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Topological Equivalence of Complex Curves and Bi-Lipschitz Homeomorphisms

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Brieskorn [4] has observed that, if we want to capture information regarding local qualitative properties of complex analytic curves in $\mathbb{C}^2$, we should not approach them by an abstract form. This statement is justified by the following result.

Proposition 0.1. *Any irreducible analytic curve in $\mathbb{C}^2$ is homeomorphic to* $(\mathbb{C}, 0)$.

This result also justifies the following equivalence relation between germs of analytic subsets in $\mathbb{C}^d$.

Definition 0.2. Let X and Y be analytic subsets of $\mathbb{C}^d$ with $x \in X$ and $y \in Y$. The germ (X, x) is *topologically equivalent to* (Y, y) if there exists a homeomorphism $F: (\mathbb{C}^d, x) \to (\mathbb{C}^d, y)$, such that $F(X) = Y$.

This equivalence relation is finer than the abstract approach in the following sense: if (X, x) is topologically equivalent to (Y, y), then (X, x) is homeomorphic to (Y, y). In fact, this approach is strictly finer than the abstract approach; for example, the germ of a complex curve $(X, 0)$ defined by $X : z^2 = w^3$ is homeomorphic but is not topologically equivalent to $(\mathbb{C}, 0)$.

The topological equivalence of plane curves has been intensively studied since the 1920s. The mathematicians K. Brauner, K. Kähler, W. Burau, and O. Zariski are responsible for the solution of this problem. An excellent survey of the techniques and related results can be found in [4] as well as in [10]. A complete solution is the following theorem.

Theorem 0.3. *Two germs of complex curves are topologically equivalent if and only if there exists a bijection between their branches preserving the sequence of characteristic exponents and the index of intersection of pairs of branches.*

The next theorem was proved in [11].

Theorem 0.4 (F. Pham, B. Teissier). *Let X and Y be germs of analytic plane complex curves. Then there exists a meromorphic bi-Lipschitz map between X and Y if only if X and Y are topologically equivalent.*

Recall that Pham obtained this theorem as a consequence of the following conclusion, obtained in [11]: *The Lipschitz equisaturation of a family of hypersurfaces*

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implies the topological triviality of this family. The reader interested in the Lipschitz saturation theory can find more information in [6; 11; 12].

Because spatial curves are not hypersurfaces, we cannot obtain (as a consequence of the conclusions of Pham) an analogous result to Theorem 0.4 for the case of spatial curves. In the face of these results, from the viewpoint of metric theory it is natural to ask:

($*$) When does there exist a bi-Lipschitz map between X and Y?

The problem of classification of complex algebraic sets modulo bi-Lipschitz homeomorphisms was studied by T. Mostowski. He proved that the family of all complex algebraic sets of complexity bounded by some number k has a finite number of bi-Lipschitz equivalence classes (see [8]). This result was extended by A. Parusinski for semialgebraic sets (see [9]).

In [2], we classify the germs of semialgebraic sets of real dimension 1, equipped with the induced Euclidean metric, modulo bi-Lipschitz homeomorphisms. In [1], L. Birbrair classifies the germs of semialgebraic sets of real dimension 2, equipped with the length metric, modulo bi-Lipschitz homeomorphisms. Birbrair pointed out the difficulty of obtaining a classification of the same germs when considering the induced metric.

Here we present a theorem that, with respect to question ($*$), is an improved version of Theorem 0.4—because it is obvious that the class of bi-Lipschitz subanalytic maps contains the class of bi-Lipschitz meromorphic maps.

THEOREM 0.5. *Let X and Y be germs of analytic complex curves in $\mathbb{C}^n$. If there exists a bi-Lipschitz subanalytic map between X and Y, then X and Y are topologically equivalent. Morever, the converse is true if and only if $n = 2$.*

Our main approach is to consider the so-called test arcs. The bi-Lipschitz invariants of algebraic curves are expressed in terms of order of contact of some test arcs. It is valuable to observe that the class of germs that we consider here is not formed by germs normally embedded in $\mathbb{R}^4$ (cf. [3]) and hence our result does not follow as a consequence of the results presented in [1].

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1. Hölder Semicomplex and Test Arcs

NOTATION. Let f and g be nonnegative functions. We write $f \lesssim g$ if for some constant C we have $f \leq Cg$. Also, $f \approx g$ if and only if $f \lesssim g$ and $g \lesssim f$. If f and g are germs of functions on (X, x_0), we write $f \ll g$ if $g^{-1}(0) \subset f^{-1}(0)$ and $\lim_{x \to x_0}[f(x)/g(x)] = 0$.

Given a germ of a semianalytic subset (X, x) of real dimension 1 in Euclidean space that is equipped with the induced Euclidean metric, the following number is defined in [2] for each pair of half-branches X_i and X_j of (X, x):

$$\mathrm{sh}_{ij}(X, x) = \mathrm{ord}_r[\mathrm{dist}(X_i \cap S_r(x), X_j \cap S_r(x))].$$

The data $\mathrm{sh}_{ij}(X, x)$ are called the *Hölder semicomplex of* (X, x).

THEOREM 1.1. *Let* (X, x) *and* (Y, y) *be germs of subsets in Euclidean spaces that are semianalytic spaces of real dimension* 1 *and equipped with the respective induced Euclidean metrics and with real half-branches* $X = \bigcup_{i \in I} X_i$ *and* $Y = \bigcup_{i \in J} X_j$. *Then there exists* $F : (X, x) \to (Y, y)$ *bi-Lipschitz if and only if there exists a bijection* $\phi : I \to J$ *such that*

$$\mathrm{sh}_{ij}(X, x) = \mathrm{sh}_{\phi(i)\phi(j)}(X, y)$$

for each pair $i \neq j \in I$.

We appeal to Theorem 1.1 of [2] because it clarifies the metric behavior of germs of semianalytic sets of real dimension 1 that are equipped with the induced metric. Our notion of test arcs demands this clarification.

DEFINITION 1.2. A *test arc* is a germ of a semianalytic set of real dimension 1, with the induced metric and with only one real half-branch.

LEMMA 1.3. *Let* (Γ_1, x) *and* (Γ_2, x) *be test arcs and let* $F : (\Gamma_1 \cup \Gamma_2, x) \to \mathbb{R}^p$ *be a germ of a subanalytic map such that its restriction to* Γ_i *consists of germs of a Lipschitz map,* $i = 1, 2$. *If*

$$|F(x_1(r)) - F(x_2(r))| \lesssim |x_1(r) - x_2(r)|,$$

where $x_i(r) = S_r(x) \cap \Gamma_i$, *then* F *is the germ of a Lipschitz map.*

Proof. Let us admit all the hypotheses of the lemma, where $x_i(r) = S_r(x) \cap \Gamma_i$, and assume that the conclusion is false. Then, by the curve selection lemma, we can find analytic curves $q_1, q_2 : [0, \delta) \to \Gamma$ such that $q_i(0) = 0$ with $q_i(s) \in \Gamma_i$ for all s and for $i = 1, 2$ and such that

$$|F(q_1(r)) - F(q_2(r))| \gg |q_1(r) - q_2(r)|.$$

We can suppose $|q_1(s)| \geq |q_2(s)|$ for all s. If not, then by analyticity we would have a contradiction with one of the hypotheses of the lemma. Taking a reparameterization if necessary, we can assume that $|q_1(s)|$ and $|q_2(s)|$ are fractional power series and $|q_1(s)| \geq |q_2(s)| = s$; that is, $q_2(s) = x_2(s)$ for all s. Thus, using the hypotheses of the lemma, we have

$$\begin{aligned}|q_1(s) - q_2(s)| &\ll |F(q_1(s)) - F(x_1(s))| + |F(x_1(s)) - F(q_2(s))| \\ &\lesssim |q_1(s) - x_1(s)| + |x_1(s) - q_2(s)|.\end{aligned}$$

By the order comparison lemma (see [2]) we have that

$$|x_1(s) - q_2(s)| \lesssim |q_1(s) - q_2(s)|.$$

Therefore,

$$|q_1(s) - q_2(s)| \ll |q_1(s) - x_1(s)|.$$

The triangle inequality then yields the following contradiction:

$$|q_1(s) - q_2(s)| \approx |q_1(s) - x_1(s)|. \qquad \square$$

2. On Irreducible Plane Curves

Let $(C, 0)$ be a germ of an irreducible analytic curve in $\mathbb{C}^2$ (*branch*). We can assume that $(C, 0)$ lies in the following normal form:

$$\begin{cases} x = t^m, \\ y = t^n + a_2 t^{n_2} + \cdots, \end{cases}$$

where m is the multiplicity of $(C, 0)$, m does not divide the number n, and $y(t) \in \mathbb{C}\{t\}$. The fractional power series $y(x^{1/m})$ is known as *Newton–Puiseux parameterization* of $(C, 0)$, and all the other Newton–Puiseux parameterizations of $(C, 0)$ are obtained from $y(x^{1/m})$ via $x^{1/m} \mapsto wx^{1/m}$ where w is an mth root of the unit. We denote by $\beta(C)$ the sequence of characteristic exponents of $(C, 0)$.

EXAMPLE 2.1. *Let $C : w^2 = z^3$ and $D : w^2 = z^5$. Then there exists no germ of the subanalytic bi-Lipschitz map $F\colon (C, 0) \to (D, 0)$.*

Proof. Let Σ_1, Σ_2, Σ_3, Σ_4 be the following real half-branches in D:

$$\Sigma_1 = \{(r, r^{5/2}) : r \geq 0\}, \qquad \Sigma_2 = \{(ri, r^{5/2}e^{i\pi/4}) : r \geq 0\},$$

$$\Sigma_3 = \{(r, -r^{5/2}) : r \geq 0\}, \qquad \Sigma_4 = \{(-ri, r^{5/2}e^{i3\pi/4}) : r \geq 0\}.$$

Let

$$\Gamma_k(r) = (re^{i\gamma_k(r)}, r^{3/2}e^{i3\gamma_k(r)/2}) \in S_r(0) \cap F(\Sigma_k), \quad k = 1, 2, 3, 4.$$

From Theorem 1.1,

$$\|\Gamma_1(r) - \Gamma_3(r)\| \approx r^{5/2}.$$

Therefore, we can choose γ_1 and γ_3 such that

$$\lim_{r\to 0}(\gamma_1(r) - \gamma_3(r)) = 4k\pi, \quad k = 0 \text{ or } 1.$$

We suppose that

$$\lim_{r\to 0}(\gamma_1(r) - \gamma_3(r)) = 0.$$

In this case, one of the following alternatives occurs:

$$\lim_{r\to 0}(\gamma_1(r) - \gamma_2(r)) = 0; \quad \text{or} \quad \lim_{r\to 0}(\gamma_1(r) - \gamma_4(r)) = 0.$$

We now suppose

$$\lim_{r\to 0}(\gamma_1(r) - \gamma_2(r)) = 0.$$

In particular,

$$\|\Gamma_1(r) - \Gamma_2(r)\| \ll r.$$

Write $\delta_k(r) = F^{-1}(\Gamma_k(r))$ for $k = 1, 2$. Then we have

$$\delta_1(r) = (f(r), f(r)^{5/2}) \quad \text{and} \quad \delta_2(r) = (g(r)i, g(r)^{5/2}e^{i\pi/4}),$$

where

$$|f(r)| \approx |g(r)| \approx r.$$

Therefore,

$$\|\delta_1(r) - \delta_2(r)\| \gtrsim |f(r) - ig(r)| \approx r.$$

On the other hand, since F is bi-Lipschitz, we have

$$\|\delta_1(r) - \delta_2(r)\| \approx \|\Gamma_1(r) - \Gamma_2(r)\|,$$

which is a contradiction.

The other cases follow by an analogous argument. □

THEOREM 2.2. *Let $(C, 0)$ and $(\tilde{C}, 0)$ be two germs of analytic branches in $\mathbb{C}^2$. There exists a germ of the subanalytic bi-Lipschitz map $F\colon (C, 0) \to (\tilde{C}, 0)$ if and only if $\beta(C) = \beta(\tilde{C})$.*

In what follows, we analyze the metric behavior of pairs of test arcs.

Let $(C, 0)$ be an analytic branch in $\mathbb{C}^2$ with multiplicity m and Puiseux characteristic pairs $(m_1, n_1), \dots, (m_g, n_g)$. Let Γ_1, Γ_2 be test arcs in $(C, 0)$ given by

$$\Gamma_j(r) = (re^{i\alpha_j(r)}, y(r^{1/m}e^{i\alpha_j(r)/m})) \in S_r(0) \cap \Gamma_j, \quad j = 1, 2,$$

where $y(x^{1/m})$ is a Newton–Puiseux parameterization of $(C, 0)$ and α_1, α_2 are angle functions. We write $g(r) = \|\Gamma_1(r) - \Gamma_2(r)\|$ and $h(r) = r\|e^{i\alpha_1(r)} - e^{i\alpha_2(r)}\|$.

LEMMA 2.3. *If*

$$\frac{1}{2\pi}\lim_{r\to 0}(\alpha_1(r) - \alpha_2(r)) = k \in \mathbb{Z},$$

then $\operatorname{ord}_r(g) = \min\{\operatorname{ord}_r(h), m_j/(n_1 \cdots n_j)\}$, *where* $j = \min\{i : (k/n_1 \cdots n_i) \notin \mathbb{Z}\}$.

Proof. It is enough to observe the following formulas:

$$\begin{aligned} y(x^{1/m}) = {} & a_{\beta_1}x^{m_1/n_1} + a_{\beta_1+e_1}x^{(m_1+1)/n_1} + \cdots + a_{\beta_1+k_1e_1}x^{(m_1+k_1)/n_1} \\ & + a_{\beta_2}x^{m_2/n_1n_2} + a_{\beta_2+e_2}x^{(m_2+1)/n_1n_2} + \cdots + a_{\beta_q}x^{m_q/(n_1n_2\cdots n_q)} \\ & + a_{\beta_q+e_q}x^{(m_q+1)/(n_1n_2\cdots n_q)} + \cdots \\ & + a_{\beta_g}x^{m_g/(n_1n_2\cdots n_g)} + a_{\beta_g+1}x^{(m_g+1)/(n_1n_2\cdots n_g)} + \cdots . \end{aligned}$$

Moreover, since $a_{\beta_j+le_j} \neq 0$ and since

$$|a_{\beta_j+le_j}(re^{i\alpha_1(r)})^{(m_j+l)/(n_1n_2\cdots n_j)} - a_{\beta_j+le_j}(re^{i\alpha_2(r)})^{(m_j+l)/(n_1n_2\cdots n_j)}|$$

is equal to

$$|a_{\beta_j+le_j}|r^{(m_j+l)/(n_1n_2\cdots n_j)}|e^{i\alpha_1(r)(m_j+l)/(n_1n_2\cdots n_j)} - e^{i\alpha_2(r)(m_j+l)/(n_1n_2\cdots n_j)}|,$$

we have that

$$\operatorname{ord}_r|a_{\beta_j+le_j}(re^{i\alpha_1(r)})^{(m_j+l)/(n_1n_2\cdots n_j)} - a_{\beta_j+le_j}(re^{i\alpha_2(r)})^{(m_j+l)/(n_1n_2\cdots n_j)}|$$

is equal to

$$\frac{m_j + l}{n_1 n_2 \cdots n_j} + \mathrm{ord}_r |e^{i\alpha_1(r)(m_j+l)/(n_1 n_2 \cdots n_j)} - e^{i\alpha_2(r)(m_j+l)/(n_1 n_2 \cdots n_j)}|.$$

In fact, if $k/(n_1 n_2 \cdots n_j) \in \mathbb{Z}$ then

$$\mathrm{ord}_r |e^{i\alpha_1(r)(m_j+l)/(n_1 n_2 \cdots n_j)} - e^{i\alpha_2(r)(m_j+l)/(n_1 n_2 \cdots n_j)}| = \mathrm{ord}_r |e^{i\alpha_1(r)} - e^{i\alpha_2(r)}|.$$

Hence, in this case we have

$$\mathrm{ord}_r |a_{\beta_j + l e_j}(re^{i\alpha_1(r)})^{(m_j+l)/(n_1 n_2 \cdots n_j)} - a_{\beta_j + l e_j}(re^{i\alpha_2(r)})^{(m_j+l)/(n_1 n_2 \cdots n_j)}| > \mathrm{ord}_r(h).$$

On the other hand, if $k/(n_1 n_2 \cdots n_j) \notin \mathbb{Z}$, then

$$\mathrm{ord}_r |e^{i\alpha_1(r) m_j/(n_1 n_2 \cdots n_j)} - e^{i\alpha_2(r) m_j/(n_1 n_2 \cdots n_j)}| = 0.$$

As a result, in this case,

$$\mathrm{ord}_r |a_{\beta_j}(re^{i\alpha_1(r)})^{m_j/(n_1 n_2 \cdots n_j)} - a_{\beta_j}(re^{i\alpha_2(r)})^{m_j/(n_1 n_2 \cdots n_j)}| = \frac{m_j}{n_1 \cdots n_j}.$$

We have therefore shown that $\mathrm{ord}_r(g) = \min\{\mathrm{ord}_r(h), m_j/(n_1 \cdots n_j)\}$, where $j = \min\{i : k/(n_1 \cdots n_i) \notin \mathbb{Z}\}$. □

Clearly, if

$$\frac{1}{2\pi} \lim_{r \to 0} (\alpha_1(r) - \alpha_2(r)) \notin \mathbb{Z},$$

then $\mathrm{ord}_r(g) = 1$.

Let us suppose that $\beta(C) = \beta(\tilde{C})$. We claim that $F\colon (C, 0) \to (\tilde{C}, 0)$ defined by

$$F(t^m, y(t)) = (t^m, z(t))$$

is the germ of a subanalytic bi-Lipschitz map, where $y(x^{1/m})$ and $z(x^{1/m})$ are Newton–Puiseux parameterizations of $(C, 0)$ and $(\tilde{C}, 0)$, respectively. In fact, using the curve selection lemma, it is enough to prove that F is bi-Lipschitz on every pair of test arcs; this follows at once from Lemma 2.3. We note that the Lipschitz constants of F depend continuously on the coefficients of the characteristic terms of $y(x^{1/m})$ and $z(x^{1/m})$.

Conversely, we suppose that $F\colon (C, 0) \to (\tilde{C}, 0)$ is the germ of a subanalytic bi-Lipschitz map. Let $n = \mathrm{multiplicity}(C, 0)$ and $\tilde{n} = \mathrm{multiplicity}(\tilde{C}, 0)$.

LEMMA 2.4. *Let Σ_1 and Σ_2 be test arcs in $(C, 0)$ given by*

$$\Sigma_j(r) = (re^{i\sigma_j(r)}, y(r^{1/n} e^{i\sigma_j(r)/n})) \in S_r(0) \cap \Sigma_j, \quad j = 1, 2.$$

Let $\Gamma_1 = F(\Sigma_1)$ and $\Gamma_2 = F(\Sigma_2)$ be test arcs in $(\tilde{C}, 0)$ given by

$$\Gamma_j(r) = (re^{i\gamma_j(r)}, z(r^{1/\tilde{n}} e^{i\gamma_j(r)/\tilde{n}})) \in S_r(0) \cap \Sigma_j, \quad j = 1, 2.$$

Then

$$\frac{1}{2\pi} \lim_{r \to 0} (\sigma_1(r) - \sigma_2(r)) \in n\mathbb{Z}$$

if and only if

$$\frac{1}{2\pi} \lim_{r \to 0} (\gamma_1(r) - \gamma_2(r)) \in \tilde{n}\mathbb{Z}.$$

Proof. Assume that the conclusion is false. Without loss of generality, let us suppose that

$$\lim_{r\to 0}(\sigma_1(r) - \sigma_2(r)) = 0;$$

that is,

$$\lim_{r\to 0} \sigma_1(r) = \sigma_1 = \lim_{r\to 0} \sigma_2(r)$$

and $\sigma_1(r) \le \sigma_2(r)$ for all r. Since

$$\lim_{r\to 0} \gamma_1(r) = \gamma_1, \quad \lim_{r\to 0} \gamma_2(r) = \gamma_2, \quad \text{and} \quad \frac{1}{2\pi}(\gamma_1 - \gamma_2) \notin \tilde{n}\mathbb{Z},$$

we have $\gamma_1(r) < \gamma_3 \le \gamma_2(r) < \gamma_4 \le \gamma_1(r) + 2\pi\tilde{n}$ such that $\frac{1}{2\pi}(\gamma_i - \gamma_3) \notin \mathbb{Z}$ and $\frac{1}{2\pi}(\gamma_i - \gamma_4) \notin \mathbb{Z}$ for $i = 1, 2$.

Now, let Γ_3 and Γ_4 be the test arcs given by:

$$\Gamma_j(r) = (re^{i\gamma_j}, z(r^{1/\tilde{n}}e^{i\gamma_j/\tilde{n}})), \quad j = 3, 4,$$

and let $\sigma - F^{-1}(\Gamma_3)$ and $\Sigma_4 = F^{-1}(\Gamma_4)$ be the test arcs given by

$$\Sigma_j(r) = (re^{i\sigma_j(r)}, z(r^{1/n}e^{i\sigma_j(r)/n})), \quad j = 3, 4.$$

Then

$$\lim_{r\to 0} \sigma_3(r) = \sigma_1 \quad \text{or} \quad \lim_{r\to 0} \sigma_4(r) = \sigma_1,$$

but this is a contradiction because F is bi-Lipschitz. □

Proposition 2.5. $n = \tilde{n}$.

Proof. Let us assume that $n > \tilde{n}$. Consider $\alpha_j = 2j\pi$ $(j = 0, \dots, n-1)$ and Σ_j the arc test in $(C, 0)$ given by

$$\Sigma_j(r) = (re^{i\alpha_j}, y(r^{1/n}e^{i\alpha_j/n})) \in S_r(0) \cap \Sigma_j, \quad j = 0, \dots, n-1.$$

Let $\tilde{\Sigma}_j = F(\Sigma_j)$ be given by

$$\tilde{\Sigma}_j(r) = (re^{i\sigma_j(r)}, y(r^{1/n}e^{i\sigma_j(r)/n})) \in S_r(0) \cap \tilde{\Sigma}_j, \quad j = 0, \dots, n-1,$$

with $\sigma_0(r) \le \cdots \le \sigma_{n-1}(r) \le \sigma_0(r) + 2\tilde{n}\pi$. Then, by Lemma 2.4,

$$\frac{1}{2\pi} \lim_{r\to 0}(\sigma_{n-1}(r) - \sigma_0(r)) \notin \tilde{n}\mathbb{Z}.$$

By the pigeonhole principle, there exists a j such that

$$\lim_{r\to 0}(\sigma_{j+1}(r) - \sigma_j(r)) = 0.$$

We use again Lemma 2.4 to get a contradiction. □

Consider $(m_1, n_1), \dots, (m_g, n_g)$ the characteristic pairs of $(C, 0)$, and let (p_1, q_1), $\dots, (p_{\tilde{g}}, q_{\tilde{g}})$ be the characteristic pairs of $(\tilde{C}, 0)$. It is valuable to observe that $n_1 \cdots n_g = n = q_1 \cdots q_{\tilde{g}}$ (cf. Lemma 2.4).

Proposition 2.6. $m_g = p_{\tilde{g}}$.

Proof. Let us assume by way of contradiction that $m_g > p_{\tilde{g}}$. Let Γ_0, Γ_1 be the test arcs in $(C, 0)$ given by

$$\Gamma_0(r) = (r, y(r^{1/n})), \qquad \Gamma_1(r) = (r, y(r^{1/n}e^{i2n_1 \cdots n_{g-1}\pi/n})).$$

Clearly, we have

$$\mathrm{ord}_r \|\Gamma_0(r) - \Gamma_1(r)\| = \frac{m_g}{n}.$$

We consider $\Sigma_j = F(\Gamma_j)$, $j = 0, 1$, as test arcs in $(\tilde{C}, 0)$ given by

$$\Sigma_j(r) = (re^{i\alpha_j(r)}, z(r^{1/n}e^{i\alpha_j(r)/n})) \in S_r(0) \cap \Sigma_j, \quad j = 0, 1.$$

Since F is bi-Lipschitz, it follows that $\mathrm{ord}_r \|\Sigma_0(r) - \Sigma_1(r)\| = m_g/n$. Therefore, from Lemma 2.3 we have

$$\frac{1}{2n\pi} \lim_{r \to 0} (\alpha_0(r) - \alpha_1(r)) \in \mathbb{Z}$$

and, by Lemma 2.4,

$$\frac{1}{2n\pi} \lim_{r \to 0} (\gamma_0(r) - \gamma_1(r)) \in \mathbb{Z}.$$

On the other hand,

$$\frac{1}{2n\pi} \lim_{r \to 0} (\gamma_0(r) - \gamma_1(r)) = \frac{n_1 \cdots n_{g-1}}{n} \notin \mathbb{Z}. \qquad \square$$

PROPOSITION 2.7. $n_g = q_{\tilde{g}}$.

Proof. Let $n_g > q_{\tilde{g}}$, let $\gamma_j = 2jn_1 \cdots n_{g-1}\pi$ $(j = 1, \ldots, n_g)$, and let Γ_j be the test arc in $(C, 0)$ given by

$$\Gamma_j(r) = (r, y(r^{1/n}e^{i2jn_1 \cdots n_{g-1}\pi/n})) \in S_r(0) \cap \Gamma_j, \quad j = 1, \ldots, n_g.$$

Clearly,

$$\mathrm{ord}_r \|\Gamma_j(r) - \Gamma_{j+1}(r)\| > 1.$$

Now consider $\Sigma_j = F(\Gamma_j)$, $j = 1, \ldots, n_g$, as test arcs in $(\tilde{C}, 0)$ given by

$$\Sigma_j(r) = (re^{i\sigma_j(r)}, z(r^{1/n}e^{i\sigma_j(r)/n})) \in S_r(0) \cap \Sigma_j, \quad j = 1, \ldots, n_g,$$

with $\sigma_0(r) \leq \cdots \leq \sigma_{n-1}(r) \leq \sigma_0(r) + 2n\pi$.

We have $n_g > q_{\tilde{g}}$ and $\mathrm{ord}_r \|\Gamma_j(r) - \Gamma_{j+1}(r)\| > 1$. Then, by the pigeonhole principle there exists a j such that

$$\lim_{r \to 0} (\sigma_j(r) - \sigma_{j+1}(r)) = 0.$$

Now, we observe that the last equation contradicts Lemma 2.4. $\square$

Note that the foregoing argument can also be used to show that $g = \tilde{g}$ and $n_i = q_i$ for all i. In fact, we proved already that the following equalities hold:

$$m(C) = n = m(D); \qquad m_g = p_{\tilde{g}} \quad \text{and} \quad n_g = q_{\tilde{g}}.$$

We again assume by contradiction that $n_{g-1} > q_{\tilde{g}-1}$. We take $n_{g-1}n_g$ points in the interval $[2n_1 \cdots n_{g-2}\pi, 2n\pi]$—namely, $\gamma_j = 2jn_1 \cdots n_{g-2}\pi$ for $j = 1, \ldots, n_{g-1}n_g$, and we let $\Gamma_1, \ldots, \Gamma_{n_{g-1}n_g}$ be test arcs in $(C, 0)$ given by

$$\Gamma_j(r) = (r, y(r^{1/n}e^{i\gamma_j/n})) \in S_r(0) \cap \Gamma_j, \quad j = 1, \ldots, n_{g-1}n_g.$$

Let $\Sigma_j = F(\Gamma_j)$ be the test arc in $(\tilde{C}, 0)$ given by

$$\Sigma_j(r) = (re^{i\sigma_j(r)}, z(r^{1/n}e^{i\sigma_j(r)/n})) \in S_r(0) \cap \Sigma_j, \quad j = 1, \ldots, n_{g-1}n_g.$$

By Lemma 2.3 we have $\operatorname{ord}_r\|\Gamma_j(r) - \Gamma_{j+1}(r)\| > 1$, and by the pigeonhole principle it follows that there exists a j such that

$$\lim_{r\to 0}(\sigma_{j+1}(r) - \sigma_j(r)) = 0.$$

Again, this contradicts Lemma 2.4. We use induction to show that $g = \tilde{g}$ and $n_i = q_i$ for all i.

To conclude the proof of Theorem 2.2, it is enough to show that $m_i = p_i$ for all i. For this purpose, we take $j = \min\{i : m_i \neq p_i\}$. We already know that $j < g$; we claim that $j > 1$. Let us assume that $m_1 > p_1$, and let Γ_0, Γ_1 be the test arcs in $(C, 0)$ given by

$$\Gamma_0(r) = (r, y(r^{1/n})), \qquad \Gamma_1(r) = (r, y(r^{1/n}e^{2i\pi/n})).$$

Since $\operatorname{ord}_r\|\Gamma_0(r) - \Gamma_1(r)\| = m_1/n_1$, we can make use of Lemma 2.3 to obtain

$$\frac{1}{2\pi}\lim_{r\to 0}(\sigma_1(r) - \sigma_1(r)) \in n_1\mathbb{Z},$$

where $\sigma_j(r)$ is such that $\Sigma_j = F(\Gamma_j)$; that is,

$$\Sigma_j(r) = (re^{i\sigma_j(r)}, z(r^{1/n}e^{i\sigma_j(r)/n})) \in S_r(0) \cap \Sigma_j, \quad j = 0, 1.$$

We know it is not possible for the previous limit to be an integer multiple of n. Hence, for each $r > 0$ small,

$$\sigma_0(r) \le \sigma_2(r) \le \sigma_1(r) \quad \text{where } \sigma_2(r) = \sigma_0(r) + 2\pi.$$

Let Σ_2 be the test arc in $(\tilde{C}, 0)$ given by

$$\Sigma_2(r) = (re^{i\sigma_2(r)}, z(r^{1/n}e^{i\sigma_2(r)/n})) \in S_r(0) \cap \Sigma_2.$$

Then $\operatorname{sh}(\Sigma_0 \cup \Sigma_2, 0) > 1$. It follows from Theorem 1.1 that $\operatorname{sh}(\Gamma_0 \cup \Gamma_2, 0) > 1$, where Γ_2 is the test arc $\Gamma_2 = F^{-1}(\Gamma_2)$ in $(C, 0)$ given by

$$\Gamma_2(r) = (re^{i\gamma_2(r)}, y(r^{1/n}e^{i\gamma_2(r)/n})) \in S_r(0) \cap \Gamma_2$$

with $0 \le \gamma_2(r) \le 2\pi$. Since $\operatorname{sh}(\Gamma_0 \cup \Gamma_2, 0) > 1$ and $0 \le \gamma_2(r) \le 2\pi$, we must have

$$\lim_{r\to 0}\gamma_2(r) = 0 \text{ or } 2\pi.$$

But that is in conflict with

$$\frac{1}{2\pi}\lim_{r\to 0}(\alpha_2(0)-\alpha_0(r))\notin n\mathbb{Z} \quad \text{and} \quad \frac{1}{2\pi}\lim_{r\to 0}(\alpha_1(0)-\alpha_2(r))\notin n\mathbb{Z}.$$

We can finally conclude that $j>1$.

Now, let us assume by contradiction that $m_j > p_j$. Consider $\gamma_0 = 2n_1\cdots n_{j-1}\pi$ and $\gamma_1 = 2\gamma_0$ as well as Γ_i, the test arcs in $(C,0)$ given by

$$\Gamma_i(r) = (r, y(r^{1/n}e^{i\gamma_i/n})) \in S_r(0)\cap\Gamma_i, \quad i=0,1.$$

Let $\Sigma_i = F(\Gamma_i)$ be the test arcs in $(\tilde{C},0)$ given by

$$\Sigma_i(r) = (re^{i\sigma_i(r)}, z(r^{1/n}e^{i\sigma_i(r)/n})) \in S_r(0)\cap\Sigma_i, \quad i=0,1.$$

Since $\|\Gamma_1(r)-\Gamma_0(r)\| = m_j/(n_1\cdots n_j)$, we can use Lemma 2.3 to obtain

$$\frac{1}{2\pi}\lim_{r\to 0}(\sigma_1(r)-\sigma_0(r))\in n_1\cdots n_j\mathbb{Z}.$$

On the other hand, we know that

$$\frac{1}{2\pi}\lim_{r\to 0}(\sigma_1(r)-\sigma_0(r))\notin n\mathbb{Z}.$$

As a result, for each $r>0$ small, we have the inequalities

$$\sigma_0(r)\le\sigma_2(r)\le\sigma_1(r) \quad \text{where } \sigma_2(r)=\sigma_0(r)+2n_1\cdots n_{j-1}\pi.$$

Clearly,

$$\mathrm{sh}(\Sigma_0\cup\Sigma_2) = \frac{p_j}{n_1\cdots n_j} = \mathrm{sh}(\Sigma_1\cup\Sigma_2)$$

and therefore

$$\mathrm{sh}(\Gamma_0\cup\Gamma_2) = \frac{p_j}{n_1\cdots n_j} = \mathrm{sh}(\Gamma_1\cup\Gamma_2),$$

where $\Gamma_2 = F^{-1}(\Sigma_2)$. Hence, if

$$\Gamma_2(r) = (re^{i\gamma_2(r)}, y(r^{1/n}e^{i\gamma_2(r)/n})) \in S_r(0)\cap\Gamma_2,$$

then

$$\frac{1}{2\pi}\lim_{r\to 0}(\gamma_i(r)-\gamma_2(r))\in\mathbb{Z}$$

and $n_1\cdots n_{j-1}$ divide this integer. Otherwise, by Lemma 2.3 we would instead have

$$\begin{aligned}\frac{p_j}{n_1\cdots n_j} &= \mathrm{sh}(\Gamma_i\cup\Gamma_2)\\ &= \min\left\{\mathrm{ord}_r\, r|e^{i\gamma_i(r)}-e^{i\gamma_2(r)}|, \frac{m_{j-1}}{n_1\cdots n_{j-1}}\right\}\\ &= \frac{p_{j-1}}{n_1\cdots n_{j-1}}\end{aligned}$$

for $i=0,1$, which is false.

Now, since $\gamma_0(r)\le\gamma_2(r)\le\gamma_1(r)$ and $|\gamma_0(r)-\gamma_1(r)| = 2n_1\cdots n_{j-1}\pi$, we must have

$$\lim_{r\to 0}(\gamma_2(r)-\gamma_1(r))=0 \quad \text{or} \quad \lim_{r\to 0}(\gamma_2(r)-\gamma_0(r))=0.$$

In either case, we get a contradiction.

Thus, the proof of Theorem 2.2 is complete. □

3. On Reducible Plane Curves

Let $(C,0)$ be the germ of an analytic complex curve in $\mathbb{C}^2$ that is endowed with the induced Euclidean metric $\mathbb{R}^4$ and with branches $C=\bigcup_{i\in I} C_i$. Let m_i be the multiplicity of $(C_i,0)$ for $i\in I$. For each pair $i\neq j\in I$, by analogy with the real case, we denote

$$\mathrm{sh}_{ij}(C)=\mathrm{ord}_r\big(\mathrm{dist}(C_i\cap S_r(0),C_j\cap S_r(0))\big).$$

Consider also

$$\mathrm{coinc}(C_i,C_j)=\max\{\mathrm{ord}_x[y_k(x^{1/m_i})-z_l(x^{1/m_j})] \mid 1\le k\le m_i,\ 1\le l\le m_j\},$$

where $\{y_k(x^{1/m_i})\}_{k=1}^{m_i}$ is the set of the Newton–Puiseux parameterization of C_i and $\{z_l(x^{1/m_j})\}_{l=1}^{m_j}$ is the set of the Newton–Puiseux parameterization of C_j. The number $\mathrm{coinc}(C_i,C_j)$ is known as *coincidence* between C_i and C_j.

LEMMA 3.1.

$$\mathrm{sh}_{ij}(C)=\mathrm{coinc}(C_i,C_j) \quad \textit{for all } i\neq j\in I.$$

Proof. Let Γ_i,Γ_j be test arcs in $C_i\cup C_j$ given by

$$\Gamma_k(r)=(r,\tilde y_k(r^{1/m_k}))\in\Gamma_k\cap S_r(0), \quad k=i,j,$$

where $\tilde y_i(x^{1/m_i})$ and $\tilde y_j(x^{1/m_j})$ are (respectively) the Newton–Puiseux parameterization of C_i and C_j such that

$$\mathrm{coinc}(C_i,C_j)=\mathrm{ord}_x[\tilde y_i(x^{1/m_i})-\tilde y_j(x^{1/m_j})].$$

Then

$$\mathrm{dist}(C_i\cap S_r(0),C_j\cap S_r(0))\le\|\Gamma_i(r)-\Gamma_j(r)\|=|\tilde y_i(r^{1/m_i})-\tilde y_j(r^{1/m_j})|.$$

Therefore, $\mathrm{sh}_{ij}(C)\ge\mathrm{coinc}(C_i,C_j)$.

On the other hand, we have

$$\tilde y_k(x^{1/m_k})=h(x^{1/m_k})+g_k(x^{1/m_k}), \quad k=i,j,$$

with

$$\mathrm{ord}_x[g_i(x^{1/m_i})-g_j(x^{1/m_j})]=\min\{\mathrm{ord}_x\, g_i(x^{1/m_i}),\mathrm{ord}_x\, g_j(x^{1/m_j})\}.$$

Let $\Gamma_i\subset C_i$ and $\Gamma_j\subset C_j$ be test arcs such that

$$\mathrm{dist}(C_i\cap S_r(0),C_j\cap S_r(0))=\mathrm{dist}(\Gamma_i\cap S_r(0),\Gamma_j\cap S_r(0)),$$

and let

$$\Gamma_k(r)=(re^{i\gamma_k(r)},\tilde y_k(r^{1/m_k}e^{i\gamma_k(r)/m_k}))\in S_r(0)\cap\Gamma_k, \quad k=i,j.$$

Now, since

$$\operatorname{ord}_r r|e^{i\gamma_i(r)} - e^{i\gamma_j(r)}| \le \operatorname{ord}_r|h(r^{1/m_i}e^{i\gamma_i(r)/m_i}) - h(r^{1/m_j}e^{i\gamma_j(r)/m_j})|,$$

we have

$$\begin{aligned}\operatorname{ord}_r\|\Gamma_i(r) - \Gamma_j(r)\| &\le \min\{\operatorname{ord}_r g_i(r^{1/m_i}e^{i\gamma_i(r)/m_i}), \operatorname{ord}_r g_j(r^{1/m_j}e^{i\gamma_j(r)/m_j})\}\\ &= \operatorname{coinc}(C_i, C_j);\end{aligned}$$

that is, $\operatorname{sh}_{ij}(C) \le \operatorname{coinc}(C_i, C_j)$. The result follows. □

Let $(C,0) \subset (\mathbb{C}^d,0)$ be a germ of a (reduced) analytic space curve, and let $p\colon \mathbb{C}^d \to C^2$ be a linear projection. We say that p is *general for C at* 0 if it has the following property: For any sequence of couples of points $(a_i,b_i) \in (C\setminus\{0\}) \times (C\setminus\{0\})$ tending to 0, the limit direction of the secant line $\overline{a_i b_i}$ (for any subsequence) is not contained in the kernel of p. Our next theorem is the topological analogue of Zariski's result (see [13, p. 888]) for the bi-Lipschitz approach.

THEOREM 3.2. *Let $(X,0)$ and $(\tilde{X},0)$ be analytic complex curves in $\mathbb{C}^2$ with branches $X = \bigcup_{i\in I} X_i$ and $\tilde{X} = \bigcup_{j\in J}\tilde{X}_j$. Then the following conditions are equivalent.*

(1) *There exists a germ of the subanalytic bi-Lipschitz map $F\colon (X,0) \to (\tilde{X},0)$.*
(2) *There exists a bijection $\sigma\colon I \to J$ such that $\beta(X_i) = \beta(\tilde{X}_{\sigma(i)})$ for all $i \in I$ and such that* $\operatorname{coinc}(X_i, X_j) = \operatorname{coinc}(\tilde{X}_{\sigma(i)}, \tilde{X}_{\sigma(j)})$ *for all $i \ne j \in I$.*
(3) *There exists a bijection $\sigma\colon I \to J$ such that $\beta(X_i) = \beta(\tilde{X}_{\sigma(i)})$ for all $i \in I$ and such that $(X_i, X_j)_0 = (\tilde{X}_{\sigma(i)}, \tilde{X}_{\sigma(j)})_0$ for all $i \ne j \in I$, where $(\cdot,\cdot)_0$ denotes the intersection multiplicity at* 0.
(4) *$(X,0)$ is topologically equivalent to $(\tilde{X},0)$.*
(5) *There exists an integer d, a germ of the curve $(C,0) \subset (\mathbb{C}^d,0)$, and two linear projections $p, \tilde{p}\colon \mathbb{C}^d \to \mathbb{C}$, both general for C at* 0 *and such that $p(C) = X$ and $\tilde{p}(C) = \tilde{X}$.*

Proof. (1) ⇒ (2) Suppose that $F\colon (X,0) \to (\tilde{X},0)$ is the germ of a subanalytic bi-Lipschitz map. Since $(X,0)$ and $(\tilde{X},0)$ are homeomorphic, we can assume $I = J$ and $F(X_i) = \tilde{X}_i$ for all $i \in I$. It follows from Theorem 2.2 that $\beta(X_i) = \beta(\tilde{X}_i)$ for all $i \in I$.

Now, we let $i \ne j \in I$ and show that $\operatorname{sh}_{ij}(X) = \operatorname{sh}_{ij}(\tilde{X})$. For this purpose, we take Γ_1, Γ_2 as test arcs in X such that $\Gamma_i \subset X_i$ $(i = 1,2)$ and

$$\operatorname{dist}(X_i \cap S_r(0), X_j \cap S_r(0)) = \operatorname{dist}(\Gamma_i \cap S_r(0), \Gamma_j \cap S_r(0)).$$

We observe that $F(\Gamma_1), F(\Gamma_2)$ are test arcs in $\tilde{X}$ such that $F(\Gamma_i) \subset \tilde{X}_i$ $(i = 1,2)$. Hence, we have

$$\operatorname{dist}(\tilde{X}_i \cap S_r(0), \tilde{X}_j \cap S_r(0)) \le \operatorname{dist}(F(\Gamma_i) \cap S_r(0), F(\Gamma_j) \cap S_r(0)).$$

Now, since F is the germ of a bi-Lipschitz map, we can use Theorem 1.1 to obtain the relation

$$\operatorname{dist}(\Gamma_i \cap S_r(0), \Gamma_j \cap S_r(0)) \approx \operatorname{dist}(F(\Gamma_i) \cap S_r(0), F(\Gamma_j) \cap S_r(0)).$$

Then

$$\operatorname{sh}_{ij}(X) \geq \operatorname{sh}_{ij}(\tilde{X}).$$

Analogously, we show that

$$\operatorname{sh}_{ij}(X) \leq \operatorname{sh}_{ij}(\tilde{X})$$

and hence $\operatorname{sh}_{ij}(X) = \operatorname{sh}_{ij}(\tilde{X})$. Therefore, by Lemma 3.1, we may conclude that $\operatorname{coinc}(X_i, X_j) = \operatorname{coinc}(\tilde{X}_i, \tilde{X}_j)$.

(2) $\Rightarrow$ (3) Let us assume that $I = J$ and $\beta(X_i) = \beta(\tilde{X}_i)$ for all $i \in I$ and that $\operatorname{coinc}(X_i, X_j) = \operatorname{coinc}(\tilde{X}_i, \tilde{X}_j)$ for all $i \neq j \in I$. Then we have $(X_i, X_j)_0 = (\tilde{X}_i, \tilde{X}_j)_0$ for all $i \neq j \in I$ (cf. [7, Prop. 2.4, pp. 106–107).

(3) $\Rightarrow$ (4) See Theorem 0.3.

(4) $\Rightarrow$ (5) See [13, p. 888].

(5) $\Rightarrow$ (6) Assume that there exist an integer d, a germ of the curve $(C, 0) \subset (\mathbb{C}^d, 0)$, and two linear projections $p, \tilde{p}\colon \mathbb{C}^d \to \mathbb{C}$, both general for C at 0 and such that $p(C) = X$ and $\tilde{p}(C) = \tilde{X}$. Then $\tilde{p} \circ p^{-1}\colon (X, 0) \to (\tilde{X}, 0)$ is a germ of a subanalytic bi-Lipschitz map. □

Theorem 3.2 is not true for spatial curves. In fact, let

$$X : x = t^6,\ y = t^{14},\ z = t^{39};$$
$$\tilde{X} : x = t^6,\ y = t^{14} + t^{17},\ z = t^{39}.$$

Then $(X, 0)$ is topologically equivalent to $(\tilde{X}, 0)$, yet a germ of the subanalytic bi-Lipschitz map $F\colon (X, 0) \to (\tilde{X}, 0)$ does not exist. However, we still can say something about spatial curves.

LEMMA 3.3. *Let $(X, 0)$ be an analytic complex curve in $\mathbb{C}^n$ $(n \geq 3)$, and let $p\colon \mathbb{C}^n \to \mathbb{C}^2$ be a linear projection that is general for X at 0. Let $\mathbb{C}^2 \times \mathbb{C}^{n-2}$ be a decomposition of $\mathbb{C}^n$ and let (x, y) be coordinates such that $p(x, y) = x$. Then there exists a germ of the subanalytic bi-Lipschitz map $\Phi\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\Phi(X) = p(X) \times \{0\}$. In particular, $(X, 0)$ is topologically equivalent to $(p(X) \times \{0\}, 0)$ in $\mathbb{C}^n$.*

Proof. It is enough to prove this for the case $n = 3$. Since p is general for X at 0, X is the graph of the subanalytic Lipschitz function $f\colon p(X) \subset \mathbb{C}^2 \to \mathbb{C}$. Let $F\colon \mathbb{C}^2 \to \mathbb{C}$ be a subanalytic Lipschitz extension of f. Then $\Phi\colon (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ as defined by $\Phi(x, y) = (x, y - F(x))$ is the germ of a subanalytic bi-Lipschitz homeomorphism such that $\Phi(X) = p(X) \times \{0\}$. □

COROLLARY 3.4. *Let $(X, 0)$ and $(\tilde{X}, 0)$ be analytic complex curves in $\mathbb{C}^n$. If there exists a germ of the subanalytic bilipschitz map $F\colon (X, 0) \to (\tilde{X}, 0)$, then $(X, 0)$ is topologically equivalent to $(\tilde{X}, 0)$ in $\mathbb{C}^n$.*

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