ON A BOUNDARY PROPERTY OF CONTINUOUS FUNCTIONS

T. J. Kaczynski

Let D be the open unit disk in the plane, and let C be its boundary, the unit circle. If x is a point of C, then an arc at x is a simple arc \( \gamma \) with one endpoint at x such that \( \gamma = \{x\} \subset D \). If \( f \) is a function defined in D and taking values in a metric space K, then the set of curvilinear convergence of \( f \) is

\[
\{ x \in C \mid \text{there exists an arc } \gamma \text{ at } x \text{ and there exists a point } p \in K \text{ such that } \lim_{z \to x, z \in \gamma} f(z) = p \}.
\]

J. E. McMillan proved that if \( f \) is a continuous function mapping D into the Riemann sphere, then the set of curvilinear convergence of \( f \) is of type \( F_{\delta \theta} \) [2, Theorem 5]. In this paper we shall provide a simpler proof of this theorem than McMillan’s, and we shall give a generalization and point out some of its corollaries.

Notation. If \( S \) is a subset of a topological space, \( \bar{S} \) denotes the closure and \( S^* \) denotes the interior of \( S \). Of course, when we speak of the interior of a subset of the unit circle, we mean the interior relative to the circle, not relative to the whole plane. Let \( K \) be a metric space with metric \( \rho \). If \( x_0 \in K \) and \( r > 0 \), then

\[
S(r, x_0) = \{ x \in K \mid \rho(x, x_0) < r \}.
\]

An arc of \( C \) will be called nondegenerate if and only if it contains more than one point.

**Lemma 1.** Let \( \mathcal{I} \) be a family of nondegenerate closed arcs of \( C \). Then \( \bigcup_{I \in \mathcal{I}} I - \bigcup_{I \in \mathcal{I}} I^* \) is countable.

Proof. Since \( \bigcup_{I \in \mathcal{I}} I^* \) is open, we can write \( \bigcup_{I \in \mathcal{I}} I^* = \bigcup_n J_n \), where \( \{J_n\} \) is a countable family of disjoint open arcs of \( C \). If

\[
x_0 \in \bigcup_{I \in \mathcal{I}} I - \bigcup_{I \in \mathcal{I}} I^*,
\]

then for some \( I_0 \in \mathcal{I} \), \( x_0 \) is an endpoint of \( I_0 \). For some \( n \), \( I_0^* \subset J_n \), so that \( x_0 \in \bar{J_n} \). But \( x_0 \notin J_n \), so that \( x_0 \) is an endpoint of \( J_n \). Thus \( \bigcup_{I \in \mathcal{I}} I - \bigcup_{I \in \mathcal{I}} I^* \) is contained in the set of all endpoints of the various \( J_n \); this proves the lemma.

In what follows we shall repeatedly use Theorem 11.8 on page 119 in [3] without making explicit reference to it. By a cross-cut we shall always mean a cross-cut of \( D \). Suppose \( \gamma \) is a cross-cut that does not pass through the point 0. If \( V \) is the component of \( D - \gamma \) that does not contain 0, let \( L(\gamma) = \bar{V} \cap C \). Then \( L(\gamma) \) is a non-degenerate closed arc of \( C \).

---

Received February 8, 1966.

313
Suppose $\Omega$ is a domain contained in $D - \{0\}$. Let $\Gamma$ denote the family of all cross-cuts $\gamma$ with $\gamma \cap D \subset \Omega$. Let

$$I(\Omega) = \bigcup_{\gamma \in \Gamma} L(\gamma), \quad I_0(\Omega) = \bigcup_{\gamma \in \Gamma} L(\gamma)^*.$$ 

Let $\text{acc}(\Omega)$ denote the set of all points on $C$ that are accessible by arcs in $\Omega$.

The following lemma is weaker than it could be, but there is no point in proving more than we need.

**Lemma 2.** The set $\text{acc}(\Omega) - I_0(\Omega)$ is countable.

**Proof.** By Lemma 1, $I(\Omega) - I_0(\Omega)$ is countable; therefore it will suffice to show that $\text{acc}(\Omega) - I(\Omega)$ is countable. If $\text{acc}(\Omega)$ has fewer than two points, we are done. Suppose, on the other hand, that $\text{acc}(\Omega)$ has two or more points. If $a \in \text{acc}(\Omega)$, then there exists $a^i \in \text{acc}(\Omega)$ with $a^i \neq a$. Let $\gamma, \gamma'$ be arcs at $a, a^i$, respectively, with $\gamma \cap D \subset \Omega, \quad \gamma' \cap D \subset \Omega.$

Let $p$ be the endpoint of $\gamma$ that lies in $\Omega$, $p'$ the endpoint of $\gamma'$ that lies in $\Omega$. Let $\gamma'' \subset \Omega$ be an arc joining $p$ to $p'$. The union of $\gamma, \gamma'$, and $\gamma''$ is an arc $\delta$ joining $a$ to $a^i$. By [4], there exists a simple arc $\delta' \subset \delta$ that joins $a$ to $a^i$. Clearly, $\delta'$ is a cross-cut with $\delta' \cap D \subset \Omega$ and $a, a^i \in L(\delta')$. Thus $a \in I(\Omega)$, and so $\text{acc}(\Omega) \subset I(\Omega)$.$\blacksquare$

**Lemma 3.** Suppose $\Omega_1$ and $\Omega_2$ are domains contained in $D - \{0\}$. If

$$I_0(\Omega_1) \cap \text{acc}(\Omega_2) \quad \text{and} \quad I_0(\Omega_2) \cap \text{acc}(\Omega_1)$$

are not disjoint, then $\Omega_1$ and $\Omega_2$ are not disjoint.

**Proof.** We assume $\Omega_1$ and $\Omega_2$ are disjoint, and we derive a contradiction. Let $a$ be a point in both of the two sets (1). Let $\gamma_1$ be a cross-cut with $\gamma_1 \cap D \subset \Omega_1$ such that $a \in L(\gamma_1)^*$ ($i = 1, 2$). Let $U_i$ and $V_i$ be the components of $D - \gamma_i$, and (to be specific), let $U_1$ be the component containing 0. Note that $\gamma_1 \cap D$ and $\gamma_2 \cap D$ are disjoint.

Suppose $\gamma_1 \cap D \subset V_2$ and $\gamma_2 \cap D \subset V_1$. Then, since $\gamma_1 \cap D \subset \overline{U}_1$, $U_1$ has a point in common with $V_2$. But $0 \in U_1 \cap U_2$, so that $U_1$ has a point in common with $U_2$ also. Since $U_1$ is connected, this implies that $U_1$ has a common point with $\gamma_2 \cap D$, which contradicts the assumption that $\gamma_2 \cap D \subset V_1$. Therefore $\gamma_1 \cap D \not\subset V_2$ or $\gamma_2 \cap D \not\subset V_1$. We conclude that either $\gamma_1 \cap D \subset U_2$ or $\gamma_2 \cap D \subset U_1$. By symmetry, we may assume that $\gamma_2 \cap D \subset U_1$.

It is possible to choose a point $b \in L(\gamma_1)^*$ that is accessible by an arc in $\Omega_2$, because $a$ is in the closure of $\text{acc}(\Omega_2)$. Let $\gamma$ be a simple arc joining $b$ to a point of $\gamma_2 \cap D$, such that $\gamma - \{b\} \subset \Omega_2$. Then $\gamma - \{b\}$ and $\gamma_1$ are disjoint. Also, $\gamma - \{b\}$ contains a point of $U_1$ (namely, the point where $\gamma$ meets $\gamma_2 \cap D$); therefore $\gamma - \{b\} \subset U_1$. Hence $b \in \overline{U}_1$. Since $b \in L(\gamma_1)^*$, this is a contradiction.$\blacksquare$

**Theorem 1** (J. E. McMillan). Let $K$ be a complete separable metric space, and let $f$ be a continuous function mapping $D$ into $K$. Let

$$X = \{x \in \mathbb{C} \mid \text{there exists an arc } \gamma \text{ at } x \text{ for which } \lim_{z \to x} f(z) \text{ exists} \}.$$
Then $X$ is of type $F_{\sigma \delta}$.

**Proof.** Let $\{p_k\}_{k=1}^{\infty}$ be a countable dense subset of $K$. Let $\{Q(n, m)\}_{m=1}^{\infty}$ be a counting of all sets of the form

$$\left\{ \text{re}^{it} \left| 1 - \frac{1}{n} < r < 1 \text{ and } \theta < t < \theta + \frac{2\pi}{n} \right. \right\},$$

where $\theta$ is a rational number. Let $\{U(n, m, k, \ell)\}_{\ell=1}^{\infty}$ be a counting (with repetitions allowed) of the components of

$$f^{-1}\left(S\left(\frac{1}{2n}, p_k\right)\right) \cap Q(n, m).$$

(We consider $\emptyset$ to be a component of $\emptyset$.) Let

$$A(n, m, k, \ell) = \text{acc}[U(n, m, k, \ell)].$$

Set

$$Y = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{\ell=1}^{\infty} I_0(U(n, m, k, \ell)) \cap A(n, m, k, \ell).$$

Since $I_0(U(n, m, k, \ell))$ is open, it is of type $F_{\sigma \delta}$. It follows that $Y$ is of type $F_{\sigma \delta}$.

I claim that $Y \subset X$. Take any $y \in Y$. For each $n$, choose $m[n], k[n], \ell[n]$ with

(2) $y \in I_0(U(n, m[n], k[n], \ell[n])) \cap A(n, m[n], k[n], \ell[n])$ (n = 1, 2, 3, ...).

For convenience, set $U_n = U(n, m[n], k[n], \ell[n])$. By (2) and Lemma 3, $U_n$ and $U_{n+1}$ have some point $z_n$ in common. For each $n$, we can choose an arc $\gamma_n \subset U_{n+1}$ with one endpoint at $z_n$ and the other at $z_{n+1}$. Then $\gamma_n \subset Q(n+1, m[n+1])$. Also,

$$y \in A(n+1, m[n+1], k[n+1], \ell[n+1]) \subset \bigcup_{n=1}^{\infty} \subset Q(n+1, m[n+1]),$$

and therefore each point of $\gamma_n$ has distance less than $\frac{2\pi + 1}{n+1}$ from $y$. Now

$$\frac{2\pi + 1}{n+1} \to 0 \text{ as } n \to \infty; \text{ hence, if we set } \gamma = \{y\} \cup \bigcup_{n=1}^{\infty} \gamma_n, \text{ then } \gamma \text{ is an arc with one endpoint at } y.$$

Since $U_n$ and $U_{n+1}$ have a point in common,

$$f^{-1}\left(S\left(\frac{1}{2n}, p_k[n]\right)\right) \text{ and } f^{-1}\left(S\left(\frac{1}{2n+1}, p_k[n+1]\right)\right)$$

have a common point, and hence

$$S\left(\frac{1}{2n}, p_k[n]\right) \text{ and } S\left(\frac{1}{2n+1}, p_k[n+1]\right)$$

have a common point. Therefore, if $\rho$ is the metric on $K$, then

$$\rho(p_k[n], p_k[n+1]) \leq \frac{1}{2n} + \frac{1}{2n+1} < \frac{1}{2n-1},$$
and therefore
\[ \rho(p_k[n], p_{k[n+r]} \leq \sum_{i=1}^{r} \rho(p_k[n+i-1], p_{k[n+i]}) < \sum_{i=1}^{r} \frac{1}{2^{n+i-2}} < \frac{1}{2^{n-2}}. \]

Thus \( \{p_k[n]\} \) is a Cauchy sequence and must converge to some point \( p \in K \). Because
\[ \gamma_n \subset U_{n+1} \subset f^{-1}\left( S\left( \frac{1}{2^n+1}, p_{k[n+1]} \right) \right) \quad \text{and} \quad p_k[n] \rightarrow p, \]
\[ \lim_{z \rightarrow \gamma} \gamma_n = \gamma \] by a simple arc \( \gamma' \subset \gamma \). Thus \( y \in Y \), and we have shown that \( Y \subset X \).

Suppose \( x \in X \). Let \( \gamma_0 \) be an arc at \( x \) such that \( f \) approaches a limit \( p^* \) along \( \gamma_0 \). Take any \( n \). Choose \( k \) with \( p^* \in S\left( \frac{1}{2^n}, p_k \right) \). Choose \( m \) so that \( x \) is in the interior of \( Q(n, m) \cap C \). Then \( \gamma_0 \) has a subarc \( \gamma'_0 \), with one endpoint at \( x \), such that
\[ \gamma'_0 \cdot \{x\} \subset Q(n, m) \cap f^{-1}\left( S\left( \frac{1}{2^n}, p_k \right) \right). \]

Hence, for some \( \ell \), \( x \in \text{acc}[U(n, m, k, \ell)] = A(n, m, k, \ell) \). This shows that
\[ X \subset \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{\ell=1}^{\infty} A(n, m, k, \ell). \]

By Lemma 2, the set
\[ A(n, m, k, \ell) - I_0(U(n, m, k, \ell)) = A(n, m, k, \ell) - [I_0(U(n, m, k, \ell)) \cap \overline{A(n, m, k, \ell)}] \]
is countable. It follows by a routine argument that
\[ \bigcap_{n} \bigcup_{m,k,\ell} A(n, m, k, \ell) - \bigcup_{n} \bigcup_{m,k,\ell} [I_0(U(n, m, k, \ell)) \cap \overline{A(n, m, k, \ell)}] \]
is countable. Because
\[ \bigcap_{n} \bigcup_{m,k,\ell} [I_0(U(n, m, k, \ell)) \cap \overline{A(n, m, k, \ell)}] \subset \bigcup_{n} A(n, m, k, \ell), \]
the set \( X - Y \) is countable, and therefore \( X \) is of type \( F_{\sigma, \delta} \).

Before stating our generalization of the foregoing theorem, we must say a few words about spaces of closed sets. If \( K \) is a bounded metric space with metric \( \rho \), let \( \mathcal{C}(K) \) denote the set of all nonempty closed subsets of \( K \). Hausdorff [1, page 146] defined a metric \( \bar{\rho} \) on \( \mathcal{C}(K) \) by setting
\[ \bar{\rho}(A, B) = \max \left\{ \sup_{a \in A} \text{dist}(a, B), \sup_{b \in B} \text{dist}(b, A) \right\}, \]
where dist \((x, E)\) denotes \(\inf_{e \in E} \rho(x, e)\). If \(K\) is compact, then \(\mathcal{C}(K)\) is a compact metric space with \(\bar{\rho}\) as metric [1, page 150].

If \(f\) maps \(D\) into \(K\) and if \(\gamma\) is an arc at a point \(x \in C\), we let \(C(f, \gamma)\) denote the cluster set of \(f\) along \(\gamma\); that is, we write

\[
C(f, \gamma) = \{ p \in K \mid \text{there exists a sequence } \{ z_n \} \subset \gamma \cap D \text{ such that } z_n \to x \text{ and } f(z_n) \to p \}.
\]

**THEOREM 2.** Let \(K\) be a compact metric space, and let \(E\) be a closed subset of \(\mathcal{C}(K)\). Let \(f : D \to K\) be a continuous function. Then

\[
\{ x \in C \mid \text{there exists an arc } \gamma \text{ at } x \text{ and there exists } E \in \mathcal{E} \text{ such that } C(f, \gamma) \subset E \}
\]

is a set of type \(F_{\sigma\delta}\).

**Proof.** If \(\varepsilon > 0\) and \(E \in \mathcal{C}(K)\), let

\[
\mathcal{J}(\varepsilon, E) = \{ a \in K \mid \text{there exists } b \in E \text{ with } \rho(a, b) < \varepsilon \}.
\]

Note that \(\mathcal{J}(\varepsilon, E)\) is open and that

\[
F \in \mathcal{C}(K), \ \bar{\rho}(E, F) < \varepsilon \Rightarrow F \subset \mathcal{J}(\varepsilon, E).
\]

Let \(\{ P(k) \}_{k=1}^{\infty} \) be a countable dense subset of \(E\) (such a subset exists, because every compact metric space is separable). Let

\[
X = \{ x \in C \mid \text{there exist an arc } \gamma \text{ at } x \text{ and an } E \in \mathcal{E} \text{ such that } C(f, \gamma) \subset E \}.
\]

Let \(\{ Q(n, m) \}_{m=1}^{\infty} \) be defined as in the proof of the preceding theorem. Let \(\{ U(n, m, k, \ell) \}_{\ell=1}^{\infty} \) be a counting (with repetitions allowed) of the components of

\[
f^{-1}\left( \mathcal{J}\left( \frac{1}{n}, P(k) \right) \right) \cap Q(n, m).
\]

Let \(A(n, m, k, \ell) = \text{acc}[U(n, m, k, \ell)]\), and set

\[
Y = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{\ell=1}^{\infty} I_0(U(n, m, k, \ell)) \cap \overline{A(n, m, k, \ell)}.
\]

Since \(I_0(U(n, m, k, \ell))\) is open, it is of type \(F_{\sigma}\). It follows that \(Y\) is of type \(F_{\sigma\delta}\).

I claim that \(Y \subset X\). Take any \(y \in Y\). For each \(n\), choose \(m[n], k[n], \ell[n]\) so that

\[(3) \quad y \in I_0(U(n, m[n], k[n], \ell[n])) \cap \overline{A(n, m[n], k[n], \ell[n])}.
\]

Set \(U_n = U(n, m[n], k[n], \ell[n])\). Since \(\mathcal{E}\) is compact, there exist a \(P \in \mathcal{E}\) and some strictly ascending sequence \(\{ n_j \}_{j=1}^{\infty} \) of natural numbers such that
By (3) and Lemma 3, \( U_{n_j} \) and \( U_{n_{j+1}} \) have some point \( z_j \) in common. For each \( j \), choose an arc \( \gamma_j \subset U_{n_{j+1}} \) with one endpoint at \( z_j \) and the other at \( z_{j+1} \). Then \( \gamma_j \subset Q(n_{j+1}, m[n_{j+1}]) \). Also,

\[
y \in A(n_{j+1}, m[n_{j+1}], k[n_{j+1}], l[n_{j+1}]) \subset U_{n_{j+1}} \subset Q(n_{j+1}, m[n_{j+1}]),
\]

and therefore each point of \( \gamma_j \) has distance less than \( \frac{2\pi + 1}{n_{j+1}} \) from \( y \). Now

\[
\frac{2\pi + 1}{n_{j+1}} \to 0 \text{ as } j \to \infty;
\]

therefore, if we set \( \gamma = \{y\} \cup \bigcup_{j=1}^{\infty} \gamma_j \), then \( \gamma \) is an arc with one endpoint at \( y \).

I claim that \( C(f, \gamma) \subset P \). Take any \( p \in C(f, \gamma) \). There exists a sequence \( \{w_s\}_{s=1}^{\infty} \) in \( \gamma - \{y\} \) such that \( w_s \to y \) and \( f(w_s) \to p \). Let \( \varepsilon \) be an arbitrary positive number. Choose \( j_0 \) so that \( \bar{p}(P(k[n_j]), P) < \varepsilon/3 \) for all \( j \geq j_0 \). Choose \( j_1 \) so that \( j \geq j_1 \) implies \( 1/n_{j+1} < \varepsilon/3 \). We can choose an \( s \) such that \( w_s \in \gamma_i \) for some \( i \geq j_0, j_1 \) and such that

\[
(4) \quad \rho(f(w_s), p) < \frac{\varepsilon}{3}.
\]

Then

\[
f(w_s) \in f(\gamma_i) \subset f(U_{n_{i+1}}) \subset \mathcal{F}\left(\frac{1}{n_{i+1}}, P(k[n_{i+1}])\right),
\]

and therefore we can choose a point \( q \in P(k[n_{i+1}]) \) with

\[
(5) \quad \rho(f(w_s), q) < \frac{1}{n_{i+1}} < \frac{\varepsilon}{3}.
\]

Moreover, because \( \bar{p}(P(k[n_{i+1}]), P) < \varepsilon/3 \), there exists some \( q' \in P \) with

\[
(6) \quad \rho(q, q') < \frac{\varepsilon}{3}.
\]

Together, (4), (5), and (6) show that \( \rho(p, q') < \varepsilon \). Since \( P \) is closed and \( \varepsilon \) is arbitrary, this proves that \( p \in P \). Hence \( C(f, \gamma) \subset P \in E \). By [4], we can if necessary replace \( \gamma \) by a simple arc \( \gamma' \subset \gamma \); it follows that \( y \in X \). Thus \( Y \subset X \).

Now suppose \( x \in X \). Choose an arc \( \gamma_0 \) at \( x \) such that \( C(f, \gamma_0) \subset P_0 \) for some \( P_0 \in E \). Take any \( n \). Choose \( k \) with \( \bar{p}(P_0, P(k)) < 1/n \). Then

\[
P_0 \subset \mathcal{F}\left(\frac{1}{n}, P(k)\right), \quad \text{hence } \mathcal{C}(f, \gamma_0) \subset \mathcal{F}\left(\frac{1}{n}, P(k)\right).
\]

Choose \( m \) so that \( x \) is in the interior of \( \bar{Q}(n, m) \cap C \).

If for each natural number \( t \) there exists a point \( z_t' \in \gamma_0 \cap \mathcal{S}\left(\frac{1}{t}, x\right) \cap D \) with \( z_t' \notin f^{-1}\left(\mathcal{F}\left(\frac{1}{n}, P(k)\right)\right) \), then
ON A BOUNDARY PROPERTY OF CONTINUOUS FUNCTIONS

\[ f(z_t^i) \in K - \mathcal{G} \left( \frac{1}{n}, P(k) \right), \]

and since \( K - \mathcal{G} \left( \frac{1}{n}, P(k) \right) \) is compact, there exist some \( a \in K - \mathcal{G} \left( \frac{1}{n}, P(k) \right) \) and a subsequence \( \{f(z_{t_i}^i)\}_{i=1}^{\infty} \) such that \( f(z_{t_i}^i) \xrightarrow{t} a \). But then \( a \in C(t, \gamma_0) \), contrary to the relation \( C(t, \gamma_0) \subset \mathcal{G} \left( \frac{1}{n}, P(k) \right) \). We conclude that there exists a natural number \( t \) for which

\[ \gamma_0 \cap S \left( \frac{1}{t}, x \right) \cap D \subset f^{-1} \left( \mathcal{G} \left( \frac{1}{n}, P(k) \right) \right). \]

It follows that \( \gamma_0 \) has a subarc \( \gamma_0' \) with one endpoint at \( x \) such that

\[ \gamma_0' - \{x\} \subset f^{-1} \left( \mathcal{G} \left( \frac{1}{n}, P(k) \right) \right) \cap Q(n, m). \]

Hence there exists an \( \ell \) such that

\[ x \in \text{acc} \left[ U(n, m, k, \ell) \right] = A(n, m, k, \ell). \]

This shows that

\[ X \subset \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} A(n, m, k, \ell). \]

By Lemma 2, the set

\[ A(n, m, k, \ell) - I_0(U(n, m, k, \ell)) = A(n, m, k, \ell) - [I_0(U(n, m, k, \ell)) \cap \overline{A(n, m, k, \ell)}] \]

is countable. It follows easily that

\[ \bigcap_{n} \bigcup_{m, k, \ell} A(n, m, k, \ell) - \bigcap_{n} \bigcup_{m, k, \ell} [I_0(U(n, m, k, \ell)) \cap \overline{A(n, m, k, \ell)}] \]

is countable. Since

\[ \bigcap_{n} \bigcup_{m, k, \ell} [I_0(U(n, m, k, \ell)) \cap \overline{A(n, m, k, \ell)}] = Y \subset X \subset \bigcap_{n} \bigcup_{m, k, \ell} A(n, m, k, \ell), \]

\( X - Y \) must be countable. Thus \( X \) is the union of an \( F_{\sigma \delta} \) - set and a countable set, and hence it is of type \( F_{\sigma \delta} \).

In each of the following four corollaries, let \( f \) denote a continuous function mapping \( D \) into the Riemann sphere.

**COROLLARY 1** (J. E. McMillan). Let \( E \) be a closed subset of the Riemann sphere. Then the set

\[ \{x \in C \mid \text{there exist an arc } \gamma \text{ at } x \text{ and a point } p \in E \]

\[ \text{such that } \lim_{z \to x} f(z) = p \}

\[ z \in \gamma \]
is of type $F_{\sigma \delta}$.

**COROLLARY 2.** Suppose $d \geq 0$. Then the set
\[
\{ x \in C \mid \text{there exists an arc } \gamma \text{ at } x \text{ such that} \\
[\text{diameter } C(f, \gamma)] \leq d \}
\]
is of type $F_{\sigma \delta}$.

**COROLLARY 3.** Let $E$ be a closed subset of the Riemann sphere. Then the set
\[
\{ x \in C \mid \text{there exists an arc } \gamma \text{ at } x \text{ with } C(f, \gamma) \subset E \}
\]
is of type $F_{\sigma \delta}$.

**COROLLARY 4.** The set
\[
\{ x \in C \mid \text{there exists an arc } \gamma \text{ at } x \text{ such that } C(f, \gamma) \text{ is an arc of a great circle} \}
\]
is of type $F_{\sigma \delta}$.

We can obtain all these corollaries by taking $E$ to be a suitable family of closed sets and applying Theorem 2. To prove Corollary 4, we need the fact that $C(f, \gamma)$ is always connected. One could go on listing such corollaries ad infinitum, but we refrain.

It is interesting to note that in Corollary 1 it is not necessary to assume that $E$ is closed. By combining Corollary 1 with Theorem 6 of [2], one can prove that the conclusion of Corollary 1 holds even if $E$ is merely assumed to be of type $G_\delta$.

**REFERENCES**


The University of Michigan