WEYL'S THEOREM FOR NONNORMAL OPERATORS

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1. INTRODUCTION

Let $\mathscr{B}(H)$ be the algebra of all bounded operators on an infinite-dimensional complex Hilbert space H, and let $\mathscr{K}$ be the closed ideal of compact operators. I write $\sigma(A)$ for the spectrum of A in $\mathscr{B}(H)$, and I define the Weyl spectrum $\omega(A)$ by

$$\omega(A) = \bigcap \sigma(A + K),$$

where the intersection is taken over all K in $\mathscr{K}$. A celebrated theorem of Weyl [7] asserts that if A is normal ($A^* A = AA^*$), then $\omega(A)$ consists precisely of all points in $\sigma(A)$ except the isolated eigenvalues of finite multiplicity.

In this paper, I show that Weyl's theorem holds for two large classes of generally nonnormal operators. The first of these is the class of hyponormal operators, which has been studied in [6]. The second class of operators for which Weyl's theorem holds is the class of Toeplitz operators, which has been studied in [2], [3], [8] and in many other papers. (References to much of the pertinent literature can be found in [2].) Finally, the present study of hyponormal operators and Toeplitz operators suggests a notion of "extremally noncompact" operators, which I examine in the last part of this paper.

2. PRELIMINARIES

Recall that an operator A is a Fredholm operator if its range R(A) is closed and both $R(A)^\perp$ and the null space N(A) are finite-dimensional. The Fredholm operators $\mathscr{F}$ constitute a multiplicative open semigroup in $\mathscr{B}(H)$. In fact [1], if π is the natural quotient map from $\mathscr{B}(H)$ to $\mathscr{B}(H)/\mathscr{K}$, then A is in $\mathscr{F}$ if and only if $\pi(A)$ is invertible. For any A in $\mathscr{F}$, the index i(A) is defined by the formula

$$i(A) = \dim N(A) - \dim R(A)^\perp,$$

and it is known that i is a continuous integer-valued function on $\mathscr{F}$.

Schechter [5] has observed that for any operator A,

$$\omega(A) = \{\lambda \mid A - \lambda \notin \mathscr{F}\} \cup \{\lambda \mid A - \lambda \in \mathscr{F} \text{ and } i(A - \lambda) \neq 0\},$$

and I shall use this characterization of $\omega(A)$. Note that by Schechter's result, $\omega(A)$ is never empty, since

$$\{\lambda \mid A - \lambda \notin \mathscr{F}\} = \sigma[\pi(A)].$$

It should also be noted that

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$$\omega(A^*) = \{\lambda \mid \bar{\lambda} \in \omega(A)\}.$$

Further, $\omega(A)$ is clearly an invariant under unitary equivalence.

3. WEYL'S THEOREM FOR HYPONORMAL OPERATORS

An operator A is *hyponormal* if $A^*A \geq AA^*$.

THEOREM (3.1). *If* A *is hyponormal, then* $\omega(A)$ *consists precisely of all points in* $\sigma(A)$ *except the isolated eigenvalues of finite multiplicity.*

Proof. We note that if A is hyponormal, then so is $A - \lambda$, for each complex λ. Thus, by virtue of Schechter's characterization, it suffices to show that A is a Fredholm operator of index zero but is not invertible if and only if 0 is an isolated eigenvalue of finite multiplicity in $\sigma(A)$. First, suppose that A is a Fredholm operator of index zero but is not invertible. Then $R(A)$ is closed and $N(A)$ and $R(A)^\perp$ are finite-dimensional, nontrivial subspaces of H. Since A is hyponormal,

$$\|Ax\| \geq \|A^*x\|$$

for all x in H. In particular,

$$N(A) \subset N(A^*) = R(A)^\perp,$$

and since $\dim N(A) = \dim R(A)^\perp$, we must have the relation

$$N(A) = R(A)^\perp.$$

Thus, $A = 0 \overset{\perp}{\oplus} B$, where B is invertible, and therefore 0 is an isolated eigenvalue of finite multiplicity in $\sigma(A)$.

For the converse, suppose 0 is an isolated eigenvalue of finite multiplicity in $\sigma(A)$. Again by hyponormality, $N(A) \subset R(A)^\perp$, so that $A = 0 \overset{\perp}{\oplus} B$, where B is one-to-one and hyponormal. If B is not invertible, then 0 must be an isolated point in $\sigma(B)$. A theorem of Stampfli [6] then shows that B has a nontrivial null space. This contradiction shows that B is invertible, and therefore A is a Fredholm operator of index zero. The proof is complete.

COROLLARY (3.2). *If* A *is hyponormal and has no isolated eigenvalues of finite multiplicity, then* $\|A\| \leq \|A + K\|$, *for each compact operator* K.

Proof. Since A is hyponormal,

$$\|A\| = \text{spectral radius } (A)$$

[4], and the desired result follows, since $\sigma(A) \subset \sigma(A + K)$ by the theorem.

COROLLARY (3.3). *Every hyponormal operator* A *can be written in the form* $A = N \overset{\perp}{\oplus} S$, *where* N *is normal and* S *is hyponormal with* $\omega(S) = \sigma(S)$.

Proof. It is well-known that eigenspaces of a hyponormal operator reduce the operator. The construction now follows if we split off the appropriate eigenspaces.

4. WEYL'S THEOREM FOR TOEPLITZ OPERATORS

Let $d\Theta$ be normalized Haar measure on the unit circle, and let H^2 be the subspace of $L^2(d\Theta)$ spanned by the functions z^n $(n = 0, 1, \cdots)$. If P is the orthogonal projection from L^2 onto H^2, then for each essentially bounded function ϕ in L^2, an operator T_ϕ on H^2 is defined by

$$T_\phi g = P(\phi g).$$

The operator T_ϕ is called the *Toeplitz operator* associated with ϕ.

It is known [2] that Toeplitz operators are in general nonnormal. Further, Widom [8] has shown that $\sigma(T_\phi)$ is always connected. Since there are no quasi-nilpotent Toeplitz operators except 0, $\sigma(T_\phi)$ can have no isolated eigenvalues of finite multiplicity, and Weyl's theorem becomes equivalent to the conjecture that $\omega(T_\phi) = \sigma(T_\phi)$.

THEOREM (4.1). *For each Toeplitz operator* T_ϕ, $\omega(T_\phi) = \sigma(T_\phi)$.

Proof. Since $T_\phi - \lambda = T_{\phi-\lambda}$, it suffices, by Schechter's characterization, to show that if T_ϕ is a Fredholm operator of index zero, then T_ϕ must be invertible. If T_ϕ is not invertible, but is a Fredholm operator of index zero, then it is easy to see that both T_ϕ and $T_\phi^* = T_{\bar\phi}$ must have nontrivial null spaces. The remainder of the proof consists of showing that this can not happen, unless $\phi = 0$ and consequently T_ϕ is the non-Fredholm operator 0.

Suppose that there exist nonzero functions ϕ, f, and g (ϕ essentially bounded, f and g in H^2) such that $T_\phi f = 0$ and $T_{\bar\phi} g = 0$. Then $P(\phi f) = 0$ and $P(\bar\phi g) = 0$, so that by standard properties of H^2 [4], there exist functions h and k in H^2 such that

$$\int h\,d\theta = \int k\,d\theta = 0 \quad \text{and} \quad \phi f = \bar h, \quad \bar\phi g = \bar k.$$

It follows from the theorem of F. and M. Riesz [4] that ϕ, f, g, h, k are all nonzero except on a set of measure zero. Thus, dividing the two sides of the equation $\overline{\phi f} = h$ by the corresponding sides of the equation $\bar\phi g = \bar k$, we see that $\bar f/g = h/\bar k$ pointwise a. e., so that

$$\bar f \bar k = gh \qquad \text{a. e.}$$

By another standard property of H^2, this is impossible unless

$$gh = 0 \qquad \text{a. e.};$$

using again the theorem of F. and M. Riesz, we conclude that either $f = 0$ a. e. or $g = 0$ a. e. This contradiction completes the proof.

COROLLARY (4.2). *If* T_ϕ *is a Toeplitz operator, then* $\|T_\phi\| \le \|T_\phi + K\|$ *for each compact operator* K.

Proof. It is known [2] that

$$\|T_\phi\| = \text{spectral radius } (T_\phi),$$

and the desired result follows, since $\sigma(T_\phi) \subset \sigma(T_\phi + K)$.

The proof of Theorem (4.1) is suggested by the result of Hartman and Wintner [3] that a Hermitian Toeplitz operator that is not a scalar has no proper values. In fact, the proof of Theorem (4.1) implies this result.

5. EXTREMALLY NONCOMPACT OPERATORS

Corollaries (3.2) and (4.2) show that a fairly large collection of operators A has the property that $\|A\| \leq \|A + K\|$ for each compact operator K. This is equivalent to the statement that

$$\|A\| = \|\pi(A)\|,$$

and it is natural to call operators with this property *extremally noncompact*. Thus, Corollaries (3.2) and (4.2) can be rephrased: *hyponormal operators without isolated eigenvalues of finite multiplicity and Toeplitz operators are extremally noncompact.*

I am grateful to C. Berger and J. Stampfli for a discussion during which the following result emerged.

THEOREM (5.1). *An operator* A *is extremally noncompact if and only if* $\|A^* A\|$ *is not an isolated eigenvalue of finite multiplicity in* $\sigma(A^* A)$.

Proof. Since $\|A\|^2 = \|A^* A\|$ and $\|\pi(A)\|^2 = \|\pi(A^* A)\|$, we see that A is extremally noncompact if and only if $A^* A$ is extremally noncompact. If $\|A^* A\|$ is not an isolated eigenvalue of finite multiplicity in $\sigma(A^* A)$, then by Theorem (3.1), $\|A^* A\| \in \sigma(A^* A + K)$ for each compact K, and so $\|A^* A\| = \|\pi(A^* A)\|$. Conversely, if $\|A^* A\|$ is an isolated eigenvalue of finite multiplicity in $\sigma(A^* A)$, then by splitting off the corresponding eigenspace, we easily get a compact operator K with $\|A^* A + K\| < \|A^* A\|$.

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