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A Quasi-Paucity Problem

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1. Introduction

A cartographer of the diophantine landscape is compelled to acknowledge the distinguished position occupied by the investigation of diophantine systems in which there are believed to be few other than the obvious solutions. When these systems are symmetric, the task of verifying such a belief has come to be called a *paucity problem*. Although the literature surrounding this topic is by now extensive when the underlying summands are perfect powers (see, for example, the sources recorded in the bibliography), little is known for more general situations. The object of this note is to establish that the number of solutions of certain systems of additive equations is dominated, in essence, by the diagonal contribution alone.

In order to state our main conclusion precisely, we require some notation. Suppose that t is a positive integer, and let $f_1(x), \dots, f_t(x)$ be polynomials with rational coefficients of respective degrees $k_1, \dots, k_t$. When P is a positive number, denote by $S_s(P; \mathbf{f})$ the number of integral solutions of the simultaneous equations

$$\sum_{i=1}^{s} (f_j(x_i) - f_j(y_i)) = 0 \quad (1 \le j \le t), \tag{1}$$

with $1 \le x_i, y_i \le P$ $(1 \le i \le s)$.

Theorem 1. *Suppose that the polynomials* $f_i(x) \in \mathbb{Q}[x]$ $(1 \le i \le t)$ *satisfy the condition that* $1, f_1, \dots, f_t$ *are linearly independent over* $\mathbb{Q}$. *Suppose also that* A *is a positive number sufficiently large in terms of* t, $\mathbf{k}$, *and the coefficients of* $f_1, \dots, f_t$. *Then, whenever* $\max\{k_1, \dots, k_t\} \ge 2$ *and* $P \ge 3$, *one has*

$$S_{t+1}(P; f_1, \dots, f_t) \ll P^{t+1}(\log P)^A.$$

Plainly, those solutions of the system (1) in which $x_1, \dots, x_s$ are simply a permutation of $y_1, \dots, y_s$ provide a contribution to $S_{t+1}(P; \mathbf{f})$ that ensures the lower bound

$$S_{t+1}(P; \mathbf{f}) \ge (t+1)!\, P^{t+1} + O_t(P^t). \tag{2}$$

Thus we may assert that the conclusion of the theorem is somewhat close to a paucity result. We remark that the bound recorded in the theorem was already

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available from work of Wooley [28] in the special case wherein $f_i(x) = x^{k_i}$ $(1 \le i \le t)$ and $1 \le k_1 < k_2 < \cdots < k_t$. Moreover, when $t = 1$ and $f_1(x)$ is a cubic polynomial, it follows from Theorem 2 of Wooley [32] that

$$S_2(P; f_1) = 2P^2 + O_\varepsilon(P^{5/3+\varepsilon}).$$

Aside from the inherent interest of paucity problems, estimates of the type presented in Theorem 1 have potential for application in the sharpest versions, due to Parsell [19], of the new iterative methods of Vaughan and Wooley (see especially [23; 24; 27; 31]) involving exponential sums over smooth numbers.

2. Preliminary Skirmishing

Before advancing to the argument described in the next section, we prepare an eliminant polynomial and discuss some associated properties. We refer to an ordered t-tuple $(f_1, \dots, f_t)$ of polynomials with rational coefficients as being *well-conditioned* when

(1) $f_i(x) \in \mathbb{Z}[x]$ $(1 \le i \le t)$,
(2) $f_i(0) = 0$ $(1 \le i \le t)$, and
(3) the degrees k_i of the polynomials $f_i(x)$ satisfy $1 \le k_1 < k_2 < \cdots < k_t$.

By substituting the polynomial $f_i(x) - f_i(0)$ in place of $f_i(x)$ $(1 \le i \le t)$ in the system (1), one may plainly suppose that $f_i(0) = 0$ $(1 \le i \le t)$. Thus, on replacing the equations (1) by suitable linear combinations thereof, it is apparent that whenever $f_i(x) \in \mathbb{Q}[x]$ $(1 \le i \le t)$ satisfy the condition that $1, f_1, \dots, f_t$ are linearly independent over $\mathbb{Q}$, then there is no loss of generality in supposing instead that $\mathbf{f}$ is a well-conditioned t-tuple. Moreover, the coefficients of the polynomials in the new system plainly depend at most on those in the original system.

It is convenient in what follows to refer to a polynomial $F(\mathbf{x}) \in \mathbb{Z}[x_1, \dots, x_t]$ as being *asymptotically definite* if there exists a number C with the property that, whenever $x_i > C$ $(1 \le i \le t)$, one has

$$|F(x_1, \dots, x_t)| \ge 1.$$

Finally, we write $f'(x)$ for the derivative of the polynomial $f(x)$. As a prerequisite to a discussion of eliminant polynomials, we introduce a generalisation of the Vandermonde determinant

$$V_t(\mathbf{x}) = \prod_{1 \le i < j \le t} (x_j - x_i) = \det(x_j^{i-1})_{1 \le i, j \le t}.$$

LEMMA 1. *Suppose that $(f_1, \dots, f_t)$ is a well-conditioned t-tuple of polynomials with respective degrees $k_1, \dots, k_t$. Then there exists an asymptotically definite polynomial $\Theta = \Theta(\mathbf{x}; \mathbf{f})$ with the property that*

$$\det(f_i'(x_j))_{1 \le i, j \le t} = V_t(\mathbf{x})\Theta(\mathbf{x}; \mathbf{f}). \tag{3}$$

Moreover, the total degree of Θ *is*

$$d = \sum_{i=1}^{t} k_i - \frac{t(t+1)}{2}.$$

Proof. We apply the theory of symmetric functions, specifically Schur functions (see Macdonald [17]). When $d_1, \dots, d_t$ are integers with

$$1 \le d_1 < d_2 < \cdots < d_t,$$

we define the polynomial $K(\mathbf{x}; \mathbf{d})$ by means of the relation

$$\det(x_j^{d_i - 1})_{1 \le i, j \le t} = K(\mathbf{x}; \mathbf{d}) V_t(\mathbf{x}). \tag{4}$$

For the sake of concision, we make use of the notation used in Macdonald [17]. Thus, by equation (3.1) of [17, Chap. I], one has $K(\mathbf{x}; \mathbf{d}) = s_\lambda$, where λ is the partition

$$(d_t - t, d_{t-1} - (t-1), \dots, d_1 - 1).$$

Yet equation (5.12) of [17, Chap. I] shows that $s_\lambda = \sum_T x^T$, where the summation is over all semi-standard tableaux T of shape λ, and here, if the weight of T is $\alpha = (\alpha_1, \dots, \alpha_t)$, then x^T is the monomial $x_1^{\alpha_1} \cdots x_t^{\alpha_t}$. Note that if $\lambda = (0, \dots, 0)$ then one adopts the convention that $s_\lambda = 1$.

Observe next that, by elementary properties of determinants, the polynomial

$$\det(f_i'(x_j))_{1 \le i, j \le t} \tag{5}$$

is a linear combination of polynomials of the shape

$$\det(x_j^{d_i - 1})_{1 \le i, j \le t} \tag{6}$$

with $1 \le d_i \le k_i$ $(1 \le i \le t)$. A moment's reflection here reveals that this linear combination contains the polynomial

$$\det(x_j^{k_i - 1})_{1 \le i, j \le t}$$

with a nonvanishing coefficient. By permuting rows within the determinants (6), there is no loss of generality in supposing that $1 \le d_1 \le d_2 \le \cdots \le d_t$. Then each such determinant contributes either 0, or else a polynomial of the shape (4), to the expansion of (5). We may therefore conclude that

$$\det(f_i'(x_j))_{1 \le i, j \le t} = \hat{K}(\mathbf{x}; \mathbf{k}) V_t(\mathbf{x}),$$

where $\hat{K}(\mathbf{x}; \mathbf{k})$ is a polynomial of total degree

$$d = \sum_{i=1}^{t} (k_i - 1) - \sum_{i=1}^{t} (i - 1),$$

in which the homogeneous part of highest degree is a nonzero multiple of s_Λ, where $\Lambda = (k_t - t, k_{t-1} - (t-1), \dots, k_1 - 1)$. But in view of the discussion concluding the previous paragraph, whenever $x_i > B > 1$ $(1 \le i \le t)$ it follows that

$s_\Lambda \geq B^d$, and thus we conclude that $\hat{K}(\mathbf{x};\mathbf{k})$ is asymptotically definite. This completes the proof of the lemma. □

Next define the polynomials $\phi_{i,s} = \phi_{i,s}(\mathbf{x};\mathbf{f})$ by taking

$$\phi_{i,s}(\mathbf{x}) = f_i(x_1) + \cdots + f_i(x_s) \quad (1 \leq i, s \leq t).$$

Our next lemma establishes the existence of an eliminant polynomial suitable for subsequent deliberations.

LEMMA 2. *Suppose that $t \geq 2$ and that $(f_1, \dots, f_t)$ is a well-conditioned t-tuple of polynomials. Then there exists a polynomial $\Psi(\mathbf{z}) \in \mathbb{Z}[z_1, \dots, z_t]$, with total degree and coefficients depending at most on t, $\mathbf{k}$, and the coefficients of $f_1, \dots, f_t$, such that*

$$\Psi(\phi_{1,t-1}(\mathbf{x}), \dots, \phi_{t,t-1}(\mathbf{x})) = 0 \tag{7}$$

and yet

$$\Psi(\phi_{1,t}(\mathbf{x}), \dots, \phi_{t,t}(\mathbf{x})) \neq 0. \tag{8}$$

Proof. The existence of a nontrivial polynomial $\Psi(\mathbf{z}) \in \mathbb{Z}[z_1, \dots, z_t]$ for which $\Psi(\phi_{1,t-1}, \dots, \phi_{t,t-1})$ is identically zero will follow by considering transcendence degrees. Let $K = \mathbb{Q}(\phi_{1,t-1}, \dots, \phi_{t,t-1})$. Then $K \subseteq \mathbb{Q}(x_1, \dots, x_{t-1})$, so that K has transcendence degree at most $t-1$ over $\mathbb{Q}$. But then the t polynomials $\phi_{i,t-1}(\mathbf{x}) \in K$ $(1 \leq i \leq t)$ cannot be algebraically independent, whence the existence of the above polynomial Ψ follows immediately.

In order to verify the condition (8), consider any nontrivial polynomial Ψ of smallest total degree for which the polynomial equation (7) holds and suppose, if possible, that $\Psi(\phi_{1,t}(\mathbf{x}), \dots, \phi_{t,t}(\mathbf{x}))$ is identically zero. Then the polynomials

$$(\partial/\partial x_i)\Psi(\phi_{1,t}(\mathbf{x}), \dots, \phi_{t,t}(\mathbf{x})) \quad (1 \leq i \leq t)$$

are also identically zero. On applying the chain rule, we therefore find that

$$\sum_{j=1}^{t} f_j'(x_i)\Psi_j(\phi_{1,t}(\mathbf{x}), \dots, \phi_{t,t}(\mathbf{x})) = 0 \quad (1 \leq i \leq t), \tag{9}$$

where we have written $\Psi_j(\mathbf{z})$ for $(\partial/\partial z_j)\Psi(\mathbf{z})$. But as a consequence of Lemma 1, the polynomial

$$\det(f_j'(x_i))_{1 \leq i,j \leq t}$$

is not identically zero, and thus it follows from (9) that each of the polynomials $\Psi_j(\phi_{1,t}(\mathbf{x}), \dots, \phi_{t,t}(\mathbf{x}))$ $(1 \leq j \leq t)$ must be identically zero. However, since $\Psi(\mathbf{z})$ is a nonconstant polynomial, at least one of the derivatives $\Psi_j(\mathbf{z})$ $(1 \leq j \leq t)$ must be nonzero. Hence there exists a nontrivial polynomial $\Psi_j(\mathbf{z}) \in \mathbb{Z}[\mathbf{z}]$ for which, in particular, one has

$$\Psi_j(\phi_{1,t-1}(\mathbf{z}), \dots, \phi_{t,t-1}(\mathbf{z})) = 0.$$

Since the latter conclusion contradicts the minimality of the total degree of Ψ, we are forced to conclude that the inequality (8) does indeed hold. □

3. Application of the Eliminant Polynomial

We are now equipped to prosecute our proof of Theorem 1. We begin by noting that the conclusion of the theorem is classical when $t = 1$. For on writing $e(z)$ for $\exp(2\pi i z)$ and setting

$$F(\alpha) = \sum_{1 \le x \le P} e(\alpha f_1(x)),$$

it follows from Hua's lemma (see e.g. [16, Thm. 4]) that, whenever $k_1 \ge 2$ and $P \ge 3$, one has

$$S_2(P;\, f_1) = \int_0^1 |F(\alpha)|^4 \, d\alpha \ll P^2 (\log P)^A,$$

where A is a positive number depending at most on k_1 and the coefficients of f_1. We may therefore suppose in what follows that $t \ge 2$, and moreover the discussion of Section 2 permits the assumption that $(f_1, \dots, f_t)$ is well-conditioned. We thus infer from Lemma 1 that there exists an asymptotically definite polynomial $\Theta(\mathbf{x}; \mathbf{f})$ with the property that the relation (3) holds. We write C for the parameter associated with $\Theta(\mathbf{x}; \mathbf{f})$ from our definition of asymptotic definiteness.

We next dispose of small solutions of (1) counted by $S_{t+1}(P; \mathbf{f})$. Write

$$G(\boldsymbol{\alpha}) = \sum_{1 \le x \le C} e(\alpha_1 f_1(x) + \cdots + \alpha_t f_t(x))$$

and

$$H(\boldsymbol{\alpha}) = \sum_{C < x \le P} e(\alpha_1 f_1(x) + \cdots + \alpha_t f_t(x)).$$

Then by orthogonality one finds that

$$\begin{aligned} S_{t+1}(P; \mathbf{f}) &= \int_{[0,1)^t} |G(\boldsymbol{\alpha}) + H(\boldsymbol{\alpha})|^{2t+2} \, d\boldsymbol{\alpha} \\ &\ll \int_{[0,1)^t} |G(\boldsymbol{\alpha})|^{2t+2} \, d\boldsymbol{\alpha} + \int_{[0,1)^t} |H(\boldsymbol{\alpha})|^{2t+2} \, d\boldsymbol{\alpha}. \end{aligned}$$

Thus, on writing $S_s^*(P; \mathbf{f})$ for the number of integral solutions of (1) with $C < x_i, y_i \le P$ $(1 \le i \le s)$, a trivial estimate for $G(\boldsymbol{\alpha})$ yields the upper bound

$$S_{t+1}(P; \mathbf{f}) \ll 1 + S_{t+1}^*(P; \mathbf{f}).$$

Let $S_{t+1}^0(P; \mathbf{f})$ denote the number of integral solutions of the system

$$\sum_{i=1}^{t} f_j(x_i) - f_j(x_{t+1}) = \sum_{i=1}^{t} f_j(y_i) - f_j(y_{t+1}) \quad (1 \le j \le t), \tag{10}$$

with $C < x_i, y_i \le P$ $(1 \le i \le t)$, satisfying the condition that $x_i = x_j$ for some $1 \le i < j \le t$. Also, let $\hat{S}_{t+1}(P; \mathbf{f})$ denote the complementary number of solutions of (10) in which $x_i \ne x_j$ for $1 \le i < j \le t$. Then plainly

$$S_{t+1}^*(P; \mathbf{f}) = S_{t+1}^0(P; \mathbf{f}) + \hat{S}_{t+1}(P; \mathbf{f}). \tag{11}$$

But it follows from a consideration of the underlying diophantine equations that

$$\begin{aligned} S^0_{t+1}(P;\mathbf{f}) &\ll \int_{[0,1)^t} H(2\boldsymbol{\alpha})H(\boldsymbol{\alpha})^{t-1}H(-\boldsymbol{\alpha})^{t+1}\,d\boldsymbol{\alpha} \\ &\ll \int_{[0,1)^t} |H(2\boldsymbol{\alpha})H(\boldsymbol{\alpha})^{2t}|\,d\boldsymbol{\alpha}, \end{aligned}$$

and thus, by Hölder's inequality, we find that

$$S^0_{t+1}(P;\mathbf{f}) \ll \left(\int_{[0,1)^t} |H(\boldsymbol{\alpha})|^{2t+2}\,d\boldsymbol{\alpha}\right)^{t/(t+1)} \left(\int_{[0,1)^t} |H(2\boldsymbol{\alpha})|^{2t+2}\,d\boldsymbol{\alpha}\right)^{1/(2t+2)}.$$

Then, on considering the underlying diophantine equations, we deduce that

$$S^0_{t+1}(P;\mathbf{f}) \ll (S^*_{t+1}(P;\mathbf{f}))^{(2t+1)/(2t+2)},$$

whence by (2) and (11) we have

$$S^*_{t+1}(P;\mathbf{f}) \ll \hat{S}_{t+1}(P;\mathbf{f}). \tag{12}$$

We now analyse the solutions of (10) counted by $\hat{S}_{t+1}(P;\mathbf{f})$. By Lemma 2, there exists a polynomial $\Psi(\mathbf{z}) \in \mathbb{Z}[z_1,\dots,z_t]$, with total degree and coefficients depending at most on t, $\mathbf{k}$, and the coefficients of $f_1,\dots,f_t$, such that whenever $u_i = u_{t+1}$ for some $1 \le i \le t$,

$$\Psi(\phi_{1,t}(\mathbf{u}) - f_1(u_{t+1}),\dots,\phi_{t,t}(\mathbf{u}) - f_t(u_{t+1})) = 0$$

and yet

$$\Psi(\phi_{1,t}(\mathbf{u}),\dots,\phi_{t,t}(\mathbf{u})) \neq 0.$$

It follows that, for some nontrivial polynomial $\Phi(\mathbf{u}) \in \mathbb{Z}[u_1,\dots,u_{t+1}]$,

$$\Psi(\phi_{1,t}(\mathbf{u}) - f_1(u_{t+1}),\dots,\phi_{t,t}(\mathbf{u}) - f_t(u_{t+1})) = \Phi(\mathbf{u})\prod_{i=1}^{t}(u_i - u_{t+1}). \tag{13}$$

For the sake of concision, we write

$$\Upsilon(\mathbf{z}) = \Phi(\mathbf{z})\prod_{i=1}^{t}(z_i - z_{t+1}).$$

Let T_1 denote the number of solutions of (10) counted by $\hat{S}_{t+1}(P;\mathbf{f})$ having the property that $\Upsilon(\mathbf{y})$ is nonzero, and let T_2 denote the corresponding number of solutions with $\Upsilon(\mathbf{y}) = 0$. Then

$$\hat{S}_{t+1}(P;\mathbf{f}) = T_1 + T_2. \tag{14}$$

Consider first a solution $(\mathbf{x},\mathbf{y})$ counted by T_1. In view of (10) and (13), one has

$$\Phi(\mathbf{x})\prod_{i=1}^{t}(x_i - x_{t+1}) = \Phi(\mathbf{y})\prod_{i=1}^{t}(y_i - y_{t+1}). \tag{15}$$

Fix a choice of $\mathbf{y}$ with $\Upsilon(\mathbf{y}) \neq 0$. Then, if $\tau(n)$ denotes the divisor function, we find from (15) that there are at most $(2\tau(|\Upsilon(\mathbf{y})|))^t$ possible choices for $x_i - x_{t+1}$

$(1 \le i \le t)$. Fixing any one such choice of the latter t quantities, we write $x_i = x_{t+1} + d_i$ $(1 \le i \le t)$. Then, on substituting these fixed choices of $\mathbf{y}$ and $\mathbf{d}$ into (10), we find that x_{t+1} satisfies the evidently nontrivial equation

$$\sum_{i=1}^{t} f_1(x_{t+1} + d_i) - f_1(x_{t+1}) = \sum_{i=1}^{t} f_1(y_i) - f_1(y_{t+1}).$$

One therefore has $O(1)$ possibilities for x_{t+1}, whence the total number of solutions of this type is

$$T_1 \ll \sum_{\mathbf{y}} (\tau(|\Upsilon(\mathbf{y})|))^t,$$

where the summation is over $\mathbf{y}$ with $1 \le y_i \le P$ $(1 \le i \le t+1)$ and $\Upsilon(\mathbf{y}) \ne 0$. We thus conclude from [16, Thm. 3] that

$$T_1 \ll P^{t+1}(\log P)^A, \tag{16}$$

where the positive number A depends at most on t, $\mathbf{k}$, and the coefficients of $f_1, \ldots, f_t$.

Next consider a solution $(\mathbf{x}, \mathbf{y})$ counted by T_2. The number of values of $\mathbf{y}$ with $1 \le y_i \le P$ $(1 \le i \le t+1)$ for which $\Upsilon(\mathbf{y}) = 0$ is $O(P^t)$ (see e.g. the proof of [28, Lemma 2]). Fix any one such choice of $\mathbf{y}$, and fix any one of the $O(P)$ possible choices for x_{t+1}. Then, on writing

$$N_j = f_j(x_{t+1}) - f_j(y_{t+1}) + \sum_{i=1}^{t} f_j(y_i) \quad (1 \le j \le t),$$

we find from (10) that

$$\sum_{i=1}^{t} f_j(x_i) = N_j \quad (1 \le j \le t). \tag{17}$$

Suppose first that $\mathbf{x}$ is a singular solution of (17). Then one has

$$\det(f_i'(x_j))_{1 \le i, j \le t} = 0,$$

whence from (3) it follows that

$$\Theta(\mathbf{x}; \mathbf{f}) \prod_{1 \le i < j \le t} (x_j - x_i) = 0. \tag{18}$$

But by hypothesis, one has $x_i \ne x_j$ for $1 \le i < j \le t$; moreover, Lemma 1 ensures that, since $x_i > C$ $(1 \le i \le t)$, one has $\Theta(\mathbf{x}; \mathbf{f}) \ne 0$. Then the equation (18) is impossible, whence there are no singular solutions $\mathbf{x}$ counted by T_2.

We complete our treatment of T_2 by considering the nonsingular points $\mathbf{x}$ satisfying (17). According to Theorem 7.7 of Hartshorne [5, Chap. 1], the number of irreducible components contained in the intersection (17) is at most $k_1 k_2 \cdots k_t$. If such a component has positive dimension, then it arises from an improper intersection and is consequently singular. Thus it follows that all the points that concern us here arise from components of the intersection having dimension 0, whence their number is also at most $k_1 k_2 \cdots k_t$. We may thus conclude that, for the fixed choice of $(x_{t+1}, \mathbf{y})$ under consideration, there are $O(1)$ permissible choices of $x_1, \ldots, x_t$. Finally, therefore, we deduce that

$$T_2 \ll P^{t+1}. \tag{19}$$

On combining (12), (14), (16), and (19), we at last arrive at the upper bound

$$S_{t+1}(P; \mathbf{f}) \ll P^{t+1}(\log P)^A,$$

and this completes the proof of our theorem. □

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