unbounded.

THEOREM 2.2. *Let Ω be a uniformly regular domain of class C^2 in $\mathbb{R}^N$ with $p > 2$ and $u_0 = \lambda\psi$ ($\psi \not\equiv 0$, $\psi \geq 0$), and consider the problem* (1.2)–(1.4). *Assume that Poincaré's inequality holds in $W_0^{1,1}(\Omega)$ and there exists some positive function $\phi \in W_0^{1,\infty}(\Omega)$ such that $\int_\Omega (1/\phi(x)^\delta)\,dx < \infty$ for $\delta = \frac{1}{p-1}$ and $\delta = \frac{1}{pN}$. Then there exists a $K = K(\Omega, p, \phi)$ such that, if $\lambda > K$, blow-up occurs.*

Notice that these conditions on Ω hold for every bounded domain and may also hold for some unbounded domain (see Remark 1.1).

For the proof of Theorem 2.2 we need the following lemma.

LEMMA 2.3. *Let the conditions of Theorem 2.2 hold. Then*:

(i) *$u(t) \in W_0^{1,r}(\Omega)$ for all $r \geq 1$;*
(ii) *there exists a constant $C = C(\Omega, \phi, p)$ such that*

$$\left(\int_\Omega u\phi\,dx\right)^p \leq C \int_\Omega |\nabla u|^p \phi\,dx.$$

Proof. (i) Since $u(t) \in W_0^{1,r}(\Omega)$ for $r > pN$ [30, Props. A_3 & A_4], it is sufficient to show that $u(t) \in W_0^{1,r}(\Omega)$ for $1 \leq r \leq pN$. By Hölder's inequality we have

$$\int_\Omega u\,dx = \int_\Omega u\phi^{1/s}\phi^{-1/s}\,dx \le \left(\int_\Omega u^s\phi\,dx\right)^{1/s}\left(\int_\Omega \phi^{-s'/s}\,dx\right)^{1/s'},$$

where $1/s + 1/s' = 1$. On the other hand, if $s = pN + 1$ then $u \in W_0^{1,s}(\Omega)$; moreover, $\phi \in W_0^{1,\infty}(\Omega)$ and $\int_\Omega \phi^{-s'/s}\,dx < \infty$, so $\int_\Omega u\,dx < \infty$. Consequently, $u \in L^1(\Omega) \cap L^{pN+1}(\Omega)$. Hence $u \in L^r(\Omega)$ for $1 \le r \le pN$. By using a similar argument, we can show that $|\nabla u| \in L^r(\Omega)$ for $1 \le r \le pN$. Therefore, $u(t) \in W_0^{1,r}(\Omega)$ for $1 \le r \le pN$.

(ii) First of all, notice that

$$\begin{aligned}\int_\Omega |\nabla u|\,dx = \int_\Omega |\nabla u|\phi^{1/p}\phi^{-1/p}\,dx &\le \left(\int |\nabla u|^p\phi\,dx\right)^{1/p}\left(\int_\Omega \phi^{-p'/p}\,dx\right)^{1/p'}\\ &\le C_1\left(\int |\nabla u|^p\phi\,dx\right)^{1/p},\end{aligned}$$

where p' is the conjugate of p and $C_1 = C_1(\Omega,\phi,p)$. By using Poincaré's inequality in $W_0^{1,1}(\Omega)$, we have

$$\begin{aligned}\int_\Omega u\phi\,dx \le \|\phi\|_\infty \int_\Omega u\,dx &\le \|\phi\|_\infty \int_\Omega |\nabla u|\,dx\\ &\le \|\phi\|_\infty C_1\left(\int_\Omega |\nabla u|^p\phi\,dx\right)^{1/p}.\end{aligned}$$

Hence

$$\left(\int_\Omega u\phi\,dx\right)^p \le C\int_\Omega |\nabla u|^p\phi\,dx,$$

where $C = C(\Omega,\phi,p)$. □

Now, we can prove Theorem 2.2.

Proof of Theorem 2.2. Let us first assume that $u_0 \in C_c^3(\Omega)$. By Propositions A_3 and A_4 in [30], it follows that for every finite $r \ge pN$ we must have $u \in C^1([0,T^*), L^r(\Omega))$ and $u(t) \in W_0^{1,r}(\Omega) \cap W^{2,r}(\Omega)$ for all $t \in [0,T^*)$.

Let $Z(t) = \int_\Omega u(t,x)\phi(x)\,dx$ and $T_1 = \min(1, T^*/2)$. Using integration by parts, we obtain

$$Z'(t) + \int_\Omega \nabla u.\nabla\phi\,dx = \int_\Omega |\nabla u|^p\phi\,dx, \quad t \in (0,T_1). \tag{2.6}$$

By considering Hölder's inequality and Young's inequality for the second term on the left-hand side, we can write

$$\begin{aligned}&\int_\Omega \nabla u.\nabla\phi\,dx\\ &\quad\le \int_\Omega |\nabla u|\phi^{1/p}|\nabla\phi|\phi^{-1/p}\,dx\\ &\quad\le \left(\int_\Omega |\nabla u|^p\phi\,dx\right)^{1/p}\left(\int_\Omega |\nabla\phi|^{p'}\phi^{-p'/p}\,dx\right)^{1/p'} \quad \text{(Hölder's inequality)}\\ &\quad\le \frac{1}{2}\int_\Omega |\nabla u|^p\phi\,dx + C_1\int_\Omega |\nabla\phi|^{p'}\phi^{-p'/p}\,dx \quad \text{(Young's inequality).}\end{aligned} \tag{2.7}$$

Let $M = C_1 \int_\Omega |\nabla\phi|^{p'}\phi^{-p'/p}\, dx$. Then (2.6) and (2.7) imply that

$$Z'(t) \geq \frac{1}{2}\int |\nabla u|^p \phi\, dx - M, \quad t \in (0, T_1). \tag{2.8}$$

On the other hand, from Lemma 2.3(ii) we have

$$\int_\Omega |\nabla u|^p \phi\, dx \geq C_2\left(\int u\phi\, dx\right)^p \tag{2.9}$$

for some positive constant C_2. From (2.8) and (2.9) we obtain

$$Z'(t) \geq C_3 Z(t)^p - M, \quad t \in (0, T_1), \tag{2.10}$$

for some positive constant C_3. Now if $Z(0) \geq M + (2M/C_3)^{1/p}$, then it follows from (2.10) that $M \leq (C_3/2)Z(t)^p$ for every $t \in (0, T_1)$. Therefore,

$$Z'(t) \geq \frac{C_3}{2} Z(t)^p, \quad t \in (0, T_1),$$

so

$$Z(t) \geq \left[(1-p)\frac{C_3}{2}t + Z(0)^{1-p}\right]^{1/(1-p)}, \quad t \in [0, T_1). \tag{2.11}$$

Consequently, the right-hand side of (2.11) becomes infinite for some finite values of t. By taking a larger value for $Z(0)$ if necessary, we must have $T_1 < 1$ and hence $T^* < 2$. By using continuous dependence of the solutions to the initial data in $W_0^{1,s}(\Omega)$ [30, Prop. A_1], we can show that this result is true for all large values of the initial data $u_0 \in W_0^{1,r}(\Omega)$, $r \geq pN$. The proof of Theorem 2.2 is complete. □

Here, we assume that the domain Ω has a finite Lebesgue measure. In this particular case it is possible to prove that, if $p > 2$ and $\|u_0\|_{L^{2(p-1)/(p-2)}}(\Omega)$ is large enough, then blow-up occurs.

THEOREM 2.4. *Let Ω be a domain with finite Lebesgue measure and smooth boundary (say C^2), and let Poincaré's inequality hold in $W_0^{1,p}(\Omega)$. Assume that $p > 2$ and consider the problem* (1.2)–(1.4). *Then there exists $K = K(\Omega, p) > 0$ such that, if $u_0(x) \geq 0$ and $\int_\Omega u_0^{2(p-1)/(p-2)}\, dx > K$, then blow-up occurs.*

Proof. Similar to the proof of Theorem 2.2, we assume $u_0 \in C_c^3(\Omega)$. Multiplying equation (1.4) by $u^{p/(p-2)}$ and integrating over Ω yields

$$\int_\Omega u_t u^{p/(p-2)}\, dx + \int_\Omega \nabla u.\nabla u^{p/(p-2)}\, dx = \int |\nabla u|^p u^{p/(p-2)}\, dx.$$

Thus

$$\frac{p-2}{2(p-1)}\frac{d}{dt}\int_\Omega u^{2(p-1)/(p-2)}\, dx + \int_\Omega |\nabla u|^2 u^{2/(p-2)}\, dx = \int |\nabla u|^p u^{p/(p-2)}\, dx. \tag{2.12}$$

For the second term on the left-hand side, we have

$$\begin{aligned}\int_\Omega |\nabla u|^2 u^{2/(p-2)}\,dx \\ &\le \left(\int_\Omega |\nabla u|^p u^{p/(p-2)}\,dx\right)^{2/p} \operatorname{mes}(\Omega)^{(p-2)/p} \quad \text{(Hölder's inequality)} \\ &\le \frac{1}{2}\int_\Omega |\nabla u|^p u^{p/(p-2)}\,dx + C_1 \operatorname{mes}(\Omega) \quad \text{(Young's inequality)}, \end{aligned} \tag{2.13}$$

where $\operatorname{mes}(\Omega)$ denotes the Lebesgue measure of the domain Ω. Therefore, from (2.12) and (2.13) we obtain

$$\frac{p-2}{2(p-1)}\frac{d}{dt}\left(\int_\Omega u^{2(p-1)/(p-2)}\,dx\right) + C_1 \operatorname{mes}(\Omega) \ge \frac{1}{2}\int_\Omega |\nabla u|^p u^{p/(p-2)}. \tag{2.14}$$

On the other hand, Poincaré's inequality yields

$$\int_\Omega |\nabla u|^p u^{p/(p-2)}\,dx = \int_\Omega |\nabla u^{(p-1)/(p-2)}|^p\,dx \ge C_2(\Omega, p)\int_\Omega u^{p(p-1)/(p-2)}\,dx.$$

But

$$\int_\Omega u^{2(p-1)/(p-2)}\,dx \le \left(\int_\Omega u^{p(p-1)/(p-2)}\,dx\right)^{2/p} \operatorname{mes}(\Omega)^{(p-2)/p}.$$

Therefore,

$$\int_\Omega |\nabla u|^p u^{p/(p-2)}\,dx \ge C_3\left(\int_\Omega u^{2(p-1)/(p-2)}\,dx\right)^{p/2}, \tag{2.15}$$

where $C_3 = C_3(\Omega, p)$ is a positive constant. By setting $Z(t) = \int_\Omega u^{2(p-1)/(p-2)}\,dx$, from (2.14) and (2.15) we obtain

$$Z'(t) + M \ge C_4 Z(t)^{p/2}, \quad t \in (0, T^*),$$

for some constants M and $C_4 = C_4(\Omega, p)$. Hence, for large values of $Z(0)$, we must have

$$Z'(t) \ge C_5 Z(t)^{p/2}, \quad t \in (0, T_1),$$

where $T_1 = \min(1, T^*)$ and $C_5 = C_5(\Omega, p)$. Similar to the proof of Theorem 2.2, it follows that $T^* < 1$ for all large values of initial data in $W_0^{1,r}(\Omega)$, $r \ge pN$. $\square$

In the previous theorems we showed that blow-up occurs for large values of initial data. However, if we add some positive constant to the right-hand side of (1.4), say,

$$u_t - \Delta u = |\nabla u|^p + \lambda, \quad x \in \Omega,\ t > 0, \tag{2.16}$$

then, by the following theorem, blow-up may occur for all $u_0 \ge 0$, $u_0 \not\equiv 0$.

THEOREM 2.5. *Let Ω and ϕ be exactly the same as in Theorem 2.2 and let u be the nonnegative classical solution of* (2.16), (1.2)–(1.3). *If $u_0 \ge 0$ ($u_0 \not\equiv 0$), then blow-up occurs for large values of λ.*

Proof. Similar to the proof of Theorem 2.2, we arrive at

$$Z'(t) \ge C_1 Z(t)^p + \lambda\int_\Omega \phi(x)\,dx - M, \quad t \in (0, T^*),$$

where $Z(t) = \int_\Omega \phi(x)u(t,x)\,dx$ and C_1 is a positive constant. Therefore, if $\lambda > M/\left(\int_\Omega \phi(x)\,dx\right)$ then

$$Z'(t) \geq C_1 Z(t)^p, \quad t \in (0, T^*).$$

Now, since $Z(0) > 0$, Z becomes infinite at a finite time, which implies that $T^* < \infty$. □

REMARK 2.6. Notice that, when Ω is a bounded domain, each solution of the problem (1.2)–(1.4) satisfies a maximum principle. Thus, the $L^\infty(\Omega)$ norm of the solution remains finite as long as the solution exists. Since the solution blows up in finite time for $p > 2$, it follows that some of its derivatives must be singular for a finite time. Moreover, as noted by many authors, if F depends only on ∇u then it follows (from the maximum principle) that the maximum values of $|\nabla u|$ must be attained on the parabolic boundary. Therefore, in this problem we have

$$\lim_{t \to T^*} \sup \sup_{x \in \partial\Omega} |\nabla u(t,x)| = \infty.$$

REMARK 2.7. All of the results of this section remain valid if u is a classical solution of $u_t - \triangle u = F(\nabla u)$ and $F(\nabla u) \geq b|\nabla u|^p$ for $p > 2$ and $b > 0$.

3. Blow-up and Global Solutions for the Case $F(u, \nabla u) = b|\nabla u|^p + au^q$

In this section we consider the problem

$$\begin{aligned} u_t - \triangle u &= b|\nabla u|^p + au^q, && x \in \Omega,\ t > 0, \\ u(x,0) &= u_0(x), && x \in \Omega, \\ u(x,t) &= 0, && x \in \partial\Omega,\ t > 0, \end{aligned} \tag{3.17}$$

where $p > 1$, $q > 1$, $a \in \mathbb{R}$, $b \in \mathbb{R}$, and Ω is a regular (and possibly unbounded) domain in $\mathbb{R}^N$. For this problem we show that, under some assumptions on a, b, p, q, Ω and the initial data, the solutions can be global or blow up in finite time. In order to start our work, we recall some results about the special cases of this problem that have been considered before.

For the equation with zero-order nonlinearity,

$$u_t - \triangle u = u^q, \quad q > 1, \tag{3.18}$$

it is well known that L^∞ blow-up occurs for large (nonnegative) initial data if Ω is bounded. It is therefore natural to ask what happens if the nonlinearity involves both zero-order and first-order source terms, such as

$$u_t - \triangle u = |\nabla u|^p + u^q, \tag{3.19}$$

or if the zero-order term is an absorption term, such as

$$u_t - \triangle u = |\nabla u|^p - u^q. \tag{3.20}$$

More generally, one may consider the equation

$$u_t - \triangle u = |\nabla u|^p + a(x)u^q, \quad x \in \Omega,\ t > 0, \tag{3.21}$$

where $a \in C^{\alpha}(\bar{\Omega})$, $0 < \alpha < 1$.

The author in [29] provided answers to some of these questions for a bounded domain by the following theorem.

THEOREM 3.1. *Assume that $p > 2$ and $p > q \geq 1$. Let $u \geq 0$ be the solution of* (3.21), (1.2)–(1.3) *where $u_0 = \lambda\psi$ ($\psi \geq 0$, $\psi \not\equiv 0$). Then there exists $\Lambda_0 = \Lambda_0(p, q, a, \Omega, \psi) > 0$ such that, for all $\lambda \geq \Lambda_0$, gradient blow-up occurs.*

Moreover, it was shown in [29] that the conclusion of Theorem 3.1 remains valid when $q = p$ and $\|a\|_{\infty}$ is sufficiently small (depending on p, Ω, and ψ).

In this section, we have the following results related to the problem (3.17).

(i) If $a > 0$ and $b > 0$, we show that blow-up occurs for large values of initial data in every regular domain in $\mathbb{R}^N$.

(ii) If $b > 0$, $a < 0$, $|a| \ll 1$, and $\Omega = (-1, 1)$ is an interval in $\mathbb{R}^1$, we show that either blow-up occurs or global solutions are unbounded for large values of initial data.

(iii) If $a > 0$, $b < 0$, and Ω is bounded, we show that the solutions decay exponentially whenever the initial data is small.

Here is our first result.

THEOREM 3.2. *Let Ω be an open (bounded or unbounded) domain in $\mathbb{R}^N$ with smooth boundary, and let $a > 0$, $b > 0$, and $p, q > 1$. If u is the nonnegative solution of the problem* (3.17) *with $u_0 = \lambda\psi$ ($\psi \geq 0$, $\psi \not\equiv 0$) then there exists a $\Lambda_0 = \Lambda_0(p, q, a, b, \Omega, \psi)$ such that, for all $\lambda \geq \Lambda_0$, blow-up occurs.*

Proof. Similar to the proof of Theorem 2.2, here we assume that $u_0 \in C_c^3(\Omega)$. Let $\Omega_0 \subseteq \Omega$ be a bounded domain such that $\psi \not\equiv 0$ on Ω_0, and let $\varphi_1 > 0$ be the first eigenfunction of the $-\triangle$ operator in $H_0^1(\Omega_0)$. Multiplying (3.17) by $\phi = \varphi_1^{\sigma}$ ($\sigma > p'$, $1/p + 1/p' = 1$) and then integrating over Ω_0 implies that

$$\int_{\Omega_0} u_t \phi\, dx + \int_{\Omega_0} \nabla u . \nabla \phi\, dx = b \int_{\Omega_0} |\nabla u|^p \phi\, dx + a \int_{\Omega_0} u^q \phi\, dx. \tag{3.22}$$

For the second term on the left-hand side, we have

$$\begin{aligned}
&\int_{\Omega_0} \nabla u . \nabla \phi\, dx \\
&\quad \leq \int_{\Omega_0} |\nabla u| \phi^{1/p} \phi^{-1/p} |\nabla \phi|\, dx \\
&\quad \leq \left(\int_{\Omega_0} |\nabla u|^p \phi\, dx \right)^{1/p} \left(\int_{\Omega_0} |\nabla \phi|^{p'} \phi^{-p'/p}\, dx \right)^{1/p'} \quad \text{(Hölder's inequality)} \\
&\quad \leq b \left(\int_{\Omega_0} |\nabla u|^p \phi\, dx \right) + C_1 \int_{\Omega_0} |\nabla \phi|^{p'} \phi^{-p'/p}\, dx. \quad \text{(Young's inequality).}
\end{aligned} \tag{3.23}$$

Now, let $Z(t) = \int_{\Omega_0} u(t,x)\phi\, dx$. From (3.22), (3.23), and Jenson's inequality we obtain

$$Z'(t) + C_1 \int_{\Omega_0} |\nabla\phi|^{p'} \phi^{-p'/p}\, dx \geq aZ(t)^q. \tag{3.24}$$

Since $\sigma > p'$, we must have $\int_{\Omega_0} |\nabla\phi|^{p'}\phi^{-p'/p}\, dx < \infty$. Let

$$M = C_1 \int_{\Omega_0} |\nabla\phi|^{p'}\phi^{-p'/p}\, dx;$$

then (3.24) becomes

$$Z'(t) + M \geq aZ(t)^q. \tag{3.25}$$

Similar to the proof of Theorem 2.2, it follows that $T^* < \infty$ for all large values of initial data in $W_0^{1,r}(\Omega)$, $r \geq N \max(p,q)$. This completes the proof of Theorem 3.2. □

In the following theorem we obtain a more interesting result by taking $\Omega = \mathbb{R}^N$, $a > 0$, and $b > 0$.

THEOREM 3.3. *Let* $\Omega = \mathbb{R}^N$ *and* $a, b > 0$. *Let* $u \geq 0$ *be a solution of the problem* (3.17) *with* $u_0 \geq 0$ ($u_0 \not\equiv 0$). *If* $1 < q < 1 + 2/N$ *and* $1 < p < 1 + 1/(N+1)$, *then* u *is nonglobal.*

Proof. Suppose that $T^* = \infty$ and $\zeta \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^N)$ is a function satisfying

$$\int_0^\infty \int_{\mathbb{R}^N} |\zeta_t|^q \zeta^{-q'/q}\, dx\, dt < \infty, \qquad \int_0^\infty \int_{\mathbb{R}^N} |\nabla\zeta|^{p'} \zeta^{-p'/p}\, dx\, dt < \infty, \tag{3.26}$$

where $1/p + 1/p' = 1/q + 1/q' = 1$. Multiplying (3.17) by $\zeta(t,x)$ and integrating over $Q = \mathbb{R}^+ \times \mathbb{R}^N$ gives

$$\int_Q u_t\zeta\, dx\, dt + \int_Q \nabla u.\nabla\zeta\, dx\, dt = b\int_Q |\nabla u|^p \zeta\, dx\, dt + a\int_Q u^q \zeta\, dx\, dt. \tag{3.27}$$

Integrating by parts yields

$$\begin{aligned} &-\int_Q u\zeta_t\, dx\, dt + \int_Q \nabla u.\nabla\zeta\, dx\, dt \\ &\qquad = b\int_Q |\nabla u|^p\zeta\, dx\, dt + a\int_Q u^q\zeta\, dx\, dt + \int_{\mathbb{R}^N} u_0(x)\zeta(x,0)\, dx. \end{aligned} \tag{3.28}$$

For the first and second terms on the left-hand side of (3.28), we have

$$\begin{aligned} &-\int_Q u\zeta_t\, dx\, dt \\ &\quad \leq \int_Q u|\zeta_t|\zeta^{1/q}\zeta^{-1/q}\, dx\, dt \\ &\quad \leq \left(\int_Q u^q\zeta\, dx\, dt\right)^{1/q} \left(\int_Q |\zeta_t|^{q'}\zeta^{-q'/q}\, dx\, dt\right)^{1/q'} \\ &\quad \leq \frac{a}{2}\int_Q u^q\zeta\, dx\, dt + C_1\int_Q |\zeta_t|^{q'}\zeta^{-q'/q}\, dx\, dt \quad \text{(Young's inequality)} \end{aligned} \tag{3.29}$$

and

$$\begin{aligned}
&\int_Q \nabla u.\nabla\zeta\,dx\,dt \\
&\quad\le \int_Q |\nabla u|\zeta^{1/p}|\nabla\zeta|\zeta^{-1/p}\,dx\,dt \\
&\quad\le \Big(\int_Q |\nabla u|^p\zeta\,dx\,dt\Big)^{1/p}\Big(\int_Q |\nabla\zeta|^{p'}\zeta^{-p'/p}\,dx\,dt\Big)^{1/p'} \\
&\quad\le \frac{b}{2}\int_Q |\nabla u|^p\zeta\,dx\,dt + C_2\int_Q |\nabla\zeta|^{p'}\zeta^{-p'/p}\,dx\,dt \quad \text{(Young's inequality)},
\end{aligned} \tag{3.30}$$

where C_1 and C_2 are positive constants. Therefore, by (3.28)–(3.30) and using the fact that $u_0 \ge 0$, we obtain

$$\begin{aligned}
&\frac{b}{2}\int_Q |\nabla u|^p\zeta\,dx\,dt + \frac{a}{2}\int_Q u^q\zeta\,dx\,dt \\
&\qquad\qquad \le C_1\int_Q |\zeta_t|^{q'}\zeta^{-q'/q}\,dx\,dt + C_2\int_Q |\nabla\zeta|^{p'}\zeta^{-p'p}\,dx\,dt.
\end{aligned} \tag{3.31}$$

Now, let $\phi \in C^\infty(\mathbb{R}^+)$ be a decreasing function with $0 \le \phi \le 1$, $r|\phi'(r)| \le c$ (c is a positive constant) for every $r > 0$, and

$$\phi(r) = \begin{cases} 1 & \text{for } r \le 1, \\ 0 & \text{for } r \ge 2. \end{cases} \tag{3.32}$$

Let

$$\zeta(t,x) = \phi^\sigma\Big(\frac{t+|x|^2}{R^2}\Big), \quad R > 0,\ \sigma \gg 1.$$

Using the change of variables $\tau = t/R^2$ and $y = x/R$, we obtain

$$\int_Q |\zeta_t|^{q'}\zeta^{-q'/q}\,dx\,dt \le cR^{\gamma_1}, \qquad \int_Q |\nabla\zeta|^{p'}\zeta^{-p'/p}\,dx\,dt \le cR^{\gamma_2}, \tag{3.33}$$

where $\gamma_1 = -2q' + 2 + N$ and $\gamma_2 = -p' + 2 + N$. On the other hand, we know that $1 < q < 1 + 2/N$ and $1 < p < 1 + 1/(N+1)$, so $\gamma_1 < 0$ and $\gamma_2 < 0$. Now, if $R \to \infty$ in (3.31), then from (3.32) and (3.33) we obtain

$$\frac{b}{2}\int_Q |\nabla u|^p\,dx\,dt + \frac{a}{2}\int_Q u^q\,dx\,dt = 0. \tag{3.34}$$

Thus $u \equiv 0$, which is a contradiction. □

For the case $b > 0$ and $a < 0$ the following theorem shows that the global solutions are unbounded whenever Ω is a bounded interval in $\mathbb{R}^1$.

THEOREM 3.4. *Let $\Omega = (-1,1)$ be an interval in $\mathbb{R}^1$. Let $q = p$, $b = 1$, and $a < 0$ with $|a|$ small. If $u \ge 0$ is the solution of the problem* (3.17) *with $u_0 = \lambda\psi$ ($\psi \ge 0$, $\psi \not\equiv 0$) and $p > 2$, then there exists a $\Lambda_0 = \Lambda_0(p,a,\Omega,\psi) > 0$ such that, for all $\lambda \ge \Lambda_0$, either blow-up occurs or $T^* = \infty$ and u is unbounded.*

Proof. Suppose that $T^* = \infty$ and $u \geq 0$ is global and uniformly bounded. Let $\varphi(x) = \sqrt{1-|x|}$. Multiplying (3.17) by φe^{-t} and integrating over $Q_1 = \mathbb{R}^+ \times \Omega$ yields

$$\int_{Q_1} e^{-t}|\nabla u|^p \varphi \, dt\, dx + a\int_{Q_1} e^{-t}u^p\varphi\, dt\, dx$$
$$= \int_{Q_1} e^{-t}u_t\varphi\, dt\, dx - \int_{Q_1} e^{-t}\triangle(u)\varphi\, dt\, dx. \quad (3.35)$$

For the second term on the right-hand side of (3.35), we have

$$-\int_{Q_1} e^{-t}(\triangle u)\varphi\, dt\, dx = \int_{Q_1} e^{-t}(\nabla u).(\nabla\varphi)\, dt\, dx. \quad (3.36)$$

Integrating by parts, for the first term on the right-hand side of (3.35) we obtain

$$\int_{Q_1} e^{-t}u_t\varphi\, dt\, dx = \int_{Q_1} e^{-t}u\varphi\, dt\, dx - \int_{\Omega} u_0(x)\varphi(x)\, dx. \quad (3.37)$$

Therefore, by (3.35)–(3.37) we have

$$\int_{Q_1} e^{-t}|\nabla u|^p \varphi \, dt\, dx + a\int_{Q_1} e^{-t}u^p\varphi\, dt\, dx$$
$$= \int_{Q_1} e^{-t}u\varphi\, dt\, dx - \int_{\Omega} u_0(x)\varphi(x)\, dx + \int_{Q_1} e^{-t}(\nabla u).(\nabla\varphi)\, dt\, dx. \quad (3.38)$$

On the other hand,

$$\int_{Q_1} e^{-t}(\nabla u).(\nabla\varphi)\, dt\, dx$$
$$\leq \int_{Q_1} |\nabla u||\nabla\varphi|\varphi^{1/p}\varphi^{-1/p}e^{-t/p}e^{-t/p'}\, dt\, dx$$
$$\leq \left(\int_{Q_1} e^{-t}|\nabla u|^p\varphi\, dt\, dx\right)^{1/p}\left(\int_{Q_1} e^{-t}\varphi^{-p'/p}|\nabla\varphi|^{p'}\, dt\, dx\right)^{1/p'}$$

(Hölder's inequality)

$$\leq \frac{1}{2}\int_{Q_1} e^{-t}|\nabla u|^p\varphi\, dt\, dx$$
$$+ C\int_{Q_1} e^{-t}\varphi^{-p'/p}|\nabla\varphi|^{p'}\, dt\, dx \quad \text{(Young's inequality)} \quad (3.39)$$

and

$$\int_{Q_1} e^{-t}u\varphi\, dt\, dx$$
$$= \int_{Q_1} u\varphi^{1/p+1/p'}e^{-t/p'}e^{-t/p}\, dt\, dx$$
$$\leq \left(\int_{Q_1} e^{-t}u^p\varphi\, dt\, dx\right)^{1/p}\left(\int_{Q_1} \varphi e^{-t}\, dt\, dx\right)^{1/p'}$$
$$\leq \frac{|a|}{2}\int_{Q_1} e^{-t}u^p\varphi\, dt\, dx + C(a) \quad \text{(Young's inequality)}, \quad (3.40)$$

where $C(a)$ is a positive constant. Hence, from (3.38)–(3.40) we get

$$\frac{1}{2}\int_{Q_1} e^{-t}|\nabla u|^p\varphi\,dt\,dx + \frac{3a}{2}\int_{Q_1} e^{-t}u^p\varphi\,dt\,dx + \int_\Omega u_0(x)\varphi(x)\,dx \le C\int_{Q_1} e^{-t}\varphi^{-p'/p}|\nabla\varphi|^{p'}\,dt\,dx + C(a). \quad (3.41)$$

Thus, if

$$\int_\Omega u_0(x)\varphi(x)\,dx \ge C\int_{Q_1} e^{-t}\varphi^{-p'/p}|\nabla\varphi|^{p'}\,dt\,dx + C(a)$$

then

$$\int_{Q_1} e^{-t}|\nabla u|^p\varphi\,dt\,dx \le 3|a|\int_{Q_1} e^{-t}u^p\varphi\,dt\,dx. \quad (3.42)$$

On the other hand, integrating by parts yields

$$\begin{aligned}
&\int_0^1 u^p(x,t)\sqrt{1-x}\,dx \\
&\quad = -\frac{2p}{3}\int_0^1 u^{p-1}(x,t)\frac{\partial u}{\partial x}(x,t)(1-x)^{3/2}\,dx - \frac{2}{3}u^p(0,t) \\
&\quad \le \frac{2p}{3}\int_0^1 u^{p-1}\big(\sqrt{1-x}\,\big)^{(p-1)/p}|\nabla u|\big(\sqrt{1-x}\,\big)^{1/p}\,dx - \frac{2}{3}u^p(0,t) \\
&\quad \le \frac{2p}{3}\left(\int_0^1 u^p\sqrt{1-x}\,dx\right)^{(p-1)/p}\left(\int_0^1|\nabla u|^p\sqrt{1-x}\,dx\right)^{1/p} - \frac{2}{3}u^p(0,t).
\end{aligned}$$

By using the same argument, we get

$$\begin{aligned}
&\int_{-1}^0 u^p(x,t)\sqrt{1+x}\,dx \\
&\quad \le \frac{2p}{3}\left(\int_{-1}^0 u^p\sqrt{1+x}\,dx\right)^{(p-1)/p}\left(\int_{-1}^0|\nabla u|^p\sqrt{1+x}\,dx\right)^{1/p} + \frac{2}{3}u^p(0,t).
\end{aligned}$$

From the foregoing inequalities we now obtain

$$\int_\Omega u^p\varphi\,dx \le \left(\frac{4p}{3}\right)^p\int_\Omega|\nabla u|^p\varphi\,dx. \quad (3.43)$$

Multiplying (3.43) by e^{-t} and integrating over $[0,\infty)$ with respect to t yields

$$\int_{Q_1} e^{-t}u^p\varphi\,dt\,dx \le \left(\frac{4p}{3}\right)^p\int_{Q_1} e^{-t}|\nabla u|^p\varphi\,dt\,dx. \quad (3.44)$$

Now, let $|a|$ be small enough that $0 < 3|a|\big(\frac{4p}{3}\big)^p < \frac{1}{2}$. Then from (3.42) and (3.44) we have

$$\int_{Q_1} e^{-t}u^p\varphi\,dt\,dx \le \frac{1}{2}\int_{Q_1} e^{-t}u^p\varphi\,dt\,dx. \quad (3.45)$$

On the other hand, u is uniformly bounded and so $0 \le \int_{Q_1} e^{-t}u^p\varphi\,dt\,dx < \infty$. Therefore, $\int_{Q_1} e^{-t}u^p\varphi\,dt\,dx = 0$, which implies that $u \equiv 0$ on Ω which is a contradiction. This completes the proof. $\square$

REMARK 3.5. The conclusion of Theorem 3.4 remains valid if $q > p$ and the remaining conditions hold.

Now we consider the equation

$$u_t - \triangle u = -\mu|\nabla u|^q + u^p, \quad x \in \Omega,\ t > 0, \tag{3.46}$$

where $\mu > 0$ is a constant. In the following theorem we show that the solutions of (3.46), (3.2)–(3.3), for small values of initial data, cannot blow up in finite time whenever Ω is a bounded domain in $\mathbb{R}^N$.

THEOREM 3.6. *Let Ω be a bounded domain. If $u_0 \geq 0$ and if u is the solution of* (3.46), (1.2)–(1.3), *then there exists an $\varepsilon > 0$ such that, if $|u_0|_{C^1(\bar{\Omega})} < \varepsilon$, then u is a global solution and decays exponentially.*

Proof. Let $\bar{\Omega}$ be in $B_R = \{x : |x| < R\}$ for some $R > 0$. We denote by $\lambda_1 > 0$ the lowest eigenvalue of $-\triangle$ in $H_0^1(B_R)$ and by φ_1 the associated eigenfunction such that

$$0 < \varphi_1 \leq 1 \ \text{ on } B_R.$$

Now observe that, for the function $W(x,t) = \varepsilon e^{-\lambda_1(t/2)}\varphi_1(x)$ with $\varepsilon > 0$ sufficiently small, we have

$$\begin{aligned} W_t - \triangle W - W^p + \mu|\nabla W|^q &= \tfrac{1}{2}\varepsilon\lambda_1\varphi_1(x)e^{-\lambda_1(t/2)} - \varepsilon^p e^{-\lambda_1(pt/2)}\varphi_1(x)^p \\ &\quad + \mu|\nabla W|^q \geq 0, \quad (x,t) \in \Omega \times (0,\infty). \end{aligned}$$

Moreover for $(x,t) \in \partial\Omega \times (0,\infty)$ we have $W > 0$.

From [30, Lemma B_1] it follows that, for u_0 with

$$0 \leq u_0(x) \leq \varepsilon \min_{\bar{\Omega}} \varphi_1(x),$$

we must have

$$0 \leq u(x,t) \leq W(x,t) \leq Ce^{-\lambda_1(t/2)},$$

where C is a positive constant. By Proposition 2.1 in [30], we must have u as a global solution. This completes the proof. □

ACKNOWLEDGMENTS. Both authors would like to thank Sharif University of Technology for supporting this research. The authors also would like to thank the referee for careful reading and useful comments.

References

[1] N. Alaa, *Solutions faibles d'equations paraboliques quasi-lineaires avec données initiales mesures,* Ann. Math. Blaise-Pascal 3 (1996), 1–15.

[2] L. Alfonsi and F. B. Weissler, *Blow-up in $\mathbb{R}^N$ for a parabolic equation with a damping nonlinear gradient term,* Progr. Nonlinear Differential Equations Appl., 7 (N. G. Lloyd et al., eds.), pp. 1–20, Birkhäuser, Boston, 1992.

[3] L. Amour and M. Ben-Artzi, *Global existence and decay for viscous Hamilton–Jacobi equations,* Nonlinear Anal. 31 (1998), 621–628.

[4] S. Benachour and P. Laurencot, *Global solutions to viscous Hamilton–Jacobi equation with irregular data,* Comm. Partial Differential Equations 24 (1999), 1999–2021.

[5] M. Ben-Artzi, *Global existence and decay for a nonlinear parabolic equation,* Nonlinear Anal. 19 (1992), 763–768.

[6] M. Ben-Artzi, J. Goodman, and A. Levy, *Remarks on a nonlinear parabolic equation,* Trans. Amer. Math. Soc. 352 (2000), 731–751.

[7] M. Ben-Artzi and H. Koch, *Decay of mass for a semilinear parabolic equation,* Comm. Partial Differential Eqations 24 (1999), 869–881.

[8] M. Ben-Artzi, Ph. Souplet, and F. B. Weissler, *Sur la non-existence et la non-unicité des solutions du probleme de Cauchy pour une équation parabolique semi-linéaire,* C. R. Acad. Sci. Paris Sér. I Math. 329 (1999), 371–376.

[9] M. Chlebik and M. Fila, *From critical exponents to blow-up rates for parabolic problems,* Rend. Mat. Appl. (7) 19 (1999), 449–470.

[10] M. Chlebik, M. Fila, and P. Quittner, *Blow-up of positive solutions of a semilinear parabolic equation with a gradient term,* Dynam. Contin. Discrete Impuls. Systems 10 (2003), 525–537.

[11] M. Chipot and F .B. Weissler, *Some blow up results for a nonlinear parabolic problem with a gradient term,* SIAM J. Math. Anal. 20 (1989), 886–907.

[12] K. Deng, *Stabilization of solutions of a nonlinear parabolic equation with a gradient term,* Math. Z. 216 (1994), 147–155.

[13] M. Fila, *Remarks on blow up for a nonlinear parabolic equation with a gradient term,* Proc. Amer. Math. Soc. 111 (1991), 795–801.

[14] A. Friedman, *Partial differential equations of parabolic type,* Prentice-Hall, Englewood Cliffs, NJ, 1964.

[15] V. A. Galaktionov and J. L. Vazquez, *Blow-up for quasilinear heat equations described by means of nonlinear Hamilton–Jacobi equations,* J. Differential Equations 127 (1996), 1–40.

[16] B. H. Gilding, M. Guedda, and R. Kersner, *The Cauchy problem for the KPZ equation,* J. Math. Anal. Appl. 284 (2003), 733–755.

[17] M. Hesaaraki and A. Moameni, *Global existence and comparison theorem for a nonlinear parabolic equation,* Bull. Austral. Math. Soc. 67 (2003), 481–492.

[18] B. Kawohl and L.A. Peletier, *Observations on blow up and dead cores for nonlinear parabolic equations,* Math. Z. 202 (1989), 207–217.

[19] M. Kirane and M. Qafsaoui, *Global nonexistence for the Cauchy problem of some nonlinear reaction-diffusion systems,* J. Math. Anal. Appl. 268 (2002), 217–243.

[20] E. Mitidieri and S. I. Pohozaev, *Absence of positive solutions for systems of quasilinear elliptic equations and inequalities in* $\mathbb{R}^N$, Dokl. Akad. Nauk 366 (1999), 13–17.

[21] O. Oleinik and S. Kruzkov, *Quasi-linear parabolic second order equations of many independent variables,* Uspekhi Mat. Nauk 16 (1961), 115–155.

[22] R. G. Pinsky, *Decay of mass for the equation* $u_t = \triangle u - a(x)|\nabla u|^q$, J. Differential Equations 165 (2000), 1–23.

[23] P. Quittner, *Blow-up for semilinear parabolic equations with a gradient term,* Math. Methods Appl. Sci. 14 (1991), 413–417.

[24] ———, *On global existence and stationary solutions for two classes of semilinear parabolic equations,* Comment. Math. Univ. Carolin. 34 (1993), 105–124.

[25] S. Snoussi, S. Tayachi, and F. B. Weissler, *Asymptotically self-similar global solutions of a semilinear parabolic equation with a nonlinear gradient terms,* Proc. Roy. Soc. Edinburgh Sect. A 129 (1999), 1291–1307.

[26] Ph. Souplet, *Résultats d'explosion en temps fini pour une équation de la chaleur non linéaire,* C. R. Acad. Sci. Paris Sér. I Math 321 (1995), 721–726.

[27] ———, *Finite time blow up for a nonlinear parabolic equation with a gradient term and applications,* Math. Methods Appl. Sci. 19 (1996), 1317–1333.

[28] ———, *Recent results and open problems on parabolic equations with gradient nonlinearities,* Electron. J. Differential Equations 20 (2001), 1–19.

[29] ———, *Gradient blow-up for multidimensional nonlinear parabolic equations with general boundary conditions,* Differential Integral Equations 15 (2002), 237–256.

[30] Ph. Souplet and F. B. Weissler, *Poincarés inequality and global solutions of a nonlinear parabolic equation,* Ann. Inst. H. Poincaré Anal. Non Linéaire 16 (1999), 337–373.

[31] S. Tayachi, *Forward self-similar solutions of a semilinear parabolic equation with a nonlinear gradient term,* Differential Integral Equations 9 (1996), 1107–1117.

M. Hesaaraki
Department of Mathematics
Sharif University of Technology
P.O. Box 11365-9415
Tehran
Iran

hesaraki@sina.sharif.ac.ir

A. Moameni
Department of Mathematics
Sharif University of Technology
P.O. Box 11365-9415
Tehran
Iran

moameni@mehr.sharif.ac.ir