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Cycles over Fields of Transcendence Degree 1

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Introduction

We work over subfields k of $\mathbb{C}$, the field of complex numbers. For a smooth variety V over k, the Chow group of cycles of codimension p is defined (see [F]) as

$$\mathrm{CH}^p(V) = \frac{Z^p(V)}{R^p(V)},$$

where (a) the group of cycles $Z^p(V)$ is the free abelian group on scheme-theoretic points of V of codimension p and (b) rational equivalence $R^p(V)$ is the subgroup generated by cycles of the form $\mathrm{div}_W(f)$, where W is a subvariety of V of codimension $p-1$ and f is a nonzero rational function on it. There is a natural cycle class map

$$\mathrm{cl}_p \colon \mathrm{CH}^p(V) \to \mathrm{H}^{2p}(V),$$

where the latter denotes the singular cohomology group $\mathrm{H}^{2p}(V(\mathbb{C}), \mathbb{Z})$ with the (mixed) Hodge structure given by Deligne (see [D]). The kernel of cl_p is denoted by $F^1\,\mathrm{CH}^p(V)$. There is an Abel–Jacobi map (see [G])

$$\Phi_p \colon F^1\,\mathrm{CH}^p(V) \to \mathrm{IJ}^p(\mathrm{H}^{2p-1}(V)),$$

where the latter is the intermediate Jacobian of a Hodge structure and defined as

$$\mathrm{IJ}^p(H) = \frac{H \otimes \mathbb{C}}{F^p(H \otimes \mathbb{C}) + H}.$$

The kernel of Φ_p is denoted by $F^2\,\mathrm{CH}^p(V)$.

Conjecture 1 (Bloch–Beilinson). *If V is a variety defined over a number field k, then $F^2\,\mathrm{CH}^p(V) = 0$.*

We (of course) offer no proof of this conjecture. However, there are examples due to Schoen and Nori (see [S]) showing that one cannot relax the conditions in this conjecture. In this paper we present these and other examples to show that $F^2\,\mathrm{CH}^2(V)$ is nonzero for V a variety over a field of transcendence degree ≥ 1 whenever it is nonzero over some larger algebraically closed field.

We begin in Section 1 with a lemma. We then apply this lemma to prove the following result (using Hodge-theoretic methods) for the second symmetric power of a curve.

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PROPOSITION 1. *Let C be a curve of genus ≥ 2 over a number field F. Then there is a nontorsion cycle ξ in $F^2\,\mathrm{CH}^2(\mathrm{Sym}^2(C)\otimes K)$, where K is the algebraic closure of the function field $F(C)$.*

In Section 2 we generalize this to surfaces other than $\mathrm{Sym}^2(C)$. In order to do this we need to use an l-adic analogue of the Abel–Jacobi map. As before, let V be a smooth projective variety over k. Let R be a finitely generated subring of k for which there is a smooth projective morphism $\pi : \mathcal{V} \to \mathrm{Spec}(R)$ whose base change to k gives V. Let $F^1\,\mathrm{CH}^p(\mathcal{V})$ denote the kernel of the natural homomorphism

$$\mathrm{CH}^p(\mathcal{V}) \to \mathrm{H}^0_{\text{ét}}(\mathrm{Spec}(R), \mathrm{R}^{2p}\,\pi_*\mathbb{Q}_l(p)).$$

We obtain a natural homomorphism

$$\Phi_{p,l,\mathcal{V}} \colon F^1\,\mathrm{CH}^p(\mathcal{V}) \to \mathrm{H}^1_{\text{ét}}(\mathrm{Spec}(R), \mathrm{R}^{2p-1}\,\pi_*\mathbb{Q}_l(p)).$$

Because any cycle (or rational equivalence) on V over k is defined over a finitely generated ring R, the direct limit of $F^1\,\mathrm{CH}^p(\mathcal{V})$ is $F^1\,\mathrm{CH}^p(V)$. For want of better notation, we define

$$\mathrm{IJ}^p_{\text{ét}}(\mathrm{H}^{2p-1}(V)) = \varinjlim_R \mathrm{H}^1_{\text{ét}}(\mathrm{Spec}(R), \mathrm{R}^{2p-1}\,\pi_*\mathbb{Q}_l(p));$$

that is, IJ^p is the corresponding direct limit of the $\mathrm{H}^1_{\text{ét}}$. An l-adic analogue of the Abel–Jacobi map is the direct limit of the $\Phi_{p,l,\mathcal{V}}$ as we vary R:

$$\Phi_{p,l,\lim} \colon F^1\,\mathrm{CH}^p(V) \to \mathrm{IJ}^p_{\text{ét}}(\mathrm{H}^{2p-1}(V)).$$

On the other hand, it is clear that we have natural homomorphisms

$$\mathrm{H}^1_{\text{ét}}(\mathrm{Spec}(R), \mathrm{R}^{2p-1}\,\pi_*\mathbb{Q}_l(p)) \to \mathrm{Ext}^1_{\mathrm{Gal}(\bar{k}/k)}(\mathbb{Q}_l(-p), \mathrm{H}^{2p-1}_{\text{ét}}(V, \mathbb{Q}_l)).$$

One can show that these homomorphisms are injective when k is the field of fractions of the ring R by finding extensions of l-adic sheaves that represent elements of $\mathrm{H}^1_{\text{ét}}$. The limiting homomorphism

$$\Phi_{p,l} \colon F^1\,\mathrm{CH}^p(V) \to \mathrm{Ext}^1_{\mathrm{Gal}(\bar{k}/k)}(\mathbb{Q}_l(-p), \mathrm{H}^{2p-1}_{\text{ét}}(V, \mathbb{Q}_l))$$

is thus another variant of the Abel–Jacobi homomorphism that has the same kernel as $\Phi_{p,l,\lim}$ when k is a finitely generated field. We use the former map to define $F^2\,\mathrm{CH}^2(V) = \bigcap_l \ker \Phi_{p,l,\lim}$. Recent results of Nori (unpublished) show that the Hodge-theoretic definition of F^2 gives a group that is not smaller than the group given by the preceding l-adic one. It is still not clear that the two definitions coincide, as conjectured by many (see [J]) and as proved by Raskind [R] for *zero cycles* by using the identification of the intermediate Jacobian with the Albanese variety in this case.

The following theorem uses the l-adic version of the definition of F^2. In the rest of the paper the two definitions of F^2 will be used without a change of notation; it will be clear from the context which definition is being employed. By the result of Raskind [R], the two definitions coincide in our case.

THEOREM 2. *Let S be any surface with $p_g(S) \neq 0$ defined over a number field F. Then there are nontrivial cycles in $F^2\,\mathrm{CH}^2(S \otimes K, \mathbb{Q})$, where K is the algebraic closure of the function field $F(T)$ in one variable T over F.*

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1. The Second Symmetric Power of a Curve

For a surface S, let $H^2(S)_{\mathrm{tr}}$ denote the orthogonal complement in $H^2(S)$ of the (Néron–Severi) subspace generated by cohomology classes of curves in S that are defined over an algebraically closed field.

LEMMA 3 (Green–Griffiths). *Let C be a smooth projective curve and let S be a smooth projective surface. Suppose Z is a homologically trivial cycle of codimension 2 in $C \times S$. Assume that C, S, and Z are defined over a subfield k of $\mathbb{C}$. Suppose that* either *of the following conditions hold*:

(1) *the component in $\mathrm{IJ}^2(\mathrm{H}^1(C) \otimes \mathrm{H}^2(S)_{\mathrm{tr}})$ of the Abel–Jacobi invariant of Z is nontorsion*; *or*
(2) *the component in $\mathrm{IJ}^2_{\text{ét}}(\mathrm{H}^1(C) \otimes \mathrm{H}^2(S)_{\mathrm{tr}})$ of the l-adic Abel–Jacobi invariant $\Phi_{2,l,\lim}(Z)$ is nontorsion.*

Then the following map is nontorsion on $F^1\,\mathrm{CH}^1(C \otimes \mathbb{C})$:

$$z = p_{2*}(p_1^*(\_) \cdot Z) \colon \mathrm{CH}^1(C \otimes \mathbb{C}) \to \mathrm{CH}^2(S \otimes \mathbb{C}).$$

Proof. Let A denote the canonical divisor of C (which is defined over k). Let g denote the generic point of C considered as a point of $C \otimes k(C)$ and hence as a point of $C \otimes \mathbb{C}$ via some inclusion of $k(C)$ in $\mathbb{C}$. If $z(\deg(A)g - A)$ is torsion in $\mathrm{CH}^2(S \otimes \mathbb{C})$, then a rational equivalence demonstrating this is defined over some finite extension of $k(C)$. Taking the norm of this equivalence, we see that $z(\deg(A)g - A)$ is torsion in $\mathrm{CH}^2(S \otimes k(C))$ as well. Thus, a multiple of $z(\deg(A)g - A)$ is a sum of the form $\sum \mathrm{div}_{\mathcal{D}_i}(f_i)$, where $\mathcal{D}_i \subset \mathrm{Spec}(k(C)) \times S$ is a curve and f_i is a function on $\mathcal{D}_i$. Let $D_i \subset C \times S$ be the Zariski closure of $\mathcal{D}_i$, and consider the cycle W obtained by summing up $\mathrm{div}_{D_i}(f_i)$. By assumption, W restricts to a multiple of $z(\deg(A)q - A)$ on $\mathrm{Spec}(k(C)) \times S$. Hence there exist closed points q_j of C over k and divisors E_j in $\mathrm{Spec}(k(q_j)) \times S$ such that, for some integer n, we have

$$\sum \mathrm{div}_{D_i}(f_i) = W = n(\deg(A)Z - C \times z(A)) + \sum_j q_j \times E_j.$$

The image in the relevant piece of IJ^2, of cycles of the form $C \times z(A)$ and of cycles of the form $\sum_j q_j \times E_j$, is zero in either case, (1) or (2). This contradicts the hypothesis on Z. $\square$

From this lemma it follows that, in order to construct a nontrivial cycle in $F^2\,\mathrm{CH}^2(S)$ over a field of transcendence degree 1, it is sufficient to construct C, S, and Z (as in the lemma) over a field of transcendence degree 0.

COROLLARY 4. *Let C be a smooth projective curve and S a smooth projective surface over a subfield F of $\bar{\mathbb{Q}}$. Let Z be a homologically trivial cycle on $C \times S$ such that the induced homomorphism*

$$z = p_{2*}(p_1^*(\_) \cdot Z)\colon \mathrm{CH}^1(C \otimes \mathbb{C}) \to \mathrm{CH}^2(S \otimes \mathbb{C})$$

is nontorsion. Then there is a nontorsion class in $F^2\,\mathrm{CH}^2(S \otimes_F \overline{F(C)})$.

Proof. From a result of Roitman [Ro] it follows that the kernel of z is a countable union of translates of proper abelian subvarieties A_i of $J(C)$. But each such translated abelian subvariety is defined over $\bar{\mathbb{Q}}$ because Z is defined over $\bar{\mathbb{Q}}$. Using a point p of C over $\bar{\mathbb{Q}}$, we embed C in $J(C)$. Since the image of C generates $J(C)$, the image of the generic point $\mathrm{Spec}(F(C)) \to C$ is not in A_i for any i. Its image under z gives us the desired cycle in $F^2\,\mathrm{CH}^2(S \otimes_F \overline{F(C)})$. □

Let C be a curve of genus at least 2 over $k = \bar{\mathbb{Q}}$. Let $S = \mathrm{Sym}^2(C)$, and consider the cycle X in $C \times C \times S$ that is the graph of the natural map $q\colon C \times C \to S = \mathrm{Sym}^2(C)$. The map q_* on $\mathrm{H}^{1,0}(C) \otimes \mathrm{H}^{1,0}(C)$ identifies the range $\mathrm{H}^{2,0}(S)$ with the second exterior power of $\mathrm{H}^{1,0}(C)$. The cohomology class $[X]$ therefore induces a nonzero map

$$p_{23*}(p_1^*(\_) \cup [X])\colon \mathrm{H}^{1,0}(C) \to \mathrm{H}^{0,1}(C) \otimes \mathrm{H}^{2,0}(S).$$

Thus the map

$$J(C \otimes \mathbb{C}) = \mathrm{IJ}^1(\mathrm{H}^1(C)) \to \mathrm{IJ}^2(\mathrm{H}^1(C) \otimes \mathrm{H}^2(S)_{\mathrm{tr}})$$

is nontorsion. Hence, the image is a nonzero abelian variety A that is a quotient of $J(C \otimes \mathbb{C})$. Since $J(C)$ is defined over $\bar{\mathbb{Q}}$, so is any quotient abelian variety. In particular, A is $B \otimes \mathbb{C}$ for some abelian variety B defined over a number field.

PROPOSITION 5. *Let $J \to B$ $(B \neq 0)$ be a surjective homomorphism of abelian varieties over a number field. Then there exist points in $J(\bar{\mathbb{Q}})$ whose images in B are nontorsion.*

Proof. Let L be the completion of the number field at a place lying over a prime p, and let M be the algebraic closure of $\mathbb{Q}$ in L. Observe that $J(L)$ and $B(L)$ are p-adic Lie groups. Consider a finite morphism $\pi\colon J \to \mathbb{P}^n$ and let $U \subset \mathbb{P}^n$ be a Zariski open set over which π is étale. It is clear that $U(M)$ is dense in $U(L)$. By the implicit function theorem, the map $J(L) \cap \pi^{-1}(U) \to U(L)$ is open; moreover, the former set is dense in $J(L)$. Since π is finite, any point in $J(L) \cap \pi^{-1}(U(M))$ is in $J(M)$. Thus $J(M)$ is dense in $J(L)$. Now $J(L) \to B(L)$ is open, so the image of $J(M)$ in $B(L)$ is dense in an open subgroup. The torsion subgroup of a p-adic Lie group is finite. The result follows. □

COROLLARY 6. *Let A be a nonzero abelian variety over a number field. Then the group $A(\bar{\mathbb{Q}})$ is nontorsion.*

By Proposition 5 and our previous discussion, we obtain a cycle Z on $C \times S$ over $\bar{\mathbb{Q}}$ as required. Applying Lemma 3, we see that we have proved Proposition 1.

2. The General Case

We use an argument of Terasoma (as explained to us by Bloch and Nori) to construct a triple (C, S, Z) as in Lemma 3 for *any* surface S with $p_g \neq 0$.

LEMMA 7. *Assume $(\tilde{C}, \tilde{S}, \tilde{Z})$ is a triple over some finitely generated subfield k of $\mathbb{C}$ that satisfies condition* (2) *of Lemma 3. Then there exists a specialization (C, S, Z) over a number field $F \subset \bar{\mathbb{Q}}$ that satisfies condition* (2) *of Lemma 3.*

Proof. Let T be an affine variety over a number field contained in k, and let $(\mathcal{C}, \mathcal{S}, \Xi)$ be a triple over S so chosen that this data restricts to $(\tilde{C}, \tilde{S}, \tilde{Z})$ under the natural map $\operatorname{Spec} k \to T$. Using the Gysin sequence, we obtain the following exact sequence of $\mathbb{Q}_l$ constructible local systems on T:

$$0 \to \mathrm{R}^1_{\text{ét}}\, \pi_{T*}(\mathcal{C}, \mathbb{Q}_l) \otimes \mathrm{R}^2_{\text{ét}}\, \pi_{T*}(\mathcal{S}, \mathbb{Q}_l)_{\text{tr}} \to \mathcal{V} \to \mathbb{Q}_l(-2)_T \to 0;$$

here $\mathrm{R}^2_{\text{ét}}\, \pi_{T*}(\mathcal{S}, \mathbb{Q}_l)_{\text{tr}}$ denotes the local system on T associated with the Galois representation $\mathrm{H}^2_{\text{ét}}(\tilde{S} \otimes \bar{k}, \mathbb{Q}_l)_{\text{tr}}$. Moreover, the extension class of this short exact sequence is naturally identified with the component in $\mathrm{H}^1_{\text{ét}}(T, \mathrm{R}^1_{\text{ét}}\, \pi_{T*}(\mathcal{C}, \mathbb{Q}_l) \otimes \mathrm{R}^2_{\text{ét}}\, \pi_{T*}(\mathcal{S}, \mathbb{Q}_l)_{\text{tr}} \otimes \mathbb{Q}_l(2))$ of the class $\Phi_{2,l,\mathcal{C}\times_T\mathcal{S}}(\Xi)$. By assumption, condition (2) of Lemma 3 means this component is nonzero. So the sequence is nonsplit.

If p is a closed point of T, let C_p, S_p, Z_p denote the specializations. By restricting the previous sequence we obtain an exact sequence of representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/k(p))$ or, equivalently, of a decomposition group of p in $\mathrm{Gal}(\bar{k}/k)$:

$$0 \to \mathrm{H}^1_{\text{ét}}(C_p \otimes \bar{\mathbb{Q}}, \mathbb{Q}_l) \otimes (\mathrm{R}^2_{\text{ét}}\, \pi_{T*}(\mathcal{S}, \mathbb{Q}_l)_{\text{tr}})_{\overline{k(p)}} \to \mathcal{V}_{\overline{k(p)}} \to \mathbb{Q}_l(-2) \to 0.$$

There are two possible problems.

(i) There may be more divisors in S_p than on the generic fibre of $\mathcal{S} \to T$, so that

$$\mathrm{H}^2_{\text{ét}}(S_p \otimes \overline{k(p)}, \mathbb{Q}_l)_{\text{tr}} \subsetneq (\mathrm{R}^2_{\text{ét}}\, \pi_{T*}(\mathcal{S}, \mathbb{Q}_l)_{\text{tr}})_{\overline{k(p)}}.$$

(ii) The preceding exact sequence may split so that the extension class becomes zero for Z_p.

These two problems are resolved by applying the following lemma to the l-adic representation

$$\mathcal{V}_{\bar{k}} \oplus \mathrm{H}^2_{\text{ét}}(S \otimes \bar{k}, \mathbb{Q}_l). \qquad \square$$

LEMMA 8 (Terasoma). *Let V be a continuous l-adic representation of the fundamental group of a variety T over a number field. Then there are infinitely many closed points p of T such that the image of* $\mathrm{Gal}(\overline{k(T)}/k(T))$ *in* $\mathrm{GL}(V)$ *is the same as the image of the decomposition group of p.*

Proof. Let G denote the image of $\mathrm{Gal}(\overline{k(T)}/k(T))$ in $\mathrm{GL}(V)$, and let D_p denote the image of a decomposition group associated with a closed point p of T. These are compact subgroups of $\mathrm{GL}(V)$. As in [T], there is a subgroup H of finite index in G such that $D_p = G$ if and only if the natural map $D_p \to G/H$ is a surjection. Equivalently, if $f\colon T' \to T$ denotes the étale cover of T' corresponding to this finite quotient of the fundamental group of T, then it is enough that $f^{-1}(p)$ is a closed point of T'. To find such a point p, we may shrink T in order to assume that T is affine and that there is an étale map $g\colon T \to \mathbb{A}^n$. Thus it is enough to find a point q in $\mathbb{A}^n$ such that $(g \circ f)^{-1}(q)$ is a closed point of T'. Such a point can be found by the Hilbert irreducibility theorem. $\square$

Now we consider a surface S over $\bar{\mathbb{Q}}$ with $p_g(S) \neq 0$; let p be a closed point on S. Murre [M] has constructed a decomposition of the diagonal of a surface S,

$$\Delta_S = p \times S + S \times p + X_{2,2} + X_{1,3} + X_{3,1},$$

for the diagonal Δ_S in $\mathrm{CH}^2(S \times S) \otimes \mathbb{Q}$; here $X_{i,j}$ is a cycle class such that its cohomology class has a nonzero Künneth component only in $\mathrm{H}^i(S) \otimes \mathrm{H}^j(S) \otimes \mathbb{Q}$. Moreover, by the Hodge index theorem we can write $nX_{2,2} = \sum_i C_i \times D_i + X_{2,2,\mathrm{tr}}$, where C_i and D_i are curves on S and $X_{2,2,\mathrm{tr}}$ has its cohomology class in $\mathrm{H}^2(S)_{\mathrm{tr}} \otimes \mathrm{H}^2(S)_{\mathrm{tr}}$.

Let $H \subset S \times \mathbb{P}^1$ be any pencil; $Y_{i,j}$ will denote the cycles on $H \times S$ that are the natural pull-backs of $X_{i,j}$ for each i and j; similarly, let $Y_{2,2,\mathrm{tr}}$ be the pull-back of $X_{2,2,\mathrm{tr}}$. Let H_0 denote the divisor class of the fibres of the pencil and let A denote the class of an ample divisor on S. The class $Y_{2,2,\mathrm{tr}}$ on $H \times S$ restricts to a homologically trivial class on every member H_t of the pencil; however, $\mathrm{H}^2(H)_{\mathrm{tr}} = \mathrm{H}^2(S)_{\mathrm{tr}}$ and so the cohomology class of $Y_{2,2,\mathrm{tr}}$ in $\mathrm{H}^2(H)_{\mathrm{tr}} \otimes \mathrm{H}^2(S)_{\mathrm{tr}}$ is nonzero.

For any open subset $U \subset \mathbb{P}^1$, let $H_U = H \times_{\mathbb{P}^1} U$; the map $\mathrm{H}^2(H)_{\mathrm{tr}} \to \mathrm{H}^2(H_U)$ is injective. Thus the pull-back of $Y_{2,2,\mathrm{tr}}$ to $H_U \times S$ is not homologically trivial for all U. By the Leray spectral sequence for the map $\pi\colon H_U \to U$, we see that $\mathrm{H}^2(H)_{\mathrm{tr}}$ maps injectively into $\mathrm{H}^1(U, \mathrm{R}^1\pi_*(\mathbb{Q}_l))$. Let g denote the generic point of $\mathbb{P}^1$. The restriction of $Y_{2,2,\mathrm{tr}}$ to $H_g \times S$ is a cycle Z_g such that the triple $(H_g, S \otimes_k k(g), Z_g)$ satisfies condition (ii) of Lemma 7 over the field $k(g)$. Applying Lemma 7, we see that we have a point t in $\mathbb{P}^1(\mathbb{Q})$ such that (H_t, S, Z_t) is a triple satisfying condition 3 of Lemma 3. Thus we have proved Theorem 2.

References

[D] P. Deligne, *Théorie de Hodge. II,* Inst. Hautes Études Sci. Publ. Math. 40 (1971), 5–57.

[F] W. Fulton, *Intersection theory,* 2nd ed., Springer-Verlag, Berlin, 1998.

[G] P. A. Griffiths, *On the periods of certain rational integrals. I, II,* Ann. of Math. (2) 90 (1969), 460–495, 496–541.

[J] U. Jannsen, *Mixed motives, motivic cohomology, and Ext-groups,* Proceedings of the International Congress of Mathematicians (Zürich, 1994), pp. 667–679, Birkhäuser, Basel, 1995.

[M] J. P. Murre, *On the motive of an algebraic surface,* J. Reine Angew. Math. 409 (1990), 190–204.

[R] W. Raskind, *Higher l-adic Abel–Jacobi mappings and filtrations on Chow groups,* Duke Math. J. 78 (1995), 33–57.

[Ro] A. A. Rojtman, *The torsion of the group of 0-cycles modulo rational equivalence,* Ann. of Math. (2) 111 (1980), 553–569.

[S] C. Schoen, *Zero cycles modulo rational equivalence for some varieties over fields of transcendence degree one,* Algebraic geometry, Bowdoin (Brunswick, ME, 1985), pp. 463–473, Amer. Math. Soc., Providence, RI, 1987.

[T] T. Terasoma, *Complete intersections with middle Picard number 1 defined over* $\mathbb{Q}$, Math. Z. 189 (1985), 289–296.

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