Michigan Math. J. 51 (2003)

Second Variation of Compact Minimal Legendrian Submanifolds of the Sphere

Francisco Urbano

1. Introduction

The second variation operator of minimal submanifolds of Riemannian manifolds (the *Jacobi operator*) carries information about stability properties of the submanifold when it is thought of as a critical point for the area functional. When the ambient Riemannian manifold is a sphere $\mathbb{S}^m$, Simons [S] characterized the totally geodesic submanifolds as the minimal submanifolds of $\mathbb{S}^m$ either with the lowest index (number of independent infinitesimal deformations that do decrease the area) or with lowest nullity (dimension of the Jacobi fields, i.e., infinitesimal deformations through minimal immersions). Other results about the index and the nullity of minimal surfaces of the sphere can be found in [E2; MU; U1]. If m is odd (i.e., if $m = 2n+1$) then one can consider n-dimensional minimal *Legendrian* submanifolds of $\mathbb{S}^{2n+1}$ (see Section 2 for the definition). These submanifolds are particulary interesting because the cones over them are special Lagrangian submanifolds of the complex Euclidean space $\mathbb{C}^{n+1}$, and as Joyce pointed out in [J, Sec. 10.2], the knowledge of their index is deeply related to the dimension of the moduli space of asymptotically conical special Lagrangian submanifolds of $\mathbb{C}^{n+1}$. This fact, joint to the characterization of minimal Legendrian submanifolds given by Lê and Wang in [LW], directed my attention to the study of the second variation of minimal Legendrian submanifolds of odd-dimensional spheres.

In Section 2 we compute the Jacobi operator of compact minimal Legendrian submanifolds of $\mathbb{S}^{2n+1}$, proving that it is an intrinsic operator on the submanifold and that it can be written in terms of the exterior differential, its codifferential operator, and the Laplacian (see formula (2)). In Section 3 we decompose the Jacobi operator as the sum of two elliptic operators and then study their indexes and nullities (Theorem 1 and Corollary 1). As a consequence we obtain a formula for the index and the nullity of compact minimal Legendrian submanifolds of $\mathbb{S}^{2n+1}$ (Corollary 2). Finally, we particularize our study to compact minimal Legendrian surfaces of $\mathbb{S}^5$ and prove the following result.

If M is an orientable compact minimal (*non–totally geodesic*) *Legendrian surface in* $\mathbb{S}^5$, *then*:

Received April 26, 2002. Revision received January 6, 2003.
Research partially supported by DGICYT Grant no. BFM2001-2967.

(a) $\operatorname{Ind}(M) \geq 8$, *and the equality holds if and only if M is the equilateral torus*; *and*

(b) $\operatorname{Nul}(M) \geq 13$, *and the equality holds if and only if M is the equilateral torus.*

If $\Pi : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ is the Hopf fibration and M^n is a minimal Legendrian submanifold of $\mathbb{S}^{2n+1}$, then $\Pi(M)$ is a minimal Lagrangian submanifold of $\mathbb{CP}^n$. It is interesting to compare our Jacobi operator with the one given in [Oh] for minimal Lagrangian submanifolds of $\mathbb{CP}^n$. It is also interesting to compare the results of this paper with those obtained by the author in [U2] for minimal Lagrangian submanifolds of $\mathbb{CP}^n$.

2. Jacobi Operator of Minimal Legendrian Submanifolds

Let $\mathbb{C}^{n+1}$ be the $(n+1)$-dimensional complex Euclidean space, $\langle\cdot,\cdot\rangle$ the Euclidean metric, and Ω the Kähler 2-form on $\mathbb{C}^{n+1}$. Then $\Omega = d\Lambda$, where Λ is the Liouville 1-form given by

$$2\Lambda(v) = \langle v, Jp\rangle,$$

where $v \in T_p\mathbb{C}^{n+1}$, $p \in \mathbb{C}^{n+1}$, and J is the complex structure on $\mathbb{C}^{n+1}$.

Let $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ be the $(2n+1)$-dimensional unit sphere. The vector space of Killing vector fields on $\mathbb{S}^{2n+1}$ can be identified with the space of skew-symmetric matrices $\mathfrak{so}(2n+2)$ in such way that the Killing field corresponding to a matrix $A \in \mathfrak{so}(2n+2)$ is given by $X_A(p) = Ap$ for any $p \in \mathbb{S}^{2n+1}$. This space can be decomposed as $\mathfrak{so}(2n+2) = \langle J\rangle \oplus \mathfrak{so}^+(2n+2) \oplus \mathfrak{so}^-(2n+2)$, where $\langle J\rangle$ is the linear line spanned by the skew-symmetric matrix J and

$$\mathfrak{so}^+(2n+2) = \{A \in \mathfrak{so}(2n+2) \mid AJ = JA \text{ and Trace } AJ = 0\},$$

$$\mathfrak{so}^-(2n+2) = \{A \in \mathfrak{so}(2n+2) \mid AJ = -JA\}.$$

This decomposition of $\mathfrak{so}(2n+2)$ is orthogonal with respect to the inner product $\langle\langle A, B\rangle\rangle = -\text{Trace } AB$ for all $A, B \in \mathfrak{so}(2n+2)$. It is clear that $\dim \mathfrak{so}^+(2n+2) = (n+1)^2 - 1$, $\dim \mathfrak{so}^-(2n+2) = n(n+1)$, and $\mathfrak{so}^+(2n+2)$ is the real representation of $\mathfrak{su}(n+1)$, so $\mathfrak{so}^-(2n+2)$ can be identified with the tangent space at the origin to the symmetric space $\mathrm{SO}(2n+2)/\mathrm{SU}(n+1)$.

An immersion $\phi : M^n \to \mathbb{S}^{2n+1}$ of an n-dimensional manifold M is called *Legendrian* if $\phi^*\Lambda = 0$. This implies that $\phi^*\Omega = 0$, and then M is an isotropic submanifold of $\mathbb{C}^{n+1}$. If $\mathbb{CP}^n$ is the n-dimensional complex projective space and $\Pi : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ the Hopf fibration, then $\phi : M^n \to \mathbb{S}^{2n+1}$ is a horizontal immersion with respect to Π and so $\Pi \circ \phi : M^n \to \mathbb{CP}^n$ is a Lagrangian immersion.

Let $\phi : M^n \to \mathbb{S}^{2n+1}$ be a Legendrian immersion, and let $T^\perp M$ denote the normal bundle of ϕ. Since $J\phi$ is a normal vector field to ϕ, it follows that $T^\perp M = J(TM) \oplus \langle J\phi\rangle$, where $\langle J\phi\rangle$ is the trivial line bundle spanned by $J\phi$. Therefore, a section $\xi \in \Gamma(T^\perp M)$ can be decomposed as

$$\xi = JX + fJ\phi,$$

where $X \in \Gamma(TM)$ and $f \in C^\infty(M)$. If $\bar\nabla$ is the connection on $\phi^*\mathbb{S}^{2n+1}$ induced by the Levi–Civita connection of $T\mathbb{S}^{2n+1}$ and if $\bar\nabla = \nabla + \nabla^\perp$ is the decomposition into tangent and normal components, then it is easy to derive

$$\begin{aligned} JA_{JX}Y &= \sigma(X,Y), \\ \nabla^\perp_X JY &= J\nabla_X Y - \langle X,Y\rangle J\phi, \qquad (1) \\ A_{J\phi} = 0, &\quad \nabla^\perp_X J\phi = JX, \end{aligned}$$

where σ is the second fundamental form of ϕ, A is the shape operator, and X, Y are vector fields tangent to M.

From now on we assume that ϕ is a *minimal* immersion. We shall mention two properties of these submanifolds that will be relevant throughout the paper. We denote by Δ the Laplacian of the induced metric by ϕ on M (which will be also denoted by $\langle\cdot,\cdot\rangle$); that is, $\Delta = \delta d + d\delta$, where δ is the codifferential operator of the exterior differential d. Then, since ϕ is minimal, *n is an eigenvalue of the Laplacian Δ acting on $C^\infty(M)$*. Also, since $\Pi\circ\phi$ is a minimal Lagrangian immersion in $\mathbb{CP}^n$, it follows that $2(n+1)$ *is also an eigenvalue of Δ acting on $C^\infty(M)$* [R, Cor. 2.11]. Lê and Wang [LW] have obtained a lower bound of the multiplicity of $2(n+1)$, characterizing the totally geodesic immersions as the only ones attaining that lower bound.

If $\phi\colon M^n \to \mathbb{S}^m$ is a minimal immersion of a compact manifold M, then the well-known *Jacobi operator* of ϕ (which we will denote by L) is an endomorphism of the space $\Gamma(T^\perp M)$ given by

$$L = \Delta^\perp + \mathcal{B} + n\mathcal{I},$$

where $\mathcal{I}$ is the identity and $\Delta^\perp$ and $\mathcal{B}$ are the operators

$$\Delta^\perp = \sum_{i=1}^n \{\nabla^\perp_{e_i}\nabla^\perp_{e_i} - \nabla^\perp_{\nabla_{e_i}e_i}\}, \qquad \mathcal{B}(\xi) = \sum_{i=1}^n \sigma(A_\xi e_i, e_i),$$

with $\xi\in\Gamma(T^\perp M)$ and $\{e_1,\dots,e_n\}$ an orthonormal reference tangent to M. Let $Q(\xi) = -\int_M \langle L\xi,\xi\rangle\, dV$ be the quadratic form associated to the Jacobi operator L. We will represent by $\mathrm{Ind}(\phi)$ and $\mathrm{Nul}(\phi)$ the index and nullity of the quadratic form Q, which are (respectively) the number of negative eigenvalues of L and the multiplicity of zero as an eigenvalue of L. It is interesting to summarize some results of Simons [S] that are the starting point of our paper.

THEOREM A [S]. *Let $\phi\colon M^n\to\mathbb{S}^m$ be a minimal immersion of a compact manifold M. Then the following statements hold.*

1. *If $a^\perp$ is the normal component of a vector $a\in\mathbb{R}^{m+1}$, then $La^\perp - na^\perp = 0$ and $\dim\{a^\perp \mid a\in\mathbb{R}^{m+1}\}\ge m-n$, holding the equality if and only if ϕ is totally geodesic. As consequence, $\mathrm{Ind}(\phi)\ge m-n$ and the equality holds if and only if ϕ is totally geodesic.*
2. *If $X_A^\perp$ is the normal component of the Killing field X_A on $\mathbb{S}^m$ with $A\in\mathfrak{so}(m+1)$, then $LX_A^\perp = 0$ and $\dim\{X_A^\perp \mid A\in\mathfrak{so}(m+1)\}\ge(n+1)(m-n)$, holding the equality if and only if ϕ is totally geodesic. As consequence, $\mathrm{Nul}(\phi)\ge (n+1)(m-n)$ and the equality holds if and only if ϕ is totally geodesic.*

Now we shall analyze the Jacobi operator L when $\phi\colon M^n\to\mathbb{S}^{2n+1}$ is a minimal Legendrian submanifold. Using (1) and writing $\xi = JX + fJ\phi$, we check easily that

$$\Delta^{\perp}\xi = J\hat{\Delta}X - JX + 2J\nabla f + (\Delta f - nf - 2\,\mathrm{div}\,X)J\phi,$$

where div is the divergence operator, ∇f is the gradient of f, and $\hat{\Delta}$ is the operator on $\Gamma(TM)$:

$$\hat{\Delta} = \sum_{i=1}^{n}\{\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i}e_i}\}$$

for $\{e_1, \dots, e_n\}$ a local orthonormal reference on M. Also from (1) we have

$$\mathcal{B}(\xi) = \sum_{i=1}^{n} JA_{\sigma(X,e_i)}e_i.$$

Now using the Gauss equation for ϕ, we finally obtain

$$L\xi = J\hat{\Delta}X - JS(X) + 2(n-1)JX + 2J\nabla f + (\Delta f - 2\,\mathrm{div}\,X)J\phi,$$

where S denotes the Ricci operator on M. Because the tangent vector field X can be identified with its dual 1-form, we will consider the following identification:

$$\Gamma(T^{\perp}M) \equiv \Omega^1(M) \oplus C^{\infty}(M),$$
$$\xi \equiv (\alpha, f),$$

where α is the dual 1-form of X (i.e., $\alpha(v) = \langle v, X\rangle = \langle Jv, \xi\rangle$ for any v tangent to M) and where, in general, $\Omega^p(M)$ denotes the space of p-forms on M. Taking into account the relation between $\hat{\Delta}$ and the Laplacian Δ acting on 1-forms finally yields the expression of the Jacobi operator L of a minimal Legendrian immersion $\phi\colon M^n \to \mathbb{S}^{2n+1}$ as follows:

$$\begin{aligned} L\colon \Omega^1(M) \oplus C^{\infty}M &\to \Omega^1(M) \oplus C^{\infty}M, \\ L(\alpha, f) &= (\Delta\alpha + 2(n-1)\alpha + 2df, \Delta f - 2\delta\alpha). \end{aligned} \tag{2}$$

It is interesting to identify the eigensections given in Theorem A in terms of 1-forms and functions. In fact, it is easy to check the following assertions.

1. If $a \in \mathbb{C}^{n+1}$, then the eigensection $a^{\perp}$ corresponding to the eigenvalue $-n$ of L is identified with

$$a^{\perp} \equiv (df, f), \quad \text{where } f = \langle J\phi, a\rangle \text{ and } \Delta f + nf = 0.$$

2. The Jacobi field $X_J^{\perp}$ is identified with

$$X_J^{\perp} \equiv (0, f), \quad \text{where } f(p) = 1\ \forall p \in M.$$

3. If $A \in \mathfrak{so}^{+}(2n+2)$, then the Jacobi field $X_A^{\perp}$ is identified with

$$X_A^{\perp} \equiv (dg, 2g), \quad \text{where } g = \langle A\phi, J\phi\rangle \text{ and } \Delta g + 2(n+1)g = 0.$$

4. If $A \in \mathfrak{so}^{-}(2n+2)$, then the Jacobi field $X_A^{\perp}$ is identified with

$$X_A^{\perp} \equiv (\alpha, 0), \quad \text{where } \alpha_p(v) = \langle A\phi(p), Jv\rangle\ \forall p \in M,\ \forall v \in T_pM$$

and where $\Delta\alpha + 2(n-1)\alpha = 0$. Moreover $\alpha = \delta\omega$, where ω is the 2-form $\omega_p(v, w) = \langle Av, Jw\rangle$ for all $p \in M$ and all $v \in T_pM$.

3. Index and Nullity of Minimal Legendrian Submanifolds

To study the index and the nullity of a minimal Legendrian immersion $\phi\colon M^n \to \mathbb{S}^{2n+1}$ of a compact orientable manifold M, we consider the Hodge decomposition

$$\Omega^1(M) = \mathcal{H}(M) \oplus dC^\infty M \oplus \delta\Omega^2(M),$$

which allows us to write in a unique way any 1-form α as $\alpha = \alpha_0 + dg + \delta\omega$, with α_0 a harmonic 1-form, g a real function, and ω a 2-form on M. The space $\mathcal{H}(M)$ of harmonic 1-forms is the kernel of Δ, and its dimension is $\beta_1(M)$, the first Betti number of M.

Hodge's decomposition of 1-forms, together with the preceding decomposition of the Jacobi fields (via the special decomposition of $\mathfrak{so}(2n+2)$ given in Section 2), suggests the split $L = L_1 \oplus L_2$, where

$$L_1\colon \mathcal{H}(M) \oplus \delta\Omega^2(M) \to \mathcal{H}(M) \oplus \delta\Omega^2(M),$$
$$L_1(\alpha) = \Delta\alpha + 2(n-1)\alpha,$$

and

$$L_2\colon dC^\infty M \oplus C^\infty(M) \to dC^\infty M \oplus C^\infty(M),$$
$$L_2(dg, f) = (\Delta dg + 2(n-1)dg + 2df, \Delta f - 2\Delta g).$$

Hence, if Q_i are the quadratic forms associated to the operators L_i $(i = 1, 2)$, then

$$\begin{aligned} \mathrm{Ind}(\phi) &= \mathrm{Ind}(Q_1) + \mathrm{Ind}(Q_2), \\ \mathrm{Nul}(\phi) &= \mathrm{Nul}(Q_1) + \mathrm{Nul}(Q_2). \end{aligned} \tag{3}$$

In the next result we study the index and the nullity of Q_1.

THEOREM 1. *If $\phi\colon M^n \to \mathbb{S}^{2n+1}$ is a minimal Legendrian immersion of a compact orientable manifold M, then*

$$\mathrm{Ind}(Q_1) = \beta_1(M) + \sum_{0<\mu_k<2(n-1)} n_{\mu_k}, \qquad \mathrm{Nul}(Q_1) = n_{2(n-1)},$$

where μ_k denotes the eigenvalues of Δ of M acting on $\delta\Omega^2(M)$ and n_{μ_k} their respective multiplicities. Moreover:

1. $\mathrm{Ind}(Q_1) = 0$ *if and only if ϕ is totally geodesic*; *and*
2. $\mathrm{Nul}(Q_1) \geq n(n+1)/2$, *and the equality holds if and only if ϕ is totally geodesic.*

Proof. The expressions for $\mathrm{Ind}(Q_1)$ and $\mathrm{Nul}(Q_1)$ come from the fact that $L_1 = \Delta + 2(n-1)I$, where I is the identity.

To prove part 1, we consider the minimal Lagrangian immersion $\psi = \Pi \circ \phi\colon M^n \to \mathbb{CP}^n$ and use a weak modification of Theorem 1 in [U2]. This theorem states that the first eigenvalue ρ_1 of Δ acting on $\Omega^1(M)$ satisfies $\rho_1 \leq 2(n-1)$ and that, if the equality holds, then ψ is totally geodesic. To obtain this result in [U2], the author used certain "test" 1-forms that actually belonged to $\mathcal{H}(M) \oplus \delta\Omega^2(M)$. So really Theorem 1 in [U2] says that the first eigenvalue ρ_1 of Δ acting on $\mathcal{H}(M) \oplus \delta\Omega^2(M)$ satisfies $\rho_1 \leq 2(n-1)$, with equality implying that ψ is totally geodesic. Hence we obtain that the first eigenvalue of L_1 is nonpositive and, if it is zero, then ϕ is totally geodesic. This means that if $\operatorname{Ind}(Q_1) = 0$ then ϕ is totally geodesic.

Conversely, if ϕ is totally geodesic then M is isometric to a unit sphere $\mathbb{S}^n$, with $2(n-1)$ as the first eigenvalue of Δ acting on $\mathcal{H}(M) \oplus \delta\Omega^2(M)$ (see [IT]).

To prove part 2 of our Theorem 1 we consider, for each $A \in \mathfrak{so}^-(2n+2)$, the 1-form γ_A on M defined by $\gamma_A(v) = \langle A\phi, Jv\rangle$, where v is a tangent vector on M. From assertion 3 at the end of Section 2, it follows that γ_A is in the nullity of Q_1 for each $A \in \mathfrak{so}^-(2n+2)$. Thus we have defined a linear map

$$F\colon \mathfrak{so}^-(2n+2) \to N(Q_1), \qquad A \mapsto \gamma_A;$$

therefore, $\operatorname{Nul}(Q_1) \geq \dim \operatorname{img} F = n(n+1) - \dim \operatorname{Ker} F$. Now, if $A \in \operatorname{Ker} F$ then $\gamma_A = 0$ and so $JA\phi$ is a section of $J(TM)$. This means that the Killing field X_A on $\mathbb{S}^{2n+1}$ is tangent to M and so X_A is also a Killing field on M. Hence we have defined a linear map

$$G\colon \operatorname{Ker} F \to \{\text{Killing fields on } M\}, \qquad A \mapsto X_A,$$

which is a monomorphism. As the dimension of the isometry group of a compact n-dimensional manifold is not greater than $n(n+1)/2$, we obtain $\dim \operatorname{Ker} F \leq n(n+1)/2$. This, together with the previous inequality, means that $\operatorname{Nul}(Q_1) \geq n(n+1)/2$. If the equality holds, the dimension of the isometry group of M is exactly $n(n+1)/2$, and since M is compact we have that M is isometric either to a sphere or to a real projective space. Considering the Hopf fibration $\Pi\colon \mathbb{S}^{2n+1} \to \mathbb{CP}^n$, we see that $\Pi \circ \phi$ defines a minimal Lagrangian immersion of a manifold of positive constant curvature. Using the main result in [E1], we obtain that $\Pi \circ \phi$ (and hence ϕ) is a totally geodesic immersion.

Conversely, if ϕ is totally geodesic then M is isometric to a unit sphere $\mathbb{S}^n$, with $2(n-1)$ as the first eigenvalue of Δ acting on $\mathcal{H}(M) \oplus \delta\Omega^2(M)$ and with multiplicity $n(n+1)/2$ (see [IT]). This completes the proof of Theorem 1. $\square$

Now we study the second operator L_2.

PROPOSITION 1. *Let $\phi\colon M^n \to \mathbb{S}^{2n+1}$ be a minimal Legendrian immersion of a compact orientable manifold M. The negative eigenvalues ρ of L_2 are given by*

$$\rho = h(\lambda) := \lambda - (n-1) - \sqrt{(n-1)^2 + 4\lambda},$$

where λ is an eigenvalue of Δ of M acting on $C^\infty(M)$ such that $0 < \lambda < 2(n+1)$. Moreover, the eigenspace V_ρ corresponding to a negative eigenvalue ρ of L_2 is given by

$$V_\rho = \left\{ \left(df, \frac{2\lambda}{\lambda - \rho} f \right) \;\middle|\; \Delta f + \lambda f = 0 \text{ and } h(\lambda) = \rho \right\}.$$

The eigenspace V_0 corresponding to the eigenvalue 0 of L_2 is

$$V_0 = \{(dg, 2g + a) \mid \Delta g + 2(n+1)g = 0,\ a \in \mathbb{R}\}.$$

REMARK 1. When $n \geq 3$, the function h is a bijection from $]0, 2(n+1)[$ onto $]-2(n-1), 0[$. So, in particular, the first eigenvalue ρ_1 of the operator L_2 satisfies $\rho_1 > -2(n-1)$.

If $n = 2$ then h is a map from $]0, 6[$ onto $[-9/4, 0[$; in particular, $\rho_1 \geq -9/4 = h(3/4)$. In this case $3/4$ is the only critical point (in fact, it is a minimum) of h on $]0, 6[$. Also, h from $[2, 6[$ onto $[-2, 0[$ is a bijection and, for each real number $y \in]-9/4, -2[$, there exist exactly $x_1 \in]0, 3/4[$ and $x_2 \in]3/4, 2[$ such that $h(x_i) = y$ for $i = 1, 2$.

Proof of Proposition 1. Let ρ be a negative eigenvalue of L_2 and let $(dg, f) \in dC^\infty(M) \oplus C^\infty(M)$ be an eigensection corresponding to ρ, that is, $L_2((dg, f)) + \rho(dg, f) = 0$. Looking at the expression of L_2, we write this equation as

$$\begin{aligned} \Delta(dg) + (2(n-1) + \rho)dg + 2df &= 0, \\ \Delta f - 2\Delta g + \rho f &= 0. \end{aligned} \tag{4}$$

If $dg = \sum_{k \geq 1} dg_k$ and $f = \sum_{k \geq 0} f_k$ are the decompositions of dg and f in eigenforms and eigenfunctions (respectively) with $\Delta dg_k + \lambda_k dg_k = 0$ and $\Delta f_k + \lambda_k f_k = 0$, then (4) yields the following equations:

$$(2(n-1) + \rho - \lambda_k)dg_k + 2df_k = 0, \quad k \geq 1; \tag{5}$$

$$2\lambda_k g_k + (\rho - \lambda_k)f_k = 0, \quad k \geq 1; \tag{6}$$

$$\rho f_0 = 0. \tag{7}$$

As $\rho < 0$ and $\lambda_k > 0$ for $k \geq 1$, from (7) and (6) we have that

$$f_0 = 0, \qquad f_k = \frac{2\lambda_k}{\lambda_k - \rho} g_k \quad \text{for any } k \geq 1. \tag{8}$$

Using (8) in (5), we obtain

$$\left(2(n-1) + \rho - \lambda_k + \frac{4\lambda_k}{\lambda_k - \rho}\right) dg_k = 0 \quad \text{for any } k \geq 1,$$

and hence either $\rho = \lambda_k - (n-1) \pm \sqrt{(n-1)^2 + 4\lambda_k}$ or $dg_k = 0$. Because the solution corresponding to the positive root is positive, we deduce that for any $k \geq 1$ either $\rho = \lambda_k - (n-1) - \sqrt{(n-1)^2 + 4\lambda_k}$ or $dg_k = 0$. Since dg is not trivial, we finally obtain that $\rho = h(\lambda)$, where λ is an eigenvalue of Δ acting on $C^\infty(M)$. It is also clear that $\rho < 0$ implies $\lambda < 2(n+1)$. From (8) it follows that

$$(dg, f) = \sum_{h(\lambda_k) = \rho} \left(dg_k, \frac{2\lambda_k}{\lambda_k - \rho} g_k\right).$$

Also, there are at most two λ_k satisfying $h(\lambda_k) = \rho$. In Remark 1 we pointed out that this one occurs only when $n = 2$ and $-9/4 < \rho < -2$.

If $\rho = 0$, reasoning as before yields

$$f_k = 2g_k \quad \text{and} \quad (2(n+1) - \lambda_k)dg_k = 0 \quad \text{for any } k \geq 1.$$

Hence there are only two posibilities for (dg, f): either $(dg, f) = (dg, 2g)$ with $\Delta g + 2(n+1)g = 0$, or $(dg, f) = (0, a)$ with $a \in \mathbb{R}$. The proof of Proposition 1 is now complete. $\square$

This result allows us to determine the index and nullity of the quadratic form Q_2.

COROLLARY 1. *Let $\phi\colon M^n \to \mathbb{S}^{2n+1}$ be a minimal Legendrian immersion of a compact orientable manifold M. Then*

$$\mathrm{Ind}(Q_2) = \sum_{0<\lambda_k<2(n+1)} m_{\lambda_k}, \qquad \mathrm{Nul}(Q_2) = 1 + m_{2(n+1)},$$

where λ_k denotes the eigenvalues of Δ of M acting on $C^\infty(M)$ and m_{λ_k} their respective multiplicities. Moreover:

1. $\mathrm{Ind}(Q_2) \geq n+1$, *and the equality holds if and only if ϕ is totally geodesic*;
2. $\mathrm{Nul}(Q_2) \geq (n+1)(n+2)/2$, *and the equality holds if and only if ϕ is totally geodesic*;
3. *if ϕ is not totally geodesic, then* $\mathrm{Ind}(Q_2) \geq 2n+2$.

Proof. The expressions of index and nullity of Q_2 are direct consequences of Proposition 1. Part 2 is exactly Theorem 1.2 in [LW]. To prove parts 1 and 3, we proceed as follows. For any $a \in \mathbb{C}^{n+1}$, let $f_a = \langle \phi, a\rangle$. Then $\Delta f_a + n f_a = 0$ and so $m_n \geq \dim V$, where $V = \{f_a \mid a \in C^{n+1}\}$. It is clear that $\dim V \leq 2n+2$. If $\dim V < 2n+2$, there exists a nonzero vector a in $\mathbb{C}^{n+1}$ such that $f_a = 0$. Deriving f_a with respect to a tangent vector v yields $0 = \langle v, a\rangle = \langle Jv, Ja\rangle$, so the normal component of Ja is $(Ja)^\perp = \langle Ja, J\phi\rangle J\phi = f_a J\phi = 0$. Hence the Hessian of f_{Ja} is given by

$$(\nabla^2 f_{Ja})(v, w) = \langle \sigma(v, w), Ja\rangle - \langle v, w\rangle \phi = -\langle v, w\rangle \phi.$$

A theorem of Obata [O] states that either M^n is isometric to a unit sphere or $f_{Ja} = 0$. In the first case, ϕ is totally geodesic and $\dim V = n+1$; in the second case we have $f_a = f_{Ja} = 0$, which implies that $a = 0$ and so contradicts the assumption. Hence it follows (a) that $\mathrm{Ind}(Q_2) \geq m_n \geq \dim V \geq n+1$, holding the equality if and only if ϕ is totally geodesic, and (b) that if ϕ is not totally geodesic then $\mathrm{Ind}(Q_2) \geq m_n \geq \dim V = 2n+2$. This completes the proof. $\square$

Taking into account (3), Theorem 1, and Corollary 1, we obtain the expressions of the index and the nullity of an orientable compact minimal Legendrian submanifold of $\mathbb{S}^{2n+1}$.

COROLLARY 2. *Let $\phi\colon M^n \to \mathbb{S}^{2n+1}$ be a minimal Legendrian immersion of a compact orientable manifold M. Then*

$$\mathrm{Ind}(\phi) = \beta_1(M) + \sum_{0<\lambda_k<2(n+1)} m_{\lambda_k} + \sum_{0<\mu_k<2(n-1)} n_{\mu_k},$$

$$\mathrm{Nul}(\phi) = 1 + m_{2(n+1)} + n_{2(n-1)},$$

where λ_k (resp., μ_k) are the eigenvalues of Δ of M acting on $C^\infty(M)$ (resp., on $\delta\Omega^2(M)$) and where m_{λ_k} (resp., n_{μ_k}) are their respective multiplicities.

Finally, we study the case in which the minimal Legendrian submanifold M is a compact orientable surface. Here the star (or duality) operator $*$ is an isomorphism from $\delta\Omega^2(M)$ onto $dC^\infty(M)$ commuting with Δ, which implies that the eigenvalues of Δ acting on $\delta\Omega^2(M)$ are the nonnull eigenvalues of Δ acting on $C^\infty(M)$ and with the same multiplicity. Hence we obtain the following result.

COROLLARY 3. *Let $\phi\colon M^2 \to \mathbb{S}^5$ be a minimal Legendrian immersion of a compact orientable surface M of genus g. Then*

$$\mathrm{Ind}(\phi) = 2g + 2\sum_{0<\lambda_k<2} m_{\lambda_k} + \sum_{2\le\lambda_k<6} m_{\lambda_k},$$
$$\mathrm{Nul}(\phi) = 1 + m_2 + m_6,$$

where λ_k denotes the eigenvalues of Δ of M acting on $C^\infty(M)$ and m_{λ_k} their respective multiplicities.

Consider the Legendrian immersion

$$\mathbb{R}^2 \to \mathbb{S}^5,$$
$$(x, y) \to \frac{1}{\sqrt{3}}(e^{ix}, e^{iy}, e^{-i(x+y)});$$

its first fundamental form is given by $g_{11} = g_{22} = 2/3$ and $g_{12} = 1/3$. It therefore induces an embedding $\phi\colon T \to \mathbb{S}^5$ from $T = \mathbb{R}^2/\Gamma$, where Γ is the lattice in $\mathbb{R}^2$ generated by $\{(1/\sqrt{2}, 1/\sqrt{6}); (0, \sqrt{2}/\sqrt{3})\}$. It is clear that the induced metric in T by ϕ is flat, that T is a minimal Legendrian surface, and that $\lambda_1(T) = 2$, $\lambda_2(T) = 6$, $m_2(T) = 6$, and $m_6(T) = 6$. This surface is usually called the *equilateral torus.* Projecting the immersion just described from $\mathbb{R}^2$ into $\mathbb{S}^5$ by the Hopf fibration $\Pi\colon \mathbb{S}^5 \to \mathbb{CP}^2$, we obtain an immersion from $\mathbb{R}^2$ into $\mathbb{CP}^2$ that defines the minimal Lagrangian embedding of the (generalized) Clifford torus. The equilateral torus is a 3-covering of the Clifford torus, and in this case 6 is the first nonnull eigenvalue of Δ acting on functions of the Clifford torus.

COROLLARY 4. *Let $\phi\colon M^2 \to \mathbb{S}^5$ be a minimal* (*non–totally geodesic*) *Legendrian immersion of a compact orientable surface M of genus g. Then*

$$\mathrm{Ind}(\phi) \ge 2 + \sum_{2\le\lambda_k<6} m_{\lambda_k},$$

where λ_k denotes the eigenvalues of Δ of M acting on $C^\infty(M)$ and m_{λ_k} their respective multiplicities. Moreover, the equality holds if and only if M is the equilateral torus.

Proof. If the genus g of the surface M is zero then $\psi = \Pi \circ \phi\colon M^2 \to \mathbb{CP}^2$ is a minimal Lagrangian immersion of a sphere, and Theorem 7 of [Y] says that ψ is totally geodesic. This implies that also ϕ is totally geodesic, which contradicts our assumptions. So $g \ge 1$ and the inequality follows. If the equality holds, then M^2 is a torus with $\lambda_1 = 2$. Using the main result in [EI], we obtain that M is the equilateral torus. $\square$

COROLLARY 5. *Let $\phi\colon M^2 \to \mathbb{S}^5$ be a minimal* (*non–totally geodesic*) *Legendrian immersion of a compact orientable surface M of genus g. Then*:

(a) $\mathrm{Ind}(\phi) \geq 8$, *and the equality holds if and only if M is the equilateral torus*; *and*

(b) $\mathrm{Nul}(\phi) \geq 13$, *and the equality holds if and only if M is the equilateral torus.*

Proof. From Corollary 4 and the proof of Corollary 1.3, it follows that $\mathrm{Ind}(\phi) \geq 2 + m_2 \geq 8$. If $\mathrm{Ind}(\phi) = 8$ then $g = 1$ and $\lambda_1 = 2$, and using again the main result in [EI] we obtain that M is the equilateral torus. Finally, if M^2 is the equilateral torus then $\mathrm{Ind}(\phi) = 8$. This proves part (a).

From the proof of Corollary 1.3 it follows that $m_2 \geq 6$. In order to estimate m_6, we proceed as in the proof of Theorem 1. We defined a linear map

$$F\colon \mathfrak{so}^+(6) \to V_6 = \{f \mid \Delta f + 6f = 0\},$$
$$A \mapsto f_A = \langle A\phi, J\phi\rangle,$$

and hence $m_6 \geq \dim \mathrm{img}\, F = 8 - \dim \mathrm{Ker}\, F$. Now, if $A \in \mathrm{Ker}\, F$ then $f_A = 0$ and so $\nabla f_A = 0$, which means that $A\phi$ is tangent to M. Hence the Killing field X_A on $\mathbb{S}^{2n+1}$ is tangent to M and so X_A is also a Killing field on M. This means that we can define a linear map

$$G\colon \mathrm{Ker}\, F \to \{\text{Killing fields on } M\},$$
$$A \mapsto X_A,$$

which is a monomorphism. Because ϕ is not totally geodesic (and so not isometric to a sphere), it follows that the dimension of the isometry group of M is not greater than 2, which in turn implies that $\dim \mathrm{Ker}\, F \leq 2$. A previous inequality states that $m_6 \geq 6$. Now Corollary 3 together with the estimation of m_2 implies that $\mathrm{Nul}\,\phi \geq 13$.

If $\mathrm{Nul}\,\phi = 13$ then (in particular) $m_6 = 6$ and the dimension of the isometry group of M is 2. This implies that the genus g of M is either 0 or 1. If $g = 0$, the result mentioned in the proof of Corollary 4 says that ϕ is totally geodesic, which contradicts the assumption. This means that M is a torus with two linear independent Killing vector fields. Hence the torus M is flat and so $\Pi \circ \phi\colon M \to \mathbb{CP}^2$ is a flat minimal Lagrangian torus. Therefore, $\Pi \circ \phi$ is a finite Riemannian covering of the Clifford torus. Thus $\phi\colon M \to \mathbb{S}^5$ is a finite Riemannian covering of the equilateral torus, but only among the finite Riemannian coverings of the equilateral torus does it satisfy $m_2 = 6$ and $m_6 = 6$. This finishes the proof. □

References

[E1] N. Ejiri, *Totally real minimal immersions of n-dimensional real space forms into n-dimensional complex space forms,* Proc. Amer. Math. Soc. 84 (1982), 243–246.

[E2] ———, *The index of minimal immersions of* $\mathbb{S}^2$ *into* $\mathbb{S}^{2n}$, Math. Z. 184 (1983), 127–132.

[EI] A. El Soufi and S. Ilias, *Riemannian manifolds admitting isometric immersions by their first eigenfunctions,* Pacific J. Math. 195 (2000), 91–99.

[IT] A. Ikeda and Y. Taniguchi, *Spectra and eigenforms of the Laplacian on* $\mathbb{S}^n$ *and* $\mathbb{P}^n(\mathbb{C})$, Osaka J. Math. 15 (1978), 515–546.

[J] D. Joyce, *Lectures on Calabi–Yau and special Lagrangian geometry,* math.DG/0108088 (2001).

[LW] H. Lê and G. Wang, *A characterization of minimal Legendrian submanifolds in* $\mathbb{S}^{2n+1}$, Compositio Math. 129 (2001), 87–93.

[MU] S. Montiel and F. Urbano, *Second variation of superminimal surfaces into self-dual Einstein four-manifolds,* Trans. Amer. Math. Soc. 349 (1997), 2253–2269.

[O] M. Obata, *Certain conditions for a Riemannian manifold to be isometric with a sphere,* J. Math. Soc. Japan 14 (1962), 333–340.

[Oh] Y. G. Oh, *Second variation and stabilities of minimal Lagrangian submanifolds in Kähler manifolds,* Invent. Math. 101 (1990), 501–519.

[R] A. Ros, *Spectral geometry of CR-minimal submanifolds in the complex projective space*, Kodai Math. J. 6 (1983), 88–99.

[S] J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. of Math. (2) 88 (1968), 62–105.

[U1] F. Urbano, *Minimal surfaces with low index in the three-dimensional sphere,* Proc. Amer. Math. Soc. 108 (1990), 989–992.

[U2] ———, *Index of Lagrangian submanifolds of* $\mathbb{CP}^n$ *and the laplacian of* 1*-forms,* Geom. Dedicata 48 (1993), 309–318.

[Y] S. T. Yau, *Submanifolds with constant mean curvature, I,* Amer. J. Math. 96 (1974), 346–366.

Departamento de Geometría y Topología
Universidad de Granada
18071 Granada
Spain

furbano@ugr.es