# APPROXIMATION BY ENTIRE FUNCTIONS

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## 1. INTRODUCTION

In 1927, Carleman [7] generalized the classical Weierstrass approximation theorem by proving that every function $Q(x)$ continuous for $-\infty < x < \infty$ can be uniformly approximated by an entire function. It is the purpose of the present paper to present several extensions of the Carleman theorem and to point out some of their applications. For the sake of completeness, a brief proof of the Carleman theorem is given in Section 2.

The first extension (Section 3) concerns the approximation of a function $Q(z)$, continuous on a subset C of a domain D, by a function $f(z)$ analytic in D; C is chosen as the union of a family of disjoint open arcs $c_n$ $(n = 1, 2, \cdots)$ approaching the boundary of D individually and as a family.

In Section 4 it is shown that if $Q(x)$ has a continuous derivative for $-\infty < x < \infty$, then an entire function $f(z)$ exists such that both $f(x) - Q(x)$ and $f'(x) - Q'(x)$ tend to zero arbitrarily rapidly as $|x| \to \infty$; in Section 5 this is applied to approximation of paths by analytic paths.

The final section is devoted to the Dirichlet problem for the unit disc; existence of a "solution" is proved for a general class of nonintegrable boundary values and indeed for arbitrary measurable boundary values.

## 2. THE CARLEMAN APPROXIMATION THEOREM

THEOREM 1. *Let* $Q(x)$ *be a continuous complex-valued function of* $x$ *for* $-\infty < x < \infty$. *Let* $E(x)$ *be continuous and positive for* $-\infty < x < \infty$. *Then there exists an entire function* $f(z)$ $(z = x + iy)$ *such that* $|f(x) - Q(x)| < E(x)$ *for* $-\infty < x < \infty$.

For this theorem, originally proved by Carleman [7], we give here a brief proof suggested by M. Brelot in a private communication.

Select constants $E_n$ $(n = 0, 1, 2, \cdots)$ such that

$$E_0 = E_1 > E_2 > E_3 > \cdots > E_n > \cdots,$$

$$0 < E_n < E(x) \text{ for } n \leq |x| \leq n + 1.$$

Let $d_n = E_{n+1} - E_{n+2}$ $(n = -1, 0, 1, \cdots)$. A sequence of polynomials $f_n(z)$ is then chosen inductively as follows. First $f_0(z)$ is chosen (in accordance with the Weierstrass theorem) so that $|f_0(x) - Q(x)| < d_0$ for $|x| \leq 1$. When $f_n(z)$ has been chosen, then a function $g_{n+1}(z)$ is defined as $f_n(z)$ for $|z| \leq n + 1$ and as $Q(x)$ for $z$ real and $n + 1 + 1/2 \leq |x| \leq n + 2$; $g_{n+1}(x)$ is also defined for $n + 1 \leq |x| \leq n + 1 + 1/2$ so as to remain continuous for $n + 1 \leq |x| \leq n + 2$ and so that $|g_{n+1}(x) - Q(x)| < d_n$ on these two intervals. By a theorem of Walsh ([14], p. 47), $f_{n+1}(z)$ can now be

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chosen so that $|f_{n+1}(z) - g_{n+1}(z)| < d_{n+1}$ for $|z| \leq n + 1$ and for $y = 0$, $n + 1 \leq |x| \leq n + 2$. The sequence of polynomials thus constructed satisfies conditions

$$|f_{n+1}(z) - f_n(z)| < d_{n+1} \quad \text{for } |z| \leq n + 1,$$

$$|f_n(x) - Q(x)| < d_{n-1} + d_n \quad \text{for } n \leq |x| \leq n + 1.$$

It follows that

$$f(z) = f_0(z) + \sum_{n=0}^{\infty} [f_{n+1}(z) - f_n(z)]$$

is entire. Also, for $n + 1 \leq |x| \leq n + 2$,

$$|Q(x) - f(x)| \leq |Q(x) - f_{n+1}(x)| + |f_{n+1}(x) - f(x)|$$

$$< d_n + d_{n+1} + d_{n+2} + \cdots = E_{n+1} < E(x).$$

Brelot has pointed out that the same proof is valid for approximation to tinuous real function $Q(x, y)$ by a function $u(x, y, z)$ harmonic in all of spa for approximation to a function $Q(x_1, \cdots, x_n)$ continuous for all $(x_1, \cdots, x_n)$ function $u(x_1, \cdots, x_{n+1})$ harmonic for all $(x_1, \cdots, x_{n+1})$. Carleman remarke Theorem 1 can be extended to approximation of a function $Q(z)$, continuous union of several paths leading to $\infty$ in the z-plane, by an entire function. T "several" was shown by Roth [13] to include families of paths of the power continuum, and important results on asymptotic paths of entire functions we tained by her. Lavrentieff and Keldych [9] characterized completely the sul of the complex plane with the property that every function $Q(z)$ continuous be approximated by an entire function $f(z)$, with $|f(z) - Q(z)|$ approaching rapidly as desired as $|z| \to \infty$.

Theorem 1 can be regarded as an extension of the Weierstrass product sentation of an entire function with prescribed zeros; the function $f(z)$ is cc to have given asymptotic behavior on the real axis, instead of preassigned z proof of Theorem 1 is also analogous to that of the Weierstrass theorem. shown by Bernstein [4] (see also [1] and [6], pp. 248-251), the rate of growtl can be related to that of the entire function $f(z)$.

## 3. APPROXIMATION IN A GENERAL DOMAIN

Just as the Weierstrass product expansion can be generalized to functio lytic in a general domain ([5], p. 295), as opposed to entire functions, so ca proof of Theorem 1 be adapted to such a case.

THEOREM 2. *Let* D *be a domain in the* z-*plane,* B *its boundary. Let* $c_m$ $(m = 1, 2, \cdots)$ *be a sequence of disjoint open arcs* $z = z_m(t)$ $(-1 < t <$ *each approaching* B *as* $|t| \to 1$. *Let at most a finite number of the* $c_m$ *int each compact subset of* D. *Let* C *be the union of the* $c_m$ *as point sets, an be a complex-valued function defined and continuous on* C; *let* $E(z)$ *be cont and positive on* C. *Then there exists a function* $f(z)$, *analytic in* D, *for wh* $|f(z) - Q(z)| < E(z)$ *on* C.

*Proof.* Choose a sequence of compact subsets $A_n$ of D such that $A_n$ is contained in the interior of $A_{n+1}$, $\bigcup_{n=1}^{\infty} A_n = D$, and $A_n$ is bounded by a set $B_n$ which is the union of a finite number of disjoint simple closed curves. For each m there exists a least index $n = n_m$ for which $A_n$ meet $c_m$. For $k \geq n_m$, let $r_{m,k}$, $s_{m,k}$ be the smallest and largest t for which $z_m(t)$ is in $A_k$; for $k = n_m$, let $c_{m,k} = \bigcup z_m(t)$ for $r_{m,k} \leq t \leq s_{m,k}$; for $k > n_m$, let $c_{m,k} = \bigcup z_m(t)$ for $r_{m,k} \leq t \leq r_{m,k-1}$ and for $s_{m,k-1} \leq t \leq s_{m,k}$. Let $C_k = \bigcup_m c_{m,k}$. Accordingly, $C_k$ is the union of a finite number of disjoint Jordan arcs, each of which is outside of $A_{k-1}$ except possibly for one endpoint, so that each component of the complement of $A_{k-1} \cup C_k$ contains at least one point of B. We can assume the $A_n$ chosen so that $C_1$ is nonvoid; then $C_2, C_3, \cdots$ are also nonvoid. We choose $E_n$ $(n = 0, 1, 2, \cdots)$ so that $E_0 = E_1$, so that $E_n > E_{n+1}$ for $n > 0$ and $E_n \to 0$ as $n \to \infty$, and so that $E_n < E(z)$ on $C_n$. Let $d_n = E_{n+1} - E_{n+2}$ $(n \geq -1)$. We can now choose a polynomial $f_0(z)$ such that $|f_0(z) - Q(z)| < d_0$ on $C_1$. Let $g_1(z)$ be defined as $f_0(z)$ in $A_1$ and as $Q(z)$ on $C_2$, except in the neighborhood of $C_2 \cap A_1$, where $g_1(z)$ is defined so as to remain continuous and so that $|g_1(z) - Q(z)| < d_0$ on $C_2$. By a theorem of Walsh ([14], p. 47) we can now choose a rational function $f_1(z)$ whose poles lie in B, and such that $|g_1(z) - f_1(z)| < d_1$ in $A_1 \cup C_2$. Proceeding inductively we obtain a sequence of rational functions $f_n(z)$, with poles in B and such that

$$|f_{n+1}(z) - f_n(z)| < d_{n+1} \quad \text{in } A_{n+1},$$

$$|f_{n+1}(z) - Q(z)| < d_n + d_{n+1} \quad \text{on } C_{n+1}.$$

Hence the function $f(z) = f_0(z) + \Sigma_0^{\infty}[f_{n+1}(z) - f_n(z)]$ is analytic in D, and on $C_{n+1}$

$$|Q(z) - f(z)| \leq |Q(z) - f_{n+1}(z)| + |f_{n+1}(z) - f(z)|$$

$$\leq d_n + d_{n+1} + \cdots = E_{n+1} < E(z).$$

*Remark.* The set C can be chosen with considerably greater generality, as in the work of Roth [13], Lavrentieff and Keldych [9] and Mergelyan [10].

## 4. SMOOTH APPROXIMATION OF SMOOTH FUNCTIONS

If $Q(x)$ is of class $C^{(n)}$ for $0 \leq x \leq 1$, then a direct application of the classical Weierstrass approximation theorem yields a polynomial $f(z)$ such that

$$|f^{(k)}(x) - Q^{(k)}(x)| < \varepsilon \quad \text{for } 0 \leq x \leq 1, \quad k = 0, 1, \cdots, n;$$

for we need only choose a polynomial $g(z)$ such that $|g(x) - Q^{(n)}(x)| < \varepsilon$ $(0 \leq x \leq 1)$. Then $f(z)$ can be chosen as that solution of the differential equation $d^n w/dz^n = g(z)$ for which $f^{(k)}(0) = Q^{(k)}(0)$ $(k = 1, \cdots, n - 1)$. A similar extension of Theorem 1 will now be established for the case $n = 1$.

THEOREM 3. *Let* $Q(x)$ *be a complex-valued function, defined for* $-\infty < x < \infty$ *and having a continuous derivative* $Q'(x)$ *over this interval. Let* $E(x)$ *be continuous and positive for* $-\infty < x < \infty$. *Then there exists an entire function* $f(z)$ $(z = x + iy)$ *such that, for* $-\infty < x < \infty$,

$$|f(x) - Q(x)| < E(x), \quad |f'(x) - Q'(x)| < E(x).$$

*Proof.* We first establish two lemmas.

LEMMA 1. *Let* $E(x)$ *be continuous and positive for* $0 \leq x < \infty$. *Then t exists a function* $E_1(x)$ *which is continuous for* $0 \leq x < \infty$ *and is such that f values of* $x$

$$0 < E_1(x) < E(x)/2, \quad \int_x^\infty E_1(t)\,dt < E(x)/2.$$

*Proof.* We first choose a function $r(x)$ of class $C'$ for $x \geq 0$ such that

$$r'(x) > 0, \quad r(x) > -\tfrac{1}{2}E(x), \quad \text{and } r(x) \to 0 \text{ as } r \to +\infty.$$

We then define

$$E_1(x) = \min[r'(x), E(x)/3].$$

Accordingly, $E_1(x) < E(x)/2$ and

$$0 < \int_x^\infty E_1(t)\,dt \leq \int_x^\infty r'(t)\,dt = -r(x) < E(x)/2.$$

LEMMA 2. *Let* $E_1(x)$ *be continuous and positive for* $-\infty < x < \infty$. *Let* $E_1(x) = E_1(-x)$, *and let*

$$k = \int_0^\infty E_1(x)\,dx$$

*be finite. Let* $A$, $B$ *be complex numbers such that* $|A - B| < 2k$. *Then the an entire function* $h(z)$ *such that*

$$|h'(x)| < E_1(x) \quad (-\infty < x < \infty),$$

$$\lim_{x \to +\infty} h(x) = B, \quad \lim_{x \to -\infty} h(x) = A.$$

*Proof.* If $A = B$, we can take $h(z) \equiv A$. Otherwise, let

$$r = \frac{|A - B|}{2k}, \quad q = \frac{1 - r}{2(1+r)},$$

so that $0 < r < 1$ and $0 < q < 1/2$. By virtue of Theorem 1, we can choose function $h_0(z)$ such that for all $x$

$$|h_0'(x) - E_1(x)| < qE_1(x).$$

Then $|h_0'(x)| < (1 + q)E_1(x)$, so that

$$\lim_{x \to +\infty} h_0(x) = \int_0^\infty h_0'(x)\,dx + h(0) = B_0,$$

$$\lim_{x \to -\infty} h_0(x) = \int_0^{-\infty} h_0'(x)\,dx + h(0) = A_0$$

exist. Furthermore, $\Re[h_0'(x)] > (1 - q)E_1(x)$, so that

$$|A_0 - B_0| = \left|\int_{-\infty}^{\infty} h_0'(x)\,dx\right| > \int_{-\infty}^{\infty} \Re[h_0'(x)]\,dx > 2k(1 - q).$$

It follows that constants a, b can be chosen such that

$$aA_0 + b = A, \quad aB_0 + b = B.$$

We now set $h(z) = ah_0(z) + b$, so that $\lim_{x \to +\infty} h(x) = B$ and $\lim_{x \to -\infty} h(x) = A$. Also

$$|a| = \frac{|A - B|}{|A_0 - B_0|} < \frac{2rk}{2k(1 - q)} = \frac{r}{1 - q}.$$

Hence

$$|h'(x)| < \frac{r}{1 - q}|h_0'(x)| < r\frac{1 + q}{1 - q}E_1(x) = \frac{r^2 + 3r}{3r + 1}E_1(x) < E_1(x).$$

We now prove Theorem 3. We can assume without loss of generality that $E(x) = E(-x)$ for all x. Let $E_1(x)$ then be chosen in accordance with Lemma 1, and define $E_1$ for negative x so that $E_1(x)$ is also even. By Theorem 1 we can choose an entire function $f_1(z)$ such that $|f_1(x) - Q'(x)| < E_1(x)$. We set

$$g(x) = \int_0^x [f_1(t) - Q'(t)]\,dt.$$

Then, by the choice of $E_1(x)$, the function $g(x)$ has finite limits B as x approaches $+\infty$ and A as x approaches $-\infty$, and

$$|A - B| < \int_{-\infty}^{\infty} E_1(x)\,dx = 2k.$$

We now choose a function $h(z)$ in accordance with Lemma 2, and let

$$f(z) = \int_0^z f_1(\zeta)\,d\zeta + Q(0) - h(z).$$

Then

$$|f'(x) - Q'(x)| = |f_1(x) - Q'(x) - h'(x)| < E_1(x) + E_1(x) < E(x),$$

$$\lim_{x \to +\infty} [f(x) - Q(x)] = \lim_{x \to \infty}\left\{\int_0^x [f_1(t) - Q'(t)]\,dt - h(x)\right\} = B - B = 0.$$

Hence for $x > 0$

$$f(x) - Q(x) = -\int_x^{\infty} [f'(t) - Q'(t)]\,dt,$$

$$|f(x) - Q(x)| < \int_x^{\infty} 2E_1(t)\,dt < E(x).$$

Similarly, this inequality holds for $x < 0$, and the theorem is proved.

*Remark.* The preceding proof carries over without essential change to of a function Q defined on a locally rectifiable simple path C from $\infty$ to $\infty$ Q has a continuous derivative with respect to arc length on C. Approximat function Q defined on several curves leads to difficulties, namely in constr analogue of h(z). However, it appears probable that the theorem can be so ized.

## 5. ANALYTIC APPROXIMATION OF CURVES

From Theorem 1 we deduce at once that a continuous path $w = Q(t)$ $(-\infty$ can be approximated as closely as desired by an entire function $w = f(z)$:

$$|f(t) - Q(t)| < E(t) \quad (-\infty < t < \infty),$$

for a given positive and continuous $E(t)$. If the path $w = Q(t)$ has unique en i.e., if $Q(t)$ has a limit (in the extended plane) as $t \to +\infty$ and as $t \to -\infty$ —a tends to 0 as $t \to \infty$ and as $t \to -\infty$, then for the entire function $f(z)$ the rea an asymptotic path in both directions. This is the basis of the work of Roth $w = Q(t)$ displays complicated behavior as $t \to \pm\infty$, then $w = f(z)$ has corres complicated behavior on the real axis. For example, $w = Q(t)$ can be chose the finite w-plane (as a Peano curve), each value w being taken on a seque points $t_n(w)$ $(|t_n(w)| \to \infty)$; the corresponding curve $w = f(t)$ will hence be d the extended plane; the real axis is then a "path of complete indeterminatio f(z). Corresponding results hold for systems of curves, as in Theorem 2 a as in the work of Roth [13], and of Bagemihl and Seidel [2], [3].

If the path function $Q(t)$ has a continuous derivative $Q'(t)$, then Theore that one can choose $f(z)$ so that $Q(t)$, $Q'(t)$ are approximated as closely as by $f(t)$, $f'(t)$. It is of interest to remark that if the path $w = Q(t)$ is a home image of the t-axis, then the same applies to the path $w = f(t)$:

THEOREM 4. *Let* $w = Q(t)$ *be a complex-valued function defined and h continuous derivative* $Q'(t) \neq 0$ for $-\infty < t < \infty$; *let* $w = Q(t)$ *define a home ism of the* t-*axis into the* w-*plane. Let* $E(t)$ *be continuous and positive for Then there exists an entire function* $w = f(z)$ *such that*

$$|f(t) - Q(t)| < E(t), \qquad |f'(t) - Q'(t)| < E(t),$$

*and the path* $w = f(t)$ $(-\infty < t < \infty)$ *is also a homeomorphic image of the* t-*a*

*Proof.* First we remark that there exist an open set D containing the $w = Q(t)$ $(w = u + iv)$ and a homeomorphism of D:

$$x = F(u, v), \qquad y = G(u, v)$$

onto an open set $D_1$ in the xy-plane, where F and G are of class C',

$$\partial(F, G)/\partial(u, v) \neq 0,$$

and $w = Q(t)$ is mapped on the point $x = t$, $y = 0$. This is proved by piecing homeomorphisms, as was done by Morse ([11], pp. 108-110).

Now along $w = Q(t)$

$$\frac{dx}{dt} = 1 = \Re\left[\left(\frac{\partial F}{\partial u} + i\frac{\partial F}{\partial v}\right)\overline{Q'(t)}\right].$$

It follows from the continuity of the functions appearing that we can choose $E_1(t)$ continuous, positive, and such that

$$|Q(t) - Q_1(t)| < E_1(t), \quad |Q'(t) - Q_1'(t)| < E_1(t) \quad (-\infty < t < \infty)$$

imply that, for each $t$, $w = Q_1(t)$ is in $D$ and

$$\Re\left[\left(\frac{\partial F}{\partial u} + i\frac{\partial F}{\partial v}\right)\overline{Q_1'(t)}\right] > 1/2,$$

where the partial derivatives are evaluated at $u + iv = Q_1(t)$; that is, $dx/dt > 1/2$ on the path $w = Q_1(t)$. Since $x$ increases steadily on the path $w = Q_1(t)$, the mapping $w = Q_1(t)$ must be a homeomorphism of the t-axis into $D$.

Let $E_2(t) = \min[E(t), E_1(t)]$. Then by Theorem 3 we can choose an entire function $f(z)$ such that

$$|f(t) - Q(t)| < E_2(t), \quad |f'(t) - Q'(t)| < E_2(t)$$

for all $t$. Hence also the path $w = f(t)$ $(-\infty < t < \infty)$ is a homeomorphic image of the t-axis.

*Remark.* The condition that the mapping $w = Q(t)$ of the t-axis be a homeomorphism permits paths of considerable complexity; e.g., the path may spiral towards a closed curve, or it may oscillate towards a segment as does the curve $y = \sin(1/x)$. If the path tends to $\infty$ as $t \to \pm\infty$, then the approximating smooth path $w = f(t)$ must do the same. Thus a domain $H$ bounded by a simple curve $w = Q(t)$ tending to $\infty$ in both directions can be arbitrarily closely approximated by a domain $H_1$ whose boundary is $w = f(t)$, where $f$ is entire; for the homeomorphism $Q(t)$ can be approximated by a homeomorphism $Q_1(t)$ of class $C'$ and then, by the theorem above, by an entire homeomorphism.

## 6. THE DIRICHLET PROBLEM

In 1925 R. Nevanlinna [12] established the following existence theorem:

THEOREM 5. *Let* $Q(x)$ *be a continuous real-valued function of* $x$ *for* $-\infty < x < \infty$. *Then there exists a function* $u(x, y)$, *harmonic for* $y > 0$, *and such that, for every* $x_0$, $u(x, y) \to Q(x_0)$ *as* $(x, y) \to (x_0, 0)$.

Thus the Dirichlet problem for a half-plane, with continuous boundary values, can always be solved. The solution is not unique, for $u(x, y) + v(x, y)$ is also a solution, provided $v$ is the imaginary part of an entire function $f(z) = \Sigma a_n z^n$, with $a_n$ real.

*Theorem* 5 *is a corollary of Theorem* 1. For we can choose an entire function $f_1(z) = u_1(x, y) + iv_1(x, y)$ such that $|f_1(x) - Q(x)| < 1$ for $-\infty < x < \infty$. Hence $Q_1(x) = Q(x) - u_1(x, 0)$ is continuous, and $|Q_1(x)| < 1$ for all $x$. We can solve the Dirichlet problem for the half-plane with boundary values $Q_1(x)$ by the Poisson integral; let $u_2(x, y)$ be the solution. Then $u(x, y) = u_1(x, y) + u_2(x, y)$ is harmonic for $y > 0$ and has boundary values $Q(x)$.

The same reasoning applies to the analogous Dirichlet problem in higher dimensions. As the following theorem shows, one can also establish the existence of solutions for more general boundary values.

THEOREM 6. *Let* C *be an open subset of the circumference* K: $|z| =$ C *is the union of disjoint open arcs* $c_n$ $(n = 1, 2, \cdots)$. *Let* $Q(z)$ *be a real function defined on* C *and integrable on each* $c_n$. *Then there exists a func harmonic for* $|z| < 1$ *and such that* $u(re^{i\theta}) \to Q(e^{i\theta})$ *as* $r \to 1$, *almost eve in* C. *Moreover,* u *can be represented as follows:*

$$(1) \qquad u(re^{i\theta}) = u_1(re^{i\theta}) + \frac{1}{2\pi}\int_0^{2\pi} Q_2(e^{i\phi})\frac{1 - r^2}{1 + r^2 - 2r\cos(\phi - \theta)}d\phi,$$

*where* $u_1$ *is harmonic for* $|z| < 1$, *for* $|z| > 1$, *and on* C, *and* $Q_2(e^{i\theta})$ *is on* $[0, 2\pi]$.

*Proof.* Since Q is integrable on each $c_n$, we can choose a function $Q_0$ fined and continuous on C, and such that the integral of $|Q(z) - Q_0(z)|$ on than $2^{-n}$. Let D be the domain consisting of the finite z-plane minus the of C relative to K. As a consequence of Theorem 2 we can choose a func harmonic in D and such that $|u_1(z) - Q_0(z)| < 1$ on C. Let $Q_2(z) = Q(z) -$ z in C, and let $Q_2(z) = 0$ for z in K - C. Then

$$\int_{c_n} |Q_2(z)|d\theta \leq \int_{c_n} |u_1(z) - Q_0(z)|d\theta + \int_{c_n} |Q_0(z) - Q(z)|d\theta \leq m(c_n)$$

Hence $Q_2(z)$ is integrable on C, and the corresponding Poisson integral r a function $u_2(z)$ harmonic for $|z| < 1$ and such that $u_2(re^{i\theta}) \to Q_2(e^{i\theta})$ alr where on C, as $r \to 1$. The function $u(z) = u_1(z) + u_2(z)$ is then harmonic and has boundary values $u_1(z) + Q_2(z) = Q(z)$ almost everywhere on C. Th the theorem.

If C has measure $2\pi$, we can state simply: $u(re^{i\theta}) \to Q(re^{i\theta})$, as r → everywhere. Even in this case the boundary function Q(z), although meas K, is not in general integrable on K. This suggests the conjecture that, f measurable function Q(z) on K, there exists a solution (not unique) of the problem: i.e., a function u(z) with radial limits equal to $Q(e^{i\theta})$ almost where. Such a solution does indeed exist. For there exists a function F(z uous for $|z| < 1$ and having Q(z) as radial limit almost everywhere. Th from the fact that arctan Q(z) is an integrable function for which the Diri lem has a solution U(z); we take F(z) = tan U(z). Let E be an $F_\sigma$ set of gory on $|z| = 1$. Then by results of Roth ([13], p. 124) and Bagemihl and S p. 188) there exists a harmonic function u(z) in $|z| < 1$ such that u(z) - radial limit 0 on E. Now E can be chosen to have measure $2\pi$. With suc u(z) - F(z) has radial limit 0 almost everywhere. Since F(z) has radial l almost everywhere, it follows that u(z) also has radial limit Q(z) almost where.

THEOREM 7. *Let* Q(z) *be a real-valued function, defined for* $|z| = 1$ *urable. Then there exists a function* u(z), *harmonic for* $|z| < 1$, *and suc* $u(re^{i\theta}) \to Q(e^{i\theta})$, *as* $r \to 1$, *for almost all* $\theta$.

The crucial idea behind the proof above was pointed out to the author t that every measurable function on $[0, 2\pi]$ is of Baire class at most 1 on a measure $2\pi$ (see [8], p. 567).

The solution of the Dirichlet problem obtained in this way is not in ge sentable in the form (1)—i.e., as a Poisson integral plus a function with s smoothness property on $|z| = 1$; in general the solution is unaffected by continuity of Q along any portion of $|z| = 1$. In particular, one can thus not assert existence of nontangential limits for u at any point of $|z| = 1$; such nontangential limits do exist almost everywhere for (1), by virtue of familiar properties of the Poisson integral. The "almost all $\theta$" in Theorem 7 refers only to a set of first category; it is an open question whether the theorem can be strengthened by enlarging this set.

In both of Theorems 6 and 7 the solution provided is not unique, and uniqueness can not be expected unless some supplementary conditions are imposed. It would be very desirable to obtain such conditions, and in general to determine a linear correspondence between boundary functions Q and solutions u. More specifically, let M be the space of all measurable functions on K, and let U be the space of all functions harmonic for $|z| < 1$. We then seek a mapping T of M into U such that $TQ = u$ is a solution of the Dirichlet problem for boundary values Q, such that $T(c_1Q_1 + c_2Q_2) = c_1TQ_1 + c_2TQ_2$ for all constants $c_1$, $c_2$, and such that TQ is bounded when Q is bounded. As remarked by R. K. Ritt, the existence of such a mapping can be established by an argument employing a Hamel basis in M. However, one would like to exhibit T explicitly and, moreover, to achieve some sort of continuity properties. It would be of considerable value if this could be achieved for the class $M_1$ corresponding to the functions of Theorem 5; one would then have the basis of an operational calculus for arbitrary continuous functions on the infinite interval $-\infty < x < \infty$.

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