Michigan Math. J. 50 (2002)

A Gluing Formula for the Seiberg–Witten Invariant along T^3

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1. Introduction

This paper is a continuation of studies initiated in [P1]. For the definition and basic properties of the Seiberg–Witten monopole invariant, we refer the reader to the bibliography in [P1]. Our hope is that these studies will ultimately yield a useful theory of Floer-type cohomology for 3-manifolds that is *infinitely* generated. The present goal of this paper is to provide a method of computing the Seiberg–Witten (SW) invariant of a smooth 4-manifold that can be decomposed into two parts along an embedded 3-torus. Under some mild assumptions, we prove a gluing formula for the SW invariant in terms of products of suitably perturbed relative SW invariants of the two end pieces whose common boundary is T^3. In particular, our formula does not require that one of the glued-up pieces be $T^2 \times D^2$, as is the case in [MMS]. We shall derive some interesting applications of this product formula and others in future work [P2].

Acknowledgments. The author would like to thank Zoltán Szabó for generously providing some much-needed guidance and inspiration. He would also like to thank Tian-Jun Li, Michael McCooey, and the referee for very helpful comments and suggestions. Part of this work was done while the author was visiting the Korea Institute for Advanced Study.

2. Perturbed Solutions over the 3-Torus

We study the Seiberg–Witten equations over the 3-manifold $Y = T^3$. We shall always view Y as the trivial S^1 bundle over the 2-torus. Let Σ be the base space T^2; that is, $Y = \Sigma \times S^1$. Note that Y is the unit circle bundle of the canonical line bundle K_Σ over Σ $(\deg(K_\Sigma) = 0)$.

Choose a constant curvature connection on the unit circle bundle Y and let $i\zeta$ denote the corresponding connection form. Let g_Σ be a constant curvature metric on the surface Σ, normalized so that the area of Σ is equal to 1. We endow Y with the metric

$$h_Y = \zeta \otimes \zeta + \pi^*(g_\Sigma),$$

where $\pi : Y \to \Sigma$ is the bundle projection map. Of course, the global 1-form ζ allows a reduction in the structure group of TY to SO(2), and the Levi–Civita

Received June 30, 2001. Revision received February 25, 2002.

connection on Σ induces a reducible connection on Y that respects this splitting. We study the moduli space of solutions to the Seiberg–Witten equations over Y, using the preceding metric and connection on TY.

We consider the following *perturbed* Seiberg–Witten equations on Y corresponding to a $\mathrm{Spin}^c(3)$ structure W:

$$\begin{aligned} F_A &= \tau(\Psi) + ir\pi^*\mu_\Sigma, \\ \partial\!\!\!/_A\Psi &= 0, \end{aligned} \tag{2.1}$$

where $r \in \mathbb{R}$ is a fixed nonzero real parameter, μ_Σ is the volume form on Σ, and $\pi: Y \to \Sigma$ is the projection map. Here, $\tau: \Gamma(Y, W) \to \Omega^2(Y, i\mathbb{R})$ is the quadratic map adjoint to the Clifford multiplication. We let $\gamma = [\{\text{point}\} \times S^1] \in H_1(Y; \mathbb{Z})$ be the Poincaré dual of $[\mu_\Sigma] \in H^2(Y; \mathbb{Z})$.

As in [MOY, Secs. 5.5–5.7], we identify the Seiberg–Witten moduli space with the moduli space of Kähler vortices on Σ. (Contrary to the hypothesis in [MOY], Y has degree 0 but the identification is still valid.) In the notation of [MOY], the vortex equations read

$$\begin{gathered} 2F_{B_0} - F_{K_\Sigma} = i(r + |\alpha_0|^2 - |\beta_0|^2)\mu_\Sigma, \\ \bar\partial_{B_0}\alpha_0 = 0 \quad \text{and} \quad \bar\partial^*_{B_0}\beta_0 = 0, \\ \alpha_0 \equiv 0 \quad \text{or} \quad \beta_0 \equiv 0. \end{gathered} \tag{2.2}$$

Here, B_0 is a connection on a Hermitian line bundle E_0 over Σ, and α_0 and β_0 are sections in $\Gamma(\Sigma, E_0)$ and $\Gamma(\Sigma, K_\Sigma^{-1} \otimes E_0)$, respectively.

For generic r, we immediately see that there is no reducible solution to the perturbed SW equations (2.1). For generic negative values of r with $|r|$ very small, there is only one $\mathrm{Spin}^c(3)$ structure on Y for which the corresponding SW moduli space of irreducible solutions is not empty. This is because we must have, by virtue of the vortex equations (2.2), $\deg(E_0) = 0$, $\beta_0 \equiv 0$, and $\alpha_0 = \text{constant}$. Thus the canonical $\mathrm{Spin}^c(3)$ structure, $\mathbb{C} \oplus K_\Sigma^{-1}$, is the only $\mathrm{Spin}^c(3)$ structure that has nonempty SW solution space. We denote this trivial $\mathrm{Spin}^c(3)$ structure by $\mathcal{L}_0$. The connections in this solution space correspond to constant sections of the trivial line bundle over T^2 and hence, after dividing by the gauge group action, $\mathcal{M}(Y) = \mathcal{M}_{\mathrm{sw}}(Y, \mathcal{L}_0, r\pi^*\mu_\Sigma) = \mathrm{Sym}^0(\Sigma) = \{\text{point}\}$.

LEMMA 1 (cf. [MOY; P1]). *The single point set $\mathcal{M}(Y)$ is smooth* (*nondegenerate*) *in the sense that it is transversally cut out by the Seiberg–Witten equations* (2.1) *modulo gauge.*

For generic small positive values of r, we similarly have $\deg(E_0) = 0$, $\alpha_0 \equiv 0$, and $\beta_0 = \text{constant}$. As in the negative case, the SW moduli space $\mathcal{M}(Y)$ consists of a single smooth point corresponding to $\mathcal{L}_0$. Note that our philosophy diverges from [MMS], wherein different choices of metric and Dirac operator were made, resulting in reducible solutions of which some were actually degenerate.

Now suppose that Y is the boundary of some smooth 4-manifold M. Then we have a distinguished subgroup of the gauge group $\mathcal{G}_0(M) \subset \mathcal{G}(Y)$ consisting of maps u that can be extended to $u\colon M \to U(1)$. Dividing out by the action of $\mathcal{G}_0(M)$ instead of the full gauge group, we obtain another moduli space $\tilde{\mathcal{M}}(Y)$. Of course this moduli space depends on M. Note that dividing by $\mathcal{G}(Y)$ gives a covering $p\colon \tilde{\mathcal{M}}(Y) \to \mathcal{M}(Y)$. The fiber of p is $H^1(Y;\mathbb{Z})/i^*(H^1(M;\mathbb{Z}))$, where $i\colon Y \hookrightarrow M$ is the inclusion map.

3. Solutions over the Cylinder

We consider the infinite cylinder $Y \times \mathbb{R}$. Given a Spin^c structure on $Y \times \mathbb{R}$, let W^+ and W^- be the associated Spin^c bundles. Clifford multiplication defines a linear map

$$\rho\colon i\Lambda^2 \to \mathrm{End}_{\mathbb{C}}(W^+)$$

whose kernel is $i\Lambda^-$. We denote $L = \det(W^+)$ and write $\mathcal{A}(L)$ for the affine space of connections on L. We pull back the perturbing form on Y of the previous section and obtain the following 4-dimensional Seiberg–Witten equations for a pair $(A,\phi) \in \mathcal{A}(L) \times \Gamma(W^+)$:

$$\begin{aligned} \partial\!\!\!/_A \phi &= 0, \\ \rho(F_A - ir\pi_1^*\pi^*\mu_\Sigma) &= q(\phi) = \phi^* \otimes \phi - \frac{|\phi|^2}{2}\,\mathrm{Id}, \end{aligned} \tag{3.1}$$

where $\pi_1\colon Y \times \mathbb{R} \to Y$ is the projection map. We identify $L = L_0 \times \mathbb{R}$, where L_0 is a complex line bundle over Y. Similarly, $W^+ = W_0 \times \mathbb{R}$, where W_0 is the Spin^c bundle over Y with respect to the Spin^c structure inherited from $Y \times \mathbb{R}$. As shown in [KM], equations (3.1) then become the gradient flow equation for the Chern–Simons–Dirac functional $C\colon \mathcal{A}(L_0) \times \Gamma(W_0) \to \mathbb{R}$ given by

$$C(A,\phi) = \int_Y (F_{A_0} + \xi) \wedge a + \frac{1}{2}\int_Y a \wedge da + \int_Y \langle \phi, \partial\!\!\!/_A \phi\rangle \, d\,\mathrm{vol},$$

where $\xi = -ir\pi_1^*\pi^*\mu_\Sigma$, A_0 is a fixed connection on L_0, and $a = A - A_0$.

Let M be a compact smooth 4-manifold whose boundary is Y. Assume that the 2-form $ir\pi^*\mu_\Sigma$ on Y extends to a closed 2-form on M. Then C descends to a real-valued function on the space $\tilde{\mathcal{B}} := (\mathcal{A}(L_0) \times \Gamma(W_0))/\mathcal{G}_0(M)$, where $\mathcal{G}_0(M) \subset \mathcal{G}(Y)$ is as in Section 2. From now on we shall always view C as a functional on $\tilde{\mathcal{B}}$ for some fixed M. Note that the set of critical points of C is the moduli space $\tilde{\mathcal{M}}(Y)$.

To ensure the compactness of the cylindrical end moduli spaces in the next section, we need to further perturb equation (3.1) using a method due to Frøyshov. This extra perturbation will allow us to treat C as if it were a *perfect* Morse function. We briefly recall the necessary definitions from [Fr]. Let $f_1\colon \mathbb{R} \to [0,\infty)$ be a smooth function supported in the interval $[-1,1]$ and satisfying $\int f_1 = 1$. Let $f_2\colon \mathbb{R} \to \mathbb{R}$ be a smooth function with compact support such that $f_2(t) = t$ on some interval containing all critical values of C. If A is any connection on L and if ϕ is a section of W^+, then we let $S = (A,\phi)$ and define a smooth function $h_S\colon \mathbb{R} \to \mathbb{R}$ by

$$h_S(T) = \int_{\mathbb{R}} f_1(t_1 - T) f_2\left(\int_{\mathbb{R}} f_1(t_2 - t_1) C(S_{t_2})\, dt_2 \right) dt_1,$$

where $S_t = S(t)$ is the restriction of S to $Y \times \{t\}$.

We choose a compactly supported 2-form $\omega \in \Omega^2(Y \times \mathbb{R})$ such that the norm $\|\omega\|_{C^k}$ is very small. (Here, k is some fixed integer that is sufficiently large.) We require the support of ω to lie in a set $Y \times \Xi$, where Ξ is the result of removing from $\mathbb{R}$ a small open interval around each critical value of C. Let $h_S^*(\omega)$ denote the pull-back of ω by the map $(\mathrm{id}_Y \times h_S) \colon Y \times \mathbb{R} \to Y \times \mathbb{R}$. We study the following translation invariant equations for $S = (A, \phi)$:

$$\begin{aligned} \partial\!\!\!/_A \phi &= 0, \\ \rho(F_A - ir\pi_1^*\pi^*\mu_\Sigma + ih^*_{(A,\phi)}(\omega)) &= q(\phi). \end{aligned} \tag{3.2}$$

Now, for a pair of nonnegative real numbers v and w, we let $L_m^{p;v,w}$ be the L_m^p Sobolev space over $Y \times \mathbb{R}$ defined using a weight e^{-vt} on the negative end and e^{wt} on the positive end. (Of course, $L_m^{p;0,0} = L_m^p$.) Let $\mathcal{B} = L_4^{2;v,w}(Y \times \mathbb{R}, i\Lambda^1 \oplus W^+)$ and

$$\mathcal{G} = \{u \colon Y \times \mathbb{R} \to U(1) \mid u \in L^2_{5,\mathrm{loc}};\ du \cdot u^{-1} \in L_4^{2;v,w}\}.$$

Let $x, y \in \tilde{\mathcal{M}}(Y)$ be critical points of C, that is, solutions to (2.1), the perturbed Seiberg–Witten equations on Y. We define the space of "perturbed flowlines" on the cylinder between x and y to be the set

$$\mathcal{F}_\omega(x, y) = \left\{ S \in \mathcal{B} \text{ satisfying (3.2)} \mid \lim_{t \to -\infty} [S_t] = x;\ \lim_{t \to \infty} [S_t] = y \right\} \Big/ \mathcal{G}.$$

Note that the elements of $\mathcal{F}_\omega(x, y)$ satisfy the gradient flow equation for C outside a compact subset of $Y \times \mathbb{R}$. As shown in [Fr, p. 380, (2.3)], there is a natural identification between the $\mathcal{F}_\omega(x, y)$ defined using different exponential weights, provided all the weights v and w are sufficiently small. (Exactly how small depends on x and y.) The following lemma shows that there are no nontrivial flowlines after the Frøyshov perturbation.

LEMMA 2. *For generic small $\omega \in C^k$, $\mathcal{F}_\omega(x, x)$ consists of a single smooth point and $\mathcal{F}_\omega(x, y)$ is empty when $x \neq y$.*

Proof. The first statement follows readily from [Fr, Lemma 4]. Although Frøyshov concentrates only on the case when the 3-manifold Y is an oriented rational homology sphere, the proof in [Fr, Apx. A] still goes through with very little modification.

Let Ω^2_Ξ denote the space of C^k 2-forms on $Y \times \mathbb{R}$ with compact support contained in $Y \times \Xi$. As in [Fr, Prop. 5], one can show that the linearization of equations (3.2) at a point (ω, A, ϕ),

$$F = F_{(\omega, A, \phi)} \colon \Omega^2_\Xi \times \mathcal{B} \to L_3^{2;v,w}(Y \times \mathbb{R},\ i\Lambda^0 \oplus i\Lambda^+ \oplus W^-),$$

is Fredholm on the slices $\{\omega\} \times \mathcal{B}$ and is surjective whenever (ω, A, ϕ) is a solution to equations (3.2). The Smale–Sard theorem then implies that $\mathcal{F}_\omega(x, y)$ is a smooth manifold for generic ω.

Now we let $\mathcal{P} = \mathcal{P}(0, \omega)$ be the space of L^2_1 maps,

$$\nu\colon [0,1] \to L^{2;\,v,w}_3(Y \times \mathbb{R}, \Lambda^2),$$

satisfying $\nu(0) = 0$ and $\nu(1) = \omega$. We define a map

$$G\colon \mathcal{B} \times \mathcal{P}(0,\omega) \times [0,1] \to L^{2;\,v,w}_3(Y \times \mathbb{R},\ i\Lambda^+ \oplus W^-)$$

by

$$G(a, \phi, \nu, t) = (F^+_{A_0+a} + \eta(t)^+ - \tau(\phi),\ \partial\!\!\!/_{A_0+a}(\phi)),$$

where $\eta(t) = ih^*_{(A_0+a,\phi)}(\nu(t)) - ir\pi_1^*\pi^*\mu_\Sigma$. One can show that the differential DG is surjective at every point (a, ϕ, ν, t) for which G vanishes. Let $\mathbf{M}$ denote the zero set $G^{-1}(0)$ modulo the weighted L^2_5 gauge transformations. Let $\mathbf{F}$ be a generic fiber of the projection $\mathbf{M} \to \mathcal{P}$ onto the second factor. Note that the boundary of $\mathbf{F}$ consists of two ends, one of which is cut out by the gradient of the Chern–Simons–Dirac functional C (whose linearization always has index 0 on the critical set). Thus, the corresponding boundary components have *expected* dimension 0. It follows that $\mathbf{F}$ is a 1-dimensional smooth manifold with boundary. Consequently, the expected dimension of the space of "perturbed flowlines" *modulo* the weighted L^2_5 gauge transformations must be zero. But recall from [Fr] that the solutions to equations (3.2) are translation invariant in the $\mathbb{R}$-direction. Hence, after dividing out by the weighted $L^2_{5,\mathrm{loc}}$ gauge, the expected dimension of $\mathcal{F}_\omega(x, y)$ is (–1), which implies that $\mathcal{F}_\omega(x, y)$ is empty for generic ω. $\square$

4. Moduli Space over a Cylindrical End Manifold

Now suppose that X is a smooth oriented 4-manifold and that the end of X is diffeomorphic to $Y \times [0, \infty)$. Assume that the intersection form of X is not negative definite and that the end perturbation $\xi = -ir\pi_1^*\pi^*\mu_\Sigma$ extends to a closed 2-form $\bar{\xi}$ over the whole manifold X. Fix a Riemannian metric h on Y (as in the previous section) and a Riemannian metric g on X such that g is equal to $h + dt^2$ at the cylindrical end of X. We look at the perturbed SW equations

$$\begin{aligned} \partial\!\!\!/_A\phi &= 0, \\ \rho(F_A + \eta) &= q(\phi), \\ \eta &= f \cdot (ih^*_{(A,\phi)}(\omega) - ir\pi_1^*\pi^*\mu_\Sigma), \end{aligned} \tag{4.1}$$

where $f\colon X \to [0,1]$ is a suitable cut-off function that vanishes away from the cylindrical end of X. Note that the perturbing 2-form η actually depends on the unknowns (A, ϕ).

We require our configuration (A, ϕ) to lie in $\mathcal{A}_{L^2_4}(\det \mathcal{L}) \times L^2_4(X, W^+(\mathcal{L}))$, where $\mathcal{A}_{L^2_4}(\det \mathcal{L})$ denotes the space of L^2_4 unitary connections on the line bundle $\det \mathcal{L}$ and $W^+(\mathcal{L})$ is the positive spinor bundle for the Spin^c structure $\mathcal{L}$. The *energy* of a solution (A, ϕ) is defined to be the total variation of the Chern–Simons–Dirac functional C over the cylindrical end $Y \times [0, \infty)$,

$$\sup\{C(S_t) - C(S_{t'}) \mid t, t' \in [0, \infty)\}.$$

As before, S_t denotes the restriction of (A, ϕ) to the slice $Y \times \{t\}$. The cylindrical end moduli space $\mathcal{M}^r_X(\mathcal{L}, g, \omega)$ is defined by dividing the space of finite energy solutions by the action of the $L^2_{5,\mathrm{loc}}$ gauge group $\mathcal{G}(X)$. Note that every solution to (4.1) is irreducible; that is, $\phi \not\equiv 0$.

LEMMA 3. *If $\mathcal{L}$ does not restrict to $\mathcal{L}_0$ on the slice Y, then $\mathcal{M}^r_X(\mathcal{L}, g, \omega)$ is empty. If $\mathcal{L}|_Y = \mathcal{L}_0$, then $\mathcal{M}^r_X(\mathcal{L}, g, \omega)$ is a smooth oriented manifold of dimension*

$$d = d(\mathcal{L}) = \tfrac{1}{4}(c_1(\det \mathcal{L})^2 - 2\,\mathrm{e}(X) - 3\,\mathrm{sign}(X)).$$

Moreover, by taking limits at the open noncompact end of the infinite cylinder $Y \times [0, \infty)$, we have a continuous map

$$\partial_\infty \colon \mathcal{M}^r_X(\mathcal{L}, g, \omega) \to \tilde{\mathcal{M}}(Y).$$

For each point $x \in \tilde{\mathcal{M}}(Y)$, the preimage $\partial^{-1}_\infty(x)$ is compact. There is a constant $v_r > 0$ such that every solution $[(A, \phi)] \in \partial^{-1}_\infty(x)$ decays exponentially to x with exponent at least v_r; that is, the L^2_4 distance between x and the restriction $(A(t), \phi(t))$ is less than $\exp(-v_r t)$ for all t large.

Proof. Suppose that $[(A, \phi)] \in \mathcal{M}^r_X(\mathcal{L}, g, \omega)$. Since (A, ϕ) has finite energy, it follows from [KM, Prop. 8] that $\mathcal{M}(Y, \mathcal{L}|_Y)$ is not empty. Now the results from Section 2 imply that $\mathcal{L}|_Y = \mathcal{L}_0$. The smoothness of $\mathcal{M}^r_X(\mathcal{L}, g, \omega)$ for a small generic 2-form ω follows from what is now a "standard" argument, which we choose to omit. As in the closed case, a cohomology orientation of the pair $(X, \partial X)$ induces an orientation of $\mathcal{M}^r_X(\mathcal{L}, g, \omega)$. The existence of the continuous map ∂_∞ follows from the arguments in [MMS]. Given a point $x \in \tilde{\mathcal{M}}(Y)$, we can calculate the formal dimension of $\partial^{-1}_\infty(x)$, and hence of $\mathcal{M}^r_X(\mathcal{L}, g, \omega)$, by the index formula of [APS], which gives

$$\dim(\partial^{-1}_\infty(x)) = \dim \mathcal{M}^r_X(\mathcal{L}, g, \omega) = \tfrac{1}{4}(c_1(\det \mathcal{L})^2 - 2\,\mathrm{e}(X) - 3\,\mathrm{sign}(X)).$$

Note that the eta invariant (or rho invariant) of the linearization of (2.1) on Y is zero. (This is because Y admits an orientation-reversing self-diffeomorphism and $\mathrm{eta}(-Y) = -\mathrm{eta}(Y)$.) Lemma 2 implies that every finite energy flowline over the cylinder $Y \times \mathbb{R}$ is *static*—that is, pulls back from Y. Hence the arguments in [KM, Lemma 4] imply that the preimage $\partial^{-1}_\infty(x)$ is compact. The statement about exponential decay can be proved as in [MMR, Chap. 5]. □

5. Relative Seiberg–Witten Invariant

Let M be a smooth oriented compact 4-manifold with boundary, and suppose that ∂M is diffeomorphic to Y. Let $\mathcal{K}(M)$ denote the set of isomorphism classes of Spin^c structures on M that restrict to $\mathcal{L}_0$ on ∂M.

DEFINITION 4. Let γ denote the homology class of the circle fiber of Y as before, and let $i \colon \partial M \hookrightarrow M$ be the inclusion map. We shall say that M is *admissible* if the following two conditions are satisfied:

(i) $m\gamma \in \mathrm{Ker}[i_*: H_1(\partial M;\mathbb{Z}) \to H_1(M;\mathbb{Z})]$ for some positive integer m.
(ii) M is not negative definite.

Condition (ii) means that the quadratic form associated to the intersection pairing on $H^2(M, \partial M; \mathbb{Z})$ is not negative definite. For an admissible M, we shall define the corresponding noncompact cylindrical end manifold $\hat{M} := M \cup_Y Y \times [0, \infty)$ and then choose a cylindrical end metric g on $\hat{M}$. Condition (i) ensures that the perturbation ξ extends to a closed 2-form $\bar{\xi}$ over $\hat{M}$. The goal of this section is to define the relative Seiberg–Witten invariant

$$\mathrm{SW}_M: \mathcal{K}(M) \times \tilde{\mathcal{M}}(\partial M) \to \mathbb{Z}$$

using moduli spaces over $\hat{M}$. Given $\mathcal{L} \in \mathcal{K}(M)$, we continue to denote the corresponding Spin^c structure on $\hat{M}$ by $\mathcal{L} \to \hat{M}$. Let $\mathcal{M}^r_{\hat{M}}(\mathcal{L}, g, \omega)$ be the cylindrical end moduli space of the previous section. Now suppose that $d(\mathcal{L}) \equiv 0 \pmod 2$. We take a geometric representative D of $\mu(pt)^{d/2}$ and define

$$\mathcal{N}_M(\mathcal{L}, x;\ r, g, \omega, D) := \mathcal{M}^r_{\hat{M}}(\mathcal{L}, g, \omega) \cap D \cap \partial_\infty^{-1}(x).$$

Note that D is a generic d-codimensional stratified set in the space of configurations, where we can choose D to be supported in a small neighborhood of the base fibration point. For the definition and properties of the μ map, we refer the reader to [Sa] or the last section of [OS2].

DEFINITION 5. Let $M, \hat{M}, \mathcal{L}, g, \eta$ be as before. For a generic D, $\mathcal{N}_M(\mathcal{L}, x;\ r, g, \omega, D)$ is a compact oriented 0-dimensional manifold; by counting its points with signs, we define

$$\mathrm{SW}_M(\mathcal{L}, x) := \#(\mathcal{N}_M(\mathcal{L}, x;\ r, g, \omega, D)).$$

If $d(\mathcal{L}) \equiv 1 \pmod 2$ then we define $\mathrm{SW}_M(\mathcal{L}, x) = 0$. As in the closed case, we say that M is of *simple type* when $\mathrm{SW}_M(\mathcal{L}, x) \neq 0$ only if $d(\mathcal{L}) = 0$.

THEOREM 6. *The function* SW_M *is independent of generic choices of* g, ω, *and* D. *Furthermore, for any orientation-preserving self-diffeomorphism* $f: M \to M$, *we have*

$$\mathrm{SW}_M(\mathcal{L}, x) = (-1)^{\varepsilon}\mathrm{SW}_M(f^*(\mathcal{L}), f^*(x)),$$

where $\varepsilon \in \mathbb{Z}/2$ *is the sign of the action of* f^* *on the cohomology orientation of the pair* $(M, \partial M)$.

Proof. The statements can be proved exactly the same way as in the closed case. □

REMARK 7. We shall see in [P2] that the function SW_M may actually depend on the sign of the real parameter $r \neq 0$ in the perturbation.

6. Stretching Out the Neck

Let M_j ($j = 1, 2$) be smooth compact oriented 4-manifolds with boundary $\partial M_j = Y = T^3$. For any orientation-reversing self-diffeomorphism $\varphi: \partial M_1 \to \partial M_2$, we define a closed oriented 4-manifold $M(\varphi) = M_1 \cup_\varphi M_2$. Given the identifications $\partial M_j = Y$, we can form the closed 4-manifolds $\bar{M}_j = M_j \cup_Y (T^2 \times D^2)$, where the circle $\gamma \subset \partial M_j$ is identified with $[\{\mathrm{pt}\} \times \partial D^2]$. We shall sometimes call $M(\varphi)$ a *fiber sum* and write it as $\bar{M}_1 \#_\varphi \bar{M}_2$. Let $(i_j)_*\colon H_1(Y) \to H_1(M_j)$ and $(i_j)^*\colon H^1(M_j) \to H^1(Y)$ be the homomorphisms induced by the inclusion maps. Let $n_j = \dim \operatorname{Ker}(i_j)_*$.

DEFINITION 8. We shall say that the triple (M_1, M_2, φ) is *admissible* if the following four conditions are satisfied.

(i) $m\gamma \in \operatorname{Ker}(i_j)_*$ for $j = 1, 2$, and $m\varphi_*(\gamma) \in \operatorname{Ker}(i_2)_*$ for some positive integer m.
(ii) $\varphi^*(\operatorname{Image}(i_2)^*) \subset \operatorname{Image}(i_1)^*$.
(iii) Let $\mathcal{H}_j = \operatorname{Coker}(i_j)^*$. Then the $\mathcal{H}_j$ are torsion-free for $j = 1, 2$.
(iv) M_j are not negative definite for $j = 1, 2$.

From now on we will concern ourselves only with admissible triples. Note that a generalized (or topological) logarithmic transformation—with $M_1 = T^2 \times D^2$ and the complement M_2 having neither a cusp nor a fishtail neighborhood—may form an admissible triple.

Given an admissible triple (M_1, M_2, φ), let m_γ denote the smallest possible positive integer satisfying condition (i). We can choose $b_j \in H_2(M_j, \partial M_j)$ such that $\partial b_1 = m_\gamma \gamma$ and $\partial b_2 = m_\gamma \varphi_*(\gamma)$. Let $\Sigma_j \subset M_j$ be smoothly embedded surfaces with boundaries, representing b_j. Let $[\mu] \in H^2(M(\varphi); \mathbb{Z})$ denote the cohomology class that is Poincaré dual to the homology class represented by the closed smooth surface $(\Sigma_1 \cup_{m_\gamma \gamma} \Sigma_2)$ in $M(\varphi)$.

We define a family of metrics on $M(\varphi)$ as follows. First we have the decomposition

$$M(\varphi) \cong M_1 \cup Y \times [-1, 1] \cup M_2.$$

Suppose we are given a metric g on $M(\varphi)$ that is of the form $h + dt^2$ on the neck $Y \times [-1, 1]$, where h is a metric on Y as in Section 2. For each $\ell \geq 1$, let $\lambda_\ell(t)$ be a positive smooth function on the interval $[-1, 1]$ that is identically equal to 1 on $[-1, -1/2] \cup [1/2, 1]$ and satisfies

$$\int_{-1}^{1} \lambda_\ell(t)\, dt = 2\ell.$$

We define a metric g_ℓ to be g on the two ends $M_1 \cup M_2$ and $h + \lambda_\ell(t)^2\, dt^2$ along the neck $Y \times [-1, 1]$. We think of the family $\{g_\ell\}$ as stretching out the neck $Y \times [-1, 1]$ isometrically into $T_\ell = Y \times [-\ell, \ell]$. We denote the Riemannian manifold $(M(\varphi), g_\ell)$ by $M(\varphi)_\ell$.

Next we construct a family of perturbing 2-forms that are supported on the neck T_ℓ. As in Section 3, we choose a compactly supported 2-form $\omega \in \Omega^2(Y \times \mathbb{R})$ such that $\|\omega\|_{L^2_k}$ is very small. Let $k_\ell\colon T_\ell \hookrightarrow Y \times \mathbb{R}$ be the inclusion map. Let W_0 denote the Spin^c bundle over Y corresponding to $\mathcal{L}_0$ and let $L_0 = \det \mathcal{L}_0 = \det W_0$. As in Section 3, we let $W^+ = W_0 \times \mathbb{R}$ and $L = \det W^+$. Suppose that

$\mathcal{L}$ is a Spin^c structure on $M(\varphi)_\ell$ that restricts to $\mathcal{L}_0$ on Y and that $W^\pm(\mathcal{L})$ are the associated Spin^c bundles. Given a pair $(A, \phi) \in \mathcal{A}(\det \mathcal{L}) \times \Gamma(W^+(\mathcal{L}))$, we define the "push-forward" $(k_\ell)_*(A, \phi) \in \mathcal{A}(L) \times \Gamma(W^+)$ as follows. We extend the restriction $(A, \phi)|_{T_\ell}$ over the whole infinite cylinder $Y \times \mathbb{R}$ by constants; that is,

$$(k_\ell)_*(A, \phi)|_{Y\times\{t\}} = \begin{cases} (A, \phi)|_{Y\times\{-\ell\}} & \text{if } t \le -\ell, \\ (A, \phi)|_{Y\times\{t\}} & \text{if } -\ell \le t \le \ell, \\ (A, \phi)|_{Y\times\{\ell\}} & \text{if } t \ge \ell. \end{cases}$$

Using the same notation as before, we define

$$\eta_\ell = f_\ell \cdot (k_\ell)^* \big(i h^*_{(k_\ell)_*(A,\phi)}(\omega) - i r \pi_1^* \pi^* \mu_\Sigma \big),$$

where $f_\ell \colon M(\varphi) \to [0, 1]$ is a suitably chosen cut-off function that vanishes away from the interior of the neck T_ℓ.

Now we consider the following perturbed Seiberg–Witten equations on the closed manifold $M(\varphi)_\ell$:

$$\begin{aligned} \partial\!\!\!/_A \phi &= 0, \\ \rho(F_A + \eta_\ell) &= q(\phi). \end{aligned} \tag{6.1}$$

The corresponding moduli space—or the set of solutions to (6.1) divided by the action of the gauge group—will be denoted by $\mathfrak{M}^r_{M(\varphi)}(\mathcal{L}, g_\ell, \omega)$.

LEMMA 9. *Suppose $S = (A, \phi)$ is a solution to the perturbed Seiberg–Witten equations* (6.1) *on $M(\varphi)_\ell$ corresponding to the* Spin^c *structure $\mathcal{L}$.*

(i) *There is a constant $K > 0$ independent of the neck length ℓ and the* Spin^c *structures such that*

$$0 \le C(S_\ell) - C(S_{-\ell}) \le K + \frac{2\pi r}{m_\gamma}[\mu] \cdot c_1(\det \mathcal{L}).$$

(ii) *There is a constant $K' > 0$ independent of the neck length ℓ (but depending on $\mathcal{L}$) such that the L^2_4 distance between the restriction $(A(t), \phi(t))$ and a static solution is less than*

$$K' \cdot \exp(-\upsilon_r \cdot \min\{t + \ell, \ell - t\})$$

for every $t \in [-\ell, \ell]$, where υ_r is the constant in Lemma 3.

Proof. These estimates can be derived exactly as in [P1] and [MST]. □

7. The Product Formula

Suppose (M_1, M_2, φ) is admissible. As before, let $\tilde{\mathcal{M}}_j(Y)$ denote the SW moduli space of Y that is obtained by dividing out the solution space of equations (2.1) by the action of the restricted gauge group $\mathcal{G}_0(M_j)$. Note that $\tilde{\mathcal{M}}_j(Y)$ is a $\mathbb{Z}^{n_j}$ affine space (i.e., there is a set-theoretic one-to-one correspondence between $\tilde{\mathcal{M}}_j(Y)$ and $\operatorname{Coker}(i_j)^* \cong \mathbb{Z}^{n_j}$). More precisely, we choose a solution $x_0 = (A_0, \Psi_0)$, where A_0 is the trivial connection. Every other solution $x \in \tilde{\mathcal{M}}_j(Y)$ is of the form

$x = u^*(x_0)$ for some gauge transformation $u\colon Y \to S^1$. Such a map u gives rise to a well-defined element $[u_x] \in H^1(Y)/(i_j)^*(H^1(M_j))$.

Let $\mathcal{K}(M(\varphi))$ denote the set of isomorphism classes of Spin^c structures on $M(\varphi)$ that restrict to $\mathcal{L}_0$ on Y. Given a Spin^c structure $\mathcal{L} \in \mathcal{K}(M(\varphi))$, let $\mathcal{L}|_j$ denote its restriction to the half M_j. Note that $c_1(\det(\mathcal{L}|_j))$ can be lifted to an element of $H^2(M_j, Y)$. We let $x_{\mathcal{L}}^j \in \tilde{\mathcal{M}}_j(Y)$ be the element satisfying $c_1(\det(\mathcal{L}|_j)) = \partial^*([u_{x_{\mathcal{L}}^j}])$, where $\partial^*\colon \mathrm{Coker}(i_j)^* \to H^2(M_j, Y)$ is the coboundary map in the long exact sequence for the pair (M_j, Y). If no such $x_{\mathcal{L}}^j$ exists then we just let $x_{\mathcal{L}}^j := x_0$ of the previous paragraph. Hence an element $\mathcal{L} \in \mathcal{K}(M(\varphi))$ gives us an identification $\tilde{\mathcal{M}}_j(Y) \cong \mathcal{H}_j$, where the base point $x_{\mathcal{L}}^j$ is identified with the zero element in $\mathcal{H}_j$.

The gluing map φ induces a homomorphism $\varphi^*\colon \mathcal{H}_2 \to \mathcal{H}_1$. Using the preceding identifications, we thus obtain a map $\varphi_{\mathcal{L}}^*\colon \tilde{\mathcal{M}}_2(Y) \to \tilde{\mathcal{M}}_1(Y)$ and define the "graph" set

$$G_{\mathcal{L}}(\varphi) := \{(x, y) \in \tilde{\mathcal{M}}_1(Y) \times \tilde{\mathcal{M}}_2(Y) \mid x = \varphi_{\mathcal{L}}^*(y)\}.$$

(In particular, if $n_1 = n_2 = n$, then $G_{\mathcal{L}}(\varphi)$ looks like the graph set of a matrix in $\mathrm{GL}_n(\mathbb{Z})$.) Given the Spin^c structures $\mathcal{L}_j \in \mathcal{K}(M_j)$, there is the obvious gluing map $P\colon \mathcal{K}(M_1) \times \mathcal{K}(M_2) \to \mathcal{K}(M(\varphi))$. We define the subset

$$\mathcal{M}_{\hat{M}_1}^{r_1}(\mathcal{L}_1, g_1, \omega_1) \times_\varphi \mathcal{M}_{\hat{M}_2}^{r_2}(\mathcal{L}_2, g_2, \omega_2) \subset \prod \mathcal{M}_{\hat{M}_j}^{r_j}(\mathcal{L}_j, g_j, \omega_j)$$

to be $\{([A_1, \phi_1], [A_2, \phi_2]) \mid \partial_\infty^1[A_1, \phi_1] = \varphi_{P(\mathcal{L}_1, \mathcal{L}_2)}^* \partial_\infty^2[A_2, \phi_2]\}$, where the maps $\partial_\infty^j\colon \mathcal{M}_{\hat{M}_j}^{r_j}(\mathcal{L}_j, g_j, \omega_j) \to \tilde{\mathcal{M}}_j(Y)$ are as in Lemma 3.

LEMMA 10. *For generic choice of parameters and all ℓ sufficiently large, there is a diffeomorphism*

$$\mathfrak{M}_{M(\varphi)}^r(\mathcal{L}, g_\ell, \omega) \xrightarrow{\cong} \coprod_{P^{-1}(\mathcal{L})} \mathcal{M}_{\hat{M}_1}^r(\mathcal{L}_1, g_1, \omega_1) \times_\varphi \mathcal{M}_{\hat{M}_2}^{-r}(\mathcal{L}_2, g_2, \omega_2).$$

Proof. The statement follows readily from the previous lemmas and the standard limiting and gluing arguments as in [MM], [T1], and [T2]. □

THEOREM 11 (Product Formula). *Given any admissible triple (M_1, M_2, φ) and* Spin^c *structure $\mathcal{L} \in \mathcal{K}(M(\varphi))$, we have*

$$\mathrm{SW}_{M(\varphi)}(\mathcal{L}) = \sum_{P^{-1}(\mathcal{L})} \sum_{G_{\mathcal{L}}(\varphi)} \mathrm{SW}_{M_1}(\mathcal{L}_1, x) \cdot \mathrm{SW}_{M_2}(\mathcal{L}_2, y),$$

where the outer sum on the right side is taken over all pairs $(\mathcal{L}_1, \mathcal{L}_2)$ in the preimage $P^{-1}(\mathcal{L})$ and where the inner sum is taken over all points $(x, y) \in G_{\mathcal{L}}(\varphi)$.

Proof. As shown in [P1], the space $\mathfrak{M}_{M(\varphi)}^r(\mathcal{L}, g_\ell, \omega)$ is smoothly cobordant to a standard Seiberg–Witten moduli space and hence can be used to compute $\mathrm{SW}_{M(\varphi)}$. □

8. Applications

Suppose X is a closed oriented smooth 4-manifold with $b_2^+(X) > 1$. Let $\mathcal{S}(X)$ be the set of isomorphism classes of Spin^c structures on X, and let $\mathbb{Z}[G]$ denote the group ring of the Abelian group $G = H_2(X;\mathbb{Z})$. The usual addition in G becomes multiplication in $\mathbb{Z}[G]$.

DEFINITION 12. We define the *Seiberg–Witten series of X* to be the element of $\mathbb{Z}[G]$ given by

$$\overline{\mathrm{SW}}_X := \sum_{\mathcal{L}\in\mathcal{S}(X)} \mathrm{SW}_X(\mathcal{L})\,\mathrm{PD}(c_1(\det\mathcal{L})),$$

where $\mathrm{PD}\colon H^2(X;\mathbb{Z}) \to G$ is the Poincaré duality isomorphism.

If $b_2^+(X) = 1$ then let $\mathcal{C}_X = \{\alpha \in H^2(X;\mathbb{R}) \mid \alpha\cdot\alpha > 0\}$. Suppose we have a smoothly embedded torus $\Sigma \hookrightarrow X$ with $[\Sigma] \neq 0 \in H_2(X;\mathbb{Z})$ and $[\Sigma]\cdot[\Sigma] = 0$. Then such a torus Σ determines a preferred component of $\mathcal{C}_X$ that contains those classes α with $\langle\alpha,[\Sigma]\rangle > 0$. As in [FS3], we can define a pair of functions

$$\mathrm{SW}_X^+\colon \mathcal{S}(X) \to \mathbb{Z}, \qquad \mathrm{SW}_X^-\colon \mathcal{S}(X) \to \mathbb{Z}.$$

Let $\mathbb{Z}[[G]]$ denote the formal power series ring of the group $G = H_2(X;\mathbb{Z})$, where the underlying set is the set of integer-valued functions on G and the multiplication rule is given by the convolution product $\left(\sum_g a_g g\right)\cdot\left(\sum_g b_g g\right) = \sum_{fg=h}(a_f\cdot b_g)h$.

DEFINITION 13. Given $b_2^+(X) = 1$ and $\Sigma \hookrightarrow X$ a torus embedding as before, we define the $[\Sigma]^\perp$*-restricted* Seiberg–Witten series of X to be the elements of $\mathbb{Z}[[G]]$ given by

$$\overline{\mathrm{SW}}^{\pm}_{X,\Sigma} := \sum_{\mathcal{L}\in\mathcal{K}(X)} \mathrm{SW}_X^{\pm}(\mathcal{L})\,\mathrm{PD}(c_1(\det\mathcal{L})),$$

where $\mathcal{K}(X) = \{\mathcal{L}\in\mathcal{S}(X) \mid \langle c_1(\det\mathcal{L}),[\Sigma]\rangle = 0\}$.

Now suppose M is a compact oriented smooth 4-manifold with boundary $\partial M = T^3 = \Sigma\times S^1$. Let $j\colon (M,\emptyset) \to (M,\partial M)$ be the inclusion map, and define $b_2^{\geq 0}(M)$ to be the dimension of the maximal submodule of $H_2(M;\mathbb{Z})$ on which the intersection form

$$H_2(M;\mathbb{Z})\otimes H_2(M;\mathbb{Z}) \xrightarrow{\mathrm{id}\otimes j_*} H_2(M;\mathbb{Z})\otimes H_2(M,\partial M;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

is positive semidefinite. When $b_2^{\geq 0}(M) = 1$, we can also define the functions

$$\mathrm{SW}_M^{\pm}\colon \mathcal{K}(M)\times\tilde{\mathcal{M}}(\partial M) \longrightarrow \mathbb{Z}$$

just as in the closed manifold case. We have the following commutative diagram:

$$\begin{tikzcd}
H_3(M,\partial M) \arrow[r, "\partial_*"] & H_2(\partial M) \arrow[r, "i_*"] & H_2(M) \arrow[r, "j_*"] & H_2(M,\partial M) \arrow[r, "\partial_*"] & H_1(\partial M) \\
H^1(M) \arrow[r, "i^*"] \arrow[u, "\mathrm{PD}"] & H^1(\partial M) \arrow[r, "\partial^*"] \arrow[u, "\mathrm{PD}"'] & H^2(M,\partial M) \arrow[r, "j^*"] \arrow[u, "\mathrm{PD}"'] & H^2(M) \arrow[r, "i^*"] \arrow[u, "\mathrm{PD}"'] & H^2(\partial M). \arrow[u, "\mathrm{PD}"']
\end{tikzcd} \tag{8.1}$$

As before, we let $\mathcal{H} = \mathrm{Coker}[i^*\colon H^1(M) \to H^1(\partial M)]$. Note that the composition $(i_* \circ \mathrm{PD})$ is well-defined on the quotient $\mathcal{H} = H^1(\partial M)/i^*H^1(M)$. Recall that there exists a canonical identification $\tilde{\mathcal{M}}(\partial M) = \mathcal{H}$. There also exists an orthogonal decomposition

$$H_2(M) = H_2^{\perp}(M) \oplus i_*(H_2(\partial M)),$$

where the image of i_* is the radical (kernel) of the intersection form. The map j_* is injective on the summand $H_2^{\perp}(M) \subset H_2(M)$ and hence we can define its inverse, $j_*^{-1}\colon \mathrm{Ker}(\partial_*) \to H_2^{\perp}(M)$. Given any Spin^c structure $\mathcal{L} \in \mathcal{K}(M)$, we have $i^*(c_1(\det \mathcal{L})) = 0$ and hence $\mathrm{PD}(c_1(\det \mathcal{L}))$ lies in the kernel of $\partial_*\colon H_2(M, \partial M) \to H_1(\partial M)$.

Definition 14. Let M be as before with a chosen factorization $\partial M = \Sigma \times S^1$. If $b_2^{\geq 0}(M) > 1$, then we define the Seiberg–Witten series of M to be the element of $\mathbb{Z}[[H_2(M;\mathbb{Z})]]$ given by

$$\overline{\mathrm{SW}}_M := \sum_{\mathcal{L}\in\mathcal{K}(M)} \sum_{x\in\mathcal{H}} \mathrm{SW}_M(\mathcal{L}, x)[(j_*^{-1} \circ \mathrm{PD})(c_1(\det \mathcal{L})) + (i_* \circ \mathrm{PD})(x)].$$

If $b_2^{\geq 0}(M) = 1$, let $\mathcal{H}(\Sigma) := \{x \in \mathcal{H} \mid (i_*[\Sigma]) \cdot (i_* \circ \mathrm{PD})(x) = 0\}$ and define the $[\Sigma]^{\perp}$-restricted Seiberg–Witten series of M to be

$$\overline{\mathrm{SW}}_M^{\pm} := \sum_{\mathcal{L}\in\mathcal{K}(M)} \sum_{x\in\mathcal{H}(\Sigma)} \mathrm{SW}_M^{\pm}(\mathcal{L}, x)(i_* \circ \mathrm{PD})(x).$$

With this notation in place, the product formula in Theorem 11 can be restated in the following form.

Corollary 15 (Product Formula II). *Suppose that (M_1, M_2, φ) is an admissible triple. Let $k_j\colon M_j \to M(\varphi)$ be the inclusion maps $(j = 1, 2)$ and let $(k_j)_*\colon H_2(M_j;\mathbb{Z}) \to H_2(M(\varphi);\mathbb{Z})$ be the induced homomorphisms. Then we have*

$$\overline{\mathrm{SW}}_{M(\varphi)} = \overline{(k_1)}_*(\overline{\mathrm{SW}}_{M_1}) \cdot \overline{(k_2)}_*(\overline{\mathrm{SW}}_{M_2}),$$

where the maps $\overline{(k_j)}_*\colon \mathbb{Z}[[H_2(M_j;\mathbb{Z})]] \to \mathbb{Z}[[H_2(M(\varphi);\mathbb{Z})]]$ are defined by $\overline{(k_j)}_*\big(\sum_g a_g g\big) = \sum_g a_g (k_j)_*(g)$. The displayed formula remains valid when the terms in it are replaced by $[\Sigma]^{\perp}$-restricted Seiberg–Witten series of the same sign.

For each integer $n > 0$, let $E(n)$ be a simply connected elliptic surface with no multiple fibers and with geometric genus $p_g = n-1$. Let F denote a generic torus fiber of $E(n)$ whose tubular neighborhood is a trivial product $F \times D^2$. We shall compute the Seiberg–Witten series of the complements $E_0(n) := E(n) - (F \times D^2)$. Recall the standard decomposition

$$E(m+n) = [E(m) - (F \times D^2)] \cup_{\varphi} [E(n) - (F \times D^2)],$$

where the gluing map $\varphi\colon F \times S^1 \to F \times S^1$ is given by $\varphi(x, \zeta) = (x, \zeta^{-1})$ for any $x \in F$ and $\zeta = e^{it} \in S^1$. Since the manifolds $E_0(n)$ are simply connected, we can easily verify that any triple $(E_0(m), E_0(n), \varphi)$ is admissible. Let $t := [F]$ denote the homology class of a fiber in $H_2(E_0(n);\mathbb{Z})$ or $H_2(E(n);\mathbb{Z})$, and let $\tilde{\gamma} \in H^1(F \times S^1;\mathbb{Z})$ denote the dual element of $\gamma = [\{\text{point}\} \times S^1] \in H_1(F \times S^1;\mathbb{Z})$ under the universal coefficient theorem. Note that $\mathcal{H}_1 = \mathcal{H}_2 = H^1(F \times S^1;\mathbb{Z})$ and $\varphi^*(\tilde{\gamma}) = -\tilde{\gamma}$. Also note that $(i_* \circ \mathrm{PD})(\tilde{\gamma}) = t$, where $i\colon (F \times S^1) \to E_0(n)$ are the inclusion maps.

Lemma 16 (cf. [FS2]). *We have* $\overline{\mathrm{SW}}_{E(n)} = (t^{-1} - t)^{n-2}$.

Let us start out with the decomposition $E(4) = E_0(2) \cup_\varphi E_0(2)$. Corollary 15 says that

$$(t^{-1} - t)^2 = \overline{(k_1)}_*(\overline{\mathrm{SW}}_{E_0(2)}) \cdot \overline{(k_2)}_*(\overline{\mathrm{SW}}_{E_0(2)}).$$

From the symmetry of the situation we can easily conclude that (up to sign)

$$\begin{aligned} \overline{(k_1)}_*(\overline{\mathrm{SW}}_{E_0(2)}) &= t^m(t^{-1} - t), \\ \overline{(k_2)}_*(\overline{\mathrm{SW}}_{E_0(2)}) &= t^{-m}(t^{-1} - t) \end{aligned} \tag{8.2}$$

or vice versa. Next we need an analogue of the adjunction inequality (cf. [FS1; KM; MST; OS1]) for a 4-manifold whose boundary is T^3.

Proposition 17 (cf. [P1; P2]). *Suppose that M is a compact oriented smooth 4-manifold with $\partial M = T^3$ and $b_2^{\geq 0}(M) > 1$. We choose a factorization $\partial M = T^2 \times S^1$ and let $\gamma = [\{\text{point}\} \times S^1] \in H_1(\partial M;\mathbb{Z})$ as before. Assume that M is admissible (in the sense of Definition 4) and of simple type. Let σ be a compact oriented smooth 2-dimensional surface inside M such that, if $\partial\sigma \neq \emptyset$, then $\partial\sigma \subset \partial M$ and furthermore the homology class $[\partial\sigma]$ equals some integer multiple $m\gamma$ inside $H_1(\partial M;\mathbb{Z})$. If $\mathrm{SW}_M(\mathcal{L}, x) \neq 0$, then we must have*

$$|[(j_*^{-1} \circ \mathrm{PD})(c_1(\det \mathcal{L})) + (i_* \circ \mathrm{PD})(x)] \cdot [\sigma]| + [\sigma] \cdot [\sigma] + \mathrm{e}(\sigma) \leq 0,$$

where $(j_*^{-1} \circ \mathrm{PD})$ and $(i_* \circ \mathrm{PD})$ are the compositions of maps in (8.1) *as before and where $\mathrm{e}(\sigma)$ is the Euler characteristic of σ.*

Now we apply Proposition 17 to a punctured section $\sigma_\circ$ of $E(2)$ inside $E_0(2)$ that satisfies

$$t \cdot \sigma_\circ = 1, \quad \sigma_\circ \cdot \sigma_\circ = -2, \quad \mathrm{e}(\sigma_\circ) = 1;$$

we conclude that $m = 0$ in (8.2). (We could instead have used a punctured torus coming from the homology class $t + [\sigma_\circ \cup D^2] \in H_2(E(2);\mathbb{Z})$ to draw the same conclusion.) We remark that $m \neq 0$ also contradicts the logarithmic transformation formula in [FS2]. Hence we have $\overline{\mathrm{SW}}_{E_0(2)} = t^{-1} - t$. From the decomposition $E(3) = E_0(2) \cup_\varphi E_0(1)$, we have

$$t^{-1} - t = (t^{-1} - t) \cdot \overline{(k_2)}_*(\overline{\mathrm{SW}}_{E_0(1)}),$$

which immediately implies that $\overline{\mathrm{SW}}_{E_0(1)} = 1$. Finally, from the recursive relation $E(n+1) = E_0(n) \cup_\varphi E_0(1)$ we conclude that

$$\overline{\mathrm{SW}}_{E(n+1)} = \overline{\mathrm{SW}}_{E_0(n)}.$$

In summary, we have proved the following.

THEOREM 18. *If $E_0(n) = E(n) - (F \times D^2)$ then we have*

$$\overline{\mathrm{SW}}_{E_0(n)} = (t^{-1} - t)^{n-1},$$

where $t = [F] \in H_2(E_0(n); \mathbb{Z})$ is the homology class of a regular fiber. We also have

$$\overline{\mathrm{SW}}^-_{T^2 \times D^2} = \frac{1}{t^{-1} - t} = t + t^3 + t^5 + \cdots = \overline{\mathrm{SW}}^-_{E(1),F},$$

$$\overline{\mathrm{SW}}^+_{T^2 \times D^2} = \frac{1}{t^{-1} - t} = -t^{-1} - t^{-3} - t^{-5} - \cdots = \overline{\mathrm{SW}}^+_{E(1),F},$$

where $t = [T^2 \times \{\text{point}\}] \in H_2(F \times D^2; \mathbb{Z})$.

Perhaps it would be instructive to see what the "graph" set $G_{\mathcal{L}}(\varphi)$ in Section 7 looks like for the decomposition $E(4) = E_0(2) \cup_\varphi E_0(2)$. Let $\mathcal{T}_m$ denote the Spin^c structure on $E(4)$ such that $c_1(\det \mathcal{T}_m)$ is Poincaré dual to mt. Since $\overline{\mathrm{SW}}_{E(4)} = t^{-2} - 2 + t^2$, the SW moduli space for $\mathcal{T}_0$ consists of two negatively oriented points and each of the moduli spaces for $\mathcal{T}_2$ and $\mathcal{T}_{-2}$ consists of a single positively oriented point. Note that $t = 0$ inside $H_2(E_0(2), \partial E_0(2))$ and so we always have $c_1(\det(\mathcal{T}_m|_{E_0(2)})) = 0$ inside $H^2(E_0(2))$. In Figures 1 and 2, we look at only the 1-dimensional submodule of $\mathcal{H}_1 = \mathcal{H}_2 \cong \mathbb{Z}^3$ generated by $\tilde{\gamma}$. In the figures, we blur the distinction between $\tilde{\gamma}$ and its Poincaré dual t. The homomorphism $\varphi^* : \mathcal{H}_2 \to \mathcal{H}_1$ restricts to multiplication by -1 on the submodule generated by $\tilde{\gamma}$. Figure 1 shows how the moduli space for $\mathcal{T}_0$ is obtained. It shows how the gluing occurs for two possible pairs of compatible lifts of the zero class $c_1(\det(\mathcal{T}_0|_{E_0(2)}))$ into $H^2(E_0(2), \partial E_0(2))$. Similarly, the moduli space for $\mathcal{T}_2$ is depicted in Figure 2, wherein two different pairs of compatible lifts of the zero class $c_1(\det(\mathcal{T}_2|_{E_0(2)}))$ into $H^2(E_0(2), \partial E_0(2))$ are shown.

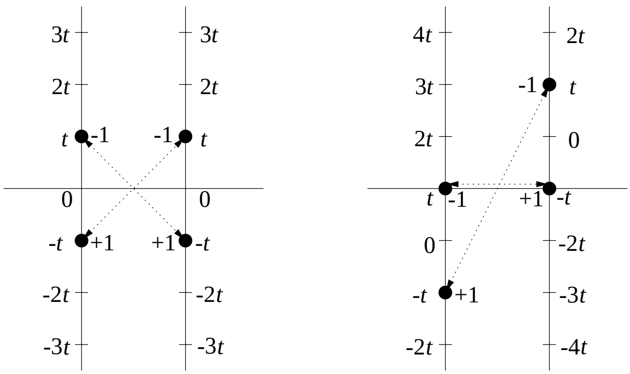

Figure 1 $(x^1_{\mathcal{T}_0}, x^2_{\mathcal{T}_0}) = (0, 0)$ and $(x^1_{\mathcal{T}_0}, x^2_{\mathcal{T}_0}) = (\tilde{\gamma}, -\tilde{\gamma})$

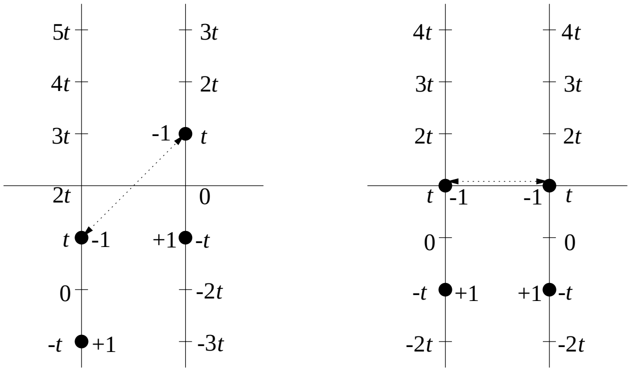

Figure 2 $(x^1_{T_2}, x^2_{T_2}) = (2\tilde{\gamma}, 0)$ and $(x^1_{T_2}, x^2_{T_2}) = (\tilde{\gamma}, \tilde{\gamma})$

COROLLARY 19. *Let Σ_g denote a Riemann surface of genus g, and define $\Sigma_g^\circ := \Sigma_g - D^2$ to be a once-punctured Riemann surface of genus g. We have $\overline{\mathrm{SW}}_{T^2\times\Sigma_g^\circ} = (t^{-1} - t)^{2g-1}$ and $\overline{\mathrm{SW}}_{T^2\times\Sigma_g} = (t^{-1} - t)^{2g-2}$, where t is the homology class $[T^2 \times \{\text{point}\}]$.*

Proof. Because T^4 is a complex surface, the adjunction inequality implies that $\overline{\mathrm{SW}}_{T^4} = 1$. Since we know that $\overline{\mathrm{SW}}^{\pm}_{T^2\times D^2} = (t^{-1}-t)^{-1}$, it follows that $\overline{\mathrm{SW}}_{T^2\times\Sigma_1^\circ} = t^{-1} - t$. Now we can proceed by induction as follows. If $g = 2n$ then

$$\overline{\mathrm{SW}}_{T^2\times\Sigma_g^\circ} = \frac{(\overline{\mathrm{SW}}_{T^2\times\Sigma_n^\circ})^2}{\overline{\mathrm{SW}}^{\pm}_{T^2\times D^2}} = (\overline{\mathrm{SW}}_{T^2\times\Sigma_n^\circ})^2 \cdot (t^{-1} - t),$$

and if $g = 2n - 1$ then

$$\overline{\mathrm{SW}}_{T^2\times\Sigma_g^\circ} = \frac{(\overline{\mathrm{SW}}_{T^2\times\Sigma_n^\circ})^2}{\overline{\mathrm{SW}}_{T^2\times\Sigma_1^\circ}} = \frac{(\overline{\mathrm{SW}}_{T^2\times\Sigma_n^\circ})^2}{t^{-1} - t}. \qquad \square$$

More generally we have the following result, whose proof is immediate (cf. [FS3, Thm. 2.1]).

COROLLARY 20. *Let X be a closed oriented smooth 4-manifold with $b_2^+(X) > 1$. Suppose we have a smoothly embedded torus $\Sigma \hookrightarrow X$ such that $[\Sigma] \neq 0 \in H_2(X;\mathbb{Z})$ and $[\Sigma]\cdot[\Sigma] = 0$. Let $\nu(\Sigma)$ denote the tubular neighborhood of Σ and let $i^*\colon H^1(X - \nu(\Sigma)) \to H^1(\partial\nu(\Sigma))$ be the homomorphism induced by the inclusion map. If $X - \nu(\Sigma)$ is admissible as in Definition 4 and if* $\mathrm{Coker}(i^*)$ *is torsion-free, then we have*

$$\overline{\mathrm{SW}}_{X-\nu(\Sigma)} = \overline{\mathrm{SW}}_X \cdot ([\Sigma]^{-1} - [\Sigma]).$$

The following is a direct generalization of [FS2, Thm. 8.6].

COROLLARY 21 (Logarithmic Transformation). *Let X and Σ be as in Corollary 20. Given any orientation-reversing diffeomorphism $\psi\colon T^2\times S^1\to\partial(X-\nu(\Sigma))$ such that the triple $(T^2\times D^2, X-\nu(\Sigma),\psi)$ is admissible, the Seiberg–Witten series of $X_\psi := (T^2\times D^2)\cup_\psi (X-\nu(\Sigma))$ is*

$$\overline{\mathrm{SW}}_{X_\psi}=\overline{\mathrm{SW}}_X\cdot\frac{[\Sigma]^{-1}-[\Sigma]}{(i\circ\psi)_*(t)^{-1}-(i\circ\psi)_*(t)},$$

where $t=[T^2\times\{\text{point}\}]\in H_2(T^2\times\partial D^2)$ and $i\colon\partial(X-\nu(\Sigma))\to(X-\nu(\Sigma))$ is the inclusion map. If $b_2^+(X)=1$, then the displayed formula remains valid when the terms in it are replaced by $[\Sigma]^\perp$-restricted Seiberg–Witten series of the same sign.

Proof. Notice that, for an admissible triple $(T^2\times D^2, X-\nu(\Sigma),\psi)$, we have that $\mathcal{H}_1\cong\mathbb{Z}$ and the homomorphism $\psi^*\colon\mathcal{H}_2\to\mathcal{H}_1$ can be viewed as a map into the 1-dimensional submodule $\mathbb{Z}\tilde{\gamma}\subset\mathcal{H}_2$. □

COROLLARY 22. *Let $\psi\colon F\times S^1\to F\times S^1$ be any orientation-reversing diffeomorphism. We define a family of closed 4-manifolds*

$$X(m,n;\psi):=E_0(m)\cup_\psi E_0(n).$$

Let $t_j:=(k_j)_*[F]\in H_2(X(m,n;\psi))$, where $k_1\colon E_0(m)\hookrightarrow X(m,n;\psi)$ and $k_2\colon E_0(n)\hookrightarrow X(m,n;\psi)$ are the inclusion maps. Then we have

$$\overline{\mathrm{SW}}_{X(m,n;\psi)}=(t_1^{-1}-t_1)^{m-1}\cdot(t_2^{-1}-t_2)^{n-1}.$$

In particular, if $m>1$, $n>1$, and $t_1\neq\pm t_2$, then the manifold $X(m,n;\psi)$ is not diffeomorphic to $E(m+n)$.

Proof. Since the manifolds $E_0(n)$ are simply connected, $(E_0(m),E_0(n),\psi)$ are always admissible. The last statement follows from the comparison of divisibilities of the basic classes. □

REMARK 23. As an example of Corollary 22, let $p>1$ be an integer and consider the self-diffeomorphism ψ_p specified by the matrix

$$\begin{pmatrix}1&0&0\\0&0&1\\0&1&p\end{pmatrix}$$

for a suitable choice of basis $\{\alpha,\beta,\gamma\}$ on $H_1(T^2\times S^1;\mathbb{Z})$. It is easy to see that $X(m,n;\psi_p)$ is homeomorphic to $E(m+n)$. Notice that the vanishing disk that bounds the cycle β in $E_0(m)$ can be glued to a punctured section in $E_0(n)$. This internal sum of disks produces a homologically nontrivial sphere σ inside $X(m,n;\psi_p)$. It is readily seen that $\sigma\cdot t_1=0$ but $\sigma\cdot t_2=1$. Hence $t_1\neq\pm t_2$ and $X(m,n;\psi_p)$ is not diffeomorphic to $E(m+n)$, provided $m,n>1$. More generally, different integer partitions

$$n=r+s\quad(s\geq r\geq 0;\ r=0,1,\dots,[n/2])$$

give rise to a pairwise nondiffeomorphic family, $X(r, s; \psi_p)$, each homeomorphic to $E(n)$. (Here, we can define $E_0(0) := F \times D^2$, and $[x]$ denotes the greatest integer less than or equal to x.) To get more new exotic smooth structures, we may form the fiber sums

$$X(r_0, s_0; \psi_{p_0}) \#_{\psi_{q_1}} X(r_1, s_1; \psi_{p_1}) \#_{\psi_{q_2}} \cdots \#_{\psi_{q_m}} X(r_m, s_m; \psi_{p_m}),$$

which are homeomorphic to $E\big(\sum(r_i + s_i)\big)$.

Now consider a smooth knot K in S^3 and let m denote a fixed meridian of K. Let M_K be the result of Dehn surgery with coefficient 0 on $K \subset S^3$; M_K has the same homology as $S^2 \times S^1$ with the class of m generating $H_1(M_K)$. In $M_K \times S^1$ we have a torus $T_m = m \times S^1$ of self-intersection 0.

COROLLARY 24. *Let $\Delta_K(t)$ be the symmetrized Alexander polynomial of a knot $K \subset S^3$ as in* [FS3]. *Then we have*

$$\overline{\mathrm{SW}}^{\pm}_{M_K \times S^1, T_m} = \frac{\Delta_K([T_m]^2)}{([T_m]^{-1} - [T_m])^2}.$$

Proof. In [FS3], Fintushel and Stern constructed fiber sums

$$E(2)_K = [(M_K \times S^1) - (T_m \times D^2)] \cup_\varphi [E(2) - (F \times D^2)],$$

where the gluing map φ identifies T_m with the fiber F. They showed that

$$\overline{\mathrm{SW}}_{E(2)_K} = \Delta_K([T_m]^2).$$

We know that the complement $[(M_K \times S^1) - \nu(T_m)]$ has the same homology as $T^2 \times D^2$ and also that $E_0(2)$ is simply connected; hence it is easily checked that $((M_K \times S^1) - \nu(T_m), E_0(2), \varphi)$ is admissible. From our product formula and Corollary 20, we must have

$$\Delta_K([T_m]^2) = \overline{\mathrm{SW}}^{\pm}_{M_K \times S^1, T_m}([T_m]^{-1} - [T_m]) \cdot ([F]^{-1} - [F]).$$

Since $b_2^-(M_K \times S^1) = 1$, [FS3, Lemma 5.1] implies that the Seiberg–Witten series $\overline{\mathrm{SW}}^{\pm}_{M_K \times S^1, T_m}$ involves only powers of $[T_m]$. Since $[F] = [T_m]$ in $E(2)_K$, our result follows. □

REMARK 25. (i) Let $X = \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$. Let H and E denote the generators of $H_2(\mathbb{CP}^2)$ and $H_2(\overline{\mathbb{CP}}^2)$, respectively. Then the homology class $H - E$ is represented by a smoothly embedded sphere in X. Attaching a trivial handle to that sphere yields a torus T representing $H - E$. Let $X_{p,q}$ denote the result of logarithmic transformations of orders p and q on two parallel copies of T, and let $\mathcal{L}$ be a Spin^c structure on X with $\mathrm{PD}(c_1(\det \mathcal{L})) = aH + bE$. If $\mathrm{SW}_X(\mathcal{L}) \neq 0$, then by the dimension formula we must have

$$c_1(\det \mathcal{L})^2 - 2\mathrm{e}(X) - 3\,\mathrm{sign}(X) = a^2 - b^2 - 8 \geq 0.$$

Thus it follows that $\mathrm{PD}(c_1(\det \mathcal{L})) \cdot [T] = a + b \neq 0$. Consequently we must have $\overline{\mathrm{SW}}^{\pm}_{X,T} = 0$, and Corollary 21 implies that $\overline{\mathrm{SW}}^{\pm}_{X_{p,q},T} = 0$ as well. We conjecture that one can prove that the $X_{p,q}$ are all diffeomorphic to X via Kirby calculus.

(ii) Let $\tilde{\mathcal{M}}_{1,2}(Y)$ denote the solution space of the 3-dimensional Seiberg–Witten equations (2.1) modulo the gauge subgroup $\mathcal{G}_0(M_1) \cap \mathcal{G}_0(M_2)$ (i.e., the subgroup of maps $u\colon Y \to S^1$ that can be extended to both M_1 and M_2). Even if (M_1, M_2, φ) is *not* admissible, we should still be able to compare the elements of the covers $\tilde{\mathcal{M}}_1(Y)$ and $\tilde{\mathcal{M}}_2(Y)$ by considering the following commutative diagram.

$$\begin{tikzcd} & \tilde{\mathcal{M}}_{1,2}(Y) \arrow[dl] \arrow[dr] & \\ \tilde{\mathcal{M}}_1(Y) \arrow[dr] & & \tilde{\mathcal{M}}_2(Y) \arrow[ll, dotted, "?"'] \arrow[dl] \\ & \mathcal{M}(Y) & \end{tikzcd}$$

We hope to generalize Theorem 11 to this situation in [P2].

(iii) While an earlier version of this paper was being refereed, Taubes published a paper [T3] that contains a product formula similar to ours. One of the main differences between his approach and ours is that he proves the compactness of cylindrical end moduli spaces by brute analytical force, whereas we have employed a simpler method of nonconstant perturbation that is more readily applicable to other 3-manifolds (cf. (3.2) and (4.1)).

9. Extension to Other 3-Manifolds

It is fairly straightforward to prove an analogue of Theorem 11 for many other 3-manifolds. For example, one can do it for a certain class of Seifert fibered spaces—in particular, S^1 bundles over 2-dimensional orbifolds whose corresponding line bundle has c_1 that is torsion. In [P2] we shall study gluing along these and other more general 3-manifolds. We plan to work with a more sophisticated version of the relative Seiberg–Witten invariant that is a function of the form

$$\mathrm{SW}_M\colon \mathrm{Spin}^c(M) \to \mathrm{Nov}(\mathcal{H}),$$

where $\mathcal{H}$ is some infinite group determined by the moduli space $\tilde{\mathcal{M}}(\partial M)$ and $\mathrm{Nov}(\mathcal{H})$ is a suitably defined Novikov ring. As alluded to in Section 1, we hope that this will be a germ of a new theory of infinitely generated Floer-type cohomology for the pairs $(M, \partial M)$.

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