Michigan Math. J. 52 (2004)

A Purity Theorem for Abelian Schemes

Adrian Vasiu

1. Introduction

Let K be the field of fractions of a discrete valuation ring O. Let Y be a flat O-scheme that is regular, and let U be an open subscheme of Y whose complement in Y is of codimension in Y at least 2. We call the pair (Y, U) an extensible pair. Let $q : \mathcal{S} \to \mathrm{Sch}_O$ be a stack over the category Sch_O of O-schemes endowed with the Zariski topology. Let $\mathcal{S}_Z$ be the fibre of q over an O-scheme Z. Answers to the following question provide information on $\mathcal{S}$.

Question 1.1. Is the pull-back functor $\mathcal{S}_Y \to \mathcal{S}_U$ surjective on objects?

Question 1.1 has a positive answer in any one of the following three cases:

(i) $\mathcal{S}$ is the stack of morphisms into the Nèron model over O of an abelian variety over K, and Y is smooth over O (see [N]);
(ii) $\mathcal{S}$ is the stack of smooth, geometrically connected, projective curves of genus at least 2 (see [M-B]);
(iii) $\mathcal{S}$ is the stack of stable curves of locally constant type, and there is a divisor DIV of Y with normal crossings such that the reduced scheme $Y \setminus U$ is a closed subscheme of DIV (see [dJO]).

Let p be a prime. If the field K is of characteristic 0, then an example of Raynaud–Gabber–Ogus shows that Question 1.1 does not always have a positive answer if $\mathcal{S}$ is the stack of abelian schemes (see [dJO, Sec. 6]). This invalidates [FaC, Chap. IV, Thms. 6.4, 6.4′, 6.8] and leads to the following problem.

Problem 1.2. Classify all those Y with the property that, for any extensible pair (Y, U) with U containing Y_K, every abelian scheme (resp., every p-divisible group) over U extends to an abelian scheme (resp., to a p-divisible group) over Y.

We call such Y a healthy (resp., p-healthy) regular scheme (cf. [V, 3.2.1(2), (9)]. The counterexample of [FaC, p. 192] and the classical purity theorem of [G, p. 275] indicate that Problem 1.2 is of interest only if K is of characteristic 0 (resp., only if O is a faithfully flat $\mathbb{Z}_{(p)}$-algebra). We shall therefore assume hereafter that O is of mixed characteristic $(0, p)$. Let $e \in \mathbb{N}$ be the index of ramification of O. If $e \le p-2$, then a result of Faltings states that Y is healthy and p-healthy regular,

Received September 16, 2002. Revision received August 15, 2003.

provided it is formally smooth over O (see [Mo, 3.6] and [V, 3.2.2(1) and 3.2.17], a correction to step B of which is implicitly achieved here by Proposition 4.1). If $p \geq 5$, then there are local O-schemes that are healthy and p-healthy regular but are not formally smooth over some discrete valuation ring (see [V, 3.2.2(5)]). The goal of this paper is to prove the following theorem.

THEOREM 1.3. *If $e = 1$, then any regular, formally smooth O-scheme is healthy and p-healthy regular.*

The case $p \geq 3$ is already known, as remarked previously. The case $p = 2$ answers a question of Deligne. In Section 2 we present complements on the crystalline contravariant Dieudonné functor. These complements are needed in Section 3 to prove Lemma 3.1, which pertains to extensions of short exact sequences of finite, flat, commutative group schemes. In Section 4 we use Lemma 3.1 and [FaC] to prove Theorem 1.3.

Milne used an analogue of Question 1.1(i) to define integral canonical models of Shimura varieties (see [Mi, Sec. 2] and [V, 3.2.3, 3.2.6]). Theorem 1.3 implies the uniqueness of such integral canonical models and extends parts of [V] to arbitrary mixed characteristic (see [V, 3.2.3.2, 3.2.4, 3.2.12, etc.]). Also one can use Theorem 1.3 and the integral models of compact, unitary Shimura varieties used in [K] to provide the first concrete examples of Nèron models (as defined in [BLR, p. 12]) of projective varieties over K whose extensions to $\bar{K}$ are not embeddable into abelian varieties over $\bar{K}$.

ACKNOWLEDGMENTS. I would like to thank G. Faltings for mentioning that his result should hold for $p = 2$ as well. I also thank W. McCallum and the referee for several suggestions.

2. The Crystalline Dieudonné Functor

Let k be a perfect field of characteristic $p > 0$. Let σ_k be the Frobenius automorphism of the Witt ring $W(k)$ of k, and let R be a regular, formally smooth $W(k)$-algebra. Let $Y := \mathrm{Spec}(R)$. Let Φ_R be a Frobenius lift of the p-adic completion $R^\wedge$ of R that is compatible with σ_k. Let $\Omega_R^\wedge$ be the p-adic completion of the R-module of relative differentials of R with respect to $W(k)$, and let $d\Phi_{R/p}$ be the differential of Φ_R divided by p. For $n \in \mathbb{N}$, the reduction modulo p^n of $d\Phi_{R/p}$ is denoted in the same way. If Z is an arbitrary $\mathbb{Z}_{(p)}$-scheme, let

$$p - \mathrm{FF}(Y)$$

be the category of finite, flat, commutative group schemes of p-power order over Z.

Let $\mathcal{MF}_{[0,1]}^{\nabla}(Y)$ be the Faltings–Fontaine category defined as follows. Its objects are quintuples

$$(M, F, \Phi_0, \Phi_1, \nabla),$$

where M is an R-module, F is a direct summand of M, both $\Phi_0 \colon M \to M$ and $\Phi_1 \colon F \to M$ are Φ_R-linear maps, and $\nabla \colon M \to M \otimes_R \Omega_R^\wedge$ is an integrable, nilpotent mod p connection on M, such that the following five axioms hold:

1. $\Phi_0(m) = p\Phi_1(m)$ for all $m \in F$;
2. M is R-generated by $\Phi_0(M) + \Phi_1(F)$;
3. $\nabla \circ \Phi_0(m) = p(\Phi_0 \otimes d\Phi_{R/p}) \circ \nabla(m)$ for all $m \in M$;
4. $\nabla \circ \Phi_1(m) = (\Phi_0 \otimes d\Phi_{R/p}) \circ \nabla(m)$ for all $m \in F$; and
5. locally in the Zariski topology of Y, M is a finite direct sum of R-modules of the form R/p^sR, where $s \in \mathbb{N} \cup \{0\}$.

A morphism $f: (M, F, \Phi_0, \Phi_1, \nabla) \to (M', F', \Phi'_0, \Phi'_1, \nabla')$ between two such quintuples is an R-linear map $f_0: M \to M'$ taking F into F' and such that the following three identities hold: $\Phi'_0 \circ f_0 = f_0 \circ \Phi_0$, $\Phi'_1 \circ f_0 = f_0 \circ \Phi_1$, and $\nabla' \circ f_0 = (f_0 \otimes_R 1_{\Omega^\wedge_R}) \circ \nabla$. We refer to M as the underlying R-module of $(M, F, \Phi_0, \Phi_1, \nabla)$. Disregarding the connections (and thus axioms 3 and 4), we obtain the category $\mathcal{MF}_{[0,1]}(Y)$. Categories like $\mathcal{MF}_{[0,1]}(Y)$ and $\mathcal{MF}^\nabla_{[0,1]}(Y)$, in the context of arbitrary smooth $W(k)$-schemes, were first introduced in [Fa] as inspired by [F] and [FL], which worked with the category $\mathcal{MF}_{[0,1]}(\mathrm{Spec}(W(k)))$. In the sequel we will need the following result of Faltings.

Proposition 2.1. *We assume that $\Omega^\wedge_R$ is a flat R-module. Then the category $\mathcal{MF}^\nabla_{[0,1]}(Y)$ is abelian, and the functor from it into the category of R-modules that takes f into f_0 is exact.*

Proof. This follows from [Fa, pp. 31–33]. Strictly speaking, in [Fa] the result is stated only for smooth $W(k)$-algebras, but the inductive arguments work also for regular, formally smooth $W(k)$-algebras. In fact, we can use Artin's approximation theorem to reduce Proposition 2.1 to the result in [Fa] as follows.

Let f and f_0 be as before. We denote also by Φ_0, Φ_1, ∇ and Φ'_0, Φ'_1, ∇' the different Φ_R-linear maps and connections obtained from them via restrictions or via natural passage to quotients (for ∇ and ∇' this makes sense because $\Omega^\wedge_R$ is a flat R-module). We need to show that the three quintuples $(\mathrm{Ker}(f_0), F \cap \mathrm{Ker}(f_0), \Phi_0, \Phi_1, \nabla)$, $(f_0(M), f_0(F), \Phi'_0, \Phi'_1, \nabla')$, and $(M'/f_0(M), F'/f_0(F), \Phi'_0, \Phi'_1, \nabla')$ are objects of $\mathcal{MF}^\nabla_{[0,1]}(Y)$ and that $f_0(F) = F' \cap f_0(M)$. Since $\Omega^\wedge_R$ is a flat R-module, axioms 3 and 4 hold and so from now on we do not mention ∇ and ∇'. Hence we are interested only in the morphism $g: (M, F, \Phi_0, \Phi_1) \to (M', F', \Phi'_0, \Phi'_1)$ of $\mathcal{MF}_{[0,1]}(Y)$ defined by f_0. We can assume that M and M' are annihilated by p^n and that R is local. Using devissage as in [Fa, p. 33, ll. 4–11], it is enough to handle the case $n = 1$. So all the R-modules involved in the three quintuples listed are in fact R/pR-modules. Thus, to check that they are free, we can also assume that R is complete. Based on [Ma, p. 268], there is a k-subalgebra k_1 of R/pR that is isomorphic to the residue field of R. We easily get that R/pR is a k-algebra of the form $k_1[[x_1, \ldots, x_d]]$, where $d \in \mathbb{N} \cup \{0\}$. Because $n = 1$, the choice of Φ_R plays no role in the study of the three quintuples and so we can also assume that k_1 is perfect.

We choose R/pR-bases $\mathcal{B}$ and $\mathcal{B}'$ of M and M' (respectively) such that their subsets are R/pR-bases of F and F'. With respect to $\mathcal{B}$ and $\mathcal{B}'$, the functions f_0, Φ_0, Φ_1, Φ'_0, and Φ'_1 involve a finite number of coordinates that are elements of R/pR. Let A_0 be the k_1-subalgebra of R/pR generated by all these coordinates, and observe that A_0 is of finite type. Hence, from [BLR, p. 91] we derive the existence of an A_0-algebra A_1 that is smooth and such that the k_1-monomorphism $A_0 \hookrightarrow R/pR$ factors through A_1. Localizing A_1, we can assume that A_1 is the reduction mod p of a smooth $W(k_1)$-algebra R_1. Now fix a Frobenius lift of the p-adic completion of R_1 that is compatible with σ_{k_1}; hence we can speak about $\mathcal{MF}_{[0,1]}(R_1)$. We get that g is the natural tensorization with R of a morphism g_1 of $\mathcal{MF}_{[0,1]}(R_1)$. Applying [Fa, pp. 31–32] to g_1 and tensoring with R, we deduce that axioms 1, 2, and 5 hold for the three quintuples and that $f_0(F) = F' \cap f_0(M)$. $\square$

CONSTRUCTION 2.2. Let $W_n(k) := W(k)/p^n W(k)$. There is a contravariant, $\mathbb{Z}_p$-linear functor

$$\mathbb{D}\colon p-\mathrm{FF}(Y) \to \mathcal{MF}^{\nabla}_{[0,1]}(Y).$$

Similar functors but with Y replaced by $\mathrm{Spec}(W(k))$ (resp., by a smooth $W(k)$-scheme and with $p > 2$) were first considered in [F] (resp. [Fa]). The existence of $\mathbb{D}$ is a modification of a particular case of [BBM, Chap. 3]. We now include the construction of $\mathbb{D}$ based in essence on [BBM] and [Fa, 7.1]. We will use Berthelot's crystalline site $\mathrm{CRIS}(Y_{W_n(k)}/\mathrm{Spec}(W(k)))$ (see [B, Chap. III, Sec. 4]) and its standard exact sequence $0 \to \mathcal{J}_{Y_{W_n(k)}/W(k)} \to \mathcal{O}_{Y_{W_n(k)}/W(k)}$ (see [BBM, p. 12]).

Let G be an object of $p-\mathrm{FF}(Y)$ that is annihilated by p^n. Let $(\tilde{M}, \tilde{\Phi}_0, \tilde{V}_0, \tilde{\nabla})$ be the evaluation of the Dieudonné crystal $\mathbb{D}(G_{Y_k}) = \mathcal{E}xt^1_{Y_k/W(k)}(\underline{G_{Y_k}}, \mathcal{O}_{Y_k/W(k)})$ (see [BBM, p. 116]) at the thickening naturally attached to the closed embedding $Y_k \hookrightarrow Y_{W_n(k)}$. Hence $\tilde{M}$ is an R-module, $\tilde{\Phi}_0$ is a Φ_R-linear endomorphism of $\tilde{M}$, $\tilde{V}_0\colon \tilde{M} \to \tilde{M} \otimes_{R\ \Phi_R} R$ is a Verschiebung map, and $\tilde{\nabla}$ is an integrable and nilpotent mod p connection on $\tilde{M}$. Identifying $\tilde{\Phi}_0$ with an R-linear map $\tilde{M} \otimes_{R\ \Phi_R} R \to \tilde{M}$, we have

$$\begin{aligned} \tilde{V}_0 \circ \tilde{\Phi}_0(x) &= px \quad \forall x \in \tilde{M} \otimes_{R\ \Phi_R} R, \\ \tilde{\Phi}_0 \circ \tilde{V}_0(x) &= px \quad \forall x \in \tilde{M}. \end{aligned} \tag{1}$$

Let $\tilde{F}$ be the direct summand of $\tilde{M}$ that is the Hodge filtration defined by the lift $G_{Y_{W_n(k)}}$ of G_{Y_k}. The triple $(\tilde{M}, \tilde{\Phi}_0, \tilde{V}_0, \tilde{\nabla})$ is also the evaluation of $\mathbb{D}(G_{Y_{W_n(k)}}) = \mathcal{E}xt^1_{Y_{W_n(k)}/W(k)}(\underline{G_{Y_{W_n(k)}}}, \mathcal{O}_{Y_{W_n(k)}/W(k)})$ at the trivial thickening of $Y_{W_n(k)}$. So $\tilde{F}$ is the image of the evaluation at this trivial thickening of the functorial homomorphism $\mathcal{E}xt^1_{Y_{W_n(k)}/W(k)}(\underline{G_{Y_{W_n(k)}}}, \mathcal{J}_{Y_{W_n(k)}/W(k)}) \to \mathcal{E}xt^1_{Y_{W_n(k)}/W(k)}(\underline{G_{Y_{W_n(k)}}}, \mathcal{O}_{Y_{W_n(k)}/W(k)})$.

To define the map $\tilde{\Phi}_1\colon \tilde{F} \to \tilde{M}$ and to check that axioms 1–5 hold for the quintuple $(\tilde{M}, \tilde{F}, \tilde{\Phi}_0, \tilde{\Phi}_1, \tilde{\nabla})$, we can work locally in the Zariski topology of Y. Hence we can assume that G is a closed subgroup of an abelian scheme A' over Y (cf. Raynaud's theorem of [BBM, 3.1.1]). Let $A := A'/G$, and let $i_G\colon A' \twoheadrightarrow A$ be the resulting isogeny. We now define $\tilde{\Phi}_1$ using the cokernel of a morphism f of $\mathcal{MF}^{\nabla}_{[0,1]}(Y)$ associated naturally to i_G.

Let $R(n) := R/p^n R$. Let $M := H^1_{\mathrm{crys}}(A_{R(n)}/R(n)) = H^1_{\mathrm{dR}}(A_{R(n)}/R(n))$ as in [BBM, 2.5]. Let F be the direct summand of M that is the reduction mod p^n of the Hodge filtration F_A of

$$H^1_{\mathrm{crys}}(A/R^\wedge) := \varprojlim_{l\in\mathbb{N}} H^1_{\mathrm{crys}}(A_{R(l)}/R(l)) = \varprojlim_{l\in\mathbb{N}} H^1_{\mathrm{dR}}(A_{R(l)}/R(l)).$$

Now let Φ_0 be the reduction mod p^n of the Φ_R-linear endomorphism Φ_A of $H^1_{\mathrm{crys}}(A/R^\wedge)$, and let Φ_1 be the reduction mod p^n of the Φ_R-linear map $F_A \to H^1_{\mathrm{crys}}(A/R^\wedge)$ taking $m \in F_A$ into $\Phi_A(m)/p$. Let ∇ be the reduction mod p^n of the Gauss–Manin connection ∇_A of $A_{R^\wedge}$. That $\mathcal{C} := (M, F, \Phi_0, \Phi_1, \nabla)$ is an object of $\mathcal{MF}^\nabla_{[0,1]}(Y)$ is implied by the fact that the quadruple $(H^1_{\mathrm{crys}}(A/R^\wedge), F_A, \Phi_A, \nabla_A)$ is the evaluation at the thickening attached naturally to the closed embedding $Y_k \hookrightarrow Y^\wedge := \mathrm{Spec}(R^\wedge)$ of a filtered F-crystal over R/pR in locally free sheaves (see [Ka, Sec. 8]). Similarly, starting from A' we construct $\mathcal{C}' = (M', F', \Phi'_0, \Phi'_1, \nabla')$. Let $f: \mathcal{C} \to \mathcal{C}'$ be the morphism of $\mathcal{MF}^\nabla_{[0,1]}(Y)$ associated naturally to i_G.

Let $f_0: M \to M'$ defining f. Let

$$\mathbb{D}(G) = (\tilde{M}, \tilde{F}, \tilde{\Phi}_0, \tilde{\Phi}_1, \tilde{\nabla}) := \mathrm{Coker}(f)$$

(cf. Proposition 2.1). Then $\tilde{M} := M'/f_0(M)$, $\tilde{F} := F'/f_0(F)$, and so forth. That the quadruple $(\tilde{M}, \tilde{F}, \tilde{\Phi}_0, \tilde{\nabla})$ is as defined previously follows from [BBM, 3.1.6, 3.2.9, 3.2.10].

The association $G \to (\tilde{M}, \tilde{F}, \tilde{\Phi}_0, \tilde{\nabla})$ is functorial. In order to check that $\tilde{\Phi}_1$ is well-defined and functorial, we can assume that R is local. To ease the notation we will check directly that $\mathbb{D}(G)$ is itself well-defined and functorial. So let $m: G \to H$ be a morphism of $p-\mathrm{FF}(Y)$. If H is a closed subgroup of an abelian scheme B' over R, then $\mathbb{D}(G \times_Y H)$ is computed via the product embedding of $G \times_Y H$ into $A' \times_Y B'$. We thus obtain $\mathbb{D}(G) \oplus \mathbb{D}(H) = \mathbb{D}(G \times_Y H)$. We now define $\mathbb{D}(m)$. If m is a closed embedding, then the construction of $\mathbb{D}(m)$ is obvious because i_G factors through the isogeny $i_H: A' \to A'/H$. In general, the homomorphism $(1_G, m): G \to G \times_Y H$ is a closed embedding. Hence $\mathbb{D}(m): \mathbb{D}(H) \to \mathbb{D}(G)$ is defined naturally via the epimorphism $\mathbb{D}(1_G, m): \mathbb{D}(G) \oplus \mathbb{D}(H) = \mathbb{D}(G \times_Y H) \twoheadrightarrow \mathbb{D}(G)$.

One easily checks that $\mathbb{D}(G)$ and $\mathbb{D}(m)$ are well-defined; that is, they depend neither on the chosen embeddings into abelian schemes nor on the choice of a power of p annihilating G and H. For instance, let G be a closed subgroup of another abelian scheme C' over Y. By embedding G diagonally into $A' \times_Y C'$ and then using the snake lemma in the context of any one of the two projections of $A' \times_Y C'$ onto its factors, we get that $\mathbb{D}(G)$ defined via $A' \times_Y C'$ is isomorphic to $\mathbb{D}(G)$ defined via A' or C'. This ends the construction of $\mathbb{D}$.

Remarks 2.3. (1) We have

$$\tilde{V}_0 \circ \tilde{\Phi}_1(x) = x \quad \forall x \in \tilde{F} \otimes_{R\ \Phi_R} R, \tag{2}$$

as this identity holds in the context of A and A'. Since $\tilde{M}$ is R-generated by the images of $\tilde{\Phi}_1$ and $\tilde{\Phi}_0$, it follows that $\tilde{V}_0$ is uniquely determined by $\tilde{\Phi}_0$ and $\tilde{\Phi}_1$. We therefore deem it appropriate to denote $(\tilde{M}, \tilde{F}, \tilde{\Phi}_0, \tilde{\Phi}_1, \tilde{\nabla})$ by $\mathbb{D}(G)$. As $\mathcal{C}$ and $\mathcal{C}'$ depend only on $A_{Y_{W_{n+1}(k)}}$ and $A'_{Y_{W_{n+1}(k)}}$ (respectively), $\mathbb{D}(G)$ also depends only on $G_{Y_{W_{n+1}(k)}}$.

(2) If $\tilde{F}$ is neither $\{0\}$ nor $\tilde{M}$, then $\tilde{V}_0$ has a nontrivial kernel and so $\tilde{\Phi}_1$ is not determined by $\tilde{V}_0$. The advantage we gain by using $\tilde{\Phi}_1$ instead of $\tilde{V}_0$ is that we can exploit axiom 5 and the exactness part of Proposition 2.1 (see the proof of Lemma 3.1).

(3) Let $Y_1 = \mathrm{Spec}(R_1)$ be an affine, regular, formally smooth $W(k)$-scheme. We assume that $R_1^\wedge$ is equipped with a Frobenius lift Φ_{R_1} compatible with σ_k and that there is a morphism $l\colon Y_1 \to Y$ whose p-adic completion $l^\wedge$ is compatible with the Frobenius lifts. Let $l^*\colon p-\mathrm{FF}(Y) \to p-\mathrm{FF}(Y_1)$ and $l^*\colon \mathcal{MF}^\nabla_{[0,1]}(Y) \to \mathcal{MF}^\nabla_{[0,1]}(Y_1)$ be the pull-back functors. Hence $l^*(G) = G \times_Y Y_1$ and

$$l^*(M, F, \Phi_0, \Phi_1, \nabla) = (M \otimes_R R_1, F \otimes_R R_1, \Phi_0 \otimes \Phi_{R_1}, \Phi_1 \otimes \Phi_{R_1}, \nabla_1),$$

where ∇_1 is the natural extension of ∇ to a connection on $M \otimes_R R_1$. These constructions then yield the equality $\mathbb{D}\circ l^* = l^* \circ \mathbb{D}$ of contravariant, $\mathbb{Z}_p$-linear functors from $p - \mathrm{FF}(Y)$ to $\mathcal{MF}^\nabla_{[0,1]}(Y_1)$.

(4) As in [Fa, 2.3], we see that the category $\mathcal{MF}^\nabla_{[0,1]}(Y)$ does not depend (up to isomorphism) on the choice of the Frobenius lift Φ_R of $R^\wedge$ compatible with σ_k. The arguments of [Fa] apply even for $p = 2$ because we are dealing with connections that are nilpotent mod p. One can use this to show that remark (3) makes sense even if Y and Y_1 are not affine or if no Frobenius lifts are fixed.

(5) If R is local, complete, and has residue field k, then one can use a theorem of Badra [Ba] on the category $p-\mathrm{FF}(Y)$ to obtain directly that $\mathbb{D}(G)$ is functorial.

3. A Lemma

In this section we prove the following lemma.

LEMMA 3.1. *Assume that* $e = 1$. *Let* (Y, U) *be an extensible pair, with* Y *a regular and formally smooth* O*-scheme of dimension* 2 *and with* U *containing* Y_K. *Then any short exact sequence* $0 \to G_{1U} \to G_{2U} \to G_{3U} \to 0$ *in the category* $p-\mathrm{FF}(U)$ *extends uniquely to a short exact sequence in the category* $p-\mathrm{FF}(Y)$.

Proof. Let $\mathcal{O}_X$ be the sheaf of rings on a scheme X. Let $j\colon U \hookrightarrow Y$ be the open embedding of U in Y. For $i \in \{1, 2, 3\}$, the $\mathcal{O}_Y$-module $\mathcal{F}_i := j_*(\mathcal{O}_{G_{iU}})$ is locally free (cf. [FaC, Lemma 6.2, p. 181]. The commutative Hopf algebra structure of the $\mathcal{O}_U$-module $\mathcal{O}_{G_{iU}}$ extends uniquely to a commutative Hopf algebra structure of $\mathcal{F}_i$. Hence there exists a unique finite, flat, commutative group scheme G_i over Y extending G_{iU}. We have to show that the natural complex

$$0 \to G_1 \to G_2 \to G_3 \to 0 \tag{3}$$

is, in fact, a short exact sequence. This is a local statement for the faithfully flat topology of Y. We may therefore assume that Y is local and complete and that its residue field k is separable closed and of characteristic p; we may also assume that U is the complement in Y of the maximal point y of Y. We write $Y = \mathrm{Spec}(R)$. From Cohen's coefficient ring theorem (see [Ma, pp. 211, 268]) we have that R is a $K(k)$-algebra, where $K(k)$ is a Cohen ring of k. Since R/pR is regular and formally smooth over O/pO (and thus also over k), we can identify $R = K(k)[[x]]$ as $K(k)$-algebras. Hence, by replacing R with the faithfully flat R-algebra $W(\bar{k})[[x]]$, we can assume that $k = \bar{k}$ and $K(k) = W(k)$ and so can use the notation of Section 2 (e.g. Φ_R, $\Omega_R^\wedge, \dots$). Since $\Omega_R^\wedge = dxR$ is a free R-module, we can also appeal to Proposition 2.1.

Let $\mathcal{O}$ be the local ring of Y, which is a discrete valuation ring that is faithfully flat over $W(k)$. Let $\mathcal{O}_1 := W(k_1)$, where k_1 is the algebraic closure of the residue field $k((x))$ of $\mathcal{O}$. We consider a Teichmüller lift $l\colon \mathrm{Spec}(\mathcal{O}_1) \to \mathrm{Spec}(R^\wedge)$ that—at the level of special fibres—induces the inclusion $k[[x]] \hookrightarrow k_1$. Hence, $\mathcal{O}_1$ has a natural structure of an $\mathcal{O}$-algebra. Let

$$0 \to \mathbb{D}(G_3) \to \mathbb{D}(G_2) \to \mathbb{D}(G_1) \to 0 \tag{4}$$

be the complex of $\mathcal{MF}_{[0,1]}^\nabla(Y)$ corresponding to (3). Let M_1, M_2, and M_3 be the underlying R-modules of $\mathbb{D}(G_1)$, $\mathbb{D}(G_2)$, and $\mathbb{D}(G_3)$, respectively. Let

$$0 \to M_3 \to M_2 \to M_1 \to 0 \tag{5}$$

be the complex of R-modules defined by (4). Let $N_{1,2}$ be the underlying R-module of $\mathrm{Coker}(\mathbb{D}(G_2) \to \mathbb{D}(G_1))$. The key point is that $\mathrm{Coker}(\mathbb{D}(G_2) \to \mathbb{D}(G_1))$ exists in the category $\mathcal{MF}_{[0,1]}^\nabla(Y)$ and the sequence $M_2 \to M_1 \to N_{1,2} \to 0$ is exact (cf. Proposition 2.1). We show that $N_{1,2} = \{0\}$. Because $N_{1,2}$ is a direct sum of R-modules of the form $R/p^sR = W_s(k)[[x]]$ for $s \in \mathbb{N} \cup \{0\}$ (cf. axiom 5), to show that $N_{1,2} = \{0\}$ it is enough to show that $N_{1,2}[1/x] = \{0\}$. It is thus enough to show that the complex

$$0 \to M_3 \otimes_{\mathcal{O}} \mathcal{O}_1 \to M_2 \otimes_{\mathcal{O}} \mathcal{O}_1 \to M_1 \otimes_{\mathcal{O}} \mathcal{O}_1 \to 0 \tag{6}$$

obtained from (5) by tensoring with $\mathcal{O}_1$ is a short exact sequence. Note that (6) is the complex obtained by pulling back (3) to $\mathrm{Spec}(\mathcal{O}_1)$, applying $\mathbb{D}$, and then taking underlying $\mathcal{O}_1$-modules (cf. Remark 2.3(4)) applied to l). But the pull-back of (3) to $\mathrm{Spec}(\mathcal{O}_1)$ is a short exact sequence (since the pull-back of (3) to U is so). Thus (6) is the complex associated via the classical contravariant Dieudonné functor to the short exact sequence $0 \to G_{1k_1} \to G_{2k_1} \to G_{3k_1} \to 0$ (cf. [BBM, pp. 179–180]). From the classical Dieudonné theory we therefore have that (6) is a short exact sequence (cf. [F, p. 128 or p. 153]). So $N_{1,2} = \{0\}$.

Hence the natural $W(k)$-linear map $j_{1,2}\colon M_2/(x)M_2 \to M_1/(x)M_1$ is an epimorphism. But $j_{1,2}$ is the $W(k)$-linear map associated via the classical contravariant Dieudonné functor to the homomorphism $G_{1k} \to G_{2k}$, so this homomorphism is a closed embedding (cf. the classical Dieudonné theory). It follows by Nakayama's lemma that G_1 is a closed subgroup of G_2. Both G_3 and G_2/G_1 are finite, flat, commutative group schemes over Y extending G_{3U} and so we have $G_3 = G_2/G_1$. Hence (3) is a short exact sequence. This completes the proof. □

REMARK 3.2. For $p > 2$, Lemma 3.1 was proved by Faltings using Raynaud's theorem [R, 3.3.3] (see [Mo, 3.6; V, 3.2.17, step B]).

4. Proof of Theorem 1.3

Let O, K, e, and Y be as in Section 1. We start with a general proposition.

PROPOSITION 4.1. *If Y is p-healthy regular then Y is also healthy regular.*

Proof. Let (Y, U) be an extensible pair with U containing Y_K, and let A_U be an abelian scheme over U. We need to show that A_U extends to an abelian scheme A over Y. Since Y is p-healthy regular, the p-divisible group D_U of A_U extends to a p-divisible group D over Y. From now on we forget that Y is p-healthy regular and will use just the existence of D to show that A exists.

Let $N \in \mathbb{N} \setminus \{1, 2\}$ be prime to p. To show that A exists, we can assume that Y is local, complete, and strictly henselian, that U is the complement of the maximal point y of Y, and that A_U has a principal polarization p_{A_U} and a level N structure $l_{U,N}$ (see [FaC, (i)–(iii), pp. 185, 186]). We write $Y = \mathrm{Spec}(R)$. Let p_{D_U} be the principal quasi-polarization of D_U defined naturally by p_{A_U}; it extends to a principal quasi-polarization p_D of D (cf. Tate's theorem [T, Thm. 4]). Let g be the relative dimension of A_U. Let $\mathcal{A}_{g,1,N}$ be the moduli scheme over $\mathrm{Spec}(\mathbb{Z}[1/N])$ parameterizing principally polarized abelian schemes over $\mathrm{Spec}(\mathbb{Z}[1/N])$ of relative dimension g and with level N structure (see [MFK, 7.9, 7.10]). Let $(\mathcal{A}, \mathcal{P}_{\mathcal{A}})$ be the universal principally polarized abelian scheme over $\mathcal{A}_{g,1,N}$.

Let $f_U : U \to \mathcal{A}_{g,1,N}$ be the morphism defined by $(A_U, p_{A_U}, l_{U,N})$. We show that f_U extends to a morphism $f_Y : Y \to \mathcal{A}_{g,1,N}$.

Let $N_0 \in \mathbb{N}$ be prime to p. From the classical purity theorem we get that the étale cover $A_U[p^{N_0}] \to U$ extends to an étale cover $Y_{N_0} \to Y$. But as Y is strictly henselian, Y has no connected étale cover different from Y. So each Y_{N_0} is a disjoint union of p^{2gN_0}-copies of Y. Hence A_U has a level N_0 structure l_{U,N_0} for any $N_0 \in \mathbb{N}$ that is prime to p.

Let $\bar{\mathcal{A}}_{g,1,N}$ be a projective, toroidal compactification of $\mathcal{A}_{g,1,N}$ such that (a) the complement of $\mathcal{A}_{g,1,N}$ in $\bar{\mathcal{A}}_{g,1,N}$ has pure codimension 1 in $\bar{\mathcal{A}}_{g,1,N}$ and (b) there is a semi-abelian scheme over $\bar{\mathcal{A}}_{g,1,N}$ extending $\mathcal{A}$ (cf. [FaC, Chap. IV, Thm. 6.7]). Let $\tilde{Y}$ be the normalization of the Zariski closure of U in $Y \times_O (\bar{\mathcal{A}}_{g,1,N})_O$. It is a projective, normal, integral Y-scheme having U as an open subscheme. Let C be the complement of U in $\tilde{Y}$ endowed with the reduced structure; it is a reduced, projective scheme over the residue field k of y. The $\mathbb{Z}$-algebras of global functions of Y, U, and $\tilde{Y}$ are all equal to R (cf. [Ma, Thm. 38] for U). So C is a connected k-scheme (cf. [H, 11.3, p. 279]).

Let $\bar{A}_{\tilde{Y}}$ be the semi-abelian scheme over $\tilde{Y}$ extending A_U. Owing to existence of the l_{U,N_0}, the Néron–Ogg–Shafarevich criterion (see [BLR, p. 183]) implies that $\bar{A}_{\tilde{Y}}$ is an abelian scheme in codimension at most 1. Therefore, since the complement of $\mathcal{A}_{g,1,N}$ in $\bar{\mathcal{A}}_{g,1,N}$ has pure codimension 1 in $\bar{\mathcal{A}}_{g,1,N}$, it follows that $\bar{A}_{\tilde{Y}}$ is an abelian scheme. So f_U extends to a morphism $f_{\tilde{Y}} : \tilde{Y} \to \mathcal{A}_{g,1,N}$. Let $p_{\bar{A}_{\tilde{Y}}} := f_{\tilde{Y}}^*(\mathcal{P}_{\mathcal{A}})$. Tate's theorem implies that the principally quasi-polarized p-divisible group of $(\bar{A}_{\tilde{Y}}, p_{\bar{A}_{\tilde{Y}}})$ is the pull-back $(D_{\tilde{Y}}, p_{D_{\tilde{Y}}})$ of (D, p_D) to $\tilde{Y}$. Hence the pull-back (D_C, p_{D_C}) of $(D_{\tilde{Y}}, p_{D\tilde{Y}})$ to C is constant; that is, it is the pull-back to C of a principally quasi-polarized p-divisible group over k.

We check that the image $f_{\tilde{Y}}(C)$ of C through $f_{\tilde{Y}}$ is a point $\{y_0\}$ of $\mathcal{A}_{g,1,N}$. Since C is connected, to check this it suffices to show that, if $\widehat{O_c}$ is the completion of the local ring O_c of C at an arbitrary point c, then the morphism $\mathrm{Spec}(\widehat{O_c}) \to \mathcal{A}_{g,1,N}$ defined naturally by $f_{\tilde{Y}}$ is constant. But as (D_C, p_{D_C}) is constant, this follows from Serre–Tate deformation theory (see [Me, Chaps. 4, 5]). So $f_{\tilde{Y}}(C)$ is a point $\{y_0\}$ of $\mathcal{A}_{g,1,N}$.

Let R_0 be the local ring of $\mathcal{A}_{g,1,N}$ at y_0. Because Y is local and $\tilde{Y}$ is a projective Y-scheme, each point of $\tilde{Y}$ specializes to a point of C. Hence each point of the image of $f_{\tilde{Y}}$ specializes to y_0 and so $f_{\tilde{Y}}$ factors through the natural morphism $\mathrm{Spec}(R_0) \to \mathcal{A}_{g,1,N}$. Since R is the ring of global functions of $\tilde{Y}$, the resulting morphism $\tilde{Y} \to \mathrm{Spec}(R_0)$ factors through a morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(R_0)$. Therefore, $f_{\tilde{Y}}$ factors through a morphism $f_Y \colon Y \to \mathcal{A}_{g,1,N}$ extending f_U. This ends the argument for the existence of f_Y. We conclude that $A := f_Y^*(\mathcal{A})$ extends A_U, which completes the proof. □

REMARK 4.2. In the proof of Proposition 4.1, the use of semi-abelian schemes can be replaced by de Jong's good reduction criterion [dJ, 2.5] as follows. If we define $\tilde{Y}$ to be the normalization of the Zariski closure of U in $Y \times_O (\mathcal{A}_{g,1,N})_O$, then [dJ] implies that the morphism $\tilde{Y} \to Y$ of O-schemes of finite type satisfies the valuative criterion of properness with respect to discrete valuation rings of equal characteristic p. Using (as in the proof of Proposition 4.1) the Néron–Ogg–Shafarevich criterion, one checks that the morphism $\tilde{Y} \to Y$ of O-schemes satisfies the valuative criterion of properness with respect to discrete valuation rings whose fields of fractions have characteristic 0. Hence the morphism $\tilde{Y} \to Y$ of O-schemes is proper. The rest of the argument is entirely the same.

CONCLUSION 4.3. We assume that $e = 1$ and that Y is formally smooth over O. Based on Proposition 4.1, in order to prove Theorem 1.3 it suffices to show that Y is p-healthy regular. So let (Y, U) be an extensible pair with U containing Y_K. We need to show that any p-divisible group D_U over U extends to a p-divisible group D over Y. This is a local statement for the faithfully flat topology, so we can assume that Y is local, complete, and strictly henselian and that U is the complement of the maximal point y of Y (see [FaC, p. 183]). Write $Y = \mathrm{Spec}(R)$, and let $d \in \mathbb{N}$ be the dimension of R/pR. We show the existence of D by induction on d.

If $d = 1$ then, for all $n, m \in \mathbb{N}$, the short exact sequence $0 \to D_U[p^n] \to D_U[p^{n+m}] \to D_U[p^m] \to 0$ in the category $p - \mathrm{FF}(U)$ extends uniquely to a short exact sequence $0 \to D_n \to D_{n+m} \to D_m \to 0$ in the category $p - \mathrm{FF}(Y)$ (cf. Lemma 3.1). Hence there is a unique p-divisible group D over Y such that $D[p^n] = D_n$. Obviously D extends D_U. For $d \geq 2$, the passage from $d - 1$ to d is entirely as in [FaC, pp. 183, 184] applied to R and any regular parameter $x \in R$

such that R/xR is formally smooth over O. This ends the induction and so establishes the existence of D, concluding the proof of Theorem 1.3.

References

[Ba] A. Badra, *Déformations des p-groupes finis commutatifs sur un corps parfait et filtration de Hodge,* C. R. Acad. Sci. Paris Sér. A-B 291 (1980), 539–542.

[B] P. Berthelot, *Cohomologie cristalline des schémas de caractéristique* $p > 0$, Lecture Notes in Math., 407, Springer-Verlag, New York, 1974.

[BBM] P. Berthelot, L. Breen, and W. Messing, *Théorie de Dieudonné crystalline, II* Lecture Notes in Math., 930, Springer-Verlag, New York, 1982.

[BLR] S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models,* Springer-Verlag, Berlin, 1990.

[dJ] J. de Jong, *Homomorphisms of Barsotti–Tate groups and crystals in positive characteristic,* Invent. Math. 134 (1998), 301–333.

[dJO] J. de Jong and F. Oort, *On extending families of curves,* J. Algebraic Geom. 6 (1997), 545–562.

[Fa] G. Faltings, *Crystalline cohomology and p-adic Galois representations,* Algebraic analysis, geometry, and number theory (Baltimore, 1988), pp. 25–80, Johns Hopkins Univ. Press, Baltimore, 1989.

[FaC] G. Faltings and C.-L. Chai, *Degeneration of abelian varieties*, Springer-Verlag, Berlin, 1990.

[F] J.-M. Fontaine, *Groupes p-divisibles sur les corps locaux,* Astérisque 47/48 (1977).

[FL] J.-M. Fontaine and G. Laffaille, *Construction de représentations p-adiques,* Ann. Sci. École Norm. Sup. (4) 15 (1982), 547–608.

[G] A. Grothendieck, *Revêtements étales et groupe fondamental,* Lecture Notes in Math., 224, Springer-Verlag, New York, 1971.

[H] R. Hartshorne, *Algebraic geometry,* Grad. Texts in Math., 52, Springer-Verlag, Berlin, 1977.

[Ka] N. Katz, *Travaux de Dwork,* Seminaire Bourbaki, 24eme annee (1971/1972), Exp. no. 409, pp. 167–200, Lecture Notes in Math., 317, Springer-Verlag, New York, 1973.

[K] R. E. Kottwitz, *Points on some Shimura varieties over finite fields,* J. Amer. Math. Soc. 5 (1992), 373–444.

[Ma] H. Matsumura, *Commutative algebra,* Benjamin, New York, 1970.

[Me] W. Messing, *The crystals associated to Barsotti–Tate groups, with applications to abelian schemes,* Lecture Notes in Math., 264, Springer-Verlag, New York, 1972.

[Mi] J. S. Milne, *The points on a Shimura variety modulo a prime of good reduction,* The zeta functions of Picard modular surfaces, pp. 153–255, Univ. Montreal Press, Montreal, Quebec, 1992.

[Mo] B. Moonen, *Models of Shimura varieties in mixed characteristics,* Galois representations in arithmetic algebraic geometry (Durham, 1996), London Math. Soc. Lecture Note Ser., 254, pp. 267–350, Cambridge Univ. Press, Cambridge, U.K., 1998.

[M-B] L. Moret-Bailly, *Un théorème de pureté pour les familles de courbes lisses,* C. R. Acad. Sci. Paris Sér. I Math. 300 (1985), 489–492.

[MFK] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory,* Springer-Verlag, New York, 1994.

[N] A. Néron, *Modèles minimaux des variétés abéliennes,* Inst. Hautes Études Sci. Publ. Math. 21 (1964).

[R] M. Raynaud, *Schémas en groupes de type* $(p, \ldots, p)$, Bull. Soc. Math. France 102 (1974), 241–280.

[T] J. Tate, *p-divisible groups*, Proceedings of a conference on local fields (Driesbergen, 1966), pp. 158–183, Springer-Verlag, Berlin, 1967.

[V] A. Vasiu, *Integral canonical models of Shimura varieties of preabelian type*, Asian J. Math. 3 (1999), 401–518.

Department of Mathematics
University of Arizona
P.O. Box 210089
Tucson, AZ 85721-0089

adrian@math.arizona.edu