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Foliation by Graphs of CR Mappings and a Nonlinear Riemann–Hilbert Problem for Smoothly Bounded Domains

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1. Introduction

Let D be a bounded, smoothly bounded domain in $\mathbb{C}^\ell$, $\ell \geq 2$, and let M be a real C^∞ submanifold of $\partial D \times \mathbb{C}^m$, $m \geq 1$. We here address the question of when there exists a mapping f such that

(RH) $f : \bar{D} \to \mathbb{C}^m$ is continuous on $\bar{D}$ and analytic on D such that the graph of f over ∂D is contained in M.

A problem where one is required to find an f satisfying (RH) is often called a *Riemann–Hilbert* problem; Riemann proposed such a question for $\ell = m = 1$ in 1851. We shall refer to the problem of finding an f satisfying (RH) as the Riemann–Hilbert problem for M. If an f exists satisfying (RH) then we shall say that the Riemann–Hilbert problem for M is *solvable* and that f is a *solution*. For $z \in \partial D$, let $M_z \equiv \{w \in \mathbb{C}^m : (z, w) \in M\}$. We will say here that the Riemann–Hilbert problem (RH) is *linear* if, for every $z \in \partial D$, M_z is a real affine subspace of $\mathbb{C}^m$. We shall say that the Riemann–Hilbert problem (RH) is *nonlinear* if it is not linear.

For the case $\ell = 1$ we mention the references [B1; B2; Fo; HMa; S; Sh1; Sh2; V; We1–We5]. See [We5] for a useful survey and reference list. For $\ell \geq 2$, see [B2; B3; BD; D1; D2].

We will first address the following more general question. Let S be a C^∞ CR manifold in $\mathbb{C}^\ell$ (e.g., a real hypersurface in $\mathbb{C}^\ell$) and let M be a real C^∞ submanifold of $S \times \mathbb{C}^m$, $m \geq 1$. Let $z^0 \in S$ and U a neighborhood of z^0 in S. Does there exist a CR map $f : U \to \mathbb{C}^m$ whose graph in $U \times \mathbb{C}^m$ is contained in M? We shall establish conditions under which the answer to this question is "yes". For the case where S is a complex manifold, this question is addressed by theorems in [F1–F3; Kr; So1; So2]. We consider the case of more general S. Applying our result to the case where S is the boundary of an open set in $\mathbb{C}^\ell$ with C^∞ boundary (so S is a real hypersurface in $\mathbb{C}^\ell$), we shall establish conditions where the Riemann–Hilbert problem for M is solvable.

For the more general question of the previous paragraph, assumptions we shall make will imply that the set M is a CR manifold. Under conditions outlined in

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Theorem 1, we will prove the existence of a special involutive subbundle of the real tangent bundle to M that possesses certain null properties relative to the Levi form of M. The Frobenius theorem will guarantee the existence of integral submanifolds of this bundle. Locally, these integral submanifolds will turn out to be graphs of CR maps from open subsets of S to $\mathbb{C}^m$. In Theorem 2 and Theorem 3 we establish conditions guaranteeing the existence of graphs of CR maps, contained in M, that are defined on all of S. In Theorem 4 we show that (under some conditions) these graphs possess a particular extremal property. In Corollary 1 and Corollary 2, we will assume that S is the boundary of a bounded, C^∞-bounded open set in $\mathbb{C}^\ell$ and establish conditions where the CR maps found in Theorem 2 and Theorem 3 extend to be analytic on D. These maps are solutions to the Riemann–Hilbert problem for M. The conditions we establish for the solvability of (RH) are not always necessary; when they are satisfied, we obtain graphs with strong properties that we shall describe.

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2. Definitions and Notation

For the reader's reference, a list of commonly used notation is provided in Section 8.

In studying subsets of $\mathbb{C}^\ell \times \mathbb{C}^m$, we shall generally label points in $\mathbb{C}^\ell$ with $z = (z_1, z_2, \ldots, z_\ell)$ and those in $\mathbb{C}^m$ with $w = (w_1, w_2, \ldots, w_m)$. If $P(z, w)$ is a function on an open subset of $\mathbb{C}^\ell \times \mathbb{C}^m$, then let ∂P denote the 1-form $\sum_{i=1}^{\ell}(\partial P/\partial z_i)dz_i + \sum_{i=1}^{m}(\partial P/\partial w_i)dw_i$ and let $\partial_w P$ denote the 1-form

$$\partial_w P = \sum_{i=1}^{m} \frac{\partial P}{\partial w_i} dw_i. \tag{1}$$

Let $[X, Y]$ denote the Lie bracket of two vector fields X, Y, and let $\langle A, B\rangle$ denote the canonical action of A on B where, for some n: A is an n-cotangent (i.e., an element of the nth exterior algebra of the cotangent space) at a point on a manifold and B is an n-tangent at that point; see [Bo, p. 10]. For $i = 1, 2, \ldots, n$, if ϕ_i is a cotangent at a point and ψ_i is a tangent at the same point then $\langle \phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_n, \psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_n\rangle$ is the determinant of the matrix whose (i, j) component is $\langle \phi_i, \psi_j\rangle$. We use the same notation to denote the action of A on B, where A is an n-form on a manifold and B is a section of the nth exterior algebra of the tangent bundle, the action being defined pointwise. If B is a vector bundle on an open subset A of a C^∞ manifold, then

$$\Gamma(A, B) \text{ will denote the set of } C^\infty \text{ sections of } B \text{ over } A. \tag{2}$$

We shall also make use of the notion of *generic* CR manifold (see [BER, p. 9]) and CR manifold of (Bloom–Graham–Kohn) *finite type* τ (see [BER, pp. 17–18]).

If $\mathcal{M}$ is a C^∞ generic CR submanifold of $\mathbb{C}^n$, then let $T\mathcal{M}$ denote the real tangent bundle to $\mathcal{M}$, $\mathbb{C}T\mathcal{M}$ the complexified tangent bundle to $\mathcal{M}$, $\mathbb{C}T_x\mathcal{M}$ the fiber of $\mathbb{C}T\mathcal{M}$ over $x \in \mathcal{M}$, $\mathbb{C}T^{1,0}\mathcal{M}$ the (1, 0)-subbundle of $\mathbb{C}T\mathcal{M}$, and $\mathbb{C}T_x^{1,0}\mathcal{M}$ the fiber of $\mathbb{C}T^{1,0}\mathcal{M}$ over x. (We use similar notation for the $(0, 1)$-subbundle.) We shall make use of the Levi map of $\mathcal{M}$, and we follow the definition in [BER]: Let $P_x : \mathbb{C}T_x\mathcal{M} \to \mathbb{C}T_x\mathcal{M}/(\mathbb{C}T_x^{1,0}\mathcal{M} \oplus \mathbb{C}T_x^{0,1}\mathcal{M})$ be projection and let the Levi map at $x \in \mathcal{M}$ be

$$\begin{gathered}\mathcal{L}_x \colon \mathbb{C}T_x^{1,0}\mathcal{M} \times \mathbb{C}T_x^{1,0}\mathcal{M} \to \mathbb{C}T_x\mathcal{M}/(\mathbb{C}T_x^{1,0}\mathcal{M} \oplus \mathbb{C}T_x^{0,1}\mathcal{M}) \\ \mathcal{L}_x(X_x, Y_x) := \frac{1}{2i} P_x([X, \bar{Y}](x)),\end{gathered} \tag{3}$$

where X, Y are C^∞ (1, 0)-vector fields on $\mathcal{M}$ near x such that $X(x) = X_x$ and $Y(x) = Y_x$. As is well known, the Levi map at x does not depend on these smooth extensions X, Y; it depends only on X_x and Y_x, so we get a smooth bundle map $\mathcal{L}\colon \mathbb{C}T^{1,0}\mathcal{M} \times \mathbb{C}T^{1,0}\mathcal{M} \to \mathbb{C}T\mathcal{M}/(\mathbb{C}T^{1,0}\mathcal{M} \oplus \mathbb{C}T^{0,1}\mathcal{M})$. Suppose that $\mathcal{M}$ has defining functions ϕ_i, $i = 1, 2, \dots, d$. Note that $\mathcal{L}_x(X_x, Y_x) = 0$ if and only if $\langle \partial\phi_i, [X, \bar{Y}]\rangle(x) = 0$ for $i = 1$ to d and for X, Y as before. By Cartan's identity (see [Bo, p. 14, Lemma 3]), these equations are equivalent to the equations $\langle \bar{\partial}\partial\phi_i, X \wedge \bar{Y}\rangle(x) = 0$ for $i = 1$ to d.

We follow the definition of CR function given in [Bo]: A CR function on a $\mathcal{M}$ is a C^1 function u such that, for every section $X \in \Gamma(\mathcal{M}, \mathbb{C}T^{0,1}\mathcal{M})$, we have $Xu = 0$.

We assume that S satisfies the following properties.

(4) Let S be a generic C^∞ CR manifold in $\mathbb{C}^\ell$ ($\ell \geq 2$) of CR codimension $c < \ell$. Specifically, we assume that S has C^∞ real-valued defining functions $p_1, p_2, \dots, p_c$ defined in a neighborhood $\mathcal{N}$ of S (i.e., for $i = 1$ to c, $p_i : \mathcal{N} \to \mathbb{R}$ is C^∞, $S = \{z \in \mathcal{N} : p_i(z) = 0 \text{ for all } i = 1, 2, \dots, c\}$, and the 1-forms $\{\partial p_i(z)\}_{i=1}^c$ are linearly independent 1-forms over $\mathbb{C}$ for all $z \in S$). Let $\mathrm{H}^S = \mathbb{C}T^{1,0}S$ be the (1, 0)-tangent bundle to S with fibers H_z^S; that is,

$$\mathrm{H}_z^S = \left\{ \sum_{i=1}^{\ell} a_i \frac{\partial}{\partial z_i} \in \mathbb{C}T_z S : a_i \in \mathbb{C},\ i = 1, 2, \dots, c \right\}.$$

We assume M satisfies the following properties.

(5) Let d be an integer, $1 \leq d \leq m$, and let M be a C^∞ CR submanifold of $S \times \mathbb{C}^m$ of CR codimension $c + d$ in $\mathbb{C}^{\ell+m}$ such that given $M_z \equiv \{w \in \mathbb{C}^m : (z, w) \in M\}$ we have, for all $z \in S$, that M_z is a nonempty, generic, Levi nondegenerate CR submanifold of CR codimension d in $\mathbb{C}^m$. Specifically, let $\mathcal{U} \subset \mathbb{C}^{\ell+m}$ be an open set meeting $S \times \mathbb{C}^m$ and let $q_i : \mathcal{U} \to \mathbb{R}$ be C^∞ for $i = 1, 2, \dots, d$ such that, for any $(z^0, w^0) \in \mathcal{U}$, the 1-forms $\{\partial_w q_i(z^0, w^0)\}$ $(i = 1, 2, \dots, d)$ are linearly independent over $\mathbb{C}$. Let $M = \{(z, w) \in \mathcal{U} \cap (S \times \mathbb{C}^m) \mid q_1(z, w) = q_2(z, w) = \dots = q_d(z, w) = 0\}$ and let $\mathrm{H}^M \equiv \mathbb{C}T^{1,0}M$ be the (1, 0)-tangent bundle to M with fibers $\mathrm{H}^M_{(z,w)}$ for $(z, w) \in M$.

Now observe that, since M_z is generic, we must have $d \leq m$. The following correspondence is convenient:

$$\mathrm{H}^M_{(z,w)} \cong \left\{ \sum_{i=1}^{\ell} a_i \frac{\partial}{\partial z_i} + \sum_{i=1}^{m} b_i \frac{\partial}{\partial w_i} \in \mathrm{H}^S_z \oplus \mathbb{C}T^{1,0}_w \mathbb{C}^m \,\middle|\, \sum_{i=1}^{\ell} a_i \frac{\partial q_k}{\partial z_i}(z,w) + \sum_{i=1}^{m} b_i \frac{\partial q_k}{\partial w_i}(z,w) = 0,\ k = 1,2,\ldots,d \right\}.$$

Note that $\{p_i\}_{i=1}^c \cup \{q_i\}_{i=1}^d$ is a set of defining functions for M. Let

$$\pi : \mathbb{C}T\mathbb{C}^{\ell+m} \to \mathbb{C}T\mathbb{C}^{\ell} \tag{6}$$

be either projection or the restriction to the corresponding mapping from H^M to H^S. We will use $\pi_{(z,w)}$ to denote projection on the individual fibers: $\pi_{(z,w)}$: $\mathbb{C}T_{(z,w)}\mathbb{C}^{\ell+m} \to \mathbb{C}T_z\mathbb{C}^{\ell}$ and $\pi_{(z,w)} : \mathrm{H}^M_{(z,w)} \to \mathrm{H}^S_z$.

We shall investigate when there exists a C^∞ CR map $f : S \to \mathbb{C}^m$ such that $q_i(z, f(z)) = 0$ for $z \in S$ and $i = 1$ to d.

Before we state our main results, we need some further definitions. Let

$$V(z,w) = \{v_{zw} \in \mathrm{H}^M_{(z,w)} : \pi_{(z,w)}(v_{zw}) = 0\}; \tag{7}$$

that is, if $v_{zw} \in V(z,w)$ then v_{zw} has no terms involving $\partial/\partial z_i$ for $i = 1, 2, \ldots, \ell$. (This is the space of "vertical" (1, 0)-tangents to M at (z,w).) By (5), $V(z,w)$ has dimension $m-d$ for all $(z,w) \in M$. For any $(z^0, w^0) \in M$,

$$V(z^0,w^0) = \{v_{z^0w^0} \in \mathbb{C}T_{(z^0,w^0)}\mathbb{C}^{\ell+m} : \pi_{(z^0,w^0)}(v_{z^0w^0}) = 0$$

and

$$\langle \partial_w q_i(z^0,w^0), v_{z^0w^0}\rangle = 0,\ i = 1,2,\ldots,d\}.$$

Thus $V(z,w)$ is calculated by solving a system of linear equations whose coefficients are C^∞ functions of (z,w) and whose rank is constant for $(z,w) \in M$ near (z^0, w^0). We may thus calculate $m-d$ C^∞ vector fields v^i $(i = 1, 2, \ldots, m-d)$ near any point $(z^0, w^0) \in M$ such that, near (z^0, w^0), $\{v^i(z,w)\}_{i=1}^{m-d}$ forms a basis for $V(z,w)$. Thus we obtain a complex $(m-d)$-dimensional C^∞ (1, 0)-vector bundle V over M. Let $\bar{V}$ denote the bundle whose fiber at (z,w) is $\overline{V(z,w)}$, and let $\mathcal{L}^M$ denote the Levi map for M. Next, let

$$N(z,w) = \{n_{zw} \in \mathrm{H}^M_{(z,w)} : \mathcal{L}^M_{(z,w)}(n_{zw}, v_{zw}) = 0\ \forall v_{zw} \in V(z,w)\}. \tag{8}$$

Because V is a bundle, any v_{zw} is the value at (z,w) of some element of $\Gamma(M,V)$, so in (8) it is equivalent to demand that $\mathcal{L}^M_{(z,w)}(n_{zw}, v(z,w)) = 0$ for all $v \in \Gamma(M,V)$.

Under reasonable conditions, the $N(z,w)$ will have dimension independent of (z,w) and hence form a vector bundle N. The conjugate of $N(z,w)$, $\overline{N(z,w)}$, will also form a bundle $\bar{N}$.

3. Main Results

Our most general theorem is a local statement.

THEOREM 1. *Suppose that S and M satisfy* (4) *and* (5), *respectively; suppose also that, at every point of S, S is of* (*Bloom–Graham–Kohn*) *finite type τ. Then the following four conditions are equivalent.*

(I) *The dimension of* $N(z,w)$ *is* $\ell - c$ *for all* $(z,w) \in M$, *and for every* $k = 1$ *to* d *we have that*

$$\langle \partial p_1 \wedge \partial p_2 \wedge \partial p_3 \wedge \cdots \wedge \partial p_c \wedge \partial q_k, F_1 \wedge F_2 \wedge F_3 \wedge \cdots \wedge F_{c+1} \rangle = 0 \tag{9}$$

for all vector fields $F_1, F_2, \ldots, F_{c+1}$ *that are commutators of length ranging from* 2 *to* $\tau + 1$ *of vector fields in* $\Gamma(M, N)$ *and* $\Gamma(M, \bar{N})$. (*Note that the lengths of* $F_1, F_2, \ldots, F_{c+1}$ *may be different*.)

(II) *The dimension of* $N(z,w)$ *is* $\ell - c$ *for all* $(z,w) \in M$, *and there exists a unique self-conjugate involutive* C^∞ *bundle* $\mathcal{T}$ *with fiber* $\mathcal{T}(z,w)$ *for* $(z,w) \in M$ *such that* $N \oplus \bar{N} \subset \mathcal{T} \subset \mathbb{C}TM$ *and* $\pi_{(z,w)} \colon \mathcal{T}(z,w) \to \mathbb{C}T_z S$ *is an isomorphism for all* $(z,w) \in M$ (*and so the complex dimension of* $\mathcal{T}(z,w)$ *is* $2\ell - c$).

(III) *For every* $(z^0, w^0) \in M$, *there exists some neighborhood* U^M *of* (z^0, w^0) *in* M *such that* U^M *is foliated by graphs of* C^∞ *CR maps defined in a neighborhood* U *of* z^0 *in* S; *for such a map* f, $f \colon U \to \mathbb{C}^m$ *and* $q_k(z, f(z)) = 0$ *for all* $z \in U$ *and all* $k = 1$ *to* d. *Furthermore, for* $i = 1$ *to* d *let* ϕ_i *be the* 1*-form* $\partial_w q_i = \sum_{j=1}^m (\partial q_i / \partial w_j) dw_j$ *and let* ϕ *be the* d*-form* $\phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \cdots \wedge \phi_d$, *which we write as* $\sum_{1 \le i_1 < i_2 < i_3 < \cdots < i_d \le m} \phi_{i_1, i_2, \ldots, i_d}(z,w) dw_{i_1} \wedge dw_{i_2} \wedge dw_{i_3} \wedge \cdots \wedge dw_{i_d}$. *Then there exists a nonzero* C^∞ *function* $C \colon U \to \mathbb{C}$ *such that the* d*-form* $C(z)\phi(z, f(z))$ *has coefficients that are CR functions on* U.

(IV) *The dimension of* $N(z,w)$ *is* $\ell - c$ *for all* $(z,w) \in M$, *and for every* $k = 1$ *to* d *we have that*

$$\langle \partial p_1 \wedge \partial p_2 \wedge \partial p_3 \wedge \cdots \wedge \partial p_c \wedge \partial q_k, F_1 \wedge F_2 \wedge F_3 \wedge \cdots \wedge F_{c+1} \rangle = 0 \tag{10}$$

for all vector fields $F_1, F_2, \ldots, F_{c+1}$ *that are commutators of* arbitrary *length of vector fields in* $\Gamma(M, N)$ *and* $\Gamma(M, \bar{N})$. (*Note: the lengths of* $F_1, F_2, \ldots, F_{c+1}$ *may be different*.)

If the preceding conditions hold, then the graphs in (III) *are integral manifolds of* $\operatorname{Re} \mathcal{T}$ *and the* $(1,0)$*-tangent space to the graph of* f *at* $(z, f(z))$ *is* $N(z, f(z))$. *Suppose* $g \colon U \to \mathbb{C}^m$ *is a CR map such that* $q_k(z, g(z)) = 0$ *for all* $z \in U$ *and all* $k = 1$ *to* d *and such that* N (*restricted to the graph of* g) *is the* $(1,0)$*-tangent bundle to the graph of* g. *Then we must have* $f = g$ *for some such* f.

In the case where S is the boundary of a domain, we obtain solutions to (RH) as follows.

Corollary 1. *Suppose that* S *and* M *satisfy* (4) *and* (5), *respectively, and suppose that* S *is connected, simply connected, and of finite type* τ *at every point. Suppose also that, for every compact* $K \subset S$, $M_K \equiv \{(z,w) \in M : z \in K\}$ *is compact. Suppose that any of properties* (I)–(IV) *of Theorem 1 hold and that, in addition,* S *is the boundary of a bounded domain* D *in* $\mathbb{C}^\ell$, $\ell \ge 2$. *Then* M *is the disjoint union of graphs of CR maps that all extend to be continuous on* $\bar{D}$ *and analytic on* D; *these extensions are solutions to* (RH).

Other results specifying the significance of the special graphs arising in Corollary 1 will be discussed later. We now prove these theorems.

4. Proof of Theorem 1: Local Results

We begin with a study of properties of the spaces $V(z,w)$ and $N(z,w)$. Suppose we have fixed $z^0 \in S$ and choose C^∞ vector fields s_i $(i = 1, 2, \ldots, \ell - c)$ defined on an open set G containing z^0 such that, for $z \in G$,

$$\{s_i(z)\}_{i=1}^{\ell-c} \tag{11}$$

is a basis for H_z^S. Let

$$G^M \equiv M \cap (G \times \mathbb{C}^m). \tag{12}$$

PROPOSITION 1. *Assume that S and M satisfy* (4) *and* (5)*, respectively. For any $(z,w) \in M$ and any value of c, the projection $\pi_{(z,w)}$ of $N(z,w)$ to H_z^S is injective. Thus, for every $(z,w) \in M$, the complex dimension of $N(z,w)$ is less than or equal to $\ell - c$. If the complex dimension of $N(z,w)$ equals $\ell - c$ for every $(z,w) \in G^M$, then N is a complex C^∞ vector bundle of dimension $\ell - c$ over G^M and $\pi_{(z,w)} \colon N(z,w) \to \mathrm{H}_z^S$ is an isomorphism. The complex dimension of $N(z,w)$ is exactly $\ell - c$ in three cases*: (i) *when $d = 1$*; (ii) *when $d = m$*; *and* (iii) *when the zero sets of q_i $(i = 2, 3, \ldots, d)$ are Levi flat in $\mathbb{C}^{\ell+m}$ near M* (*i.e., the Levi form of those surfaces is totally degenerate.*)

Proof. If $n_{zw} \in N(z,w)$ satisfies the property that $\pi_{(z,w)}(n_{zw}) = 0$, then $n_{zw} \in V(z,w)$. It follows that

$$\langle \bar\partial\partial q_i(z,w), n_{zw} \wedge \bar v_{zw} \rangle = 0 \tag{13}$$

for all $v_{zw} \in V(z,w)$ and $i = 1$ to d. Note that $V(z,w)$ may be regarded as the $(1,0)$-tangent space at w to M_z, which is Levi nondegenerate by (5). Then (13) implies that n_{zw} is in the null space of the Levi form of M_z, so $n_{zw} = 0$. This proves that $\pi_{(z,w)} \colon N(z,w) \to \mathrm{H}_z^S$ is injective and that the complex dimension of $N(z,w)$ is less than or equal to the dimension of H_z^S, which is $\ell - c$. Now fix an arbitrary $(z^0, w^0) \in M$ and fix $a^0 = (a_1^0, a_2^0, \ldots, a_\ell^0) \in \mathbb{C}^\ell$ such that $\sum_{i=1}^{\ell} a_i^0 (\partial p_j/\partial z_i)(z^0) = 0$ for $j = 1$ to c. For $i = 1$ to ℓ, let a_i be a C^∞ complex function defined on S near z^0 such that $a_i(z^0) = a_i^0$ and $\sum_{i=1}^{\ell} a_i(z)(\partial p_j/\partial z_i)(z) = 0$ for $j = 1$ to c and z near $z^0 \in S$ (i.e., $A(z) \equiv \sum_{i=1}^{\ell} a_i(z)(\partial/\partial z_i) \in \mathrm{H}_z^S$). For $(z,w) \in M$ near (z^0, w^0), the necessary and sufficient conditions for the existence of a $\mathbb{C}^m$-valued mapping $b = (b_1, b_2, \ldots, b_m)$ on M such that $B(z,w) \equiv \sum_{i=1}^{\ell} a_i(z)(\partial/\partial z_i) + \sum_{i=1}^{m} b_i(z,w)(\partial/\partial w_i)$ belongs to $N(z,w)$ (and so $\pi_{(z,w)}(B(z,w)) = A(z)$) are:

$$\sum_{i=1}^{\ell} a_i(z) \frac{\partial q_k}{\partial z_i}(z,w) + \sum_{i=1}^{m} b_i(z,w) \frac{\partial q_k}{\partial w_i}(z,w) = 0 \tag{14}$$

for $k = 1$ to d; and

$$\left\langle \bar\partial\partial p_j, \left(\sum_{i=1}^{\ell} a_i \frac{\partial}{\partial z_i} + \sum_{i=1}^{m} b_i \frac{\partial}{\partial w_i} \right) \wedge \left(\sum_{i=1}^{m} \bar v_i \frac{\partial}{\partial \bar w_i} \right) \right\rangle (z,w) = 0 \tag{15}$$

and

$$\left\langle \bar{\partial}\partial q_k, \left(\sum_{i=1}^{\ell} a_i \frac{\partial}{\partial z_i} + \sum_{i=1}^{m} b_i \frac{\partial}{\partial w_i}\right) \wedge \left(\sum_{i=1}^{m} \bar{v}_i \frac{\partial}{\partial \bar{w}_i}\right)\right\rangle(z, w) = 0 \tag{16}$$

for $j = 1$ to c, $k = 1$ to d, and for all vertical (1, 0)-vector fields $\sum_{i=1}^{m} v_i(\partial/\partial w_i) \in \Gamma(M, V)$. Condition (15) is vacuous: $\bar{\partial}\partial p_j$ contains only terms involving $d\bar{z}_{i_1} \wedge dz_{i_2}$ but $\sum_{i=1}^{m} \bar{v}_i(\partial/\partial \bar{w}_i)$ contains no terms with $\partial/\partial \bar{z}_i$ or $\partial/\partial z_i$, so the left side of (15) is automatically zero.

Thus conditions (14) and (16) impose the requirement that $b_1(z, w), b_2(z, w), \ldots, b_m(z, w)$ must satisfy a nonhomogeneous system of linear equations. By (5), the dimension of $V(z, w)$ is $m-d$ for all $(z, w) \in M$. For a small open set $U \subset G^M$ containing (z^0, w^0), we may choose elements $v^j = \sum_{i=1}^{m} v_i^j(\partial/\partial w_i) \in \Gamma(U, V)$, $j = 1, 2, \ldots, m-d$, such that $\{v^j(z, w)\}_{j=1}^{m-d}$ is a basis for $V(z, w)$ for all $(z, w) \in U$. For equations (14) and (16) to hold for $(z, w) \in U$, it is equivalent for the following system of equations to hold for $(z, w) \in U$:

$$\sum_{i=1}^{\ell} a_i(z) \frac{\partial q_k}{\partial z_i}(z, w) + \sum_{i=1}^{m} b_i(z, w) \frac{\partial q_k}{\partial w_i}(z, w) = 0 \tag{17}$$

for $k = 1$ to d; and

$$\left\langle \bar{\partial}\partial q_k, \left(\sum_{i=1}^{\ell} a_i \frac{\partial}{\partial z_i} + \sum_{i=1}^{m} b_i \frac{\partial}{\partial w_i}\right) \wedge \left(\sum_{i=1}^{m} \bar{v}_i^j \frac{\partial}{\partial \bar{w}_i}\right)\right\rangle(z, w) = 0 \tag{18}$$

for $k = 1$ to d and $j = 1$ to $m - d$.

Our system has d equations from (17) and $d(m - d)$ equations from (18) (not necessarily independent), so it is one of $d + d(m - d)$ equations in m unknowns $b_1(z, w), \ldots, b_m(z, w)$. (Note that if $d = 1$ or $d = m$ then $d + d(m - d) = m$.)

Suppose that $N(z, w)$ has complex dimension $\ell - c$ for all $(z, w) \in G^M$. Since $\pi_{(z,w)} \colon N(z, w) \to \mathrm{H}_z^S$ is an injection and H_z^S also has complex dimension $\ell - c$, it follows that $\pi_{(z,w)} \colon N(z, w) \to \mathrm{H}_z^S$ is an isomorphism. Thus, given any element $A \in \Gamma(G, \mathrm{H}^S)$, there exists a unique $B \in \Gamma(G^M, \mathrm{H}^M)$ such that $\pi_{(z,w)}(B(z, w)) = A(z)$. That means that, for $(z, w) \in G^M$, the system (17), (18) has precisely one solution for $b(z, w)$ given a fixed $A(z)$, so there exists a subsystem of m linear equations in the $b_j(z, w)$ whose solution is the same as for (17), (18) for (z, w) near some point $(z^1, w^1) \in G^M$. This implies that the b_j are all C^∞ functions near (z^1, w^1), since they are the unique solution to a system of m equations in m unknowns with coefficients that are C^∞ in (z, w). This holds near an arbitrary $(z^1, w^1) \in G^M$, so B is C^∞ (and we write $B \in \Gamma(G^M, \mathrm{H}^M)$). Let n_i be the unique C^∞ (1, 0)-vector field defined on G^M such that $n_i(z, w) \in N(z, w)$ for $(z, w) \in G^M$ and

$$\pi_{(z,w)}(n_i(z, w)) = s_i(z). \tag{19}$$

Then $\{n_i(z, w)\}$ is a basis for $N(z, w)$ for all $(z, w) \in G^M$ because $\pi_{(z,w)}\colon N(z, w) \to \mathrm{H}_z^S$ is an isomorphism. We conclude that the $N(z, w)$ form a C^∞ bundle over G^M; since sets such as G^M cover M, we have a bundle on all of M that we call N.

The system of equations associated to (17), (18) that is homogeneous in the $b_i(z,w)$ (i.e., $A(z)=0$) is

$$\sum_{i=1}^{m} b_i(z,w)\frac{\partial q_k}{\partial w_i}(z,w)=0 \tag{17$'$}$$

for $k=1$ to d and

$$\left\langle \bar{\partial}\partial q_k, \left(\sum_{i=1}^{m} b_i \frac{\partial}{\partial w_i}\right)\wedge\left(\sum_{i=1}^{m} \bar{v}_i^{\,j}\frac{\partial}{\partial \bar{w}_i}\right)\right\rangle(z,w)=0 \tag{18$'$}$$

for $k=1$ to d and $j=1$ to $m-d$. As noted previously, $b(z,w)=0$ is the unique solution to the homogeneous system because M_z is Levi nondegenerate for all $z\in S$. If $d=1$ or $d=m$ then the number of equations in the b_j is $d+d(m-d)=m$, so then the nonhomogeneous system (17), (18) has exactly one solution for the $b_j(z,w)$, $j=1,2,\dots,m$. Then what we have just shown is that, for $d=1$ or m and $(z,w)\in M$, every element $A(z)$ in H_z^S is the image under $\pi_{(z,w)}$ of exactly one element $B(z,w)$ in $N(z,w)$; that is, the projection $\pi_{(z,w)}\colon N(z,w)\to \mathrm{H}_z^S$ is bijective and the complex dimension of $N(z,w)$ is equal to $\ell-c$. That proves (i) and (ii).

In order to prove (iii), we must determine the number of independent equations in the b_i arising from (17) and (18). From (17) we obtain d of them (one for each q_i). From (18) we obtain $m-d$ equations arising from the second-order equations involving q_1, since the rank of V is $m-d$; all other second-order equations are vacuous by the Levi flatness associated with the other q_i. Thus we have a total of m equations in $(b_1,b_2,\dots,b_m)$; since (as before) the associated homogeneous system (17$'$), (18$'$) has exactly one solution (by the Levi nondegeneracy of M_z again), the nonhomogeneous system (17), (18) does also. Reasoning as in the end of the argument for (i) and (ii), we conclude that the dimension of $N(z,w)$ is $\ell-c$ for all $(z,w)\in M$. □

If B is a vector bundle over a manifold A, then we say that a set of vector fields $\{L_i\}_{i\in I}$ is a *local basis* for B near a point in A if, for all x in A near that point, $\{L_i(x)\}_{i\in I}$ is a basis for the fiber B_x of B over x.

Proposition 2 will establish conditions where the spaces $N(z,w)$ together compose an involutive bundle over M. However, we first need the following lemma. We say that a vector field is a commutator of length $\sigma\geq 2$ if it has the form $[Y_1,[Y_2,[Y_3,\dots,[Y_{\sigma-1},Y_\sigma]]]\dots]$ for vector fields Y_i, $i=1,2,3,\dots,\sigma$.

LEMMA 1. *Let T be any commutator of the vector fields $n_1,n_2,\dots,n_{\ell-c},\bar{n}_1,\bar{n}_2,\dots,\bar{n}_{\ell-c}$. Then, for all $v_1,v_2\in\Gamma(G^M,V)$,*

$$[T, v_1+\bar{v}_2]\in\Gamma(G^M, V\oplus\bar{V}). \tag{20}$$

(*Furthermore,* (20) *holds if T is a linear combination of commutators of the $n_i,\bar{n}_i$.*)

Proof. The parenthetical sentence of Lemma 1 follows from the second sentence because (20) is linear in T. We prove the second sentence by induction on the length of T. If the length of T is 1 then $T = n_i$ or $T = \bar{n}_i$ for some $i = 1, 2, \dots, \ell - c$. If $T = n_i$ then

$$[n_i, v_1 + \bar{v}_2] = [n_i, v_1] + [n_i, \bar{v}_2]. \tag{21}$$

The first term on the right-hand side of (21) belongs to $\Gamma(G^M, \mathrm{H}^M)$ by involutivity of H^M. By (19), the coefficients of n_i in $\partial/\partial z_j$ depend only on $z \in S$ for $j =$ 1 to ℓ, so v_1 annihilates these coefficients. Thus $[n_i, v_1]$ has no terms involving $\partial/\partial z_j$, so $\pi_{(z,w)}([n_i, v_1](z, w)) = 0$ and $[n_i, v_1] \in \Gamma(G^M, V)$ by definition of V. The second term on the right-hand side of (21) belongs to $\Gamma(G^M, \mathrm{H}^M \oplus \bar{\mathrm{H}}^M)$ by definition of N. Write $[n_i, \bar{v}_2] = h^1 + \bar{h}^2$ for $h^j \in \Gamma(G^M, \mathrm{H}^M)$, $j = 1, 2$. For the same reason as with the first term of (21), $\pi_{(z,w)}([n_i, \bar{v}_2](z, w)) = 0$; that is, $\pi_{(z,w)}(h^1(z, w) + \bar{h}^2(z, w)) = 0$. Thus $\pi_{(z,w)}(h^1(z, w))$ and $\pi_{(z,w)}(\bar{h}^2(z, w))$ are negatives of each other, but the former belongs to H^S_z and the latter to $\bar{\mathrm{H}}^S_z$. Since these two spaces meet only in $\{0\}$, we have $\pi_{(z,w)}(h^i(z, w)) = 0$ for $i = 1, 2$. Thus $h^1(z, w)$ and $h^2(z, w)$ belong to $V(z, w)$, so $[n_i, \bar{v}_2] \in \Gamma(G^M, V \oplus \bar{V})$ as desired. We conclude that the right-hand side of (21) belongs to $\Gamma(G^M, V \oplus \bar{V})$. If $T = \bar{n}_i$ then $[T, v_1 + \bar{v}_2] = [\bar{n}_i, v_1 + \bar{v}_2] = \overline{[n_i, \bar{v}_1 + v_2]}$, which belongs to $\Gamma(G^M, V \oplus \bar{V})$ by what we just showed in the case $T = n_i$. This proves the lemma if the length of T is 1.

For the remainder of the proof, the commutators referred to as T, T', T'' will be commutators of vector fields in the set $\{n_i, \bar{n}_i : i = 1, 2, \dots, \ell - c\}$. Now suppose that Lemma 1 is true for all commutators T of length less than or equal to λ. Then we must show that if T is a commutator of length $\lambda + 1$, it satisfies (20). Then $T = [n_i, T']$ or $T = [\bar{n}_i, T']$ for some commutator T' of length λ. We claim that it suffices to prove (20) for all T of the form $[n_i, T']$, where T' is a commutator of length λ; once this is done, if we write $T'' = [\bar{n}_i, T']$, then $[T'', v_1 + \bar{v}_2]$ $= [[\bar{n}_i, T'], v_1 + \bar{v}_2] = \overline{[[n_i, \bar{T}'], \bar{v}_1 + v_2]}$. The expression $[[n_i, \bar{T}'], \bar{v}_1 + v_2]$ belongs to $\Gamma(G^M, V \oplus \bar{V})$ because $\bar{T}'$ is a commutator of length λ. Thus $[T'', v_1 + \bar{v}_2] = \overline{[[n_i, \bar{T}'], \bar{v}_1 + v_2]}$ also belongs to $\Gamma(G^M, V \oplus \bar{V})$.

Now assume that $T = [n_i, T']$ for some commutator T' of length λ. Then, using Jacobi's identity,

$$\begin{aligned}[T, v_1 + \bar{v}_2] &= [[n_i, T'], v_1 + \bar{v}_2] \\ &= -[[T', v_1 + \bar{v}_2], n_i] - [[v_1 + \bar{v}_2, n_i], T'] \\ &= [n_i, [T', v_1 + \bar{v}_2]] - [T', [n_i, v_1 + \bar{v}_2]]. \end{aligned} \tag{22}$$

The vector fields $[T', v_1 + \bar{v}_2]$ and $[n_i, v_1 + \bar{v}_2]$ both belong to $\Gamma(G^M, V \oplus \bar{V})$ by the induction hypothesis. Hence (22) does as well, again by applying the induction hypothesis to each term of (22). This implies that $[T, v_1 + \bar{v}_2] \in \Gamma(G^M, V \oplus \bar{V})$ also, so T satisfies (20). By induction, the second sentence of Lemma 1 is proven. □

PROPOSITION 2. *Assume that S and M satisfy* (4) *and* (5), *respectively. If $N(z, w)$ has dimension $\ell - c$ for all $(z, w) \in M$, then N is an involutive C^∞ subbundle of* H^M.

Proof. By Proposition 1, N is a C^∞ bundle. To see that N is involutive, it suffices to verify this in a neighborhood of each point $(z^0, w^0) \in M$. Furthermore, it suffices to check the condition of involutivity on the local basis $\{n_i\}$ from (19): we must show that $[n_i, n_j](z, w) \in N(z, w)$ for all (z, w) near (z^0, w^0) in M. Let $v_{(z^0,w^0)} \in V(z^0, w^0)$ and $v \in \Gamma(M, V)$ be such that $v(z^0, w^0) = v_{(z^0,w^0)}$. Then, by Lemma 1, $[[n_i, n_j], \bar{v}] \in \Gamma(G^M, V \oplus \bar{V}) \subset \Gamma(G^M, \mathrm{H}^M \oplus \bar{\mathrm{H}}^M)$. We already know that H^M is involutive, so $[n_i, n_j](z, w) \in \mathrm{H}^M_{(z,w)}$. Thus $\mathcal{L}^M_{(z,w)}([n_i, n_j](z, w), v_{(z,w)}) = 0$ for all $v_{(z,w)} \in V(z, w)$ and so, by the definition of N, we have $[n_i, n_j](z, w) \in N(z, w)$. Thus N is involutive. $\square$

Note that Proposition 1 implies $V(z, w) \cap N(z, w) = \{0\}$. If the dimension of $N(z, w)$ is $\ell - c$, it also implies that $V(z, w) \oplus N(z, w) = \mathrm{H}^M_{(z,w)}$: we know that the dimension of $V(z, w)$ is $m - d$ and the dimension of $\mathrm{H}^M_{(z,w)}$ is $(\ell + m) - (c + d)$. Thus the dimension of $\mathrm{H}^M_{(z,w)}$ is the sum of the dimensions of $V(z, w)$ and $N(z, w)$; since $V(z, w) \cap N(z, w) = \{0\}$, we must have $V(z, w) + N(z, w) = \mathrm{H}^M_{(z,w)}$ also.

Our intention is to construct a map $f : S \to \mathbb{C}^m$ whose graph lies in M and passes through a point $(z^0, w^0) \in M$ such that the (1, 0)-tangent space to the graph of f at $(z, f(z))$ is $N(z, f(z))$. This will imply that f is a CR mapping on S; see the proof of Theorem 1. We say that a real C^∞ manifold H of real dimension a is *foliated* by a class of submanifolds $\mathcal{C}$ of real dimension b near a point $y \in H$ if there exists a C^∞ diffeomorphism from $Q \equiv \{(x_1, x_2, \dots, x_a) \in \mathbb{R}^a : 0 < x_i < 1, i = 1, 2, \dots, a\}$ onto a neighborhood of y in H such that, for any constants c_i between 0 and 1 ($i = b+1, b+2, \dots, a$), the image of the set $\{(x_1, x_2, \dots, x_a) \in Q : x_i = c_i, i = b+1, b+2, \dots, a\}$ is an open subset of some member of $\mathcal{C}$. We shall say that H is foliated by the manifolds in $\mathcal{C}$ if H is foliated by $\mathcal{C}$ near each of its points.

Lemma 2 will be needed in part of the proof of Theorem 1.

LEMMA 2. *Suppose that $\phi_1, \phi_2, \dots, \phi_c, \phi_{c+1}$ are elements of the cotangent space $\mathbb{C}T_0^*(\mathbb{R}^c \times \mathbb{C}^k)$ and that $\langle \phi_i, T\rangle = 0$ for all tangent vectors $T \in \mathbb{C}T_0(\{0\} \times \mathbb{C}^k)$. Then $\phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \cdots \wedge \phi_{c+1} = 0$.*

Proof. Assume that $\mathbb{R}^c$ has coordinates $x_1, x_2, \dots, x_c$ and $\mathbb{C}^k$ has coordinates $\zeta_1, \zeta_2, \dots, \zeta_k$. Then we may write $\phi_j = \sum_{i=1}^c \alpha_i^j dx_i + \sum_{i=1}^k \beta_i^j d\zeta_i + \sum_{i=1}^k \gamma_i^j d\bar{\zeta}_i$. Since $\langle \phi_i, T\rangle = 0$ for all tangent vectors $T \in \mathbb{C}T_0(\{0\} \times \mathbb{C}^k)$, we have $\beta_i^j = \gamma_i^j = 0$ for all $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, c + 1$. Thus the $\phi_j = \sum_{i=1}^c \alpha_i^j dx_i$ may be regarded as elements of $\mathbb{C}T_0^*\mathbb{R}^c$ for $j = 1$ to $c + 1$; taking the wedge product of all $c + 1$ of them is identically zero, since $c + 1 > c$. (The wedge product of linearly dependent vectors is zero, and any $c + 1$ vectors in a c-dimensional space are linearly dependent.) $\square$

Before proving Theorem 1, we make a simple observation.

NOTE. Since from (5) the $\partial_w q_i(\cdot, \cdot)$ are linearly independent over $\mathbb{C}$ at every point of M, we have that the form $\phi(z, w)$ is never zero. If $m = d$ then $\phi(z, w)$ is of the form $\phi_{1,2,3,\dots,m}(z, w) dw_1 \wedge dw_2 \wedge \cdots \wedge dw_m$. In this case, the property that

$\phi(z, w)$ satisfies is always automatic: let $C(z) = 1/\phi_{1,2,3,\ldots,m}(z, f(z))$. Thus, in the case $d = m$, condition (III) is merely a statement that M is locally foliated by graphs of CR mappings.

Proof of Theorem 1. It is obvious that (IV) implies (I). We shall prove (I) $\Rightarrow$ (II) $\Rightarrow$ (III) $\Rightarrow$ (IV). Before proving any of these implications, we note that in each of them we shall obtain the fact that, for every $(z, w) \in M$, the complex dimension of $N(z, w)$ is $\ell - c$ and so $\pi_{(z,w)} \colon N(z, w) \to \mathrm{H}_z^S$ is a (complex) linear isomorphism. We have this fact as an assumption in the first two implications, and in the third we shall prove it. (We already know this mapping is injective from Proposition 1; if the dimension of $N(z, w)$ is $\ell - c$, then $\pi_{(z,w)} \colon N(z, w) \to \mathrm{H}_z^S$ is also surjective since the complex dimension of H_z^S is $\ell - c$.) In the proof of each implication, we shall make use of the fact that the dimension of $N(z, w)$ is $\ell - c$ in order to construct a set of vector fields as follows. Take $(z, w) \in M$ and let $\mathcal{T}(z, w)$ be the complex vector space generated by the set of values $T(z, w)$, where T is a commutator of vector fields in $\Gamma(M, N)$ and $\Gamma(M, \bar{N})$ of length less than or equal to τ. (Recall that τ is the type of S; see the definitions early in Section 2.) We have that $\mathcal{T}(z, w)$ is the same as the complex vector space generated by the set of values $T(z, w)$, where T is instead a commutator of vector fields in $\Gamma(U^M, N)$ and $\Gamma(U^M, \bar{N})$ of length less than or equal to τ and where U^M is some open neighborhood of (z, w) in M. Given a point $z_0 \in S$, choose a neighborhood G of z^0 in S with $s_1, s_2, \ldots, s_{\ell-c} \in \Gamma(G, \mathrm{H}^S)$ such that $\{s_j(z)\}_{j=1}^{\ell-c}$ forms a basis for H_z^S for $z \in G$. Because S is of finite type τ at z^0, we may choose vector fields $t_1, t_2, \ldots, t_c$, which are linear combinations of commutators of the $s_i, \bar{s}_i$ such that the length of each term of each t_i is less than or equal to τ and such that

$$\{s_1, s_2, s_3, \ldots, s_{\ell-c}\} \cup \{\bar{s}_1, \bar{s}_2, \bar{s}_3, \ldots, \bar{s}_{\ell-c}\} \cup \{t_1, t_2, \ldots, t_c\} \tag{23}$$

constitutes a local basis for $\mathbb{C}TS$ near z^0. Furthermore, by appropriate change of basis we may assume that each t_i is a real tangent to S; we assume by shrinking G that they constitute a local basis in G. Let G^M be as in (12) and let $n_1, n_2, \ldots, n_{\ell-c}$ be the C^∞ vector fields in $\Gamma(G^M, \mathrm{H}^M)$ determined by (19). Then define T_i ($i = 1, 2, \ldots, c$) in the following manner. Recall that t_i is a linear combination of Lie brackets of vector fields in the set $\{s_j\}_{j=1}^{\ell-c} \cup \{\bar{s}_j\}_{j=1}^{\ell-c}$. For every appearance of an element s_j ($j = 1, 2, \ldots, \ell-c$) in that expression, replace it with n_j to form T_i. (For example, if $t_i = [s_1, \bar{s}_2] + [s_2, [s_3, \bar{s}_4]] + s_2$ then $T_i = [n_1, \bar{n}_2] + [n_2, [n_3, \bar{n}_4]] + n_2$.) Thus we have defined $T_i \in \Gamma(G^M, \mathbb{C}TM)$ and now claim that

$$\pi_{(z,w)}(T_i(z, w)) = t_i(z). \tag{24}$$

Suppose we select $r_1, r_2, \ldots$ in the set $\{s_j\}_{j=1}^{\ell-c} \cup \{\bar{s}_j\}_{j=1}^{\ell-c}$, and suppose that $R_1, R_2, \ldots$ are the corresponding elements in $\Gamma(G^M, N)$ and $\Gamma(G^M, \bar{N})$ such that $\pi_{(z,w)}(R_j(z, w)) = r_j(z)$ for all $(z, w) \in G^M$ and $j = 1, 2, 3, \ldots$ (see (19)). To prove (24), it will suffice to prove that

$$\pi_{(z,w)}([R_j, [R_{j-1}, [\ldots[R_2, R_1]]\ldots]](z, w)) = [r_j, [r_{j-1}, [\ldots[r_2, r_1]]\ldots]](z) \tag{25}$$

for $j \geq 1$ and $(z, w) \in G^M$, since the T_i are linear combinations of elements of the form $[R_j, [R_{j-1}, [\ldots[R_2, R_1]]\ldots]]$. We do this by induction on j. If $j = 1$

then (25) follows from the definition of the n_i in (19). Now assume (25) is true for commutators $R = [R_k, [R_{k-1}, [\dots[R_2, R_1]]\dots]]$ of length equal to k. Then $\pi_{(z,w)}(R(z,w))$ depends only on z by the induction hypothesis, so we let $r(z) = \pi_{(z,w)}(R(z,w))$. Let $R' = [R_{k+1}, R]$. By the induction hypothesis, the coefficients of R_{k+1} and R in $\partial/\partial z_\sigma$ $(\sigma = 1, 2, \dots, \ell)$ depend only on $z \in \mathbb{C}^\ell$. Thus, when calculating the Lie bracket of R_{k+1} and R, the $\partial/\partial w_j$ terms of each annihilate the $\partial/\partial z_\sigma$ coefficients of the other. Thus $\pi_{(z,w)}([R_{k+1}, R](z,w)) = [r_{k+1}, r](z)$, as desired, which implies that (25) holds for $j = k+1$. By induction, (25) holds for all $j = 1, 2, 3, \dots$ and so (24) holds as well. Since $\pi_{(z,w)}(n_i(z,w)) = s_i(z)$ and $\pi_{(z,w)}(\bar{n}_i(z,w)) = \bar{s}_i(z)$ for $i = 1$ to $\ell - c$ and since $\pi_{(z,w)}(T_i(z,w)) = t_i(z,w)$ for $i = 1$ to c, we find that $\pi_{(z,w)} \colon \mathcal{T}(z,w) \to \mathbb{C}T_z S$ is surjective. Moreover, since (23) is a local basis for $\mathbb{C}TS$ in G, the set of vectors

$$\{n_i(z,w)\}_{i=1}^{\ell-c} \cup \{\bar{n}_i(z,w)\}_{i=1}^{\ell-c} \cup \{T_i(z,w)\}_{i=1}^{c} \tag{26}$$

is linearly independent in $\mathbb{C}T_{(z,w)}M$ for all $(z,w) \in G^M$ as the inverse image under $\pi_{(z,w)}$ of the basis $\{s_i(z)\}_{i=1}^{\ell-c} \cup \{\bar{s}_i(z)\}_{i=1}^{\ell-c} \cup \{t_i(z)\}_{i=1}^{c}$ for $\mathbb{C}T_z S$.

Now we assume (I). There we simply assume that the dimension of $N(z,w)$ is $\ell - c$, so the vector fields $\{n_i\}_{i=1}^{\ell-c}$ and $\{T_i\}_{i=1}^{c}$ can be constructed. We shall show that the quotient of $\mathcal{T}(z,w)$ by $N(z,w) \oplus \bar{N}(z,w)$ has complex dimension c, which we recall is the CR codimension of S in $\mathbb{C}^\ell$. We will show that the quotient $\mathcal{T}(z,w)/(N(z,w) \oplus \bar{N}(z,w))$ is, in fact, generated by the image in the quotient of $\{T_i(z,w)\}_{i=1}^{c}$; this will show that the quotient has dimension c, as desired. The (surjective) mapping $\pi_{(z,w)} \colon \mathcal{T}(z,w) \to \mathbb{C}T_z S$ can be composed with the (surjective) quotient mapping from $\mathbb{C}T_z S \to \mathbb{C}T_z S/(\mathrm{H}_z^S \oplus \bar{\mathrm{H}}_z^S)$ to form another surjective mapping $\tilde{\pi}_{(z,w)} \colon \mathcal{T}(z,w) \to \mathbb{C}T_z S/(\mathrm{H}_z^S \oplus \bar{\mathrm{H}}_z^S)$. Since $\tilde{\pi}_{(z,w)}(n_i(z,w)) = s_i(z) + \mathrm{H}_z^S \oplus \bar{\mathrm{H}}_z^S = 0 + \mathrm{H}_z^S \oplus \bar{\mathrm{H}}_z^S$ and since (for similar reasons) $\tilde{\pi}_{(z,w)}(\bar{n}_i(z,w))$ is zero, we find that $\tilde{\pi}_{(z,w)}$ factors through the quotient $\mathcal{T}(z,w)/(N(z,w) \oplus \bar{N}(z,w))$ to form a surjective mapping from $\mathcal{T}(z,w)/(N(z,w) \oplus \bar{N}(z,w))$ to $\mathbb{C}T_z S/(\mathrm{H}_z^S \oplus \bar{\mathrm{H}}_z^S)$. Because the complex dimension of the latter space is c and the mapping is surjective, the dimension of $\mathcal{T}(z,w)/(N(z,w) \oplus \bar{N}(z,w))$ is greater than or equal to c. We proceed to show it does not exceed c.

LEMMA 3. *Assume that the conditions of Theorem 1 hold and that* (I) *holds. Let $T(z,w)$ be a commutator of vector fields in the set $\bigcup_{i=1}^{\ell-c}\{n_i, \bar{n}_i\}$. We claim that there exist C^∞ functions $a_i \colon G \to \mathbb{C}$ $(i = 1, 2, \dots, c)$ such that*

$$T + \sum_{i=1}^{c} a_i T_i \in \Gamma(G^M, N \oplus \bar{N}). \tag{27}$$

(Note that, by the definition of G^M in (12), we may regard a_i as a function on G^M as well.)

Proof. We first show that, for some functions a_i,

$$T + \sum_{i=1}^{c} a_i T_i \in \Gamma(G^M, \mathrm{H}^M \oplus \bar{\mathrm{H}}^M). \tag{28}$$

To do this, recall that the coefficients of n_i in $\partial/\partial z_k$ depend only on z for all $i = 1$ to $\ell - c$ and $k = 1$ to ℓ, so from (25) the same may be said of the coefficients of $T(z, w)$; thus $\pi_{(z,w)}(T(z, w))$ is a well-defined function of z, say $s(z)$. Because $s(z) \in \mathbb{C}T_z S$ and owing to the existence of the local basis (23), we may choose C^∞ functions $a_i : G \to \mathbb{C}$ $(i = 1, 2, \dots, c)$ such that $s(z)+\sum_{i=1}^{c} a_i(z)t_i(z) \in \mathrm{H}_z^S \oplus \bar{\mathrm{H}}_z^S$ for all $z \in G$. Then consider the vector field $T + \sum_{i=1}^{c} a_i T_i \in \Gamma(G^M, \mathbb{C}TM)$. We have that, for all $j = 1, 2, \dots, c$,

$$\begin{aligned}\Big\langle \partial p_j, T + \sum_{i=1}^{c} a_i T_i \Big\rangle(z, w) &= \Big\langle \partial p_j(z), \pi_{(z,w)}\Big(T(z, w) + \sum_{i=1}^{c} a_i(z)T_i(z, w)\Big)\Big\rangle \\ &= \Big\langle \partial p_j, s + \sum_{i=1}^{c} a_i t_i \Big\rangle(z) = 0 \qquad (29)\end{aligned}$$

for all $(z, w) \in G^M$, since p_j depends only on z and since $s(z)+\sum_{i=1}^{c} a_i(z)t_i(z) \in \mathrm{H}_z^S \oplus \bar{\mathrm{H}}_z^S$ for all $z \in G$.

Hence we claim also that

$$\Big\langle \partial q_k, T + \sum_{i=1}^{c} a_i T_i \Big\rangle(z, w) = 0 \qquad (30)$$

for $k = 1, 2, \dots, d$ and $(z, w) \in G^M$. We now use the definition of the action of a $(c + 1)$-form on an element of the $(c + 1)$th exterior algebra of the tangent space (see [Bo, p. 10]) to calculate

$$\Big\langle \partial p_1 \wedge \partial p_2 \wedge \partial p_3 \wedge \cdots \wedge \partial p_c \wedge \partial q_k, T_1 \wedge T_2 \wedge T_3 \wedge \cdots \wedge T_c \wedge \Big(T + \sum_{i=1}^{c} a_i T_i\Big)\Big\rangle(z, w)$$

for $(z, w) \in G^M$. Any term involving $\big\langle \partial p_i, T + \sum_{i=1}^{c} a_i T_i \big\rangle$ is zero by (29), so we have from (9) that

$$\begin{aligned}0 &= \Big\langle \partial p_1 \wedge \partial p_2 \wedge \partial p_3 \wedge \cdots \wedge \partial p_c \wedge \partial q_k, \\ &\qquad T_1 \wedge T_2 \wedge T_3 \wedge \cdots \wedge T_c \wedge \Big(T + \sum_{i=1}^{c} a_i T_i\Big)\Big\rangle(z, w) \\ &= \langle \partial p_1 \wedge \partial p_2 \wedge \partial p_3 \wedge \cdots \wedge \partial p_c, \\ &\qquad T_1 \wedge T_2 \wedge T_3 \wedge \cdots \wedge T_c\rangle \Big\langle \partial q_k, T + \sum_{i=1}^{c} a_i T_i \Big\rangle(z, w)\end{aligned}$$

for $(z, w) \in G^M$. We claim that $\langle \partial p_1 \wedge \partial p_2 \wedge \partial p_3 \wedge \cdots \wedge \partial p_c, T_1 \wedge T_2 \wedge T_3 \wedge \cdots \wedge T_c\rangle(z, w) \neq 0$ for $(z, w) \in G^M$; this will show that (30) holds, as desired. To see why the claim holds, note that $\langle \partial p_1 \wedge \partial p_2 \wedge \partial p_3 \wedge \cdots \wedge \partial p_c, T_1 \wedge T_2 \wedge T_3 \wedge \cdots \wedge T_c\rangle(z, w)$ is equal to the determinant of the $c \times c$ matrix whose (i, j) component is $\langle \partial p_i, T_j\rangle(z, w)$. If at some $(z^0, w^0) \in G^M$ this determinant is zero, then the columns are linearly dependent over $\mathbb{C}$; hence for some $\zeta_j \in \mathbb{C}$ (which are not all zero) and all $i = 1$ to c we have $\sum_{j=1}^{c} \zeta_j \langle \partial p_i, T_j\rangle(z^0, w^0) = 0$, so

$\left\langle \partial p_i, \sum_{j=1}^{c} \zeta_j T_j \right\rangle (z^0, w^0) = 0$ and $\left\langle \partial p_i, \sum_{j=1}^{c} \zeta_j t_j \right\rangle (z^0) = 0$ for $i = 1$ to c. By definition of $\mathrm{H}_{z^0}^S$ we have $\sum_{j=1}^{c} \zeta_j t_j(z^0) \in \mathrm{H}_{z^0}^S \oplus \bar{\mathrm{H}}_{z^0}^S$, so for some $a_j, b_j \in \mathbb{C}$ ($j = 1, 2, \ldots, \ell - c$) we obtain $\sum_{j=1}^{c} \zeta_j t_j(z^0) = \sum_{j=1}^{\ell-c} a_j s_j(z^0) + \sum_{j=1}^{\ell-c} b_j \bar{s}_j(z^0)$. Since the set $\{s_i(z^0)\}_{i=1}^{\ell-c} \cup \{\bar{s}_i(z^0)\}_{i=1}^{\ell-c} \cup \{t_i(z^0)\}_{i=1}^{c}$ is linearly independent, we find that $\zeta_j = 0$ for $j = 1$ to c. This contradicts the definition of the ζ_j, so the claim holds.

By (29) and (30) we conclude that (28) holds. We proceed to show that (27) also holds. To see this, we observe that there exists an $h \in \Gamma(G^M, \mathrm{H}^M)$ such that $T + \sum_{i=1}^{c} a_i T_i + \bar{h} \in \Gamma(G^M, \mathrm{H}^M)$. Then, for $v \in \Gamma(G^M, V)$, it follows that $\left[T + \sum_{i=1}^{c} a_i T_i + \bar{h}, \bar{v}\right] = \left[T + \sum_{i=1}^{c} a_i T_i, \bar{v}\right] + [\bar{h}, \bar{v}]$. The first of these last two terms belongs to $\Gamma(G^M, \mathrm{H}^M \oplus \bar{\mathrm{H}}^M)$ by Lemma 1, and the second belongs to $\Gamma(G^M, \mathrm{H}^M \oplus \bar{\mathrm{H}}^M)$ because $\bar{\mathrm{H}}^M$ is involutive.

Recalling that every element of $V(z, w)$ is the value at (z, w) of some such $v \in \Gamma(M, V)$ and recalling also the definition of N, we may write $\mathcal{L}_{(z,w)}^M\big(T(z, w) + \sum_{i=1}^{c} a_i(z) T_i(z, w) + \bar{h}(z, w), v_{zw}\big) = 0$ for all $v_{zw} \in V(z, w)$ and all $(z, w) \in M$. Thus we may write that $T + \sum_{i=1}^{c} a_i(z) T_i + \bar{h} = n' \in \Gamma(G^M, N)$. If we solve for h here then we can use a similar argument to show that $h \in \Gamma(G^M, N)$, so (27) holds and Lemma 3 is proven. $\square$

Lemma 3 shows that the quotient of $\mathcal{T}(z, w)$ by $N(z, w) \oplus \bar{N}(z, w)$ has complex dimension less than or equal to c. We already know the dimension is greater than or equal to c, so it is exactly c. The set in (26) is linearly independent for $(z, w) \in G^M$ (as observed earlier), and Lemma 3 shows that it spans $\mathcal{T}(z, w)$ for $(z, w) \in G^M$. Since the T_i are chosen smoothly in the neighborhood G, the set of vector fields

$$\{n_i\}_{i=1}^{\ell-c} \cup \{\bar{n}_i\}_{i=1}^{\ell-c} \cup \{T_i\}_{i=1}^{c}$$

is a local basis for $\mathcal{T}$ over G^M. Since the sets G^M cover M, it follows that the $\mathcal{T}(z, w)$ form a C^∞ bundle on M of complex dimension $2\ell - c$, which we call $\mathcal{T}$.

Next we show that $\mathcal{T}$ is involutive. Suppose we select an arbitrary $(z^0, w^0) \in M$, an open neighborhood U^M of (z^0, w^0) in M, and sections $R_1, R_2, \ldots, R_{2\ell-c}$ of $\Gamma(U^M, \mathcal{T})$ such that $\{R_i(z, w)\}_{i=1}^{2\ell-c}$ is a basis for $\mathcal{T}(z, w)$ for $(z, w) \in U^M$. Then, to show that $\mathcal{T}$ is involutive, it suffices to show that Lie brackets of the R_i belong to $\Gamma(U^M, \mathcal{T})$. In fact, we can let U^M be the G^M defined in (12) and let the set $\{R_i\}_{i=1}^{2\ell-c}$ be $\{n_i\}_{i=1}^{\ell-c} \cup \{\bar{n}_i\}_{i=1}^{\ell-c} \cup \{T_k\}_{k=1}^{c}$ in G^M. It will be enough to show that, for $(z, w) \in G^M$, we have that $[n_i, n_j](z, w)$, $[n_i, \bar{n}_j](z, w)$, $[T_k, n_i](z, w)$, and $[T_k, \bar{n}_i](z, w)$ all belong to $\mathcal{T}(z, w)$. These are all consequences of Lemma 3. Thus $\mathcal{T}$ is involutive. We have that $\mathcal{T}$ is self-conjugate because the local basis $\{n_i(z, w)\}_{i=1}^{\ell-c} \cup \{\bar{n}_i(z, w)\}_{i=1}^{\ell-c} \cup \{T_i(z, w)\}_{i=1}^{c}$ for $\mathcal{T}(z, w)$ is self-conjugate.

We need to show that $\pi_{(z,w)} \colon \mathcal{T}(z, w) \to \mathbb{C}T_z S$ is an isomorphism for every $(z, w) \in M$. Once again it will suffice to assume that $(z, w) \in G^M$, where G and G^M are as before. We have that $\pi_{(z,w)}(n_i(z, w)) = s_i(z)$ ($i = 1$ to $\ell - c$), $\pi_{(z,w)}(\bar{n}_i(z, w)) = \bar{s}_i(z)$ ($i = 1$ to $\ell - c$), and $\pi_{(z,w)}(T_i(z, w)) = t_i(z)$ ($i = 1$ to c). Since

$$\{n_i(z, w)\}_{i=1}^{\ell-c} \cup \{\bar{n}_i(z, w)\}_{i=1}^{\ell-c} \cup \{T_i(z, w)\}_{i=1}^{c}$$

and

$$\{s_i(z)\}_{i=1}^{\ell-c} \cup \{\bar{s}_i(z)\}_{i=1}^{\ell-c} \cup \{t_i(z)\}_{i=1}^{c}$$

are bases (respectively) for $\mathcal{T}(z,w)$ and $\mathbb{C}T_zS$, the map $\pi_{(z,w)}\colon \mathcal{T}(z,w) \to \mathbb{C}T_zS$ is indeed an isomorphism.

Now we prove the uniqueness statement of (II). Every involutive bundle $\mathcal{T}'$ on G^M between $N \oplus \bar{N}$ and $\mathbb{C}TM$ must contain $T_i(z,w)$ in its fiber over (z,w) (for $i = 1$ to c), since the T_i are linear combinations of commutators of the n_j, $\bar{n}_j$. Thus $\mathcal{T}'(z,w)$ must contain $\mathcal{T}(z,w)$. For $\pi_{(z,w)}\colon \mathcal{T}'(z,w) \to \mathbb{C}T_{(z,w)}S$ to be an isomorphism, $\mathcal{T}'(z,w)$ must have have complex dimension $2\ell - c$ and so must be no bigger than $\mathcal{T}$ over G^M, since the complex dimension of $\mathcal{T}$ is $2\ell - c$. This shows that $\mathcal{T}'(z,w) = \mathcal{T}(z,w)$ for $(z,w) \in G^M$. Because open sets such as G^M cover M, we must have $\mathcal{T}'(z,w) = \mathcal{T}(z,w)$ for $(z,w) \in M$; this proves the uniqueness statement of (II) and concludes the proof that (I) implies (II).

Now we show that (II) $\Rightarrow$ (III). We have that $\pi_{(z,w)}\colon N(z,w) \to \mathrm{H}_z^S$ is an isomorphism and the complex dimension of $N(z,w)$ is $\ell - c$. Fix $(z^0,w^0) \in M$ and let $\{s_i\}_{i=1}^{\ell-c}$, $\{n_i\}_{i=1}^{\ell-c}$, and the sets G and G^M be as defined in (11), (12), and (19). Because the dimension of $N(z,w)$ is $\ell - c$, we may define $\{t_i\}$ as in (23) and $\{T_i\}$ as in (24). Because $\pi_{(z,w)}\colon \mathcal{T}(z,w) \to \mathbb{C}T_zS$ is an isomorphism and $\{s_i(z)\}_{i=1}^{\ell-c} \cup \{\bar{s}_i(z)\}_{i=1}^{\ell-c} \cup \{t_i(z)\}_{i=1}^{c}$ is a basis for $\mathbb{C}T_zS$, the set $\{n_i(z,w)\}_{i=1}^{\ell-c} \cup \{\bar{n}_i(z,w)\}_{i=1}^{\ell-c} \cup \{T_i(z,w)\}_{i=1}^{c}$ is a basis for $\mathcal{T}(z,w)$ for $(z,w) \in G^M$ and G sufficiently small. Let us define $\operatorname{Re}\mathcal{T}(z,w)$ to be the vector space $\{R_{zw} + \overline{R_{zw}} \mid R_{zw} \in \mathcal{T}(z,w)\}$. We check that $\operatorname{Re}\mathcal{T}$ constitutes a real involutive vector bundle over M of real dimension $2\ell - c$. It will suffice to check that $\operatorname{Re}\mathcal{T}$ constitutes a real involutive vector bundle over G^M of real dimension $2\ell - c$ for every G^M defined previously, since such G^M cover M. Since $\{n_j(z,w) : j = 1,2,\ldots,\ell - c\} \cup \{\bar{n}_j(z,w) : j = 1,2,\ldots,\ell - c\} \cup \{T_j(z,w) : j = 1,2,\ldots,c\}$ is a complex basis for $\mathcal{T}(z,w)$ for $(z,w) \in G^M$ (recalling that the T_j are real vector fields), it follows that $\{n_j(z,w) + \bar{n}_j(z,w) : j = 1,2,\ldots,\ell - c\} \cup \{in_j(z,w) + \overline{in_j(z,w)} : j = 1,2,\ldots,\ell - c\} \cup \{T_j(z,w) : j = 1,2,\ldots,c\}$ is a real basis for $\operatorname{Re}\mathcal{T}(z,w)$, so the (real) dimension of $\operatorname{Re}\mathcal{T}(z,w)$ is $2\ell - c$ and $\operatorname{Re}\mathcal{T}$ is a bundle. Also, $\operatorname{Re}\mathcal{T}$ is involutive as the real part of an involutive bundle. By the Frobenius theorem (see [War]) M is foliated near $(z^0,w^0) \in M$ by C^∞ integral manifolds for $\operatorname{Re}\mathcal{T}$. The complexified tangent space to such a manifold at $(z,w) \in M$ must equal $\mathcal{T}(z,w)$.

We know that $\pi_{(z,w)}$ is injective on $\mathcal{T}(z,w)$; hence it is injective on $\operatorname{Re}\mathcal{T}(z,w)$ also. By the inverse function theorem, near (z^0,w^0) the integral manifolds of $\operatorname{Re}\mathcal{T}$ are graphs over a fixed open subset of S, so we write that a neighborhood of (z^0,w^0) is foliated by graphs of mappings on an open subset U of S where $z^0 \in U$. The complexified tangent bundles to these manifolds must equal $\mathcal{T}$. We use the following notation.

(31) Let f be the function on an open subset U of S such that $z^0 \in U$ and such that the graph of f is the integral manifold of $\operatorname{Re}\mathcal{T}$ through (z^0,w^0). Use M^f to denote the graph of f.

The (1, 0)-tangent space to M^f at $(z, f(z))$ includes $N(z, f(z))$ (since the whole complexified tangent space to M^f at $(z, f(z))$ is $\mathcal{T}(z, f(z))$). Any (1, 0)-tangent to M^f at $(z, f(z))$ projects to H_z^S; since projection is injective on $\mathcal{T}(z, f(z)) = \mathbb{C}T_{(z,f(z))}M^f$, that tangent must belong to $N(z, f(z))$. This shows that the (1, 0)-tangent space to M^f at $(z, f(z))$ is $N(z, f(z))$.

Then the differential of the mapping $F\colon U \to M^f$ such that $F(z) = (z, f(z))$ from U to M^f carries H^S to N: for $s'_z \in \mathrm{H}_z^S$, if $d_zF(s'_z)$ is not in $N(z, f(z))$ then $\pi_{(z,f(z))}(d_zF(s'_z)) = s'_z$ is not in H_z^S (contradiction), since $\pi_{(z,f(z))}$ maps $N(z, f(z))$ onto H_z^S and is injective on $\mathcal{T}(z, f(z))$. Thus F must be a CR mapping on U, so f is as well.

We now proceed to prove the property that (III) asserts for the d-form ϕ. Let $s \in \Gamma(U, \mathrm{H}^S)$. If K is a function on S, we write $s_z\{K\}$ to denote the action of s on K at $z \in S$. (Then we regard s_z as an element of H_z^S and identify $s(z)$ with s_z; we use $\bar{s}_z$ similarly.) For convenience we note that, since f is CR, it follows that

$$n(z) \equiv s(z) + \sum_{j=1}^{m} s_z\{f_j\}\frac{\partial}{\partial w_j} \tag{32}$$

belongs to $N(z, f(z))$ as the element of the (1, 0)-tangent space $N(z, f(z))$ to the graph of f at $(z, f(z))$ that projects to $s(z)$.

We note that the function $\phi_{i_1,i_2,\dots,i_d}(z, w)$ (defined in property (III)) is the determinant of the $d \times d$ matrix with (j, k) entry $(\partial q_j/\partial w_{i_k})(z, w)$. Since at every $(z, w) \in M$ the set $\{\partial_w q_i(z, w) : i = 1, 2, \dots, d\}$ is linearly independent (by (5)), $\phi(z, w)$ is nonzero for $(z, w) \in M$. Thus, for some integers $i_1, i_2, \dots, i_d$ ($1 \le i_1 < i_2 < \cdots < i_d \le m$), we have that $\phi_{i_1,i_2,\dots,i_d}(z^0, w^0) \neq 0$. We claim that in property (III) it suffices to take $C(z) = 1/\phi_{i_1,i_2,\dots,i_d}(z, f(z))$ for z in a possibly shrunken neighborhood U of $z^0 \in S$. Assume without loss of generality that C is defined on all of U (by shrinking U). Thus we will show that, for $\bar{s} \in \Gamma(U, \bar{\mathrm{H}}^S)$ and integers j_k, $1 \le j_1 < j_2 < j_3 < \cdots < j_d \le m$,

$$\bar{s}_z\left\{\frac{\begin{vmatrix} \frac{\partial q_1}{\partial w_{j_1}}(\cdot, f(\cdot)) & \frac{\partial q_1}{\partial w_{j_2}}(\cdot, f(\cdot)) & \cdots & \frac{\partial q_1}{\partial w_{j_d}}(\cdot, f(\cdot)) \\ \frac{\partial q_2}{\partial w_{j_1}}(\cdot, f(\cdot)) & \frac{\partial q_2}{\partial w_{j_2}}(\cdot, f(\cdot)) & \cdots & \frac{\partial q_2}{\partial w_{j_d}}(\cdot, f(\cdot)) \\ \vdots & & \vdots & \vdots \\ \frac{\partial q_d}{\partial w_{j_1}}(\cdot, f(\cdot)) & \frac{\partial q_d}{\partial w_{j_2}}(\cdot, f(\cdot)) & \cdots & \frac{\partial q_d}{\partial w_{j_d}}(\cdot, f(\cdot)) \end{vmatrix}}{\begin{vmatrix} \frac{\partial q_1}{\partial w_{i_1}}(\cdot, f(\cdot)) & \frac{\partial q_1}{\partial w_{i_2}}(\cdot, f(\cdot)) & \cdots & \frac{\partial q_1}{\partial w_{i_d}}(\cdot, f(\cdot)) \\ \frac{\partial q_2}{\partial w_{i_1}}(\cdot, f(\cdot)) & \frac{\partial q_2}{\partial w_{i_2}}(\cdot, f(\cdot)) & \cdots & \frac{\partial q_2}{\partial w_{i_d}}(\cdot, f(\cdot)) \\ \vdots & & \vdots & \vdots \\ \frac{\partial q_d}{\partial w_{i_1}}(\cdot, f(\cdot)) & \frac{\partial q_d}{\partial w_{i_2}}(\cdot, f(\cdot)) & \cdots & \frac{\partial q_d}{\partial w_{i_d}}(\cdot, f(\cdot)) \end{vmatrix}}\right\} \tag{33}$$

is identically zero for $z \in U$. Let $A(z)$ be the matrix whose (σ, k) entry is $(\partial q_\sigma/\partial w_{j_k})(z, f(z))$, let $B(z)$ be the matrix with (σ, k) entry $(\partial q_\sigma/\partial w_{i_k})(z, f(z))$, and let $\det A(z)$, $\det B(z)$ be their (respective) determinants. Then (33) is equal to

$$\frac{\det B(z)\bar{s}_z\{\det A\} - \det A(z)\bar{s}_z\{\det B\}}{[\det(B(z))]^2}. \tag{34}$$

If we note that $\phi_{j_1,j_2,j_3,\ldots,j_d}(z, f(z)) = \det A(z)$ and that $\phi_{i_1,i_2,i_3,\ldots,i_d}(z, f(z)) = \det B(z)$, then we see what needs to be proved is the following result, which is stated as a lemma because we will use it later.

LEMMA 4. *Suppose that*: *the conditions of Theorem 1 are satisfied*; (II) *holds*; $U \subset S$; $f : U \to \mathbb{C}^m$ *is a CR map whose graph is an integral manifold for* $\operatorname{Re}\mathcal{T}$; *and* $s \in \Gamma(U, \mathrm{H}^S)$. *Then, for all* $z \in U$ *and for any* d*-tuples* $\{i_k\}_{k=1}^d$ *and* $\{j_k\}_{k=1}^d$ *such that* $1 \le i_1 < i_2 < \cdots < i_d \le m$ *and* $1 \le j_1 < j_2 < \cdots < j_d \le m$, *we have*

$$\begin{aligned}\phi_{i_1,i_2,i_3,\ldots,i_d}&(z, f(z))\bar{s}_z\{\phi_{j_1,j_2,j_3,\ldots,j_d}(\cdot, f(\cdot))\}\\ &- \phi_{j_1,j_2,j_3,\ldots,j_d}(z, f(z))\bar{s}_z\{\phi_{i_1,i_2,i_3,\ldots,i_d}(\cdot, f(\cdot))\} = 0,\end{aligned} \tag{35}$$

where $\phi_{i_1,i_2,i_3,\ldots,i_d}, \phi_{j_1,j_2,j_3,\ldots,j_d}$ *are the functions defined in* (III).

Proof. We let A, B be the matrices defined previously. Then we must show that

$$\det B(z)\bar{s}_z\{\det A\} - \det A(z)\bar{s}_z\{\det B\} = 0. \tag{36}$$

We have

$$\bar{s}_z\{\det A\} = \sum_{i=1}^{d} \begin{vmatrix} A_{11}(z) & A_{12}(z) & \cdots & A_{1d}(z) \\ A_{21}(z) & A_{22}(z) & \cdots & A_{2d}(z) \\ \vdots & & \vdots & \vdots \\ \bar{s}_z\{A_{i1}\} & \bar{s}_z\{A_{i2}\} & \cdots & \bar{s}_z\{A_{id}\} \\ \vdots & & \vdots & \vdots \\ A_{d1}(z) & A_{d2}(z) & \cdots & A_{dd}(z) \end{vmatrix}, \tag{37}$$

where we calculate

$$\bar{s}_z\{A_{i,k}\} = \sum_{\sigma=1}^{\ell} \frac{\partial q_i}{\partial \bar{z}_\sigma \partial w_{j_k}}(z, f(z))\bar{s}_z\{\bar{z}_\sigma\} + \sum_{\sigma=1}^{m} \frac{\partial q_i}{\partial \bar{w}_\sigma \partial w_{j_k}}(z, f(z))\bar{s}_z\{\bar{f}_\sigma\}.$$

Observe that if we let

$$v^\sigma(z) = \det \begin{pmatrix} A_{11}(z) & A_{12}(z) & \cdots & A_{1d}(z) \\ A_{21}(z) & A_{22}(z) & \cdots & A_{2d}(z) \\ \vdots & & \vdots & \vdots \\ \frac{\partial}{\partial w_{j_1}} & \frac{\partial}{\partial w_{j_2}} & \cdots & \frac{\partial}{\partial w_{j_d}} \\ \vdots & & \vdots & \vdots \\ A_{d1}(z) & A_{d2}(z) & \cdots & A_{dd}(z) \end{pmatrix}$$

(where the row with the $\partial/\partial w_{j_k}$, $k = 1$ to d, is the σth row), then for $i = 1$ to d with $i \neq \sigma$ we have

$$\langle \partial q_i(z, f(z)), v^\sigma(z)\rangle = \det\begin{pmatrix} A_{11}(z) & A_{12}(z) & \cdots & A_{1d}(z) \\ A_{21}(z) & A_{22}(z) & \cdots & A_{2d}(z) \\ \vdots & & \vdots & \vdots \\ \frac{\partial q_i}{\partial w_{j_1}}(z, f(z)) & \frac{\partial q_i}{\partial w_{j_2}}(z, f(z)) & \cdots & \frac{\partial q_i}{\partial w_{j_d}}(z, f(z)) \\ \vdots & & \vdots & \vdots \\ A_{d1}(z) & A_{d2}(z) & \cdots & A_{dd}(z) \end{pmatrix} = 0 \qquad (38)$$

for all $z \in U$ because the ith and σth rows of the determinant are the same. For $i = \sigma$,

$$\langle \partial q_i(z, f(z)), v^\sigma(z)\rangle = \det A(z). \qquad (39)$$

Write $v^\sigma = \sum_{k=1}^d v_k^\sigma(\partial/\partial w_{j_k})$. Combining these observations, we find that (37) guarantees that

$$\begin{aligned} \bar{s}_z\{\det A\} &= \sum_{i=1}^d \sum_{k=1}^d \bar{s}_z\{A_{i,k}\} v_k^i \\ &= \sum_{i=1}^d \sum_{k=1}^d \Bigg(\sum_{\sigma=1}^{\ell} \frac{\partial q_i}{\partial \bar{z}_\sigma \partial w_{j_k}}(z, f(z)) \bar{s}_z\{\bar{z}_\sigma\} \\ &\qquad\qquad + \sum_{\sigma=1}^{m} \frac{\partial q_i}{\partial \bar{w}_\sigma \partial w_{j_k}}(z, f(z)) \bar{s}_z\{\bar{f}_\sigma\} \Bigg) v_k^i \\ &= \sum_{i=1}^d \Bigg(\sum_{k=1}^d \sum_{\sigma=1}^{\ell} \frac{\partial q_i}{\partial \bar{z}_\sigma \partial w_{j_k}}(z, f(z)) \bar{s}_z\{\bar{z}_\sigma\} v_k^i \\ &\qquad + \sum_{k=1}^d \sum_{\sigma=1}^{m} \frac{\partial q_i}{\partial \bar{w}_\sigma \partial w_{j_k}}(z, f(z)) \bar{s}_z\{\bar{f}_\sigma\} v_k^i \Bigg) \\ &= \sum_{i=1}^d \langle \bar{\partial}\partial q_i(z, f(z)), \overline{n(z)} \wedge v^i(z)\rangle, \end{aligned} \qquad (40)$$

where $n(z)$ was defined in (32). A similar argument shows that

$$\bar{s}_z\{\det B\} = \sum_{i=1}^d \langle \bar{\partial}\partial q_i(z, f(z)), \overline{n(z)} \wedge u^i(z)\rangle, \qquad (41)$$

where

$$u^\sigma(z) = \det\begin{pmatrix} B_{11}(z) & B_{12}(z) & \cdots & B_{1d}(z) \\ B_{21}(z) & B_{22}(z) & \cdots & B_{2d}(z) \\ \vdots & & \vdots & \vdots \\ \frac{\partial}{\partial w_{i_1}} & \frac{\partial}{\partial w_{i_2}} & \cdots & \frac{\partial}{\partial w_{i_d}} \\ \vdots & & \vdots & \vdots \\ B_{d1}(z) & B_{d2}(z) & \cdots & B_{dd}(z) \end{pmatrix}$$

and where, once again, the row with the $\partial/\partial w_{i_k}$ is the σth row. As before,

$$\langle \partial q_j(z, f(z)), u^\sigma(z)\rangle = 0 \tag{42}$$

if $j \neq \sigma$ and

$$\langle \partial q_j(z, f(z)), u^\sigma(z)\rangle = \det B(z) \tag{43}$$

if $j = \sigma$. Then (36) is equal to

$$\sum_{i=1}^{d} \langle \bar\partial \partial q_i(z, f(z)), \overline{n(z)} \wedge ((\det B(z))v^i(z) - (\det A(z))u^i(z))\rangle, \tag{44}$$

where we note that, for $j = 1$ to d,

$$\begin{aligned} &\langle \partial q_j(z, f(z)), (\det B(z))v^i(z) - (\det A(z))u^i(z)\rangle \\ &\quad = \det B(z)\langle \partial q_j(z, f(z)), v^i(z)\rangle - \det A(z)\langle \partial q_j(z, f(z)), u^i(z)\rangle \\ &\quad = 0 \end{aligned}$$

by (38), (39), (42), and (43). By definition of $V(z, f(z))$, $(\det B(z))v^i(z) - (\det A(z))u^i(z) \in V(z, f(z))$; this implies that (44) is identically zero for $z \in U$, since $n(z) \in N(z, f(z))$. Thus (36) holds and hence (35) holds also, as desired. $\square$

Now that we know Lemma 4 holds, we have that (34) and (33) are also zero for all $z \in U$, as desired. This proves the last component of (III) and hence we now know that (II) implies (III).

Now we assume (III) and prove (IV). We must show that the dimension of $N(z, w)$ is $\ell - c$ for $(z, w) \in M$. We know there exists a nonzero complex-valued function $C(z)$ defined on U such that

$$\begin{aligned} &C(z)\phi(z, f(z)) \\ &\quad = \sum_{1 \le i_1 < i_2 < i_3 < \cdots < i_d \le m} C(z)\phi_{i_1, i_2, \ldots, i_d}(z, f(z))\, dw_{i_1} \wedge dw_{i_2} \wedge \cdots \wedge dw_{i_d} \end{aligned}$$

is a d-form with coefficients that are CR functions for $z \in U$. If $s \in \Gamma(U, \mathrm{H}^S)$ then $\bar s_z\{C(\cdot)\phi(\cdot, f(\cdot))\} = 0$ for all $z \in U$. Then, using the product rule for the differentiation of wedge products, we have

$$
\begin{aligned}
0 &= \bar{s}_z\{C\}\phi(z, f(z)) \\
&\quad + C(z)\bar{s}_z\Bigg\{\bigg(\sum_{j=1}^{m}\frac{\partial q_1}{\partial w_j}(\cdot, f(\cdot))dw_j\bigg) \wedge \bigg(\sum_{j=1}^{m}\frac{\partial q_2}{\partial w_j}(\cdot, f(\cdot))dw_j\bigg) \wedge \cdots \\
&\qquad\qquad \wedge \bigg(\sum_{j=1}^{m}\frac{\partial q_d}{\partial w_j}(\cdot, f(\cdot))dw_j\bigg)\Bigg\} \\
&= \bar{s}_z\{C\}\phi(z, f(z)) \\
&\quad + C(z)\sum_{i=1}^{d}\Bigg(\bigg(\sum_{j=1}^{m}\frac{\partial q_1}{\partial w_j}(z, f(z))dw_j\bigg) \wedge \bigg(\sum_{j=1}^{m}\frac{\partial q_2}{\partial w_j}(z, f(z))dw_j\bigg) \wedge \cdots \\
&\qquad\qquad \wedge \bigg(\sum_{j=1}^{m}\frac{\partial q_{i-1}}{\partial w_j}(z, f(z))dw_j\bigg) \\
&\qquad\qquad \wedge \bigg(\sum_{j=1}^{m}\sum_{\sigma=1}^{\ell}\frac{\partial^2 q_i}{\partial \bar{z}_\sigma \partial w_j}(z, f(z))\bar{s}_z\{\bar{z}_\sigma\}dw_j \\
&\qquad\qquad\qquad + \sum_{j,\sigma=1}^{m}\frac{\partial^2 q_i}{\partial \bar{w}_\sigma \partial w_j}(z, f(z))\bar{s}_z\{\bar{f}_\sigma\}dw_j\bigg) \\
&\qquad\qquad \wedge \bigg(\sum_{j=1}^{m}\frac{\partial q_{i+1}}{\partial w_j}(z, f(z))dw_j\bigg) \wedge \cdots \\
&\qquad\qquad \wedge \bigg(\sum_{j=1}^{m}\frac{\partial q_d}{\partial w_j}(z, f(z))dw_j\bigg)\Bigg).
\end{aligned} \tag{45}
$$

For fixed $z \in U$, we claim that there exist $v^1, v^2, \ldots, v^d$ in $\mathbb{C}T_{f(z)}(M_z)$ such that, for $i, j = 1, 2, \ldots, d$,

$$\langle \partial q_i(z, f(z)), v^j \rangle = \delta_{ij}, \tag{46}$$

where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$. Recall from (5) that M_z is a generic CR manifold in $\mathbb{C}^m$ of CR codimension d. The 1-form $\partial q_i(z, f(z))$ induces a linear functional on $\mathbb{C}T_{f(z)}(M_z)$ in the following manner. For $x \in \mathbb{C}T_{f(z)}(M_z)$, map $x \mapsto L_i(x) \equiv \langle \partial q_i(z, f(z)), x\rangle$. Note that $V(z, f(z)) = \mathbb{C}T^{1,0}_{f(z)}(M_z)$. Since for all $i = 1$ to d we have $L_i(x) = 0$ for $x \in V(z, f(z)) \oplus \overline{V(z, f(z))} \subset \mathbb{C}T_{f(z)}(M_z)$, the mapping L_i factors through the quotient $\mathbb{C}T_{f(z)}(M_z)/(V(z, f(z)) \oplus \overline{V(z, f(z))})$ to produce a linear functional $\tilde{L}_i$. Now $\mathbb{C}T_{f(z)}(M_z)/(V(z, f(z)) \oplus \overline{V(z, f(z))})$ is a complex vector space of dimension d since M_z is generic. We claim that the induced linear functionals $\tilde{L}_i$ are a basis of the dual space of $\mathbb{C}T_{f(z)}(M_z)/(V(z, f(z)) \oplus \overline{V(z, f(z))})$. If there exist $\zeta_i \in \mathbb{C}$ (not all zero) such that $\sum_{i=1}^{d}\zeta_i\tilde{L}_i$ is identically zero on $\mathbb{C}T_{f(z)}(M_z)/(V(z, f(z)) \oplus \overline{V(z, f(z))})$, then $\sum_{i=1}^{d}\zeta_i L_i$ is identically zero as a linear functional on $\mathbb{C}T_{f(z)}(M_z)$; that is, $\big\langle \sum_{i=1}^{d}\zeta_i\partial q_i(z, f(z)), x\big\rangle = 0$ for all

$x \in \mathbb{C}T_{f(z)}(M_z)$. Let $\mathbf{i}$ be the imaginary unit and let $J\colon T_{f(z)}\mathbb{C}^m \to T_{f(z)}\mathbb{C}^m$ be the complex structure mapping on $T_{f(z)}\mathbb{C}^m$. Then, extending J to $\mathbb{C}T_{f(z)}(\mathbb{C}^m)$, we recall that $J(\partial/\partial w_j) = \mathbf{i}(\partial/\partial w_j)$ and $J(\partial/\partial \bar{w}_j) = -\mathbf{i}(\partial/\partial \bar{w}_j)$ for all j. We find that

$$\left\langle \sum_{j=1}^{d} \zeta_j \partial q_j(z, f(z)), J(x) \right\rangle = \mathbf{i}\left\langle \sum_{j=1}^{d} \zeta_j \partial q_j(z, f(z)), x \right\rangle = 0 \qquad \text{for all } x \in \mathbb{C}T_{f(z)}(M_z)$$

because $\sum_{j=1}^{d} \zeta_j \partial q_j(z, f(z))$ is a (1, 0)-form. Thus $\sum_{j=1}^{d} \zeta_j \partial q_j(z, f(z))$ is zero as a linear functional on $J(\mathbb{C}T_{f(z)}(M_z))$. Because M_z is generic, $\mathbb{C}T_{f(z)}(M_z) + J(\mathbb{C}T_{f(z)}(M_z)) = \mathbb{C}T_{f(z)}\mathbb{C}^m$ (see [BER, p. 14]), so $\sum_{j=1}^{d} \zeta_j \partial q_j(z, f(z))$ is zero as a linear functional on $\mathbb{C}T_{f(z)}\mathbb{C}^m$:

$$\sum_{j=1}^{d} \zeta_j \partial q_j(z, f(z)) = 0.$$

This is impossible because M_z is generic. Thus the d functionals $\{\tilde{L}_i\}$ described previously are linearly independent in the dual space of $\mathbb{C}T_{f(z)}(M_z)/(V(z, f(z)) \oplus \overline{V(z, f(z))})$ and thus are a basis of that dual space (which has dimension d). Let $\{\tilde{v}^i\}_{i=1}^d$ be a basis of $\mathbb{C}T_{f(z)}(M_z)/(V(z, f(z)) \oplus \overline{V(z, f(z))})$ that is dual to $\{\tilde{L}_i\}$ and, for every $i = 1$ to d, let v^i be an element of $\mathbb{C}T_{f(z)}(M_z)$ whose image in the quotient $\mathbb{C}T_{f(z)}(M_z)/(V(z, f(z)) \oplus \overline{V(z, f(z))})$ is $\tilde{v}^i$. The $\{v^i\}$ satisfy (46), as desired. Note that this implies that $\langle \partial_w q_i(z, f(z)), v^j \rangle = \delta_{ij}$, since v^j has no terms involving $\partial/\partial z_k$ for any k.

Let v be an arbitrary element of $V(z, f(z))$. If R is the rightmost side of (45) then we calculate $\langle R, v \wedge v^1 \wedge v^2 \wedge \cdots \wedge \hat{v}^k \wedge \cdots \wedge v^d \rangle$, where $\hat{v}^k$ indicates that v^k is not in the wedge product. Since $\langle \partial_w q_j(z, f(z)), v \rangle = \langle \partial q_j(z, f(z)), v \rangle = 0$ for all $j = 1$ to d, we have $\langle \bar{s}_z\{C\}\phi(z, f(z)), v \wedge v^1 \wedge v^2 \wedge \cdots \wedge \hat{v}^k \wedge \cdots \wedge v^d \rangle = 0$, as every term of the expansion contains a factor of the form $\langle \partial_w q_j(z, f(z)), v \rangle$. Write the other term in (45) as $C(z) \sum_{i=1}^{d} R_i$ and consider the ith term R_i of this summation. If $i \neq k$ then $\langle R_i, v \wedge v^1 \wedge v^2 \wedge \cdots \wedge \hat{v}^k \wedge \cdots \wedge v^d \rangle$ is zero, because every term in the expansion contains a factor of the form $\left\langle \sum_{j=1}^{m} (\partial q_k/\partial w_j)(z, f(z)) dw_j, X \right\rangle$, where X is one of $v, v^1, v^2, v^3, \ldots, v^{k-1}, v^{k+1}, \ldots, v^d$ and every such factor is zero by (46) and the definition of v. Thus

$$\begin{aligned} &\langle R, v \wedge v^1 \wedge v^2 \wedge \cdots \wedge \hat{v}^k \wedge \cdots \wedge v^d \rangle \\ &\quad = C(z) \left\langle \sum_{i=1}^{d} R_i, v \wedge v^1 \wedge v^2 \wedge \cdots \wedge \hat{v}^k \wedge \cdots \wedge v^d \right\rangle \\ &\quad = C(z) \langle R_k, v \wedge v^1 \wedge v^2 \wedge \cdots \wedge \hat{v}^k \wedge \cdots \wedge v^d \rangle. \end{aligned}$$

Since (46) holds and $\langle \partial_w q_j(z, f(z)), v \rangle = 0$ for $j = 1$ to d, the only nonzero term in the expansion of $C(z)\langle R_k, v \wedge v^1 \wedge v^2 \wedge \cdots \wedge \hat{v}^k \wedge \cdots \wedge v^d \rangle$ is

$$(-1)^{k-1}C(z)\left\langle\left(\sum_{j=1}^{m}\frac{\partial q_1}{\partial w_j}(z,\,f(z))dw_j\right),v^1\right\rangle\left\langle\left(\sum_{j=1}^{m}\frac{\partial q_2}{\partial w_j}(z,\,f(z))dw_j\right),v^2\right\rangle\cdots$$

$$\left\langle\left(\sum_{j=1}^{m}\sum_{\sigma=1}^{\ell}\frac{\partial^2 q_k}{\partial\bar{z}_\sigma\partial w_j}(z,\,f(z))\bar{s}_z\{\bar{z}_\sigma\}dw_j\right.\right.$$
$$\left.\left.+\sum_{j,\sigma=1}^{m}\frac{\partial^2 q_k}{\partial\bar{w}_\sigma\partial w_j}(z,\,f(z))\bar{s}_z\{\bar{f}_\sigma\}dw_j\right),v\right\rangle\cdots$$
$$\left\langle\left(\sum_{j=1}^{m}\frac{\partial q_d}{\partial w_j}(z,\,f(z))dw_j\right),v^d\right\rangle;$$

by (46), this equals

$$\begin{aligned}(-1)^{k-1}C(z)\bigg\langle&\sum_{j=1}^{m}\sum_{\sigma=1}^{\ell}\frac{\partial^2 q_k}{\partial\bar{z}_\sigma\partial w_j}(z,\,f(z))\bar{s}_z\{\bar{z}_\sigma\}dw_j\\&+\sum_{j,\sigma=1}^{m}\frac{\partial^2 q_k}{\partial\bar{w}_\sigma\partial w_j}(z,\,f(z))\bar{s}_z\{\bar{f}_\sigma\}dw_j,v\bigg\rangle\\&=(-1)^{k-1}C(z)\left\langle\bar{\partial}\partial q_k(z,\,f(z)),\,\overline{\left(\sum_{j=1}^{\ell}s_z\{z_j\}\frac{\partial}{\partial z_j}+\sum_{j=1}^{m}s_z\{f_j\}\frac{\partial}{\partial w_j}\right)}\wedge v\right\rangle.\end{aligned}$$

By (45) this quantity is zero, and this holds for $k=1$ to d. Since $C(z)$ is never zero, we find for all $k=1$ to d that

$$\left\langle\bar{\partial}\partial q_k(z,\,f(z)),\,\overline{\left(\sum_{j=1}^{\ell}s_z\{z_j\}\frac{\partial}{\partial z_j}+\sum_{j=1}^{m}s_z\{f_j\}\frac{\partial}{\partial w_j}\right)}\wedge v\right\rangle=0.$$

It is also true that, for $i=1$ to c,

$$\left\langle\bar{\partial}\partial p_i(z,\,f(z)),\,\overline{\left(\sum_{j=1}^{\ell}s_z\{z_j\}\frac{\partial}{\partial z_j}+\sum_{j=1}^{m}s_z\{f_j\}\frac{\partial}{\partial w_j}\right)}\wedge v\right\rangle=0,$$

since p_i depends only on z and since v has no terms involving any $\partial/\partial z_j$. By definition of $V(z,\,f(z))$ and since v was chosen arbitrarily in $V(z,\,f(z))$, this implies that

$$\sum_{i=1}^{\ell}s_z\{z_i\}\frac{\partial}{\partial z_i}+\sum_{i=1}^{m}s_z\{f_i\}\frac{\partial}{\partial w_i}\in N(z,\,f(z)).$$

This is true for all $s_z\in \mathrm{H}_z^S$, so $\pi_{(z,\,f(z))}$ maps $N(z,\,f(z))$ onto H_z^S. We already know from Proposition 1 that this mapping is injective, so $\pi_{(z,\,f(z))}\colon N(z,\,f(z))\to \mathrm{H}_z^S$ is an isomorphism and $N(z,\,f(z))$ must have the same complex dimension as H_z^S, which is $\ell-c$. The foregoing argument holds for an arbitrary $z\in U$. Since any point (z,w) in M is on the graph of an f defined on some such $U\subset S$, we find that the complex dimension of $N(z,w)$ is $\ell-c$ for all $(z,w)\in M$, as desired.

Now we must prove (10). Once again we fix $(z^0, w^0) \in M$ through which there exists the graph of a CR map $f: U \to \mathbb{C}^m$ with the properties in (III) for some neighborhood U of z^0 in S. We note that every element of $N(z, f(z))$ for $z \in U$ has the form $s_z + \sum_{i=1}^m s_z\{f_i\}(\partial/\partial w_i)$ for some $s_z \in \mathrm{H}_z^S$: if $n_z \in N(z, f(z))$, then $\pi_{(z,f(z))}(n_z) \in \mathrm{H}_z^S$. Let $s_z = \pi_{(z,f(z))}(n_z)$. Then $s_z + \sum_{i=1}^m s_z\{f_i\}(\partial/\partial w_i) \in N(z, f(z))$ and $\pi_{(z,f(z))}(n_z) = \pi_{(z,f(z))}\big(s_z + \sum_{i=1}^m s_z\{f_i\}(\partial/\partial w_i)\big) = s_z$. By Proposition 1, $\pi_{(z,f(z))}: N(z, f(z)) \to \mathrm{H}_z^S$ is injective, so

$$n_z = s_z + \sum_{i=1}^m s_z\{f_i\}\frac{\partial}{\partial w_i}$$

as claimed.

It suffices to prove (10) at points on the graph of a particular such f, since the graphs of such f foliate M locally. The dimension of $N(z, w)$ is $\ell - c$ for all (z, w) in M. For all $z \in U$ we have $N(z, f(z)) \subset \mathbb{C}T_{(z,f(z))}M^f$, so the restriction of $N \oplus \bar{N}$ to M^f is a subbundle of $\mathbb{C}TM^f$. For a shrunken U, we may construct the local basis for $\mathbb{C}TS$ near z^0 as in (23); we write $G = U$. Then we may construct the vector fields T_i near (z^0, w^0) (see (24)) as at the beginning of the proof. Since the T_i are commutators of vector fields whose values on M^f are tangent to M^f, the values of the T_i are also tangent to M^f. In fact, any commutator of vector fields in $\Gamma(G^M, N)$ and $\Gamma(G^M, \bar{N})$ has values on M^f that are tangent to M^f.

We conclude that, to prove (10) for the vector fields F_i indicated there, it suffices to prove that if $z \in G$ then

$$\begin{aligned}\langle \partial p_1(z, f(z)) \wedge \partial p_2(z, f(z)) \wedge \cdots \wedge \partial p_c(z, f(z))&\\ \wedge\, \partial q_k(z, f(z)), F_z^1 \wedge F_z^2 \wedge \cdots \wedge F_z^{c+1}\rangle &= 0 \qquad (47)\end{aligned}$$

for all $F_z^i \in \mathbb{C}T_{(z,f(z))}M^f$ with $i = 1, 2, \ldots, c+1$. Now fix $z \in G$ so that $(z, f(z))$ is in the graph of f. There is a complex linear isomorphism

$$I: \mathbb{C}T_0(\mathbb{R}^c \times \mathbb{C}^{\ell-c}) \to \mathbb{C}T_{(z,f(z))}M^f$$

that maps $\mathbb{C}T_0^{1,0}(\{0\} \times \mathbb{C}^{\ell-c})$ onto $\mathbb{C}T_{(z,f(z))}^{1,0}M^f = N(z, f(z))$ and also maps $\mathbb{C}T_0^{0,1}(\{0\} \times \mathbb{C}^{\ell-c})$ onto $\mathbb{C}T_{(z,f(z))}^{0,1}M^f = \bar{N}(z, f(z))$. (Suppose that $\mathbb{R}^c$ has coordinates x_i, $i = 1, 2, \ldots, c$, and that $\mathbb{C}^{\ell-c}$ has coordinates ζ_i, $i = 1, 2, \ldots, \ell - c$; then simply let $I(\partial/\partial\zeta_i) = n_i(z, f(z))$, $I(\partial/\partial\bar{\zeta}_i) = \bar{n}_i(z, f(z))$, and $I(\partial/\partial x_i) = T_i(z, f(z))$, where n_i and T_i are defined as in (19) and (24), respectively.) The map I induces an isomorphism on the cotangent spaces

$$I^*: \mathbb{C}T^*_{(z,f(z))}M^f \to \mathbb{C}T_0^*(\mathbb{R}^c \times \mathbb{C}^{\ell-c}).$$

Note that all $\partial p_i(z, f(z))$ and $\partial q_i(z, f(z))$ may be regarded as elements of $\mathbb{C}T^*_{(z,f(z))}M^f$ by virtue of the restriction of those forms to $\mathbb{C}T_{(z,f(z))}M^f$. Let ψ_i be the cotangent in $\mathbb{C}T_0^*(\mathbb{R}^c \times \mathbb{C}^{\ell-c})$ given by $\psi_i = I^*(\partial p_i(z, f(z)))$ for all $i = 1$ to c; more precisely, we have that if $\tilde{F}_z \in \mathbb{C}T_0(\mathbb{R}^c \times \mathbb{C}^{\ell-c})$ then $\langle \psi_i, \tilde{F}_z\rangle = \langle I^*(\partial p_i(z, f(z))), \tilde{F}_z\rangle = \langle \partial p_i(z, f(z)), I(\tilde{F}_z)\rangle$. Also write $\xi_i = I^*(\partial q_i(z, f(z)))$ for all $i = 1, 2, \ldots, d$, so then $\langle \xi_i, \tilde{F}_z\rangle = \langle \partial q_i(z, f(z)), I(\tilde{F}_z)\rangle$. We know that if

$\tilde{F}_z \in \mathbb{C}T_0^{1,0}(\{0\} \times \mathbb{C}^{\ell-c})$ then $\langle \psi_i, \tilde{F}_z \rangle = \langle \partial p_i(z, f(z)), I(\tilde{F}_z) \rangle = 0$ for all $i = 1$ to c, since $I(\tilde{F}_z)$ is a $(1,0)$-tangent to M^f at $(z, f(z))$. Similarly, $\langle \xi_i, \tilde{F}_z \rangle = 0$ for all $i = 1$ to d. We also have that $\langle \psi_i, \tilde{F}_z \rangle = 0$ and $\langle \xi_i, \tilde{F}_z \rangle = 0$ for all ψ_i, ξ_i and for $\tilde{F}_z \in \mathbb{C}T_0^{0,1}(\{0\} \times \mathbb{C}^{\ell-c})$, since then $I(\tilde{F}_z)$ is a $(0,1)$-tangent to M^f at $(z, f(z))$. By Lemma 2, $\psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \cdots \wedge \psi_c \wedge \xi_k = 0$ as a $(c+1)$-cotangent in $\mathbb{C}T_0^*(\mathbb{R}^c \times \mathbb{C}^{\ell-c})$; hence, for all $\tilde{F}_z^1, \tilde{F}_z^2, \ldots, \tilde{F}_z^{c+1} \in \mathbb{C}T_0(\mathbb{R}^c \times \mathbb{C}^{\ell-c})$,

$$\langle \psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \cdots \wedge \psi_c \wedge \xi_k, \tilde{F}_z^1, \wedge \tilde{F}_z^2 \wedge \cdots \wedge \tilde{F}_z^{c+1} \rangle = 0. \tag{48}$$

Temporarily writing $\psi_{c+1} = \xi_k$, we find from (48) that

$$\begin{aligned} 0 &= \sum_\sigma \operatorname{sgn}(\sigma) \prod_{i=1}^{c+1} \langle \psi_i, \tilde{F}_z^{\sigma(i)} \rangle \\ &= \sum_\sigma \operatorname{sgn}(\sigma) \langle \partial q_k(z, f(z)), I(\tilde{F}_z^{\sigma(c+1)}) \rangle \prod_{i=1}^{c} \langle \partial p_i(z, f(z)), I(\tilde{F}_z^{\sigma(i)}) \rangle, \end{aligned} \tag{49}$$

where σ ranges over all permutations of $\{1, 2, 3, \ldots, c+1\}$ and $\operatorname{sgn}(\sigma)$ is the sign of such a permutation σ. Then (49) implies that

$$\begin{aligned} \langle \partial p_1(z, f(z)) \wedge \partial p_2(z, f(z)) \wedge \cdots \wedge \partial p_c(z, f(z)) \wedge \partial q_k(z, f(z)), \\ I(\tilde{F}_z^1) \wedge I(\tilde{F}_z^2) \wedge \cdots \wedge I(\tilde{F}_z^{c+1}) \rangle = 0 \end{aligned} \tag{50}$$

for all $\tilde{F}_z^1, \tilde{F}_z^2, \ldots, \tilde{F}_z^{c+1} \in \mathbb{C}T_0(\mathbb{R}^c \times \mathbb{C}^{\ell-c})$. Because I is an isomorphism it follows that, as the $\tilde{F}_z^i$ range over $\mathbb{C}T_0(\mathbb{R}^c \times \mathbb{C}^{\ell-c})$, $I(\tilde{F}_z^i)$ ranges over $\mathbb{C}T_{(z, f(z))} M^f$. Thus we conclude from (50) that (47) holds for all $F_z^i \in \mathbb{C}T_{(z, f(z))} M^f$, $i = 1, 2, \ldots, c+1$. As observed earlier, this implies (10). The proof that (III) implies (IV) is therefore now complete.

Now we must prove that if (I)–(IV) hold then the last statements of the theorem hold. While proving that (II) implies (III), we proved that the graphs in (III) arise as integral manifolds of Re $\mathcal{T}$ and that the $(1,0)$-tangent space to such a graph f at $(z, f(z))$ is $N(z, f(z))$. Finally, suppose a CR map g exists as stated in the theorem, and let M^g denote its graph. Then the complexified tangent space $\mathbb{C}T_{(z, g(z))} M^g$ to the graph of g contains $N(z, g(z))$ (as assumed); hence it contains $N(z, g(z)) \oplus \bar{N}(z, g(z))$ (since the graph of g is a real manifold, $\mathbb{C}TM^g$ is self-conjugate) and so it contains $\mathcal{T}(z, g(z))$ (since $\mathbb{C}TM^g$ is involutive, $\mathbb{C}TM^g_{(z, g(z))}$ must contain $T_i(z, g(z))$ for $i = 1$ to c, where T_i is defined in (24)). In fact, for all z in the domain of g, $\mathbb{C}T_{(z, g(z))} M^g$ equals $\mathcal{T}(z, g(z))$ because each has complex dimension $2\ell - c$. Thus the graph of g is an integral manifold for Re $\mathcal{T}$ and so, by the Frobenius theorem, it must be one of the graphs from (III). □

5. Global Results

The next theorem is a statement about the existence of CR maps whose graphs lie in M and whose domains are all of S.

THEOREM 2. *Suppose that S and M satisfy* (4) *and* (5) (*respectively*) *and that S is connected, simply connected, and of finite type τ at every point. Suppose also that $M_K \equiv \{(z,w) \in M : z \in K\}$ is compact for every compact $K \subset S$. Suppose that any of properties* (I)–(IV) *of Theorem 1 hold. Then for every $(z^0, w^0) \in M$ there exists a unique CR map $f\colon S \to \mathbb{C}^m$ whose graph is an integral manifold of* $\operatorname{Re}\mathcal{T}$ *and which passes through (z^0, w^0). Let $\phi_{i_1,i_2,\dots,i_d}(z,w)$ be the function defined in* (III) *of Theorem 1. If there exist CR functions $h_{i_1,i_2,\dots,i_d}(z)$ defined for $z \in S$ such that*

$$Q(z) \equiv \sum_{1\le i_1<i_2<\cdots<i_d\le m} h_{i_1,i_2,\dots,i_d}(z)\phi_{i_1,i_2,\dots,i_d}(z, f(z)) \neq 0 \tag{51}$$

for all $z \in S$, then

$$z \mapsto C(z)\phi_{i_1,i_2,\dots,i_d}(z, f(z)) \text{ is a CR function on } S \text{ for integers } i_j, \tag{52}$$

where $C(z) = 1/Q(z)$ and $1 \le i_1 < i_2 < \cdots < i_d \le m$. If $d = m$ then such $h_{i_1,i_2,\dots,i_d}$ will exist.

NOTE. Theorems 3 and 4 give natural circumstances where functions $h_{i_1,i_2,\dots,i_d}$ exist such that property (51) holds.

Proof of Theorem 2. By Theorem 1, properties (I), (II), (III), and (IV) are all equivalent. Hence there exists an involutive subbundle $\operatorname{Re}\mathcal{T}$ of the real tangent bundle to M whose integral manifolds are locally graphs of CR maps on open subsets of S. Pick $(z^0, w^0) \in M$ and let L be the maximal connected integral manifold of $\operatorname{Re}\mathcal{T}$ that passes through (z^0, w^0). (For existence of such a manifold, see [War, Thm. 1.64].) We claim that L is the graph of a CR map $f\colon S \to \mathbb{C}^m$. (Note that L is the union of submanifolds that are graphs over open subsets of S; the topology we use for L is that generated by the topologies of these graphs taken together.) First we claim that the projection mapping P from L to S is a covering map (we use the definition from [M, p. 118]). We show that it is surjective and toward this end we show that $P(L)$ is open in S. Fix $z \in S$ in $P(L)$; then there exist $(z,w) \in L$ and an integral manifold of $\operatorname{Re}\mathcal{T}$ passing through (z,w) that is the graph of a CR function in a neighborhood U of z. This graph is contained in L because the maximal integral manifold L of $\operatorname{Re}\mathcal{T}$ contains any integral manifold of $\operatorname{Re}\mathcal{T}$ passing through a point of L (see [War, Thm. 1.64]). Thus $P(L)$ contains U. This shows that the projection of L to S is open in S. If $P(L)$ is not all of S then, since S is connected and $P(L)$ is open, there exists a C^0 path $\gamma\colon [0,1] \to S$ such that $\gamma([0,1)) \subset P(L)$ and $\gamma(0) = z^0$ but $\gamma(1) \notin P(L)$. We claim that there exists a continuous $\gamma'\colon [0,1] \to L$ such that $\gamma'(0) = (z^0, w^0)$ and $P \circ \gamma' = \gamma$.

Before proving this, suppose that we have two continuous paths γ', γ'' such that $\gamma'\colon [0,\varepsilon'] \to L$, $\gamma''\colon [0,\varepsilon''] \to L$, $\gamma'(0) = \gamma''(0) = (z^0, w^0)$, $\varepsilon' \le \varepsilon''$, and $P \circ \gamma' = P \circ \gamma'' = \gamma$ on $[0,\varepsilon']$. Then, in fact, $\gamma' = \gamma''$ on $[0,\varepsilon']$. To see this, note that $\gamma'(0) = \gamma''(0)$, so the domain of coincidence of γ', γ'' is nonempty. The set where $\gamma' = \gamma''$ is closed in $[0,\varepsilon']$ because γ' and γ'' are continuous. It is also open: if t_1 is a point where $\gamma'(t_1) = \gamma''(t_1)$ then, in a neighborhood of that point in L, it follows that L is a graph over an open neighborhood of $P(\gamma'(t_1))$ in S and so P maps a neighborhood of $\gamma'(t_1)$ in L homeomorphically to a neighborhood of $P(\gamma'(t_1))$ in S. For t near t_1, both $\gamma'(t)$ and $\gamma''(t)$ lie in that neighborhood in L of $\gamma'(t_1)$ and so $\gamma'(t) = \gamma''(t)$ for t near t_1. Thus the set where $\gamma' = \gamma''$ in $[0, \varepsilon']$ is open, closed, and nonempty; hence it equals $[0, \varepsilon']$, as desired.

Now let $\gamma'(0) = (z^0, w^0)$. Near $\gamma'(0)$ in L, L is a graph over a neighborhood of $\gamma(0)$; hence, for some $\varepsilon > 0$ we may define a continuous $\gamma' \colon [0, \varepsilon] \to L$ such that $P \circ \gamma' = \gamma$ on $[0, \varepsilon]$. Let t_0 be the supremum of all $t \in [0, 1]$ such that we may define a continuous $\gamma' \colon [0, t] \to L$ such that $\gamma'(0) = (z^0, w^0)$ and $P \circ \gamma' = \gamma$. Any two such paths coincide on their common domains by the previous paragraph. Then there exists a continuous $\gamma' \colon [0, t_0) \to L$ such that $P \circ \gamma' = \gamma$. Let $K = \gamma([0, t_0])$ and let (z, w) be a limit point in M_K of the open path $\gamma'([0, t_0))$ of the form $\lim_{n\to\infty} \gamma'(t_n)$, where $t_n \uparrow t_0$. (Recall that M_K is compact.) By Theorem 1, there exists a neighborhood U^M of (z, w) in M that is foliated by graphs of CR functions on an open subset U of S, where the graphs are integral manifolds of Re $\mathcal{T}$. For t near t_0, $\gamma(t)$ lies in U and, for large n, $\gamma'(t_n)$ lies in U^M on one of the foliating graphs $f^n \colon U \to \mathbb{C}^m$. Since the image of γ' is connected and since $\gamma(t)$ lies in U for t near t_0, it follows that $\gamma'(t)$ lies only on the graph of a particular CR mapping $f \colon U \to \mathbb{C}^m$ for t near t_0. The graph of f is an integral manifold of Re $\mathcal{T}$ for some points in L (the points on the path $\gamma'(t)$ for t near t_0). Since L is maximal, L contains the graph of f. Furthermore, we may use this fact to extend γ' to a neighborhood of t_0 in $[0, 1]$. If $t_0 < 1$, this contradicts the maximality of t_0, so $t_0 = 1$ and we have the path γ' as desired. However, this implies that $P(L)$ contains $\gamma(1)$—a contradiction—so the assumption that $P(L)$ is not all of S is false: we thus have $P(L) = S$.

We recall again from Theorem 1 that, given any $(z, w) \in M$, there exist an open neighborhood U of z in S and an open neighborhood of U^M of (z, w) in M such that U^M is the disjoint union of graphs of CR maps $f \colon U \to \mathbb{C}^m$, where the graphs are all integral manifolds of Re $\mathcal{T}$. Fixing $z = z'$, we find such open sets U_w, U_w^M for every $w \in \mathbb{C}^m$ such that $(z', w) \in M$. Since $M_{z'}$ is compact, finitely many of the U_w^M cover $M \cap \{(z, w) \in M : z = z'\}$. Take the (finite) intersection of all associated U_w and let U be the path component of it that contains z'. Then U is open (as a path component of the finite intersection of open sets) and, given any point of the form $(z', w) \in M$, there exists a CR map $f \colon U \to \mathbb{C}^m$ whose graph contains (z', w) and which is an integral manifold for Re $\mathcal{T}$. (That is, given fixed $z' \in S$, there exists a fixed U for all $w \in M_{z'}$ such that such an f exists.) Thus, given $z' \in S$, we have found a path-connected neighborhood U of z' in S such that $P^{-1}(U)$ is the union of disjoint CR graphs over U, each of which is a path-connected open subset of L and each of which projects onto U through P. This proves that $P \colon L \to S$ is a covering map.

Next we claim that $P \colon L \to S$ is injective; this will show that L is a graph. Suppose it is not injective; then there exist $z^0 \in S$, $(z^0, w^0) \in L$, and $(z^0, w^1) \in L$ with $w^0 \neq w^1$. Since L is a connected manifold, it is path connected, so there exists a path in L from (z^0, w^0) to (z^0, w^1) that projects to a path in S from z^0 to z^0. By the path lifting lemma (see [M, Lemma 3.3]), since S is simply connected and since L is a covering space of S, we have $(z^0, w^0) = (z^0, w^1)$ as desired.

Thus $P\colon L \to S$ is injective (as desired) and L is the graph of a mapping $f\colon S \to \mathbb{C}^m$. Then f is CR because L is the union of graphs of CR mappings defined on open subsets of S. If another mapping $\tilde{f}\colon S \to \mathbb{C}^m$ exists with the properties that f has in Theorem 2, then its graph is an integral manifold of $\operatorname{Re}\mathcal{T}$ that passes through (z^0, w^0). Since the graph of f is L (i.e., the maximal connected integral manifold of $\operatorname{Re}\mathcal{T}$ that passes through (z^0, w^0)), it follows that the graph of $\tilde{f}$ is contained in the graph of f and so $f = \tilde{f}$, as desired.

We need to show that $z \mapsto C(z)\phi_{j_1, j_2, \dots, j_d}(z, f(z))$ is CR; if $\bar{s}_z \in \bar{\mathrm{H}}^S_z$ then, for all integers $j_1, j_2, \dots, j_m$ such that $1 \le j_1 < j_2 < \cdots < j_d \le m$, we have

$$\begin{aligned}
&\bar{s}_z C(\cdot)\phi_{j_1, j_2, \dots, j_d}(\cdot, f(\cdot)) \\
&\quad = C(z)^2 \Bigg(\sum_{1 \le i_1 < i_2 < \cdots < i_d \le m} h_{i_1, i_2, \dots, i_d}(z)\phi_{i_1, i_2, \dots, i_d}(z, f(z)) \\
&\qquad\qquad \times \bar{s}_z\{\phi_{j_1, j_2, \dots, j_d}(\cdot, f(\cdot))\} - \phi_{j_1, j_2, \dots, j_d}(z, f(z)) \\
&\qquad\qquad \times \sum_{1 \le i_1 < i_2 < \cdots < i_d \le m} h_{i_1, i_2, \dots, i_d}(z)\bar{s}_z\{\phi_{i_1, i_2, \dots, i_d}(\cdot, f(\cdot))\} \Bigg) \\
&\quad = C(z)^2 \sum_{1 \le i_1 < i_2 < \cdots < i_d \le m} h_{i_1, i_2, \dots, i_d}(z) \\
&\qquad \times \big(\bar{s}_z\{\phi_{j_1, j_2, \dots, j_d}(\cdot, f(\cdot))\}\phi_{i_1, i_2, \dots, i_d}(z, f(z)) \\
&\qquad\quad - \bar{s}_z\{\phi_{i_1, i_2, \dots, i_d}(\cdot, f(\cdot))\}\phi_{j_1, j_2, \dots, j_d}(z, f(z)) \big) \\
&\quad = 0
\end{aligned}$$

from Lemma 4. (Note that $\bar{s}_z\{h_{i_1, i_2, \dots, i_d}\} = 0$ for all $z \in S$, since $h_{i_1, i_2, \dots, i_d}$ is CR.) This being true for all $\bar{s}_z \in \bar{\mathrm{H}}^S_z$ and all $z \in S$, we find that $z \mapsto C(z)\phi_{j_1, j_2, \dots, j_d}(z, f(z))$ is CR as desired.

If $d = m$ then the note before the proof of Theorem 1 applies: there is only one $\phi_{i_1, i_2, \dots, i_d} = \phi_{1,2,3,\dots,m}$ and we may simply let $C(z) = 1/\phi_{1,2,3,\dots,m}(z, f(z))$, which is never zero for $z \in S$. $\square$

If the conditions of Theorem 2 hold, then:

(53) $\mathcal{F}$ is the set of CR mappings f whose graphs lie in M and are maximal integral manifolds of $\operatorname{Re}\mathcal{T}$.

Corollary 1. *Suppose that S and M satisfy* (4) *and* (5) (*respectively*) *and that S is connected, simply connected, and of finite type τ at every point. Suppose also that $M_K \equiv \{(z, w) \in M : z \in K\}$ is compact for every compact $K \subset S$. Suppose that any of properties* (I)–(IV) *of Theorem 1 hold and that, in addition, S is the boundary of a bounded domain D in $\mathbb{C}^\ell$, $\ell \ge 2$. Then M is the disjoint union of graphs of CR maps* (*in $\mathcal{F}$*), *all of which extend to be continuous on $\bar{D}$ and analytic on D*; *these extensions are solutions to* (RH).

Proof. The CR maps that arise from Theorem 2 are C^∞ on S; they extend to be analytic on D by the global CR extension theorem. (This is the theorem commonly known as Bochner's extension theorem; we choose not to use this attribution because of the conclusions stated in [Ra].) □

Corollary 1 provides circumstances where the Riemann–Hilbert problem (RH) for M is solvable. If $f \in \mathcal{F}$ then we let $\hat{f} = (\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_m)$ denote the analytic extension of f to D (if the extension exists).

THEOREM 3. *Suppose that S and M satisfy* (4) *and* (5) (*respectively*) *and that S is connected, simply connected, and of finite type τ at every point. Suppose also that M_K is compact for every compact K in S and that, for all $z \in S$, M_z is a convex hypersurface that encloses the origin in $\mathbb{C}^m$. Finally, suppose that any of properties* (I)–(IV) *of Theorem 1 hold. Then M is the disjoint union of CR maps $f : S \to \mathbb{C}^m$ such that there exists a nonzero complex-valued function $C(z)$ defined for $z \in S$ for which*

$$z \mapsto C(z) \frac{\partial q_1}{\partial w_i}(z, f(z)) \tag{54}$$

is a CR function on S for $i = 1, 2, 3, \ldots, m$.

Proof. From Theorem 2 we may conclude that, for any point in M, there exists a CR mapping $f : S \to \mathbb{C}^m$ whose graph is contained in M and passes through that point. As (54) is nothing more than condition (52) in the case $d = 1$, we need only show the existence of h_i $(i = 1, 2, \ldots, m)$ such that (51) holds in the case $d = 1$. That is, we must show that

$$\sum_{i=1}^{m} h_i(z) \frac{\partial q_1}{\partial w_i}(z, f(z)) \neq 0 \tag{55}$$

for all $z \in S$, where $h = (h_1, h_2, \ldots, h_m)$. If $f = (f_1, f_2, \ldots, f_m)$ then it will suffice to let $h_i = f_i$ for $i = 1$ to m. The reason for this is that, by convexity of the surfaces M_z for all $z \in S$, the complex tangent plane to M_z in $\mathbb{C}^m$ at $f(z)$ does not pass into the region enclosed by M_z and so does not pass through the origin of $\mathbb{C}^m$. This complex tangent plane has the form

$$\left\{ v = (v_1, v_2, v_3, \ldots, v_m) \in \mathbb{C}^m : \sum_{i=1}^{m} \frac{\partial q_1}{\partial w_i}(z, f(z)) v_i = \zeta_z \right\}$$

for some complex constant ζ_z. We cannot have $\zeta_z = 0$ because the plane does not pass through the origin. Thus $\sum_{i=1}^{m} (\partial q_1 / \partial w_i)(z, f(z)) f_i(z) = \zeta_z \neq 0$ for all $z \in S$, since $f(z)$ belongs to the tangent plane to M_z at $f(z)$. Thus (55) holds and Theorem 3 holds. □

NOTE. Since all that is required in Corollary 2 is that the *complex* tangent planes to M_z not pass into the interior of the region enclosed by M_z, the theorem need only require that M_z enclose a region that is *lineally convex* or *hypoconvex*. (See [K], [Wh], or [Hö, p. 290].)

COROLLARY 2. *Suppose S and M satisfy all the conditions of Theorem 3 and that, in addition, S is the boundary of a bounded domain D in $\mathbb{C}^\ell$, $\ell \geq 2$. Then* *the CR maps in $\mathcal{F}$ that arise from Theorem 3 all extend to be continuous on $\bar{D}$ and analytic on D; these extensions are solutions to* (RH).

Proof. This is again an application of the global CR extension theorem. □

6. Further Global Results. Extremal Properties

If Y is any compact subset of $\mathbb{C}^n$ then the *polynomial hull* of Y is the set

$$\hat{Y} = \left\{ z \in \mathbb{C}^n \;\middle|\; |P(z)| \le \sup_{w \in Y} |P(w)| \text{ for all polynomials } P \text{ on } \mathbb{C}^n \right\}.$$

We say that Y is *polynomially convex* if $\hat{Y} = Y$. Theorem 4 shows that, under some circumstances, the properties that f possesses in (III) of Theorem 1 guarantee that f possesses an extremal property.

THEOREM 4. *Suppose that S and M satisfy* (4) *and* (5) *and that S is a hypersurface bounding a bounded strictly pseudoconvex open set D such that $\bar{D}$ is polynomially convex. Moreover, let S be connected and simply connected. Suppose that, in* (5), *the defining function $q_d(z, w)$ satisfies the property that, for all $z \in S$, $q_d(z, w)$ is strictly convex as a function of w in a neighborhood of M_z. Suppose that, for $i = 1, 2, 3, \dots, d-1$, the defining function q_i has the form*

$$q_i(z, w) = \operatorname{Re}\left(\sum_{j=1}^{m} \alpha_j^i(z) w_j \right)$$

for some matrix $(\alpha_j^i)_{i,j=1}^m$ of functions analytic in a neighborhood of $\bar{D}$, where the determinant of (α_j^i) is never zero on $\bar{D}$. Let $\tilde{M} = \{(z, w) \in \bar{D} \times \mathbb{C}^m : q_i(z, w) = 0,\ i = 1, 2, \dots, d-1\}$ and $\tilde{M}_z = \{w : (z, w) \in \tilde{M}\}$. Assume that M is compact and that, for all $z \in S$, the origin of $\mathbb{C}^m$ is in the bounded (*convex*) *component of $\tilde{M}_z \setminus M_z$. Suppose that any of the properties* (I)–(IV) *of Theorem 1 hold. Then the set $\mathcal{F}$ is well-defined and, for all $f \in \mathcal{F}$, the graph of $\hat{f}$ in $\bar{D} \times \mathbb{C}^m$ lies in the boundary of $\hat{M}$ as a subset of $\tilde{M}$. In particular, given $f \in \mathcal{F}$ and some $z^0 \in D$, we have $k = \hat{f}$ as the only continuous mapping $k \colon \bar{D} \to \mathbb{C}^m$ such that k is analytic in D, $k(z^0) = \hat{f}(z^0)$, and $k(z)$ belongs to the convex hull of M_z for all $z \in S$.*

NOTE. By Proposition 1(iii), in order to verify that (I) of Theorem 1 holds we need only check that (9) holds.

Proof of Theorem 4. Since D is strictly pseudoconvex, S is of type 2. Hence the conditions of Theorem 2 and Corollary 1 are satisfied, so $\mathcal{F}$ exists. By an analytic change of variable that is linear in w, we may assume without loss of generality that $q_i(z, w) = 2 \operatorname{Re} w_i$ for $i = 1, 2, 3, \dots, d-1$. Fix $f \in \mathcal{F}$. Then $\operatorname{Re} f_i(z) = 0$ for $i = 1, 2, \dots, d-1$ and for $z \in S = \partial D$, so $\operatorname{Re} \hat{f}_i(z) = 0$ for $z \in \bar{D}$ and the graph of $\hat{f}$ over $\bar{D}$ is contained in $\tilde{M}$. Then, letting ϕ denote the d-form defined in Theorem 1, we have

$$\phi(z,w) = \sum_{j=d}^{m} \frac{\partial q_d}{\partial w_j}(z,w)dw_1 \wedge dw_2 \wedge dw_3 \wedge \cdots \wedge dw_{d-1} \wedge dw_j,$$

where $\phi_{1,2,3,\ldots,d-1,j}(z,w) = (\partial q_d/\partial w_j)(z,w)$ for $j = d$ to m. Furthermore, $\sum_{j=d}^{m}(\partial q_d/\partial w_j)(z, f(z)) f_j(z)$ is nonzero for all $z \in S$ because M_z is a convex hypersurface in $\{w \in \mathbb{C}^m : \operatorname{Re} w_i = 0,\ i = 1, 2, 3, \ldots, d-1\}$ such that M_z encloses the origin. (The reasoning is similar to that used in Theorem 3.) This means that (51) holds, where $h_{1,2,3,\ldots,d-1,j} = f_j$ for $j = d$ to m, so $C(z) = 1/\sum_{j=d}^{m} f_j(z)(\partial q_d/\partial w_j)(z, f(z))$. Therefore, conclusion (52) of Theorem 2 holds. Let $g_j(z) = C(z)(\partial q_d/\partial w_j)(z, f(z))$ for $z \in S$ and for $j = d, d+1, \ldots, m$. For $j = 1$ to $d-1$, let g_j be the zero function on $\bar{D}$. Then, by (52), g_j is CR on S. Let $g\colon S \to \mathbb{C}^m$ be given by $g = (g_1, g_2, \ldots, g_m) = (0, 0, \ldots, 0, g_d, g_{d+1}, \ldots, g_m)$. Then, by the global CR extension theorem, g extends continuously to $\bar{D}$ and analytically to D to produce a mapping $\hat{g} = (\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_m)$. Since $\sum_{i=1}^{m} f_i(z)g_i(z) = 1$ for all $z \in S$ and since f and g both extend analytically to D, neither $\hat{f}$ nor $\hat{g}$ can be zero anywhere on $\bar{D}$. Let $B_m(R)$ be the open ball of radius R about the origin in $\mathbb{C}^m$. Since $\bar{D}$ is polynomially convex, so is $\bar{D} \times \overline{B_m(R)}$ for any $R > 0$. Choose $R > 0$ so large that $\hat{M} \subset \bar{D} \times \overline{B_m(R)}$. Since $\hat{g}$ is never zero on $\bar{D}$, the set

$$W(z,t) \equiv \left\{ w \in \mathbb{C}^m : \sum_{i=1}^{m} \hat{g}_i(z) w_i = t \right\}$$

is a complex hyperplane in $\mathbb{C}^m$ for any $z \in \bar{D}$ and $t > 0$. For any $z \in \bar{D}$ and $t > 1$, this hyperplane comes no closer than t/K to the origin, where K is the maximum modulus of $\hat{g}$ on $\bar{D}$. Fix t_0 so large that t_0/K is greater than R, and let

$$W(t) \equiv \left\{ (z,w) \in \bar{D} \times \mathbb{C}^m : \sum_{i=1}^{m} \hat{g}_i(z) w_i = t \right\}.$$

Then, if $t = t_0$, $W(t_0)$ does not meet $\hat{M}$. Let t_1 be the infimum of all t such that $W(t)$ does not meet $\hat{M}$, and suppose that $t_1 > 1$. Then the function $F_t(z,w) = 1/\big(-t + \big(\sum_{i=1}^{m} \hat{g}_i(z) w_i\big)\big)$ is defined in a neighborhood of $\hat{M}$ in $\bar{D} \times \mathbb{C}^m$ for $t_1 < t \le t_0$. Since D is strictly pseudoconvex, $\hat{g}_i$ is uniformly approximable by functions analytic in a neighborhood of $\bar{D}$. (This theorem comes from [Li; He].) Since $\bar{D}$ is polynomially convex, the Oka–Weil theorem in turn guarantees that $\hat{g}_i$ is uniformly approximable by polynomials on $\bar{D}$ for $i = 1, 2, \ldots, m$, so F_t is the uniform limit on $\hat{M}$ of functions analytic in a neighborhood of $\hat{M}$ in $\mathbb{C}^\ell \times \mathbb{C}^m$ for $t_1 < t \le t_0$. Again by the Oka–Weil theorem, for $t_1 < t \le t_0$ we have that F_t is uniformly approximable by polynomials on $\hat{M}$, so the supremum of F_t on $\hat{M}$ is less than or equal to the supremum of F_t on M (by the definition of polynomial hull). For the same t, we can see that F_t is uniformly bounded on M: for every $z \in S$, M_z is a strictly convex hypersurface in $\tilde{M}_z \equiv \{w \in \mathbb{C}^m : (z,w) \in \tilde{M}\}$ and $W(z,1) \cap \tilde{M}_z$ is tangent to M_z in $\tilde{M}_z$ at $f(z)$. Thus, for all $t > 1$, $W(z,t)$ is a dilation of $W(z,1)$ away from the origin of $\mathbb{C}^m$ (which is contained in the convex hull of M_z), so $W(z,t)$ is disjoint from M_z. For $t > t_1 > 1$, the distance from $W(z,t)$

to M_z is then bounded below in $z \in \partial D$, so F_t is uniformly bounded on M for $t > t_1 > 1$. We conclude, from the observation that the F_t are uniformly approximable by polynomials on $\hat{M}$ for $t_1 < t < t_0$, that the F_t are uniformly bounded on $\hat{M}$ for $t_1 < t \leq t_0$. This is impossible because, for t near t_1, the singularity set $W(t)$ of F_t approaches $\hat{M}$ by definition of t_1. Thus we must have $t_1 = 1$ and that $W(t)$ is external to $\hat{M}$ for $t > 1$. Note that the graph of $z \mapsto t\hat{f}(z)$ lies in $\tilde{M}$ and also lies in $W(t)$, since $\sum_{i=1}^m \hat{g}_i(z)\hat{f}_i(z) = 1$ for all $z \in \bar{D}$. Thus the graph of $t\hat{f}$ lies external to $\hat{M}$ but inside $\tilde{M}$. This being true for all $t > 1$, we find that the graph of $\hat{f}$ is in the boundary of $\hat{M}$ as a subset of $\tilde{M}$, since every point $(z, \hat{f}(z))$ on the graph of $\hat{f}$ over $\bar{D}$ is the limit point of a set of points $\{(z, t\hat{f}(z)) : t > 1\}$ in $\tilde{M}$ that is external to $\hat{M}$.

Finally, suppose k exists as indicated in the theorem. Then the function $z \mapsto \sum_{i=1}^m \hat{g}_i(z)k_i(z) - t$ is nonzero for $t > 1$ and $z \in \partial D$ because, as we have shown, $W(z,t)$ is disjoint from M_z (and so disjoint also from the convex hull of M_z) for all $z \in \partial D$ and $t > 1$. Thus $z \mapsto \sum_{i=1}^m \hat{g}_i(z)k_i(z) - t$ is nonzero for $z \in D$ as well. If we take a limit as $t \to 1^+$, we find from Hurwitz's theorem that $z \mapsto \sum_{i=1}^m \hat{g}_i(z)k_i(z) - 1$ is either identically zero on D or nonzero for $z \in D$. The latter cannot be the case because $\sum_{i=1}^m \hat{g}_i(z^0)k_i(z^0) - 1 = \sum_{i=1}^m \hat{g}_i(z^0)\hat{f}_i(z^0) - 1 = 0$. Since $W(z,1) \cap \tilde{M}_z$ is a tangent plane to M_z in $\tilde{M}_z$ at $f(z)$ for all $z \in \partial D$, the only point w on both $W(z,1)$ and the convex hull of M_z is $f(z)$. But since $\sum_{i=1}^m \hat{g}_i(z)k_i(z) - 1 = 0$ for all $z \in \partial D$, it follows that $k(z)$ is in $W(z,1)$; it is in the convex hull of M_z by assumption. Thus $k(z) = \hat{f}(z)$ for all $z \in \partial D$ and hence for all $z \in \bar{D}$, as desired. □

When M has fibers M_z that are real hypersurfaces in $\mathbb{C}^m$ (i.e., $d = 1$), the following re-statement of Theorem 4 is of interest.

COROLLARY 3. *Suppose that S and M satisfy* (4) *and* (5) *and that S is a hypersurface bounding a bounded strictly pseudoconvex open set D such that $\bar{D}$ is polynomially convex. Let S be connected and simply connected. Suppose that M is compact and that, for every $z \in S$, M_z is a hypersurface enclosing a strictly convex open set in $\mathbb{C}^m$ such that the origin of $\mathbb{C}^m$ lies in that open set. Suppose that any of the properties* (I)–(IV) *of Theorem 1 hold. Then the set $\mathcal{F}$ is well-defined and, for all $f \in \mathcal{F}$, the graph of $\hat{f}$ in $\bar{D} \times \mathbb{C}^m$ lies in the boundary of $\hat{M}$. In particular, given $f \in \mathcal{F}$ and some $z^0 \in D$, we have $k = \hat{f}$ as the only mapping $k\colon \bar{D} \to \mathbb{C}^m$ such that k is analytic in D, $k(z^0) = \hat{f}(z^0)$, and $k(z)$ belongs to the convex hull of M_z for all $z \in S$.*

Proof. This is simply the case $d = 1$ of Theorem 4. □

In Corollary 4, we consider the case where the functions α_j^i are defined only for $i = 1$ to $d - 1$.

COROLLARY 4. *Suppose that S and M satisfy* (4) *and* (5) *and that S is a hypersurface bounding a strictly pseudoconvex bounded open set D such that $\bar{D}$ is polynomially convex. Let S be connected and simply connected. Suppose that, in* (5), *the defining function $q_d(z,w)$ satisfies the property that, for all $z \in S$, $q_d(z,w)$ is strictly convex as a function of w in a neighborhood of M_z. Suppose further that, for $i = 1, 2, 3, \ldots, d-1$, the defining function q_i has the form*

$$q_i(z,w) = \mathrm{Re}\left(\sum_{j=1}^{m} \alpha_j^i(z) w_j\right)$$

for some matrix $(\alpha_j^i)_{j=1}^{m}{}_{i=1}^{d-1}$ of functions analytic in a neighborhood of $\bar{D}$. Let $\tilde{M}$ be as before, and assume that M is compact and that the origin of $\mathbb{C}^m$ is in the bounded (convex) component of $\tilde{M}_z \setminus M_z$ for all $z \in S$. Suppose that any of the properties (I)–(IV) of Theorem 1 hold, that $m \geq 2\ell$, and that $d-1 \leq \ell$. Then the set $\mathcal{F}$ is well-defined and, for all $f \in \mathcal{F}$, the graph of $\hat{f}$ in $\bar{D} \times \mathbb{C}^m$ lies in the boundary of $\hat{M}$ as a subset of $\tilde{M}$. In particular, given $f \in \mathcal{F}$ and some $z^0 \in D$, we have $k = \hat{f}$ as the only continuous mapping $k\colon \bar{D} \to \mathbb{C}^m$ such that k is analytic in D, $k(z^0) = \hat{f}(z^0)$, and $k(z)$ belongs to the convex hull of M_z for all $z \in S$.

Proof. The assumptions are different from Theorem 4 in that the matrix (α_j^i) is defined only for $i = 1$ to $d-1$ and we require $d-1 \leq \ell$ and $m \geq 2\ell$. Since (by (5)) the $\partial_w q_i$ are pointwise linearly independent on M ($i = 1$ to $d-1$), the matrix $(\alpha_j^i(z))$ has maximal rank for all $z \in S$ and so for all $z \in \bar{D}$. By [SW, Thm. 2.2], the matrix (α_j^i) can be extended to be defined for $i = 1$ to m as in Theorem 4, so that the determinant of (α_j^i) is nonzero on $\bar{D}$. Corollary 4 then follows immediately from Theorem 4. □

7. Examples

We now present some examples. Let B_n be the open unit ball in $\mathbb{C}^n$.

EXAMPLE 1. Let $g = (g_1, g_2, \ldots, g_m)$ and $k = (k_1, k_2, \ldots, k_m)$ be $\mathbb{C}^m$-valued mappings analytic in a neighborhood of the closed unit ball $\bar{B}_\ell$ in $\mathbb{C}^\ell$. Suppose that, for each $i = 1, 2, \ldots, m$, g_i is never zero on $\bar{B}_\ell$. In (4), let us suppose that S is the unit sphere in $\mathbb{C}^\ell$, so $c = 1$ and $p_1(z) = \|z\|_\ell^2 - 1$, where $\|\cdot\|_n$ denotes Euclidean length in $\mathbb{C}^n$, $n \geq 1$. In (5), let $d = 1$ and $q_1(z,w) = \sum_{i=1}^m |g_i(z)w_i - k_i(z)|^2 - 1$, so $M = \{(z,w) \in S \times \mathbb{C}^m : \sum_{i=1}^m |g_i(z)w_i - k_i(z)|^2 - 1 = 0\}$. We claim that the mappings determined by Corollary 1 are all mappings of the form

$$\begin{aligned} &\bar{B}_\ell \to \mathbb{C}^m, \\ &z \mapsto f(z) \equiv \left(\frac{k_1(z)+a_1}{g_1(z)}, \frac{k_2(z)+a_2}{g_2(z)}, \ldots, \frac{k_m(z)+a_m}{g_m(z)}\right), \end{aligned}$$

where $a = (a_1, a_2, \ldots, a_m) \in \mathbb{C}^m$ is a constant of modulus 1. The graphs of these maps in $S \times \mathbb{C}^m$ clearly foliate M. Furthermore, $(\partial q_1/\partial w_i)(z,w) = g_i(z)(\overline{g_i(z)w_i - k_i(z)})$ for $i = 1, 2, \ldots, m$, so $(\partial q_1/\partial w_i)(z, f(z)) = \bar{a}_i g_i(z)$ for all such i. Since $\bar{a}_i$ is constant, it is CR on S and, if we let $C(z) \equiv 1$, then (III) of Theorem 1 is satisfied.

Now suppose that $\ell = m$, $g_i = 1$, and $h_i = 0$ for $i = 1, 2, \ldots, m$. Let $T = (T_1, T_2, \ldots, T_\ell)$ be an automorphism of $\bar{B}_\ell$. Then the graph of T over ∂B_ℓ in

$\partial\bar{B}_\ell \times \bar{B}_\ell$ lies in M. However, such a graph is not in the boundary of the polynomial hull of M (which is $\bar{B}_\ell \times \bar{B}_\ell$) and hence, by Corollary 3, these graphs do not arise from elements of $\mathcal{F}$. Also, $(\partial q_1/\partial w_i)(z, T(z)) = \overline{T_i(z)}$. Given any $z^0 \in \partial B_\ell$, there is no nonzero $C(z)$ defined on ∂B_ℓ such that, for all i, $C(z)\overline{T_i(z)}$ is CR in z near z^0; if there were, then $\sum_{i=1}^m C(z)\overline{T_i(z)}T_i(z) = \sum_{i=1}^m C(z)|T_i(z)|^2 = C(z)$ would be CR near z^0. Thus, for all i, $\overline{T_i(z)}$ would be CR in z near z^0, which is impossible since this would imply that the derivative of T on the sphere is degenerate near z^0.

EXAMPLE 2. Let $g = (g_1, g_2)$ be a $\mathbb{C}^2$-valued analytic mapping defined in a neighborhood of $\bar{B}_\ell$. We suppose that $S = \partial B_\ell$ with $p_1(z) = \|z\|_\ell^2 - 1$. As for M, we let $d = 2$ in (5), so that M is defined in $S \times \mathbb{C}^2$ by two real-valued functions q_1, q_2 defined as follows. Let $h = (h_1, h_2)$ be a $\mathbb{C}^2$-valued mapping analytic in a neighborhood of $\bar{B}_\ell$ that is never zero on $\bar{B}_\ell$. For $(z, w) \in S \times \mathbb{C}^2$, let $q_1(z, w) = \|w - g(z)\|_2^2 - \|h(z)\|_2^2$ and $q_2(z, w) = 2\,\mathrm{Re}\big(\sum_{i=1}^2 h_i(z)(w_i - g_i(z))\big)$. Consider the class of functions

$$\begin{aligned} &S \to \mathbb{C}^m,\\ &z \mapsto g(z) + e^{i\theta}(-h_2(z), h_1(z)), \end{aligned}$$

where θ ranges over the real numbers. Then we claim that the graphs of these functions foliate M and in fact satisfy (III) of Theorem 1. We let θ be an arbitrary real number and let $f(z) = g(z) + e^{i\theta}(-h_2(z), h_1(z))$. Then we calculate

$$\begin{aligned} q_1(z, f(z)) &= \|g(z) + e^{i\theta}(-h_2(z), h_1(z)) - g(z)\|_2^2 - \|h(z)\|_2^2\\ &= \|(-h_2(z), h_1(z))\|_2^2 - \|h(z)\|_2^2 = 0 \end{aligned}$$

for all $z \in S$ and

$$\begin{aligned} q_2(z, f(z)) &= 2\,\mathrm{Re}\left(\sum_{i=1}^2 h_i(z)(f_i(z) - g_i(z))\right)\\ &= 2\,\mathrm{Re}[(e^{i\theta})(-h_1(z)h_2(z) + h_2(z)h_1(z))] = 0, \end{aligned}$$

as desired. Furthermore, we also calculate $\partial_w q_1(z, w) = (\bar{w}_1 - \bar{g}_1(z))dw_1 + (\bar{w}_2 - \bar{g}_2(z))dw_2$ and $\partial_w q_2(z, w) = h_1(z)dw_1 + h_2(z)dw_2$. Then

$$\begin{aligned} &\partial_w q_1(z, f(z)) \wedge \partial_w q_2(z, f(z))\\ &\quad = \big(h_2(z)(\overline{f_1(z)} - \overline{g_1(z)}) - h_1(z)(\overline{f_2(z)} - \overline{g_2(z)})\big)dw_1 \wedge dw_2\\ &\quad = -e^{-i\theta}(|h_1(z)|^2 + |h_2(z)|^2)dw_1 \wedge dw_2\\ &\quad = -e^{-i\theta}\|h(z)\|_2^2 dw_1 \wedge dw_2. \end{aligned}$$

Letting $C(z) \equiv 1/\|h(z)\|_2^2$ in (III) of Theorem 1, we find that the properties of (III) are satisfied for f and C. (The key fact is that h is never zero on the closed ball.)

These examples lack certain features that more general examples would possess. In each case, there are affine maps $A\colon M_{z_1} \to M_{z_2}$ such that $A(f(z_1)) = A(f(z_2))$, where f is any one of the CR mappings whose graph is contained in M. We do not expect this kind of structure in general; indeed, we could perturb M near a point but not elsewhere. We would then lose CR graphs in M near that point but would still have other CR graphs in M because our graphs arose principally from local properties of M.

8. Notation Used Frequently

Notation	Where defined	Notation	Where defined	Notation	Where defined
c	(4)	$\mathrm{H}^S, \mathrm{H}^S_z$	(4)	p_i	(4)
$C(z)$	Thm. 1(III)	ℓ	(4)	q_i	(5)
d	(5)	$\mathcal{L}^M$	§1, (3)	$s_z, s(z)$	near (32)
D	Cor. 1	m	(5)	s_i	(11)
$\partial_w P$	(1)	M, M_z	(5)	S	(4)
ϕ, ϕ_i	Thm. 1	M_K	Cor. 1	t_i	(23)
f	(31)	$\tilde{M}$	Thm. 4	T_i	(24)
$\mathcal{F}$	(53)	M^f	(31)	τ	Thm. 1
G, G^M	near (11), (12)	$N, \bar{N}, N(z, w)$	(8)	$\mathcal{T}$	Thm. 1 (proof)
$\Gamma(A, B)$	(2)	n_i	(19)	U	Thm. 1(III)
$\mathrm{H}^M, \mathrm{H}^M_{(z,w)}$	(5)	$\pi, \pi_{(z,w)}$	(6)	$V, \bar{V}, V(z, w)$	(7)

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