ON LOCAL BALANCE AND N-BALANCE IN SIGNED GRAPHS

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A *signed graph* or *s-graph* [2] is obtained from a linear graph when some of its lines are regarded as positive and the remaining lines as negative. The *sign of a cycle* is the product of the signs of its lines. An s-graph is *balanced* if all its cycles are positive. Two characterizations of balanced s-graphs were given in [2], Theorems 2 and 3. The definitions of all terms used here may be found in [2].

For certain applications of the theory of signed graphs to problems in social psychology, one is interested only in the cycles through a designated point. For other psychological considerations, one considers only cycles of length not exceeding N. These viewpoints lead to the definitions of local balance and N-balance in s-graphs. Some properties of these kinds of balance will be derived in this note. A detailed discussion of the relevance of the notion of balance of s-graphs to psychological theory is given in [1].

An s-graph G is *locally balanced at the point* P, or briefly, G is *balanced at* P, if all cycles containing P are positive. Theorem 1 below shows the interdependence of local balance and articulation points. An *articulation point* of a connected graph is a point whose removal results in a disconnected graph. We first require an extension of the sign of a path or cycle to any set of lines of G. Let L_1 be a subset of L, the set of all lines of G. The *sign* of L_1 is the product of the signs of the lines of L_1. The previous definitions of the sign of a path or a cycle are of course specializations of this one. If L_1, L_2 are subsets of L, then $L_1 \oplus L_2$ denotes the symmetric difference, or set union modulo 2, of L_1 and L_2. Let $s(L_1)$ denote the sign of L_1. It is convenient to prove two lemmas before taking up the theorem on local balance.

LEMMA 1. $s(L_1 \oplus L_2 \oplus \cdots \oplus L_n) = s(L_1) \cdot s(L_2) \cdot \cdots \cdot s(L_n)$.

Proof. For $n = 1$, the lemma is trivial. When $n = 2$, we make use of the usual formula $L_1 + L_2 = (L_1 - L_2) \cup (L_2 - L_1)$, which expresses $L_1 \oplus L_2$ as a union of disjoint sets. By definition of the sign of L_1, we have $s(L_1) = \Pi_{\lambda \in L_1} s(\lambda)$. Now L_1 can be expressed as the union of two disjoint sets:

$$L_1 = (L_1 - L_2) \cup (L_1 \cap L_2).$$

Thus

$$s(L_1) = s(L_1 - L_2) \cdot s(L_1 \cap L_2) \quad \text{and} \quad s(L_2) = s(L_2 - L_1) \cdot s(L_1 \cap L_2).$$

Hence

$$\begin{aligned} s(L_1) \cdot s(L_2) &= s(L_1 - L_2) \cdot s(L_2 - L_1) \cdot (s(L_1 \cap L_2))^2 \\ &= s(L_1 - L_2) \cdot s(L_2 - L_1) \\ &= s(L_1 \oplus L_2). \end{aligned}$$

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The proof of the inductive step is immediate when one writes

$$L_1 \oplus L_2 \oplus \cdots \oplus L_k \oplus L_{k+1} = (L_1 \oplus L_2 \oplus \cdots \oplus L_k) \oplus L_{k+1}$$

and applies both the inductive hypothesis and the result for $n = 2$.

LEMMA 2. *If* z *and* z' *are any two cycles of a linear graph* G, *regar sets of lines, then* $K = z \oplus z'$ *is the union of pairwise disjoint cycles.*

Proof. Case (i). We first consider the case in which each point on both z', is a point of a common line of z and z'. For this case, one can show t line λ in K lies in a unique cycle $y(\lambda)$ all of whose lines are in K, by con this cycle.

If $\lambda \in K$, then $\lambda \in z$ or $\lambda \in z'$, but not both; say $\lambda \in z$. Let α_0 be the maximal length in z containing λ but no lines of z'. Then the distinct end A_0 and A_1 of α_0 are points through which both cycles z, z' pass. Let α_1 path of maximal length in z' which has A_1 as one endpoint and contains no z. Let A_2 be the other endpoint of α_1. If $A_2 = A_0$, then $\alpha_0 \cup \alpha_1$ is the cyc containing λ. If $A_2 \neq A_0$, form the path α_2 of maximal length in z which h one endpoint and is disjoint from z'. Let A_3 be the other endpoint of α_2. $A_3 \neq A_0, A_1$; for otherwise α_2 would not be of maximal length. Similarly, the maximal path α_3 in z' with endpoints A_3 and A_4 such that $\alpha_3 \cap z$ is e Then $A_4 \neq A_1, A_2$, because of the maximality of α_3. If $A_4 = A_0$, then

$$\alpha_0 \cup \alpha_1 \cup \alpha_2 \cup \alpha_3$$

is the cycle in K containing λ. If $A_4 \neq A_0$, continue this process. Since th G is finite, there exists a smallest positive even integer k such that $A_k =$ $y(\lambda) = \alpha_0 \cup \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_{k-1}$ is a cycle in K containing λ. Clearly, this c tion defines an equivalence relation on the lines of K such that the lines in equivalence class form a cycle. Hence the cycle $y(\lambda)$ is the unique cycle i taining λ, and K is the union of pairwise disjoint cycles.

Case (ii). In general, however, the cycles z and z' may pass through which do not lie on a line of $z \cap z'$. For each such point P, there exist fou points Q_1, Q_2, R_1, R_2 such that PQ_1, PQ_2 are lines of z, and PR_1, PR_2 are z'. This case (ii) can be transformed to case (i) by splitting each of these into two points P_1 and P_2 and adding the additional line P_1P_2 to both cycle The points Q_1, R_1 are then joined to P_1 by a line, and the points Q_2, R_2 ar to P_2. Applying the result of case (i), and then identifying each pair of poi we obtain a separation of K into pairwise disjoint cycles. We note that thi tion need not be unique, since each new common line P_1P_2 can be introduc essentially different ways.

The s-graph in Figure 1 (in which the dashed line is negative) shows th hypothesis that Q is not an articulation point is needed in the following the

THEOREM 1. *If the connected* s-*graph* G *is balanced at* P, Q *is a po cycle* z *passing through* P, *and* Q *is not an articulation point, then* G *is b at* Q.

Proof. Assume that G is not balanced at Q. Then there exists a nega cycle z' through Q. Since G is balanced at P, the cycle z is positive. W sider separately the cases in which $z \cap z'$ is empty or not empty, where ea cycles z, z' is regarded as a set of lines.

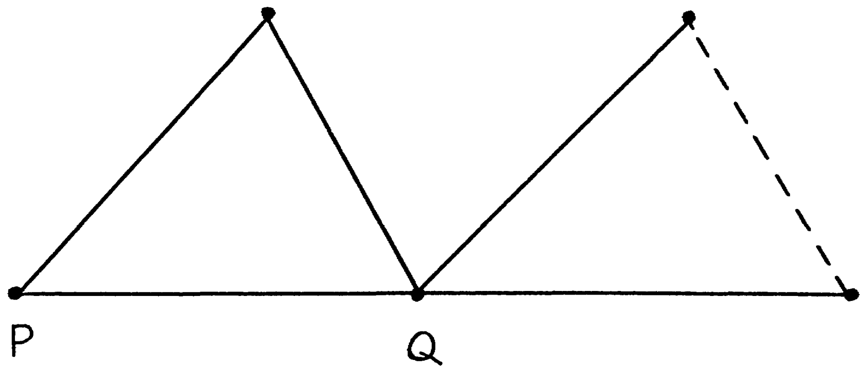

Fig. 1

Case 1. $z \cap z'$ *is not empty.* Consider the set of lines $K_1 = z \oplus z'$. It follows from Lemma 1 that K_1 is negative, and from Lemma 2 that K_1 can be written as the union of pairwise disjoint cycles $z_{11}, z_{12}, \cdots, z_{1r_1}$ $(r_1 \geq 1)$. Since K_1 is negative and $K_1 = z_{11} \oplus z_{12} \oplus \cdots \oplus z_{1r_1}$, Lemma 1 shows that at least one of these cycles is negative. Now z' does not pass through P, since z' is negative and G is balanced at P. Therefore exactly one of the cycles in K_1, say z_{11}, passes through P. If $r_1 = 1$, then z_{11} is negative and we have a contradiction to the hypothesis that G is balanced at P. If $r_1 > 1$, then z_{11} is positive since it passes through P, and one of the other cycles in K_1, say z_{1r_1}, is negative.

For any two cycles x, y, let n(x, y) be the number of connected components of the subgraph $x \cap y$ of G. Each such component is either a path of maximal length all of whose lines are in $x \cap y$, or it consists of a single point. Then

$$n(z, z') = n(z, z_{11}) + n(z, z_{12}) + \cdots + n(z, z_{1r_1}),$$

for the right-hand member is the number of connected components of the subgraph $z - z'$ (set difference) of G. Clearly each cycle z_{1j} has a line in common with z, so that $n(z, z_{1j}) > 0$ for all j. Since $r_1 > 1$, we see that $n(z, z_{1r_1}) < n(z, z')$. This fact provides the basis for an inductive proof of Case 1.

We continue this process by forming the set

$$K_2 = z \oplus z_{1r_1} = z_{21} \oplus z_{22} \oplus \cdots \oplus z_{2r_2} \quad (r_2 \geq 1).$$

Since K_2 is negative, we have a contradiction if $r_2 = 1$. Otherwise, let z_{2r_2} be a negative cycle and note that $n(z, z_{2r_2}) < n(z, z_{1r_1})$. Eventually one must necessarily obtain a set K_s for which $r_s = 1$. Then $K_s = z_{s1}$ is a negative cycle through P, which is a contradiction.

Case 2. $z \cap z'$ *is empty.* By hypothesis, Q is not an articulation point of G. Hence, for each point $R_i \neq Q$ on z', there exists a path $\rho(R_i)$ joining R_i with P which does not pass through Q. It is clear that there exists a point R on z' for which the path $\rho(R)$ passes through no point of z' other than R. Let ρ be the path $\rho(R)$. Let ϕ denote a fixed one of the two paths joining Q and R along the cycle z'. Let S be the first point of z on the path ρ in the direction from R to P.

There are two possibilities: (i) S = P, (ii) S ≠ P. (i) If S = P, let σ be either of the two paths joining P and Q along the cycle z, and form the cycle $z'' = \rho \cup \sigma \cup \phi$.

(ii) If $S \neq P$, let ρ_1 be the subpath of ρ joining R and S; let ρ_2 be that patl S and P along the cycle z which does not pass through Q; and let ρ_3 be th joining P and Q along z on which S does not lie. Then form the cycle

$$z'' = \rho_1 \cup \rho_2 \cup \rho_3 \cup \phi .$$

In either of the two possibilities (i) or (ii), z" is a cycle through P suc $z'' \cap z' = \phi$. Since G is balanced at P, z" is positive. Therefore $\bar{z} = z'' \oplus$ negative cycle through P, since z' is negative and z" is positive. Since th contradiction, G is balanced at Q.

It was shown in [2] that the following condition (C) is necessary and suff an s-graph G to be balanced:

(C) The set of all points of G can be separated into two disjoint subset: that each positive line of G joins two points of the same subset and each ne line joins points of different subsets.

A *subgraph* of G is a graph all of whose points and lines are in G. A *b graph* G is a maximal connected subgraph containing no articulation points In these terms, the theorem can be restated:

THEOREM 1′. *An* s-*graph* G *is balanced at* P *if and only if each block containing* P *is balanced.*

Thus to determine whether a given s-graph G is balanced at a designat P, one tests each block of G containing P for balance, using condition (C).

An s-graph G is called N-*balanced* if each cycle of G whose length doe ceed N is positive. We obtain a characterization of N-balanced s-graphs. simplicity, we discuss the case N = 3. A 3-*cycle* is a cycle of length 3. Gi cycles z, z' of G, we say that z' is 3-*reachable* from z if there exists a (of 3-cycles $z_1, z_2, \cdots, z_n$ such that $z_1 = z$, $z_2 \neq z_1$ and $z_2 \cap z_1$ is not empty, $\cdots, z_{k+1} \neq z_1, z_2, \cdots, z$ and $z_{k+1} \cap (z_1 \cup z_2 \cup \cdots \cup z_k)$ is not empty, $\cdots, z_n$ = Obviously, 3-reachability is an equivalence relation on the set of all 3-cycl The union of all cycles in an equivalence class of 3-reachability is a subgr: a 3-*cluster*. Similarly one can define the equivalence relation of N-reachal all r-cycles $(r \leq N)$ and N-clusters.

In Theorem 2 on N-balance, we require a lemma on cycle bases. A cy(*pends* on a set of cycles $\{z_1, z_2, \cdots, z_m\}$ if it can be written in the form

$$z = \varepsilon_1 z_1 + \varepsilon_2 z_2 + \cdots + \varepsilon_m z_m ,$$

where ε_i denotes 0 or 1, $0z_i$ is the empty set, and $1z_i$ is z_i. A set of cy *independent* if each cycle in the set does not depend on the remaining ones. *basis* of a graph is a maximal collection of independent cycles.

LEMMA 3. *An* s-*graph is balanced if and only if all the cycles in each basis are positive.*

Proof. The necessity is immediate. The sufficiency follows from Lem the fact that each cycle of a graph depends on each cycle basis.

THEOREM 2. *An* s-*graph is* N-*balanced if and only if each* N-*cluster i*

Proof. We give the proof for N = 3; that for N > 3 is analogous.

The sufficiency is trivial, for each 3-cycle is contained in a 3-cluster.

To prove the necessity we need to show that each 3-cluster Y is balanced under the hypothesis that all 3-cycles are positive. It remains to show that all cycles of Y of length greater than 3 are positive. Since Y is a 3-cluster, any maximal collection of independent cycles from the set of all 3-cycles of Y constitutes a cycle basis for Y. Thus Y has a cycle basis consisting entirely of 3-cycles, which are positive. Hence by Lemma 3, Y is balanced.

One can combine the notions of local balance and N-balance, and it is this combination which may be fruitful for the psychological study of large structures. An s-graph G is N-*balanced at* P if each cycle of length not greater than N through P is positive. We state without proof two theorems on local N-balance, since their proofs are similar to those of Theorems 1 and 2.

I. *If* G *is* N-*balanced at* P, *and if* Q *is a point on an* N-*cluster containing* P, *and is not an articulation point of the subgraph* G_N *of* G *formed by the union of all the* N-*clusters of* G, *then* G *is* N-*balanced at* Q.

II. *The* s-*graph* G *is* N-*balanced at* P *if and only if all* N-*clusters containing* P *are balanced.*

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