RELATIONS BETWEEN INTEGRAL AND MODULAR REPRESENTATIONS

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1. INTRODUCTION

Throughout this paper, let R denote a noetherian complete local integral domain, with maximal ideal P, residue class field $\overline{R} = R/P$, and field of quotients K. For example, one possible choice for R might be a valuation ring in some p-adic field. Let A be an R-algebra with unity element 1, finitely generated as R-module. An A-module will mean a left A-module, finitely generated over R, on which 1 acts as identity operator.

Set $\overline{A} = A/PA$, a finite-dimensional $\overline{R}$-algebra. To each A-module M there corresponds an $\overline{A}$-module $\overline{M} = M/PM$. As is well known, the mapping $M \to \overline{M}$ gives a one-to-one isomorphism-preserving correspondence between projective A-modules M and projective $\overline{A}$-modules $\overline{M}$. One of the main results of the present work is a generalization of this theorem for the special case in which A is the group ring RG of a finite group G. This permits us to establish some relationships between representation algebras of RG-modules and those of $\overline{R}$G-modules.

Section 2 is devoted to the necessary preliminaries concerning R-algebras. Most of the results given there are already known but not readily available in any single reference. We have therefore outlined a few of the proofs, for the convenience of the reader.

In Section 3, after some easy results on A-modules, we restrict ourselves to the case A = RG, and obtain the above-mentioned generalization. The paper concludes in Section 4 with various propositions concerning the behavior of modules under ground ring extension. One of these gives a necessary and sufficient condition that an A-module be absolutely indecomposable. Another asserts that if $\overline{R}$ is a finite field and M an indecomposable A-module, then for each suitably restricted ring S containing R, the $S \otimes A$-module $S \otimes M$ splits into a direct sum of indecomposable submodules, no two of which are isomorphic.

2. ALGEBRAS OVER COMPLETE LOCAL RINGS

In this section we list a number of results about algebras over complete local rings. We draw heavily from Jacobson [8]; but we simplify his proofs, because we do not need his results in the full generality with which he presents them. Other relevant references are Azumaya [1], Borevič and Faddeev [2], Conlon [3], Curtis and Reiner [4], Green [5], Swan [13].

If A is an arbitrary ring with 1, denote by rad A its Jacobson radical (see Jacobson [8, Chapter I]). Then rad A is a two-sided ideal of A, and the factor ring A/rad A has zero radical.

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PROPOSITION 1. *Each of the following three statements characterizes the Jacobson radical.*

i) rad A *is the intersection of the annihilators of the irreducible left* A-*modules.*

ii) rad A *is the intersection of the maximal left ideals of* A.

iii) rad A *consists of all elements* $x \in A$ *such that for each* $a, b \in A$, *the element* $1 - axb$ *is a unit in* A.

(The element $u \in A$ is called a *unit* if and only if there exists $v \in A$ such that $uv = vu = 1$.)

From iii) one easily obtains the following result.

COROLLARY. *Each ring epimorphism* $A \to B$ *induces a ring epimorphism* $A/\text{rad } A \to B/\text{rad } B$.

Finally, we remark that if A is a ring with minimum condition, then rad A is nilpotent.

Suppose now that A is an R-algebra. For all $a \in A$ and $p \in P$, we have the relation

$$(1 - pa)b = b(1 - pa) = 1, \quad \text{where } b = 1 + pa + p^2a^2 + \cdots.$$

Thus $1 - pa$ is a unit in A, which shows that $PA \subset \text{rad } A$. The epimorphism $A \to \bar{A}$ induces an epimorphism $A/\text{rad } A \to \bar{A}/\text{rad } \bar{A}$. But also

$$\frac{A}{\text{rad } A} \cong \frac{A/PA}{(\text{rad } A)/PA} \cong \frac{\bar{A}}{\bar{C}}$$

for some two-sided ideal $\bar{C}$ of $\bar{A}$. Thus $\bar{A}$ maps onto $A/\text{rad } A$, inducing an epimorphism $\bar{A}/\text{rad } \bar{A} \to A/\text{rad } A$. Since both $A/\text{rad } A$ and $\bar{A}/\text{rad } \bar{A}$ are finite-dimensional $\bar{R}$-algebras, we have the following result.

PROPOSITION 2. $A/\text{rad } A \cong \bar{A}/\text{rad } \bar{A}$. *Furthermore,* rad $\bar{A}$ *is nilpotent, and hence there exists an integer* n *for which*

$$(\text{rad } A)^n \subset PA. \tag{1}$$

By an *idempotent* $e \in A$ we shall mean a nonzero element such that $e^2 = e$. Two idempotents e and e' are *orthogonal* if $ee' = e'e = 0$. We call the idempotent e *primitive* if e cannot be expressed as a sum of orthogonal idempotents. Then each decomposition

$$1 = e_1 + \cdots + e_k \tag{2}$$

of the unity element $1 \in A$ into a sum of orthogonal primitive idempotents $\{e_i\}$ yields a decomposition

$$A = Ae_1 \oplus \cdots \oplus Ae_k \tag{3}$$

into indecomposable left ideals $\{Ae_i\}$ (called the *components* of A), and all possible such decompositions of A are obtainable in this manner.

PROPOSITION 3 (Jacobson [8, p. 51]). *Let* e_i *and* e_j *be idempotents in an arbitrary ring* A. *Then* $Ae_i \cong Ae_j$ *as left* A*-modules if and only if there exist elements* $x, y \in A$ *such that*

$$e_i x e_j = x, \quad e_j y e_i = y, \quad xy = e_i, \quad yx = e_j. \tag{4}$$

PROPOSITION 4. *Let* A *be an* R*-algebra, and* B *a two-sided ideal of* A *contained in* rad A. *Set* $A' = A/B$, *and let* $a \to a'$ *under the map* $A \to A'$. *Then every decomposition*

$$1 = e_1 + \cdots + e_k \tag{5}$$

into primitive orthogonal idempotents in A *gives a decomposition*

$$1' = e_1' + \cdots + e_k' \tag{6}$$

into primitive orthogonal idempotents in A'. *Conversely, each decomposition* (6) *can be lifted to a decomposition* (5).

The correspondence $Ae_i \to A'e_i'$ *preserves isomorphism, that is,* $Ae_i \cong Ae_j$ *if and only if* $A'e_i' \cong A'e_j'$. *Furthermore, for each* i,

$$(e_i A e_i)/\text{rad}(e_i A e_i) \cong (e_i' A' e_i')/\text{rad}(e_i' A' e_i'). \tag{7}$$

Proof. Let the idempotent $e \in A$ map onto $e' \in A'$. Then $(e')^2 = e'$; if $e' = 0$, then $e \in B \subset \text{rad}\, A$, which is impossible. Thus e' is also idempotent. We claim that if e is primitive, then so is e'. This can be established by the method of "lifting idempotents" (see for example [4, Section 77]). In order to apply the method to the present situation, we must verify that for $b \in B$, the sequence $\{b, b^2, b^3, \cdots\}$ approaches 0 in the P-adic topology of A. This is indeed the case, since by (1), $b^n \in PA$. The process of lifting idempotents also shows that there is a one-to-one correspondence between decompositions (5) and (6).

To verify (7), we note the ring epimorphisms

$$e_i A e_i \to e_i' A' e_i', \qquad e_i' A' e_i' \to e_i A e_i/\text{rad}(e_i A e_i).$$

Then we use the corollary to Proposition 1.

Finally, from $Ae_i \cong Ae_j$ it follows by Proposition 3 that $A'e_i' \cong A'e_j'$. Conversely, suppose that the latter isomorphism holds. Then there exist $x, y \in A$ such that equations (4) hold when each symbol therein is primed. Replace x by $e_i x e_j$ and y by $e_j y e_i$, and call the new elements x and y once more. Then $xy = e_i - b$, $b \in B$; set $c = e_i + b + b^2 + \cdots$, so that

$$xyc = e_i, \qquad (ycx)^2 = ycx.$$

But $ycx = e_j - d$ for some $d \in B$, and thus $(e_j - d)^2 = e_j - d$. Multiplying the last equation on the left by $e_j + d + d^2 + \cdots$, we deduce that $d = 0$. Therefore equations (4) hold with yc in place of y, and consequently $Ae_i \cong Ae_j$.

We shall say that the rings A and A' are *interrelated* if there is a correspondence between the components of A and those of A', as described in Proposition 4. Thus A and $\bar{A}$ are interrelated, as are A and A/rad A.

Throughout the rest of the paper, we shall use the following notation. For an A-module M, we set

$$E(M) = \mathrm{Hom}_A(M, M), \qquad \hat{E}(M) = E(M)/\mathrm{rad}\, E(M). \tag{8}$$

If it becomes necessary to indicate the operator domain explicitly, we shall write $E_A(M)$ rather than E(M).

If e is any idempotent in A, then it is easily verified that $E(Ae) \cong eAe$, and thus

$$\hat{E}(Ae) \cong eAe/\mathrm{rad}\,(eAe).$$

The isomorphism in (7) may be written more concisely as

$$\hat{E}_A(Ae_i) \cong \hat{E}_{A'}(A'e_i'),$$

and it is in this form that we use it hereafter.

Now let E be an arbitrary ring with unity, and let N be the set of non-units of E. Call E *completely primary* if N is a two-sided ideal of E; in this case, necessarily N = rad E.

PROPOSITION 5. *For an* R*-algebra* E, *the following conditions are equivalent.*

(i) E *is completely primary.*

(ii) E/rad E *is a skewfield.*

(iii) E *contains no idempotent except* 1.

Proof. Conditions (i) and (ii) are equivalent for arbitrary rings (see Jacobson [8, p. 58]), and we take this fact for granted here. To show that (i) implies (iii), let $e \in E$ be an idempotent different from 1. Since $e(1 - e) = 0$, both e and $1 - e$ are non-units. Hence if (i) holds, the sum $e + (1 - e)$ is also a non-unit, which is impossible.

Conversely, (iii) implies $\hat{\ }$(ii). For let $\hat{E} = E/\mathrm{rad}\, E \cong \overline{E}/\mathrm{rad}\, \overline{E}$; then $\hat{E}$ is a semisimple algebra over $\overline{R}$. If $\hat{E}$ is not a skewfield, then some nonzero $x \in \hat{E}$ is not a unit, and so either $\hat{E}x$ or $x\hat{E}$ is a proper ideal of $\hat{E}$. Hence (since $\hat{E}$ is semisimple) $\hat{E}$ has more than one component, and so it contains an idempotent different from 1. Therefore also E contains an idempotent different from 1. Hence if (ii) is false, so is (iii), which completes the proof.

PROPOSITION 6. *Let* A *be an* R*-algebra,* M *an* A*-module. Then* M *is indecomposable if and only if* E(M) *is completely primary, that is,* $\hat{E}(M)$ *is a skewfield.*

Proof. Evidently E(M) is also an R-algebra, and M is indecomposable if and only if E(M) contains no idempotent except the identity automorphism of M.

PROPOSITION 7 (Krull-Schmidt Theorem). *Let* A *be an* R*-algebra,* M *an* A*-module. Then* M *is decomposable into a direct sum of indecomposable* A*-modules, called the components of* M, *which are uniquely determined by* M, *up to isomorphism and order of occurrence.*

Proof. The proofs in [8, p. 58] or [4, Section 14] carry over virtually unchanged. Thus, Lemma 14.3 in [4] remains valid; so does Lemma 14.4 [4], since an automorphism of M is the same thing as a unit in E(M). Finally, the entire proof of of Theorem 14.5 [4] holds in the present situation, except the part showing that $M' = M$. This equality will follow once we know that $M_1 \subset M'$. But for $x \in N_1$,

$$\mu_1 x = x - \mu_2 x - \cdots - \mu_h x \in M'.$$

Hence $M_1 = \mu_1(N_1) \subset M'$, as desired.

PROPOSITION 8 (Fitting). *Let* M *be an* A-*module, where* A *is an* R-*algebra, and set* E = E(M). *Then* M *and* E *are interrelated; that is, if*

$$E = Ee_1 \oplus \cdots \oplus Ee_k$$

is a decomposition of E *into indecomposable left ideals, then*

$$M = e_1 M \oplus \cdots \oplus e_k M$$

is a decomposition of M *into indecomposable* A-*submodules. Furthermore,* $e_i M \cong e_j M$ *as left* A-*modules if and only if* $Ee_i \cong Ee_j$ *as left* E-*modules. Finally,*

$$\hat{E}(Ee_i) \cong \hat{E}(e_i M) \qquad (9)$$

for each i.

Proof. For each i, consider Ee_i as left E-module, and $e_i M$ as left A-module. Since E is an R-algebra and Ee_i is an indecomposable E-module, it follows from Proposition 6 that $\hat{E}(Ee_i)$ is a skewfield. Thus, once (9) is proved, it will follow that $e_i M$ is also indecomposable.

We have remarked previously that

$$\mathrm{Hom}_E(Ee_i, Ee_i) \cong e_i E e_i.$$

In order to establish (9), it therefore suffices to show that

$$\mathrm{Hom}_A(e_i M, e_i M) \cong e_i E e_i. \qquad (10)$$

Given $f \in \mathrm{Hom}_A(e_i M, e_i M)$, define $f' \in E$ by letting f' coincide with f on $e_i M$, and letting $f'(e_j M) = 0$ $(j \neq i)$. Then f maps onto the element $e_i f' e_i \in e_i E e_i$. In the other direction, each $g \in e_i E e_i$, upon restriction to $e_i M$, yields an A-endomorphism of $e_i M$. This proves (10), and thus also (9).

Next, suppose that $Ee_i \cong Ee_j$, so that equations (4) hold for some elements $x, y \in E$. Define the homomorphisms

$$\eta: e_i M \to e_j M, \qquad \xi: e_j M \to e_i M$$

by the formulas

$$\eta u = yu, \quad \xi v = xv \qquad (u \in e_i M,\ v \in e_j M).$$

Then $\eta\xi = 1$, $\xi\eta = 1$, whence $e_i M \cong e_j M$.

Conversely, let $f: e_i M \cong e_j M$. The composition of the maps

$$M \to e_i M \xrightarrow{f} e_j M \subset M$$

defines an element $y \in E$. Likewise, define $x \in E$ by the maps

$$M \to e_j M \xrightarrow{f^{-1}} e_i M \subset M.$$

For this choice of x and y in E, equations (4) are valid, and therefore $Ee_i \cong Ee_j$.

We have seen in Proposition 4 that the rings E(M) and $\hat{E}(M)$ are interrelated. Hence, as a consequence of Proposition 8, we have the following result.

PROPOSITION 9. *Let* M *be an* A-*module, where* A *is an* R-*algebra. The module* M *is interrelated with the semisimple* $\overline{R}$-*algebra* $\hat{E}(M)$. *If*

$$1 = \delta_1 + \cdots + \delta_t$$

(where $\{\delta_i\}$ *is a set of orthogonal primitive idempotents) is a decomposition of* $1 \in E(M)$, *then there exists a decomposition*

$$M = M_1 \oplus \cdots \oplus M_t,$$

where each M_i *is indecomposable, with the following properties:*

$$\hat{E}(M_i) \cong \delta_i \hat{E}(M) \delta_i \quad (1 \leq i \leq t),$$

and $M_i \cong M_j$ *if and only if* $\hat{E}(M)\delta_i \cong \hat{E}(M)\delta_j$.

3. REPRESENTATIONS OF GROUPS

Throughout this section, A denotes an R-algebra, M an A-module, and E(M) and $\hat{E}(M)$ are defined by (8). If Aut(M) denotes the set of A-automorphisms of M, then Aut(M) is precisely the set of units of E(M).

PROPOSITION 10. *Each* $f \in E(M)$ *induces an element* $\overline{f} \in E(\overline{M})$, *and the map* $f \to \overline{f}$ *gives a ring homomorphism* $E(M) \to E(\overline{M})$. *If* $f \in E(M)$, *then* $f \in \mathrm{Aut}(M)$ *if and only if* $\overline{f} \in \mathrm{Aut}(\overline{M})$.

Proof. Only the last statement requires proof, and in one direction it is obvious. Suppose now that f in E(M) is such that $\overline{f} \in \mathrm{Aut}(\overline{M})$, and let us show that $f \in \mathrm{Aut}(M)$.

Now $\overline{M}$ is a vector space over $\overline{R}$; since R is a local ring, any $\overline{R}$-basis of $\overline{M}$ can be lifted to a set of generators of M. Thus we may write $M = \sum Rm_i$, where $\{\overline{m}_1, \cdots, \overline{m}_k\}$ is an $\overline{R}$-basis for $\overline{M}$. Then

$$f(m_i) = \sum_{j=1}^{k} \alpha_{ij} m_j \quad (1 \leq i \leq k),$$

for some choice of coefficients $\alpha_{ij} \in R$. Consequently,

$$\overline{f}(\overline{m}_i) = \sum \overline{\alpha}_{ij} \overline{m}_j \quad (1 \leq i \leq k).$$

Since $\overline{f}$ is an automorphism, the matrix $(\overline{\alpha}_{ij})$ is invertible over $\overline{R}$. Therefore the matrix (α_{ij}) is invertible over R, and f is an R-automorphism of M. But since by hypothesis f is an A-homomorphism, it follows that f is an A-automorphism, as claimed.

The following is an easy consequence of Proposition 10.

PROPOSITION 11. *Let* λ: $E(M) \to \hat{E}(\overline{M})$ *be the ring homomorphism defined by* $E(M) \to E(\overline{M}) \to \hat{E}(\overline{M})$. *Then* $\ker \lambda \subset \operatorname{rad} E(M)$. *In particular, if* $\overline{M}$ *is indecomposable, then* λ *induces an* $\overline{R}$*-algebra monomorphism*

$$\hat{E}(M) \to \hat{E}(\overline{M}). \tag{11}$$

Hence if $\hat{E}(\overline{M}) \cong \overline{R}$, *then also* $\hat{E}(M) \cong \overline{R}$.

Proof. Let $f \in E(M)$ lie in $\ker \lambda$, so that $\overline{f} \in \operatorname{rad} E(\overline{M})$. If $f \notin \operatorname{rad} E(M)$, then there exist elements $u, v \in E(M)$ such that $1 - ufv \notin \operatorname{Aut}(M)$. Proposition 10 then shows that $\overline{1} - \overline{u}\overline{f}\overline{v} \notin \operatorname{Aut}(\overline{M})$, which contradicts the assumption that $f \in \operatorname{rad} E(\overline{M})$.

Suppose now that $\overline{M}$ is indecomposable. Then $E(\overline{M})$ is completely primary, and we claim that $\operatorname{rad} E(M) \subset \ker \lambda$. Indeed, if the inclusion does not hold, then $\lambda f \neq 0$ for some $f \in \operatorname{rad} E(M)$. Thence $\overline{f} \notin \operatorname{rad} E(\overline{M})$, so that $\overline{f} \in \operatorname{Aut}(\overline{M})$, whence $f \in \operatorname{Aut}(M)$. This is impossible, and so (11) gives a monomorphism. The proposition is thus established.

Suppose now that G is a finite group with a normal subgroup H, and let RH denote the group ring of H over R. For each RH-module M and each element $x \in G$, define the *conjugate* RH-module $M^{(x)}$ as follows: $M^{(x)}$ has the same elements as M, and each $h \in H$ acts on $M^{(x)}$ in the same way that xhx^{-1} acts on M. The *stabilizer* of M in G is defined as

$$S = \{x \in G: M^{(x)} \cong M \text{ as RH-modules}\}.$$

Then S is a subgroup of G, and $S \supset H$.

Let us form the induced RG-module M^G, defined as $RG \otimes_{RH} M$. Conlon [3] and Tucker [14] to [16] have investigated the components of M^G, especially when M is indecomposable. As a matter of fact, their work deals with $\overline{R}H$-modules rather than RH-modules, but since they use only the analogues of Propositions 6 to 8, their proofs carry over virtually unchanged to the present situation. We shall state several of their results in Proposition 12 below.

Starting with an indecomposable RH-module M, let S be its stabilizer in G. Choose coset representatives $g_1, \cdots, g_t \in G$ with $g_1 = 1$, such that

$$G = \bigcup_{i=1}^{t} g_i H, \qquad S = \bigcup_{i=1}^{s} g_i H.$$

Then

$$M^G = \sum_{i=1}^{t}{}^{\oplus} g_i \otimes M,$$

and each $g_i \otimes M$, when viewed as RH-module, is a conjugate of M. There is an R-module isomorphism

$$E(M^G) = \operatorname{Hom}_{RG}(M^G, M^G) \cong \sum_{i=1}^{t}{}^{\oplus} \operatorname{Hom}_{RH}(M, g_i \otimes M).$$

This isomorphism maps $f \in E(M^G)$ onto the t-tuple $\{\phi_1, \cdots, \phi_t\}$, where

$$f(1 \otimes m) = \sum_{i=1}^{t} \phi_i(m), \qquad \phi_i(m) \in g_i \otimes M.$$

For each i $(1 \leq i \leq s)$, the RH-modules M and $g_i \otimes M$ are isomorphic. Therefore there is an R-isomorphism

$$\mathrm{Hom}_{RH}(M, g_i \otimes M) \cong \mathrm{Hom}_{RH}(M, M) = E(M).$$

An element $\phi \in \mathrm{Hom}_{RH}(M, g_i \otimes M)$ is an isomorphism if and only if its image in E(M) is an automorphism of M.

Let us set

$$E'(M^G) = E(M^G) \cap \mathrm{rad}\, \mathrm{Hom}_{RH}(M^G, M^G),$$

where in the expression after the intersection sign, M^G is viewed as RH-module. Define

$$E^*(M^G) = E(M^G)/E'(M^G).$$

Conlon and Tucker showed the following.

PROPOSITION 12. $E'(M^G)$ *is a two-sided ideal contained in* rad $E(M^G)$, *whence* $E(M^G)$ *and* $E^*(M^G)$ *are interrelated. Given* $f \in E(M^G)$, *we have* $f \in E'(M^G)$ *if and only if none of* $\phi_1, \cdots, \phi_s$ *is an isomorphism. Therefore (as* $\bar{R}$*-modules)*

$$E^*(M^G) \cong \hat{E}(M) \dotplus \cdots \dotplus \hat{E}(M) \qquad (s \textit{ copies}).$$

Hence there is a ring isomorphism

$$E^*(M^G) \cong E^*(M^S),$$

and so M^G *and* M^S *are interrelated.*

In particular, if $\hat{E}(M) \cong \bar{R}$, *then there exists a certain twisted group algebra* $\{S/H; \bar{R}\}$ *of* S/H *over* $\bar{R}$, *such that*

$$E^*(M^G) \cong \{S/H; \bar{R}\}.$$

If S/H *is a* p*-group, and* $\bar{R}$ *has characteristic* p, *then* $\{S/H; \bar{R}\}$ *is just the ordinary group algebra of* S/H *over* $\bar{R}$.

We may now give one of the main results of the present paper.

THEOREM 1. *Let* M *be an* RH*-module such that* $\bar{M}$ *is indecomposable. Assume that* $\hat{E}(M) \cong \hat{E}(\bar{M})$, *and that* M *and* $\bar{M}$ *have the same stabilizer* S *in* G. *Then there is a one-to-one isomorphism-preserving correspondence* $X \to \bar{X}$ *between the components* $\{X\}$ *of* M^G *and the components* $\{\bar{X}\}$ *of* $\bar{M}^G$, *with* $\bar{X} = X/PX$.

In fact, the modules M^G, $\bar{M}^G$, M^S, $\bar{M}^S$ *are interrelated. Corresponding to any decomposition into components*

$$M^S = \sum_{i=1}^{n}{}^{\oplus} L_i,$$

there are decompositions into components

$$\overline{M}^S = \sum_{i=1}^{n}{}^{\oplus} \overline{L}_i, \quad M^G = \sum_{i=1}^{n}{}^{\oplus} L_i^G, \quad \overline{M}^G = \sum_{i=1}^{n}{}^{\oplus} \overline{L}_i^G .$$

Furthermore,

$$L_i \cong L_j \Leftrightarrow \overline{L}_i \cong \overline{L}_j \Leftrightarrow L_i^G \cong L_j^G \Leftrightarrow \overline{L}_i^G \cong \overline{L}_j^G .$$

Finally, for each i,

$$\hat{E}(L_i) \cong \hat{E}(\overline{L}_i) \cong \hat{E}(L_i^G) \cong \hat{E}(\overline{L}_i^G).$$

Proof. Apply Proposition 12 to both M and $\overline{M}$. There is a commutative diagram

$$\begin{array}{ccc} E(M^G) & \cong & \sum_{i=1}^{t}{}^{\oplus} \operatorname{Hom}_{RH}(M, g_i \otimes M) \\ \tau \downarrow & & \downarrow \\ E(\overline{M}^G) & \cong & \sum_{i=1}^{t}{}^{\oplus} \operatorname{Hom}_{\overline{R}H}(\overline{M}, g_i \otimes \overline{M}) \end{array},$$

with the vertical maps given by $f \to \overline{f}$. Since $\phi_i \in \operatorname{Hom}_{RH}(M, g_i \otimes M)$ is an isomorphism if and only if $\overline{\phi}_i$ is an isomorphism, it follows at once that $\tau f \in E'(\overline{M}^G)$ if and only if $f \in E'(M^G)$. Thus τ induces an $\overline{R}$-monomorphism $E^*(M^G) \to E^*(\overline{M}^G)$.

On the other hand, $\hat{E}(M) \cong \hat{E}(\overline{M})$ by hypothesis, and thus both $E^*(M^G)$ and $E^*(\overline{M}^G)$ are $\overline{R}$-isomorphic to a direct sum of s copies of $\hat{E}(M)$. This shows that $E^*(M^G) \cong E^*(\overline{M}^G)$ as $\overline{R}$-algebras. Therefore M^G and $\overline{M}^G$ are interrelated, and the theorem is proved.

Remarks. (1) Let us show that the hypotheses of Theorem 1 cannot be weakened, in general. Let Z_3^* be the ring of 3-adic integers in the 3-adic completion Q_3 of the rational field, and let ω be a primitive cube root of unity over Q_3. The 3-adic valuation of Q_3 extends uniquely to the field $K = Q_3(\omega)$, and its valuation ring is $R = Z_3^*[\omega]$. The maximal ideal P of R is given by $P = (1 - \omega)R$, and $\overline{R} = R/P$ is a finite field with three elements.

Example (i). Choose $G = S_3$, the symmetric group generated by a and b, where $a^2 = b^3 = 1$, $aba = b^2$. Let H be the cyclic subgroup generated by b. For M we take the RH-module consisting of the elements of R, with the action of H given by $bm = \omega m$ $(m \in M)$. Then $\overline{M}$ is the trivial $\overline{R}H$-module $\overline{R}$. Obviously, the stabilizer of M in G is H itself, whereas $\overline{M}$ has stabilizer G.

We may write $M^G = 1 \otimes M \oplus a \otimes M$, and relative to the R-basis $\{1 \otimes 1, a \otimes 1\}$, M^G affords the representation

$$a \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b \to \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}.$$

Since this representation is irreducible over K, it is surely indecomposable.

On the other hand, $\overline{M}^G$ affords the $\overline{R}$-representation

$$a \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since 2 is a unit in $\overline{R}$, this representation decomposes into a direct sum of two 1-dimensional components.

We have therefore shown that if M and $\overline{M}$ have different stabilizers, then the number of components of M^G and $\overline{M}^G$ need not be the same.

Example (ii). We shall next show that if $\hat{E}(M)$ and $\hat{E}(\overline{M})$ are different, then M^G and $\overline{M}^G$ may decompose differently. Choose R as before, and this time take H to be the generalized quaternion group of order 12, generated by elements x, y satisfying $x^3 = y^2$, $yxy^{-1} = x^{-1}$. Let G be the direct product $C \times H$, where C is cyclic of order 4. As M we pick the RH-module that affords the representation

$$y \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad x \to \begin{pmatrix} -\omega & 0 \\ 0 & -\omega^2 \end{pmatrix}.$$

Since $\lambda^2 + 1$ is irreducible in $K[\lambda]$ and also in $\overline{R}[\lambda]$, we see that both M and $\overline{M}$ are indecomposable. Both have stabilizer G, since C commutes elementwise with H.

It is easily verified that

$$\hat{E}(M) \cong \overline{R}, \quad \hat{E}(\overline{M}) \cong \overline{R}[i], \quad i^2 + 1 = 0.$$

Hence M^G is interrelated with $\overline{R}C$, and thus it splits into three components. On the other hand, by using the procedure of Conlon and Tucker, one finds that $\overline{M}^G$ is interrelated with $\overline{R}[i]C$. Thus $\overline{M}^G$ splits into four components, so that the decomposition of $\overline{M}^G$ is not interrelated with that of M^G.

(2) In the special case where $H = \{1\}$ and M is the trivial RH-module R, the hypotheses of the theorem are satisfied. In this case, M^G is precisely RG, and so we obtain an interrelation between the components of RG and those of $\overline{R}G$. This hardly constitutes a new proof of that fact, however, since our proof of Theorem 1 already uses the lifting theorem for idempotents.

(3) If H is a cyclic p-group, where p is the characteristic of $\overline{R}$, then each indecomposable $\overline{R}H$-module $\overline{M}$ is a cyclic module. It follows readily that $\hat{E}(\overline{M}) \cong \overline{R}$, and that $\overline{M}$ has stabilizer G. In general, we may observe that the stabilizer of $\overline{M}$ contains the stabilizer of M.

(4) Suppose that $\overline{R}$ has characteristic p, and let S/H be a p-group. Conlon and Tucker proved that $\{S/H; \overline{R}\}$ is isomorphic to the ordinary group algebra of S/H over $\overline{R}$, hence is completely primary. Furthermore,

$$\hat{E}(\{S/H; \overline{R}\}) \cong \overline{R}$$

in this case. Therefore both M^S and M^G are indecomposable, and

$$\hat{E}(M^S) \cong \hat{E}(M^G) \cong \overline{R}.$$

By applying this result repeatedly, we obtain the following conclusion.

PROPOSITION 13. *Let* G *be a* p*-group, where* $\overline{R}$ *has characteristic* p. *Let* G_0 *be any subgroup of* G, *and let* N *be an* RG_0*-module for which* $\hat{E}(\overline{N}) \cong \overline{R}$. *Then* N^G *is indecomposable, and* $\hat{E}(N^G) \cong \overline{R}$.

This improves slightly a result due to Green [5].

Theorem 1 is useful in the study of representation algebras. Recall that the *representation ring* a(RG) consists of all finite formal sums $\sum a_i[M_i]$, where each a_i is a rational integer, and each symbol $[M_i]$ corresponds to an R-free RG-module M_i. The rules for addition and multiplication in a(RG) are

$$[M] + [N] = [M \oplus N], \qquad [M][N] = [M \otimes_R N].$$

The *representation algebra* A(RG) is defined to be $\Omega \otimes_Z$ a(RG), where Ω is the complex field, Z the ring of rational integers. Similar definitions apply for $a(\overline{R}G)$ and $A(\overline{R}G)$ (see [6], [10], [11]).

Since the representation algebra $A(\overline{R}G)$ is easier to handle than A(RG), it is desirable to establish some relations between them. Suppose that H is a normal subgroup of G, and let $\{L_1, \cdots, L_t\}$ be a collection of R-free RH-modules such that

(I) $\overline{L}_1, \cdots, \overline{L}_t$ are distinct indecomposable $\overline{R}H$-modules, no two of which are conjugate,

(II) for each i, $\hat{E}(\overline{L}_i) \cong \hat{E}(L_i)$, and L_i and $\overline{L}_i$ have the same stabilizer in G.

Denote by a(RG; $\{L_i\}$) the additive subgroup of a(RG) generated by the components of the induced modules $\{L_i^G: 1 \le i \le t\}$. The map $a(RG) \to a(\overline{R}G)$ determined by $[M] \to [\overline{M}]$ yields an additive homomorphism

$$a(RG; \{L_i\}) \to a(\overline{R}G; \{\overline{L}_i\}). \tag{12}$$

Let us set $A(RG; \{L_i\}) = \Omega \otimes_Z a(RG; \{L_i\})$; then there is a homomorphism (as Ω-spaces)

$$A(RG; \{L_i\} \to A(\overline{R}G; \{\overline{L}_i\}). \tag{13}$$

THEOREM 2. *The mappings in* (12) *and* (13) *are isomorphisms.*

Proof. It suffices to show that the map in (12) is an isomorphism. Let $\{M_{ij}: 1 \le j \le v_i\}$ be a full set of non-isomorphic components of L_i^G, and let $M_{ij}^{(u)}$ denote the direct sum of u copies of M_{ij}. Then we may write

$$L_i^G \cong \sum_{j=1}^{v_i}{}^{\oplus} M_{ij}^{(u_{ij})} \qquad (1 \le i \le t),$$

for some positive integers $\{u_{ij}\}$. By Theorem 1, the component decomposition of $\overline{L}_i^G$ is given by the corresponding formula in which each M_{ij} is replaced by $\overline{M}_{ij}$. Since $\overline{M}_{ij} \cong \overline{M}_{i\ell}$ if and only if $M_{ij} \cong M_{i\ell}$, it follows from the choice of the modules $\{M_{ij}\}$ that the former isomorphism holds only when $j = \ell$.

On the other hand, suppose that $i \ne k$, and let us show that $\overline{M}_{ij} \ncong \overline{M}_{k\ell}$ for any j, ℓ. Each $\overline{R}G$-module X yields, by restriction to H, an $\overline{R}H$-module X_H. By the Mackey subgroup theorem [4, Theorem 44.2], $(\overline{L}_i^G)_H$ is a direct sum of conjugates of $\overline{L}_i$. Since $\overline{L}_i$ is indecomposable, so is each of its conjugates. But $\overline{M}_{ij}$ is a component of $\overline{L}_i^G$, whence $(\overline{M}_{ij})_H$ is a direct sum of conjugates of $\overline{L}_i$. For $i \ne k$, no conjugate of $\overline{L}_i$ can be a conjugate of $\overline{L}_k$, because of hypothesis (I). Hence the isomorphism $\overline{M}_{ij} \cong \overline{M}_{k\ell}$ is impossible.

We have thus shown that the modules $\{\overline{M}_{ij}: 1 \le j \le v_i,\ 1 \le i \le t\}$ are distinct indecomposable $\overline{R}G$-modules. But $a(\overline{R}G)$ is a free Z-module having as Z-basis the collection of all indecomposable modules, one from each isomorphism class. Hence, if

$$\sum_{i,j} a_{ij}[\overline{M}_{ij}] = 0$$

in $a(\overline{R}G)$, where the $\{a_{ij}\}$ are rational integers, then $a_{ij} = 0$ for each i, j.

We are now ready to prove that the mapping (12) is an isomorphism. Since the first part of our proof shows that it is an epimorphism, we need only prove that it is monic. Each element $X \in a(RG;\ \{L_i\})$ is expressible as

$$X = \sum_{i,j} a_{ij}[M_{ij}],$$

with coefficients $\{a_{ij}\}$ that are rational integers. Under the mapping (12), X is mapped onto $\overline{X} = \sum a_{ij}[\overline{M}_{ij}]$. Thus if $\overline{X} = 0$, then also $X = 0$, and so the theorem is proved.

COROLLARY. *Let* R *be the ring of* p*-adic integers in the* p*-adic completion of the rational field, where* $p > 2$. *Suppose that* G *has a normal* p*-Sylow subgroup* H *of order* p. *Then the mapping* $A(RG) \to A(\overline{R}G)$ *is monic.*

Proof. By [7], there are precisely three indecomposable R-free RH-modules, say L_1, L_2, L_3, where L_1 is the trivial RH-module R, L_2 is the augmentation ideal of RH, and $L_3 = RH$. Then $\overline{L}_1$, $\overline{L}_2$, $\overline{L}_3$ are indecomposable $\overline{R}H$-modules of dimensions 1, p - 1, p, respectively. Since $p > 2$, no two of them can be conjugate. For each i, $E(\overline{L}_i) \cong \overline{R}$, and both L_i, $\overline{L}_i$ have stabilizer G. The hypotheses of Theorem 2 are thus satisfied, and so we may conclude that

$$A(RG;\ \{L_1, L_2, L_3\}) \cong A(\overline{R}G;\ \{\overline{L}_1, \overline{L}_2, \overline{L}_3\}) \subset A(\overline{R}G).$$

On the other hand, since H is a p-Sylow subgroup of G, it follows as in [4, (63.8)] that every indecomposable R-free RG-module is a component of some induced module L^G, where L is an indecomposable R-free RH-module. Therefore

$$A(RG) = A(RG;\ \{L_1, L_2, L_3\});$$

this completes the proof of the corollary.

The above corollary was proved by other methods in [11].

As mentioned earlier, two projective RG-modules M and N are isomorphic if and only if $\overline{M} \cong \overline{N}$. We shall conclude this section with an easy generalization of this. Suppose that H denotes an arbitrary subgroup of G, not necessarily normal. Recall that an R-free RG-module M is said to be (G, H)-*projective* if M has the property that every exact sequence of RG-modules

$$0 \to X \to Y \to M \to 0$$

that splits as RH-sequence also splits as RG-sequence. The (G, $\{1\}$)-projective modules are then the usual RG-projective modules.

If M is (G, H)-projective, it follows from the above that for each RG-module N, the map

$$\mathrm{Ext}^1_{RG}(M, N) \to \mathrm{Ext}^1_{RH}(M_H, N_H)$$

is a monomorphism. Since [H : 1] annihilates the right-hand group, we may conclude that

$$[H:1]\, \mathrm{Ext}^1_{RG}(M, N) = 0 \tag{14}$$

whenever M is an R-free RG-module that is (G, H)-projective.

Suppose now that R is a discrete valuation ring, and let $P = \pi R$. Set $[H:1] = \pi^e s$, where s is a unit in R.

PROPOSITION 14. *Let* M *be a* (G, H)*-projective module, and let* N *be any* R*-free* RG*-module. Set*

$$M' = M/\pi^{e+1}M, \quad N' = N/\pi^{e+1}N, \quad R' = R/P^{e+1}.$$

If $M' \cong N'$ *as* R'G*-modules, then* $M \cong N$ *as* RG*-modules.*

Proof. The R'G-isomorphism $\phi: M' \cong N'$ can be lifted to an R-isomorphism $f: M \cong N$. Then for each $x \in G$,

$$(fx - xf)M \subset \pi^{e+1}N.$$

Define $g_x \in \mathrm{Hom}_R(M, N)$ by

$$g_x = \pi^{-(e+1)}(fx - xf) \qquad (x \in G).$$

Then $x \to g_x$ is a 1-cocycle from G into $\mathrm{Hom}_R(M, N)$, and so (by (14)) $\pi^e g$ is inner, that is, there exists an element $t \in \mathrm{Hom}_R(M, N)$ such that

$$\pi^e g_x = xt - tx \qquad (x \in G).$$

But then

$$fx - xf = \pi(xt - tx) \qquad (x \in G),$$

and therefore $f + \pi t \in \mathrm{Hom}_{RG}(M, N)$. Since f is an R-isomorphism of M onto N, so is $f + \pi t$. This proves that $M \cong N$ as RG-modules.

4. CHANGE OF GROUND RING

We shall now consider the behavior of indecomposable modules under ground ring extension. Several earlier results may be found in [9] and [12]. Throughout this section, A denotes an R-free R-algebra, and all A-modules are assumed R-free. By S we denote a complete noetherian local ring containing R, with maximal ideal P' and residue class field $\overline{S} = S/P'$. We assume always that $S \cap K = R$, $P' \cap K = P$, so that $\overline{R}$ may be viewed as subfield of $\overline{S}$. For an A-module M, define $SM = S \otimes_R M$. Then SM is an S-free SA-module, where $SA = S \otimes_R A$.

PROPOSITION 15. *The* SA*-module* SM *is interrelated with the* $\overline{S}$*-algebra*

$$\overline{S} \otimes_{\overline{R}} \hat{E}(M).$$

Furthermore,

$$\hat{E}(SM) \cong \overline{S} \otimes_{\overline{R}} \hat{E}(M)$$

if $\overline{S}$ *is a finite separable extension of* $\overline{R}$.

Proof. By Proposition 9, SM is interrelated with $\hat{E}(SM)$. But $\hat{E}(SM) \cong \Lambda/\text{rad}\,\Lambda$, where

$$\Lambda = E(SM)/P' \cdot E(SM).$$

Therefore SM is interrelated with Λ/Λ_0, where Λ_0 is any two-sided ideal of Λ contained in rad Λ.

Now

$$E(SM) = \text{Hom}_{SA}(SM, SM) \cong S \otimes_R E(M),$$

and this readily implies that

$$\Lambda \cong \overline{S} \otimes_{\overline{R}} \overline{E}(M),$$

where $\overline{E}(M) = E(M)/P \cdot E(M)$. Choose Λ_0 so that in the above isomorphism, $\Lambda_0 \cong \overline{S} \otimes_{\overline{R}} \text{rad}\,\overline{E}(M)$. Then clearly $\Lambda_0 \subset \text{rad}\,\Lambda$, and indeed $\Lambda_0 = \text{rad}\,\Lambda$ whenever $\overline{S}$ is a finite separable extension of $\overline{R}$. Furthermore,

$$\frac{\Lambda}{\Lambda_0} \cong \frac{\overline{S} \otimes \overline{E}(M)}{\overline{S} \otimes \text{rad}\,\overline{E}(M)} \cong \overline{S} \otimes \frac{\overline{E}(M)}{\text{rad}\,\overline{E}(M)}.$$

Since $\hat{E}(M) \cong \overline{E}(M)/\text{rad}\,\overline{E}(M)$, we see that $\Lambda/\Lambda_0 \cong \overline{S} \otimes \hat{E}(M)$, and the proposition is proved.

COROLLARY. *If* $\overline{S} = \overline{R}$, *and* M *is indecomposable, then so is* SM. *Thus, indecomposable* A-*modules remain indecomposable under a totally ramified extension of the ground ring* R.

Green [5] has called M *absolutely indecomposable* if SM is indecomposable for every S.

PROPOSITION 16. *Let* R *be a complete discrete valuation ring. An* A-*module* M *is absolutely indecomposable if and only if* $\hat{E}(M)$ *is a field that is purely inseparable over* $\overline{R}$. *In particular, if* $\overline{R}$ *is a perfect field, then* M *is absolutely indecomposable if and only if* $\hat{E}(M) \cong \overline{R}$.

Proof. The first statement in the proposition clearly implies the second; for if $\overline{R}$ is perfect, the only purely inseparable field extension of $\overline{R}$ is $\overline{R}$ itself.

Suppose now that $\hat{E}(M)$ is a field that is purely inseparable over $\overline{R}$, and let us show that for each field $\overline{S}$ containing $\overline{R}$, the ring $\overline{S} \otimes_{\overline{R}} \hat{E}(M)$ is completely primary. If $\overline{R}$ has characteristic p, the field $\hat{E}(M)$ is obtainable from $\overline{R}$ by successive adjunctions of p-th roots. A simple induction argument shows that it suffices to prove that $\overline{S} \otimes_{\overline{R}} F$ is completely primary, where

$$F \cong \overline{R}[x]/(x^p - a) \qquad (a \in \overline{R},\ a \notin \overline{R}^p).$$

But this is clear, since

$$\bar{S} \otimes_{\bar{R}} F \cong \bar{S}[x]/(x^p - a),$$

and over $\bar{S}$ the polynomial $x^p - a$ is either irreducible or else is of the form $(x - \alpha)^p$ for some $\alpha \in \bar{S}$.

Conversely, suppose that M is absolutely indecomposable. Then $\hat{E}(M)$ is a division algebra whose center C is a finite field extension of $\bar{R}$. We claim that C must be purely inseparable over $\bar{R}$; if it is not, then $C \supset C_0 \supsetneq \bar{R}$, where C_0 is a separable field extension of $\bar{R}$. Choose a complete discrete valuation ring $S \supset R$ such that $\bar{S} = C_0$. Then

$$\bar{S} \otimes_{\bar{R}} \hat{E}(M) \cong (C_0 \otimes_{\bar{R}} C_0) \otimes_{C_0} \hat{E}(M).$$

But $C_0 \otimes_{\bar{R}} C_0$ is a direct sum of two or more fields, and hence $\bar{S} \otimes_{\bar{R}} \hat{E}(M)$ is not completely primary, whence SM is decomposable.

Finally, if C is purely inseparable over $\bar{R}$ but $\hat{E}(M) \neq C$, then $\hat{E}(M)$ is not a field. We may then choose S so that $\bar{S} \supset C$, and $\bar{S}$ splits $\hat{E}(M)$; that is, $\bar{S} \otimes_C \hat{E}(M)$ is a full matrix algebra of (say) $q \times q$ matrices over $\bar{S}$, with $q > 1$. Since C is purely inseparable over $\bar{R}$, it follows easily that $\bar{S} \otimes_{\bar{R}} C$ is completely primary, and $(\bar{S} \otimes C)/\mathrm{rad}\,(\bar{S} \otimes C) \cong \bar{S}$. But

$$\bar{S} \otimes_{\bar{R}} \hat{E}(M) \cong (\bar{S} \otimes_{\bar{R}} C) \otimes_C \hat{E}(M),$$

and thus SM is interrelated with $\bar{S} \otimes_C \hat{E}(M)$. This latter ring has q components, however, whence SM also has q components. Thus, if $\hat{E}(M)$ is not a field, then M is not absolutely indecomposable.

PROPOSITION 17. *Suppose that* R *is a complete local ring, and that* S *is* R-*free of finite rank. Let* M *and* N *be* A-*modules. Then* SM *and* SN *have a common component if and only if* M *and* N *have a common component. Furthermore,* $M \cong N$ *if and only if* $SM \cong SN$.

Proof. Let S have R-rank q, and regard A as embedded in SA. Each SA-module becomes an A-module by restriction of the operator domain to A. Let $M^{(q)}$ denote the direct sum of q copies of M. Then $SM \cong M^{(q)}$ as A-modules. Hence if SM and SN have a common component, so do the A-modules $M^{(q)}$ and $N^{(q)}$, and therefore M and N must have a common component.

Finally, if $SM \cong SN$, then $M^{(q)} \cong N^{(q)}$, which at once implies that $M \cong N$.

Remark. An analogue of the preceding proposition is given in [12], where R and S are taken to be valuation rings in algebraic number fields. However, in that case S need not be of finite R-rank, as was mistakenly asserted. The result and proof in [12] remain essentially correct, nevertheless, since this wrong assertion was never used. Indeed, the only fact needed was that S/P^kS is free of finite rank as (R/P^k)-module, and this is true even if S is not of finite R-rank.

THEOREM 3. *Let* R *be a complete local ring such that* $\bar{R}$ *is finite, and let* A *be an* R-*algebra,* M *an indecomposable* A-*module. Then for any* S, *the module* SM *is a direct sum of* k *non-isomorphic components, where* k *is the number of components of* $\bar{S} \otimes_{\bar{R}} \hat{E}(M)$. *Indeed, we may write*

$$\bar{S} \otimes_{\bar{R}} \hat{E}(M) \cong F_1 \oplus \cdots \oplus F_k, \tag{15}$$

where each F_i *is a finite field extension of* $\bar{S}$. *There is then a corresponding decomposition*

(16) $$S \otimes_R M = L_1 \oplus \cdots \oplus L_k,$$

where $L_1, \cdots, L_k$ *are non-isomorphic indecomposable* SA-*modules, and where*

(17) $$\hat{E}(L_i) \cong F_i \quad (1 \leq i \leq k),$$

as $\bar{S}$-*algebras.*

Proof. Since M is indecomposable, $\hat{E}(M)$ is a skewfield of finite $\bar{R}$-dimension. But $\bar{R}$ is a finite field, and so, by Wedderburn's Theorem, $\hat{E}(M)$ is itself a field. Thus the commutative semisimple algebra $\bar{S} \otimes_{\bar{R}} \hat{E}(M)$ splits into a direct sum of (say) k fields $F_1, \cdots, F_k$, no two of which are isomorphic as $\bar{S} \otimes \hat{E}(M)$-modules. The theorem therefore follows from Proposition 15 and the results of Section 2.

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