THE CONJUGATE SPACE OF THE SPACE OF SEMIPERIODIC SEQUENCES

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We are primarily concerned with the presentation of a construction for the conjugate space of the space of semiperiodic sequences. In the somewhat expository introduction, we consider the general question of construction of conjugate spaces of algebras of almost periodic functions on the semigroup of positive integers, and we motivate the construction of the particular conjugate spaces that we undertake in Sections 2 and 3.

1. INTRODUCTION

We let Z^+ denote the semigroup of positive integers, and Z the group of all integers.

If $\{x(n)\}$ is a complex sequence, we say that $\{x(n)\}$ is *periodic of period* p if $x(n+p) = x(n)$ for all $n \in Z^+$. We say that a sequence is *periodic* if there exists a $p \in Z^+$ such that the sequence has period p. We denote by P the space of periodic sequences, and by Q the closure of P in the supremum norm. We call Q the space of *semiperiodic sequences*, and it is largely with this space that we are concerned.

The space Q can be made into a Banach algebra with the obvious coordinatewise operations. As such, $Q = C(\bar{\omega})$, where the compact group $\bar{\omega}$ is the dual of the group of rationals modulo 1 in the discrete topology. That is, Q is the algebra of all continuous complex-valued functions on $\bar{\omega}$.

We can also describe the group $\bar{\omega}$ as follows. We define a metric ρ on Z^+ by the rule

$$\rho(x, y) = 1/n!, \qquad \text{where } n! \mid (x - y) \text{ and } (n+1)! \nmid (x - y),$$

with $\rho(x, x) = 0$. Then $\bar{\omega}$ is the completion of Z^+, with addition inherited from Z and extended by continuity. The proofs of these assertions can be found in [2]. In [1], H. Anzai and S. Kakutani called $\bar{\omega}$ the *universal monothetic Cantor Group*.

We denote the conjugate space of Q by Q^*; that is, Q^* is the Banach space of all continuous linear functionals defined on Q.

It is of interest to consider Q and Q^* in the context of the theory of almost periodic functions.

Let S be a locally compact abelian semigroup. We require S to have jointly continuous addition, but we do not require an identity. If $f \in C(S)$ and $a \in S$, define $T_a f$, the *translate* of f by a, by $T_a f(x) = f(x + a)$ for $x \in S$. We call $\{T_a f \mid a \in S\}$ the *orbit* of f. If the orbit of f is conditionally compact, we say that f is an *almost periodic* function on S, and we write $f \in AP(S)$.

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If a Banach algebra A of almost periodic functions on S contains an identity, is closed under complex conjugation, and is closed under translation by members of S, we call it an AP(S) *subalgebra.* We denote the carrier space of such an A by S^A; that is, $A = C(S^A)$. It can be shown that S^A is a compact semigroup with addition inherited from S and extended by continuity. The first paragraph of Section 6 of K. DeLeeuw and I. Glicksberg [3] makes this clear when we note that in their proof any AP(S) subalgebra will serve as well as AP(S) itself.

If A is an $AP(Z^+)$ subalgebra, we can see that $A = A_p + A_0$, where A_p is an AP(Z) subalgebra with the functions restricted to Z^+, and where A_0 is an $AP(Z^+)$ subalgebra that is also a subspace of c_0. It is easy to see that either $A_0 = c_0$ (that is, A_0 is the space of all sequences converging to 0) or else $A_0 = C^n$ for some n (in other words, A_0 is the space of all sequences with support on the integers $1, \cdots, n$); indeed, this follows from Section 9.28 of Hewitt and Ross [5] if we observe that Z^{+A} is a compact monothetic semigroup. For more general theorems concerning such decompositions of AP(S) subalgebras, see DeLeeuw and Glicksberg [3], [4].

The existence of the decomposition described above shows that in the construction of the conjugate space of A, an $AP(Z^+)$ subalgebra, the real difficulty lies in constructing the conjugate space of A_p, the corresponding AP(Z) subalgebra.

This difficulty is in general substantial. Since we can write $AP(Z) = C(B \oplus \bar{\omega})$, where B is the Bohr compactification of the reals, we see that the difficulty in constructing, for example, $[AP(Z)]^*$ is as great as that of constructing $[AP(R)]^*$. The Riesz representation theorem characterizes the dual space of an AP(Z) subalgebra A in terms of regular countably additive measures on the Borel sets of Z^A; but this is not sufficiently concrete.

The problem is that the characters of Z are the natural elements on which one might attempt to define a continuous functional; because linear combinations of characters behave badly in the supremum norm, this turns out to be difficult.

In the case where the $AP(Z^+)$ subalgebra in question is Q, we can present a satisfactory construction. It is clear that if a matrix A sums each sequence in Q, that is, if $\lim_{n\to\infty} (Ax)(n)$ exists for each $x \in Q$, then A defines a member of Q^*. For each $f \in Q^*$ we present a matrix A_f such that

$$\lim_{n\to\infty} (A_f x)(n) = f(x) \quad \text{for } x \in Q.$$

Moreover, the norm of A_f as a matrix is equal to the norm of f as a functional.

In the cases $Q + C^n$ and $Q + c_0$, (J. D. Hill and W. T. Sledd [6] call $Q + c_0$ the space of *ultimately semi-periodic sequences*) we can present similar matrix realizations of the conjugate spaces.

The existence of such matrix realizations of Q^*, $(Q + C^n)^*$, and $(Q + c_0)^*$ is a consequence of the separability of Q.

2. THE DUAL SPACE OF Q

THEOREM. *Let a matrix* $A = (a_{ij})$ $(i, j = 1, 2, \cdots)$ *satisfy the conditions*

$$a_{ij} = 0 \qquad (j > i!),$$

(1)
$$\sum_{j = p \pmod{i!}} a_{i+1,j} = a_{i,p} \qquad (i = 1, 2, \cdots \text{ and } p \leq i!),$$

(2)
$$\sup_i \sum_j |a_{ij}| = \|A\| = M < \infty.$$

Then the relation $f_A(x) = \lim Ax$ *defines a functional* $f_A \in Q^*$ *with* $\|f_A\| = M$. *Moreover, if* $g \in Q^*$, *then there exists exactly one matrix* A_g *satisfying* (1) *and* (2) *such that* $g(x) = \lim A_g x$ *for* $x \in Q$. *Finally, the map* $g \to A_g$ *is linear, and hence it is an isometry between* Q^* *and the space of matrices satisfying* (1) *and* (2).

Proof. We first show that Ax converges for each x in Q. We note that if $x \in P$ with period $p!$, then by condition (1),

$$Ax(n) = \sum_j a_{pj}\, x(j) \qquad \text{for } n \geq p.$$

Since there exists an x of period $p!$ and of norm 1 such that

$$\sum_j a_{pj}\, x(j) = \sum_j |a_{pj}|,$$

and since $\sum_j |a_{pj}|$ is an increasing function of p, we see that

$$\sup_{x \in P,\ \|x\| = 1} |\lim Ax| = \|A\|.$$

But since $\overline{P} = Q$, condition (2) implies that $\lim Ax$ exists for $x \in Q$, and that

$$\sup_{x \in Q,\ \|x\| = 1} |\lim Ax| = \|A\|.$$

Hence, if for $x \in Q$ we define $f_A(x) = \lim Ax$, we see that $f_A \in Q^*$ and $\|f_A\| = \|A\|$.

We now show that each element of Q^* can be realized by such a matrix.

Let $\delta_{j,i!}$ $((i-1)! < j \leq i!)$ denote the sequence of period $i!$ defined by

$$\delta_{j,i!}(p) = \begin{cases} 1 & \text{if } p = j \pmod{i!}, \\ 0 & \text{otherwise}. \end{cases}$$

For notational convenience, we let $(i-1)! = 0$ when $i = 1$. The collection

$$H = \{\delta_{1,1};\ \delta_{2,2!};\ \delta_{3,3!}, \cdots, \delta_{6,3!};\ \delta_{7,4!}; \cdots, \delta_{24,4!};\ \delta_{25,5!}; \cdots\}$$

forms a Hamel basis for P. Indeed, for any n, the first n! sequences in H all have period n!, and they are clearly independent; hence they form a basis for the space of sequences of period n!.

Now let f belong to Q^*. We define a matrix A by setting

$$a_{i,j} = f(\delta_{j,i!}) \qquad (i = 1, 2, \cdots \text{ and } (i-1)! < j \leq i!)$$

and requiring that A satisfy condition (1). Then

$$(A\delta_{j,i!})(n) = f(\delta_{j,i!}) \qquad (n \geq i).$$

Hence, $\lim Ax = f(x)$ for $x \in P$. But since the functional f is bounded on P, the matrix A is also bounded, as we showed in the first part of the proof. Hence, $\lim Ax = f(x)$ for $x \in \overline{P} = Q$. We see, therefore, that each bounded linear functional f on Q is realized by exactly one matrix A_f satisfying (1) and (2).

The linearity and consequent isometry of the mapping $f \to A_f$ follow immediately. This completes the proof of the theorem.

It is illuminating to consider our theorem in a measure-theoretic context. Let A be a matrix satisfying the conditions of the theorem. We recall from the introduction that $Q = C(\bar{\omega})$ and that $\bar{\omega}$ is a compactification of Z^+. For each set $S \subset \bar{\omega}$, let

$$\mu_i(S) = \sum_{j \in S \cap Z^+} a_{ij} \qquad (i = 1, 2, \cdots).$$

Then each row of A defines a measure μ_i on the Borel sets of $\bar{\omega}$, and the regular, countably additive measure μ_A on the Borel sets of $\bar{\omega}$ defined by

$$\int_{\bar{\omega}} f\, d\mu_A = \lim_i \int_{\bar{\omega}} f\, d\mu_i = \lim Af \qquad \text{for } f \in Q$$

is merely the weak-star limit of these measures.

Denote an $\bar{\omega}$-disk of radius r with center x by D(x, r). That is, let

$$D(x, r) = \{y \mid y \in \bar{\omega} \text{ and } \rho(x, y) \leq r\},$$

where ρ is the metric defined in the Introduction. Then, by condition (1) of the theorem, $\mu_A D(x, r) = \mu_i D(x, r)$ for each disk D(x, r) and for all i such that $1/i! \leq r$. Condition (2) is, of course, necessary for the weak-star convergence on Borel sets of $\bar{\omega}$ in general.

Finally, we note that the Haar measure on $\bar{\omega}$ is given by the matrix A satisfying (1) and (2), where

$$a_{ij} = 1/i! \quad (j \leq i!), \qquad a_{ij} = 0 \quad (j > i!).$$

As a functional on Q, this is equivalent to the Cesàro matrix, and $f_A(x)$ is the von Neumann mean of x.

3. THE DUALS OF $Q + E_n$ AND $Q + c_0$

COROLLARY. *Let the matrix* $\tilde{A}$ *satisfy the conditions in the theorem. Define the matrix* A *by*

$$a_{ij} = 0 \quad (j \leq i!) \qquad a_{i,j+i!} = \tilde{a}_{i,j}.$$

Let $\{\beta_m\}$ belong to ℓ_1 (respectively, to C^n). *Let* $B = (b_{ij})$, *where*

$$b_{ij} = \begin{cases} \beta_j & (0 < j \leq i!), \\ -\beta_{j-i!} & (i! < j \leq 2i!), \\ 0 & (2i! < j). \end{cases}$$

Then the matrix $C = A + B$ *defines* f *in* $(Q + c_0)^*$ *(respectively, in* $(Q + C^n)^*$*) by the relationship* $f(x) = \lim Cx$ *for* $x \in Q + c_0$ *(respectively, for* $Q + C^n$*). Every member of* $(Q + c_0)^*$ *(respectively, of* $(Q + C^n)^*$*) can be realized by exactly one such matrix.*

The linear map $f \to C_f$ *(with the obvious notation) is an isometry.*

Proof. The construction is self-explanatory. We merely note that if $x = x' + x''$ ($x' \in Q$ and $x'' \in c_0$), then

$$\lim Cx = \lim (A + B)(x' + x'') = \lim (Ax' + Bx' + Ax'' + Bx'')$$

$$= \lim Ax' + 0 + 0 + \lim Bx''.$$

Since each h in Q^* is realized uniquely by such an A and each g in c_0^* is realized uniquely by such a B, the corollary is clear except possibly for the claim that the map $f \to C_f$ is an isometry. To show that this map is an isometry requires a nontrivial but elementary argument, which we omit. This completes the proof of the corollary.

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