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Geometric Properties of Pluricomplex Green Functions with One or Several Poles in $\mathbb{C}^n$

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0. Introduction and Statement of Results

In this paper we study the infinitesimal behavior near poles and the boundary behavior of pluricomplex Green functions with one or several logarithmic poles.

On the one hand, we prove a min-max principle for the Azukawa pseudometric that is related to the pluricomplex Green function. On the other hand, we find a new proof of effective formulas for the pluricomplex Green function with two poles of equal weights in the unit ball in $\mathbb{C}^2$. With these formulas, we show that the sublevel sets of this function are not (lineally) convex, no matter how close to the boundary they are situated. This fact is surprising, especially since this convexity property is lost in the case of several poles even when the domain is the unit ball (the sublevel sets of the pluricomplex Green function with one pole of a bounded convex domain are always convex). Moreover, this provides a counterexample to a recently published statement.

Let us recall first the definition of the *pluricomplex Green function* with one or several logarithmic poles in a domain D in $\mathbb{C}^n$. Let m be a positive integer and let $P = \{(p_1, c_1), \dots, (p_m, c_m)\}$ be a set of m distinct poles p_j in D with positive weights c_j, $j = 1, \dots, m$. Following Lelong (see [Le1] and [Le2]), the pluricomplex Green function with poles in P is defined on D by

$$g_D(P, z) = \sup\{u(z) : u \in \mathrm{PSH}(D, [-\infty, 0[) \text{ and } u(z) - c_j \log\|z - p_j\| \text{ is bounded from above for } z \text{ near } p_j,\ j = 1, \dots, m\},$$

where $\mathrm{PSH}(D)$ denotes the set of plurisubharmonic (psh) functions on D. If $m = 1$ and $c_1 = 1$, then $g_D(p, \cdot)$ is the well-known pluricomplex Green function with one logarithmic pole in p, introduced by Klimek (see [K1]). The pluricomplex Green function has a connection to the complex Monge–Ampère operator. This operator acts on locally bounded psh functions (see [BT1] and [BT2]) and it applies also for psh functions u such that $u^{-1}(-\infty)$ is relatively compact (see [C; D1; Ki; Si]).

Let us consider the following Dirichlet problem for the complex Monge–Ampère operator:

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$$\begin{cases} u \in \mathrm{PSH}(D, [-\infty, 0]) \cap \mathcal{C}(\bar{D}), \\ u|_{bD} \equiv 0, \\ (dd^c u)^n = 0 \text{ on } D \setminus P, \\ \text{for any } j = 1, \dots, m,\ u(z) - c_j \log\|z - p_j\| \text{ is bounded from above} \\ \quad \text{for } z \text{ near } p_j, \end{cases}$$

where $d = \partial + \bar{\partial}$ and $d^c = i(\bar{\partial} - \partial)$. If D is bounded, then a necessary and sufficient condition to obtain a unique solution to this problem is that D must be hyperconvex (i.e., there exists a continuous psh exhaustion function $\varrho\colon D \to]-\infty, 0[$). This result was obtained by Demailly (see [D2] and also [Le1; Le2]). In this case $(dd^c u)^n = (2\pi)^n \sum_{j=1}^m c_j^n \delta_{p_j}$, where δ_{p_j} is the Dirac measure at p_j. The resolution of this Dirichlet problem has been also studied in details by Lempert [L1], who obtained this result with regularity properties when D is strictly convex with smooth boundary. Lempert's method uses the study of extremal analytic discs in D for the Kobayashi metric.

In Section 1 we study the behavior of a pluricomplex Green function with one logarithmic pole locally near its pole in a bounded hyperconvex domain in $\mathbb{C}^n$. More precisely, we look for a connection between the Azukawa pseudometric associated to the pluricomplex Green function and the Kobayashi–Royden pseudometric. For that we will use Poletsky's definition of the pluricomplex Green function. This problem is the dual version of another problem solved in [N2], where it was proved that the sequence of kth Reiffen pseudometrics, generalizing the Carathéodory–Reiffen one, converges to the Azukawa pseudometric. We will now prove a min-max principle for the Azukawa pseudometric.

Recall the classical definitions of these different pseudometrics. Let w be a point in a domain D in $\mathbb{C}^n$ and let X be a "tangent vector" in $\mathbb{C}^n$. Then Azukawa [A1] has introduced $A_D(w, X)$ as

$$\log A_D(w, X) = \limsup_{\lambda \to 0} (g_D(w, w + \lambda X) - \log|\lambda|).$$

This A_D is a pseudometric on D; that is, A_D is $[0, +\infty)$-valued on $\mathbb{C}^n$ satisfying $A_D(w, \lambda X) = |\lambda| A_D(w, X)$ and is called in [JP] the *Azukawa pseudometric.* At about the same time, Lelong [Le1; Le2] introduced A_D^{-1} as the capacitative indicatrice of D. In fact, $\varrho_D(w, X) = \log A_D(w, X)$ is the Robin function of D, introduced by Bedford and Taylor in [BT3].

The *Carathéodory–Reiffen pseudometric* C_D is defined by

$$C_D(w, X) = \sup\{|f_{(1)}(w)X| : f \in \mathcal{O}(D, \Delta),\ f(w) = 0\},$$

where $\Delta = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ is the open unit disc in $\mathbb{C}$, $\mathcal{O}(D, \Delta)$ is the space of holomorphic functions in D bounded by 1 in modulus, and

$$f_{(1)}(w)X = f^{(1)}(w)X = \sum_{|\nu|=1} \frac{1}{\nu!} D^{(\nu)} f(w) X^\nu.$$

The *Kobayashi–Royden pseudometric* K_D is defined by

$$K_D(w, X) = \inf\{|t|^{-1} : \exists \varphi \in \mathcal{O}(\Delta, D) \text{ s.t. } \varphi(0) = w,\ \varphi'(0) = tX\},$$

where $\mathcal{O}(\Delta, D)$ is the space of analytic dics in D. The metric A_Δ coincides with the Poincaré metric on Δ, which implies (see [A1]) that

$$C_D \le A_D \le K_D \ \text{ on } D \times \mathbb{C}^n.$$

Now let us define two sequences of pseudometrics. For all integer p greater than 1, the pth *Reiffen pseudometric* γ_p of D, introduced in [N1] and [JP], is defined by

$$\gamma_p(w, X) = \sup\{|f_{(p)}(w)X|^{1/p} : f \in \mathcal{E}_p(w, D)\},$$

where $f_{(p)}(w)X = \frac{1}{p!} f^{(p)}(w)X = \sum_{|\nu|=p} \frac{1}{\nu!} D^{(\nu)} f(w) X^\nu$ and $\mathcal{E}_p(w, D)$ is the following compact set of $\mathcal{O}(D)$:

$$\mathcal{E}_p(w, D) = \{f \in \mathcal{O}(D, \Delta) : D^{(\nu)} f(w) = 0 \ \forall \nu \in \mathbb{N}^n \text{ with } |\nu| \le p-1\}.$$

If we denote $\mathcal{F}_p(w, X)$ the subset of $\mathcal{O}(\Delta, D)$ defined by

$$\mathcal{F}_p(w, X) = \{\varphi \in \mathcal{O}(\Delta, D) : \varphi(0) = w,\ \varphi^{(k)}(0) = 0,\ 1 \le k \le p-1,$$
$$\varphi^{(p)}(0) = p!\, tX, \text{ where } t \in \mathbb{C}\},$$

then the pth *Royden pseudometric* Γ_p is defined by

$$\Gamma_p(w, X) = \inf\left\{ \frac{\left|\prod_{\lambda \ne 0, \varphi(\lambda)=w} \lambda\right|}{|t|} : \exists \varphi \in \mathcal{F}_p(w, X),\ \frac{1}{p!}\varphi^{(p)}(0) = tX \right\}.$$

In the 1-dimensional case, we always take X equal to 1 and write $A_D(w)$ instead of $A_D(w, 1)$, and so on. Then we obtain the following theorem.

THEOREM 1 (Min-max Principle for the Pseudometrics). *Let D be a bounded hyperconvex domain in $\mathbb{C}^n$, w a point in D, and X a vector in $\mathbb{C}^n$.*

(1) *If D is strictly hyperconvex then we have, quasi-everywhere in $\mathbb{C}^n$ (i.e., except for a pluripolar set),*

$$A_D(w, X) = \sup_p \gamma_p(w, X) = \lim_p \gamma_p(w, X).$$

(2) *If $n \ge 2$ and D is strictly hyperconvex then we have, everywhere in $\mathbb{C}^n$,*

$$A_D(w, X) = \Gamma_p(w, X) \quad \forall p \ge 1.$$

If $n = 1$, there exists $\varphi \in \mathcal{F}_1(w)$ such that

$$\frac{\left|\prod_{\lambda \ne 0, \varphi(\lambda)=0} \lambda\right|}{|\varphi'(0)|} = \Gamma_1(w) = A_D(w).$$

We recall that a bounded domain D in $\mathbb{C}^n$ is said to be "strictly hyperconvex" if there exists a bounded domain Ω and a function $\varrho \in \mathcal{C}(\Omega,]-\infty, 1[) \cap \mathrm{PSH}(\Omega)$ such that $D = \{z \in \Omega : \varrho(z) < 0\}$, ϱ is exhaustive for Ω, and the open set $\{z \in \Omega : \varrho(z) < c\}$ is connected for all real numbers $c \in [0, 1]$.

Let us remark that Poletsky [PS] considered another generalization of the Kobayashi–Royden pseudometric for any domain D in $\mathbb{C}^n$:

$$\tilde{\Gamma}_p(w,X)=\inf\left\{|t|^{-1}:\exists\varphi\in\mathcal{F}_p(w,X),\ \frac{1}{p!}\varphi^{(p)}(0)=tX\right\};$$

$\tilde{\Gamma}_1 = K_D$ and $A_D \le \Gamma_p \le \tilde{\Gamma}_p \le K_D$ on $D\times\mathbb{C}^n$. Unfortunately, the sequence $(\tilde{\Gamma}_p)_p$ does not, in general, have good properties of convergence to A_D, as illustrated in the following proposition and corollary.

PROPOSITION 2. *If D is a Dirichlet domain in $\mathbb{C}$, then $\tilde{\Gamma}_1(w)=\tilde{\Gamma}_p(w)$ for all $p\ge 1$. In addition, $A_D(w)=\tilde{\Gamma}_1(w)$ if and only if D is simply connected.*

COROLLARY 3. *There exist bounded hyperconvex domains in $\mathbb{C}^n$ $(n>1)$ that are lineally convex and such that* (i) *their pluricomplex Green function with one logarithmic pole is symmetric with respect to the pole and the variable and* (ii) *nevertheless $(\tilde{\Gamma}_p(w,X))_p$ does not converge to $A_D(w,X)$ everywhere on $D\times\mathbb{C}^n$.*

Finally, let us remark that we always have the following inequalities on $D\times\mathbb{C}^n$:

$$C_D=\gamma_1\le\gamma_q\le A_D\le\Gamma_q\le\tilde{\Gamma}_q\le\tilde{\Gamma}_1=K_D.$$

For some domains, all these inequalities are equalities—for example, when D is convex or strictly lineally convex. Indeed, this is an immediate consequence of a very strong Lempert's theorem [L1; L2] concerning the pluricomplex Green function with one logarithmic pole in strictly convex or strictly lineally convex domains with smooth boundary ($\mathcal{C}^\infty$ or real analytic). The foregoing min-max principle (Theorem 1) generalizes the following.

THEOREM 4. *If D is a convex or strictly lineally convex domain in $\mathbb{C}^n$, then for all integers $q\ge 1$ we have*

$$C_D=\gamma_1=\gamma_q=A_D=\Gamma_q=\tilde{\Gamma}_q=\tilde{\Gamma}_1=K_D\ \text{ on } D\times\mathbb{C}^n.$$

We recall that a bounded domain D in $\mathbb{R}^m$ is strictly convex if there exists a neighborhood U of ∂D and a $\mathcal{C}^2$-function $r: U\to\mathbb{R}$ such that $U\cap D=\{r<0\}$, $U\cap\partial D=\{r=0\}$, $\operatorname{grad} r\ne 0$ on U, and

$$\sum_{j,k=1}^{m}\frac{\partial^2 r}{\partial x_j\partial x_k}(x)\xi_j\xi_k>0 \quad \text{for } x\in U \text{ and } \xi\in(\mathbb{R}^m)^*.$$

Moreover, a $\mathcal{C}^2$-bounded domain D in $\mathbb{C}^n$ is strictly lineally convex if the complex tangent hyperplanes of D to ∂D are disjoint from $\bar{D}$ save for the unique point of contact. These tangents have no higher than first-order contact with ∂D; that is, the distance to ∂D, restricted to a complex tangent, has a nondegenerate critical point at the point of contact.

In Section 2 we study the boundary behavior of some pluricomplex Green functions with two logarithmic poles. We know [L2] that if D is a bounded convex domain in $\mathbb{C}^n$ then the sublevel sets $\{z\in D: g_D(w,z)<c\}$ of $g_D(w,\cdot)$ are again convex for any $c\le 0$. If, in addition, D is strictly convex with smooth boundary

($\mathcal{C}^\infty$ or real analytic), then $g_D(w,\cdot)$ has the same regularity on $D \setminus \{w\}$ as the boundary.

In fact, this situation is quite special, and it is very different when the pluricomplex Green function has several poles. Indeed, there exist strictly convex domains in $\mathbb{C}^2$ (respectively in $\mathbb{C}^n$) with pluricomplex Green functions g with two (or more) logarithmic poles such that any connected sublevel set of g is not lineally convex. Recall that the notion of lineal convexity has been introduced by Martineau (see [H; M]) and that a domain D in $\mathbb{C}^n$ is called *lineally convex* if, for every $z \in \mathbb{C}^n \setminus D$, there exists an affine complex hyperplane Π such that $z \in \Pi \subset \mathbb{C}^n \setminus D$. Obviously, any convex domain is lineally convex.

We also calculate explicitly (with other methods than used by Coman [Co]) the pluricomplex Green function $g_{B_2(0,1)}$ in $B_2(0,1)$, the open unit ball in $\mathbb{C}^2$, with two logarithmic poles of weight 1. Then the sublevel sets of this function are not lineally convex, no matter how close to the boundary they are situated.

If $\beta \in]0,1[$ and $w=(w_1,w_2)$ is any point in $\mathbb{C}^2$, denote by $\mathcal{C}_w$ the open complex cone with vertex at w:

$$\mathcal{C}_w = \{z=(z_1,z_2)\in\mathbb{C}^2 : \beta|z_2-w_2| > |z_1-w_1|\}.$$

THEOREM 5 [Co]. *Let $\beta \in]0,1[$. Let $p=(\beta,0)$ and $q=(-\beta,0)\in B_2(0,1)$. Then the following formula holds for the pluricomplex Green function on $B_2(0,1)$ with the two logarithmic poles p and q of weight* 1:

$$g_{B_2(0,1)}(p,q;z) = \begin{cases} \frac{1}{2}\log\left(\frac{|\beta-z_1|^2+(1-\beta^2)|z_2|^2}{|1-\beta z_1|^2}\right) & \text{if } z\in\bar{\mathcal{C}}_p\cap B_2(0,1),\\ \frac{1}{2}\log\left(\frac{|\beta+z_1|^2+(1-\beta^2)|z_2|^2}{|1+\beta z_1|^2}\right) & \text{if } z\in\bar{\mathcal{C}}_q\cap B_2(0,1),\\ \frac{1}{2}\log\left(\frac{|\beta^2-z_1^2|^2+\beta^4|z_2|^4+2(1-\beta^4)|z_2|^2+\sqrt{\Delta(z_1^2,z_2)}}{2|1-\beta^2 z_1^2|^2}\right) & \text{if } z\in B_2(0,1)\setminus(\mathcal{C}_p\cup\mathcal{C}_q), \end{cases}$$

where

$$\Delta(w) = (\beta^4|w_2|^4-|\beta^2-w_1|^2)^2+4(1-\beta^4)|w_2|^2|\beta^2|w_2|^2-(\beta^2-w_1)|^2.$$

While writing this paper the author learned that these formulas have also been recently obtained, with the same method, by Edigarian and Zwonek [EZ].

Let $c\in]-\infty,0]$ and let $B_c=\{z\in B_2(0,1) : g_{B_2(0,1)}(p,q;z)<c\}$, a sublevel set of the pluricomplex Green function $g_{B_2(0,1)}(p,q;\cdot)$ of Theorem 5. The set B_c is connected if and only if $2\log\beta < c \le 0$ and B_c has two connected components if $c\le 2\log\beta$. Finally, we obtain the following proposition, which contradicts a recent result of Einstein-Matthews [EM, Prop. 3.7].

PROPOSITION 6. *If $2\log\beta < c < 0$, then B_c is connected and not lineally convex.*

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1. Infinitesimal Behavior of a Pluricomplex Green Function in a Neighborhood of Its Unique Logarithmic Pole

Let D be a domain in $\mathbb{C}^n$, w a point in D, and $g_D(w,\cdot)$ the pluricomplex Green function on D with one logarithmic pole in w of weight 1. There are two ways to reconstruct this function with holomorphic maps. The first method uses holomorphic functions on D with values in Δ and is, in fact, a precise version of Lelong and Bremermann's theorem for this function; the second is a theorem of Poletsky that uses analytic discs of D, that is, holomorphic maps on Δ with values in D. With the first (resp. second) method, we put in relation the Carathéodory–Reiffen (resp. Kobayashi–Royden) pseudometric with the Azukawa pseudometric associated to this pluricomplex Green function.

Let us recall briefly the first method using holomorphic functions on D valued in Δ, with a zero in w of "large order". For any positive integer p, define on D the psh Hartogs function $h_p(w,\cdot)$ by

$$h_p(w,z)=\sup\left\{\frac{1}{p}\log|f(z)| : f\in\mathcal{E}_p(w,D)\right\}.$$

We call $\exp(h_p)$ the pth *Möbius function* (introduced in [N1] and [JP]), and $\log\gamma_p(w,\cdot)$ is the *Robin function* of $h_p(w,\cdot)$. We have obtained in [N2] the following theorem, which is the first part of our min-max principle. In this paper we also find some applications of this theorem.

THEOREM 1.1. *If D is a strictly hyperconvex domain in $\mathbb{C}^n$ then, for every $w\in D$,*

$$g_D(w,z)=\lim_{p\to\infty}h_p(w,z)=\sup_{p\geq 1}h_p(w,z)\ \text{ on } D,$$

$$A_D(w,X)=\lim_{p\to\infty}\gamma_p(w,X)=\sup_{p\geq 1}\gamma_p(w,X)\ \text{ quasi-everywhere on } \mathbb{C}^n.$$

This result generalizes a previous result, which is an immediate consequence of a very strong theorem of Lempert (see [L1; L2]) concerning strictly convex or strictly lineally convex domains with smooth boundary.

THEOREM 1.2. *If D is a bounded strictly convex domain or a strictly lineally convex domain in $\mathbb{C}^n$ with smooth boundary ($\mathcal{C}^\infty$ or real analytic), then*

$$g_D(w,z)=h_p(w,z)\ \text{ on } D\times D,$$

$$A_D(w,X)=\gamma_p(w,X)\ \text{ on } D\times\mathbb{C}^n.$$

Note that we can obtain the same result as in Theorem 1.2 for any convex domain and for any strictly lineally convex domain in $\mathbb{C}^n$, since the former can be approximated internally by a sequence of strictly convex domains and the latter by a sequence of strictly lineally convex domains with analytic boundary. However, it is not possible in general to approximate internally a lineally convex domain by a sequence of strictly lineally convex domains (see [H] and [Zn]).

Now our question is as follows.

QUESTION 1.3. Does there exist a sequence of pseudometrics on $D \times \mathbb{C}^n$ that generalizes the Kobayashi–Royden one and such that it converges in some sense to the Azukawa pseudometric?

This problem is dual to the previous one, solved in Theorem 1.1. There already exists a sort of generalization $\tilde{\Gamma}_p(w, X)$ of the Kobayashi–Royden pseudometric, introduced by Poletsky [PS]. In Section 1.1 we study its properties and show that it is not a good choice for our problem. In Section 1.2 we introduce another generalization $\Gamma_p(w, X)$ of the Kobayashi–Royden pseudometric and prove that this one is a good choice for our problem.

1.1

First we remark that $\tilde{\Gamma}_1(w, X) = K_D(w, X)$. According to the Schwarz lemma we can easily prove that, for all integers p and q greater than 1 and for all X in $\mathbb{C}^n$, $\gamma_p(w, X) \le \tilde{\Gamma}_q(w, X)$. Let $k_{q,D}(w, X, \cdot)$ be the following function defined on Δ by $k_{q,D}(w, X, \lambda) = \sup\{g_D(w, \varphi \circ \theta_q(\lambda)) : \varphi \in \mathcal{F}_q(w, X), \theta_q \in \Theta_q\}$, where Θ_q is the set of all continuous determinations θ_q of the power $1/q$ defined on an open set $\Delta \setminus S$ in $\mathbb{C}$, where S is a closed line segment of length 1 with origin 0. Then we obtain the following lemma.

LEMMA 1.4. *Let D be a bounded hyperconvex domain in $\mathbb{C}^n$, let w be a point in D, and let r, $R > 0$ be such that $B(w, r) \subset D \subset B(w, R)$. Then $k_{q,D}(w, X, \cdot)$ is a subharmonic function on Δ, continuous on $\bar{\Delta}$, with values in* $[-\infty, 0]$. *On $\bar{\Delta}$, these functions* (p *and* q *are two integers greater than* 1) *verify*

$$\log|\lambda| + \log r - \log R \le k_{q,D}(w, X, \lambda) \le k_{pq,D}(w, X, \lambda) \le \log|\lambda|.$$

In addition, for all $X \in \mathbb{C}^n$, $A_D(w, X) \le \tilde{\Gamma}_{pq}(w, X) \le \tilde{\Gamma}_q(w, X)$ and

$$\limsup_{\lambda \to 0} (k_{q,D}(w, X, \lambda) - \log|\lambda|) = \log\left(\frac{A_D(w, X)}{\tilde{\Gamma}_q(w, X)}\right).$$

Proof. In order to prove the first part of this lemma, we just use the continuity and the maximality of $g_D(w, \cdot)$ and the fact that any bounded hyperconvex domain in $\mathbb{C}^n$ is taut. A domain D in $\mathbb{C}^n$ is *taut* if $\mathcal{O}(\Delta, D)$ is normal—that is, if whenever we start with a sequence $(\varphi_j)_j \subset \mathcal{O}(\Delta, D)$ there exists a subsequence (φ_{j_ν}) with φ_{j_ν} that converges in $\mathcal{O}(\Delta, D)$ to $\varphi \in \mathcal{O}(\Delta, D)$ or there exists a subsequence (φ_{j_ν}) that diverges uniformly on compact sets (i.e., for any two compact sets $K \subset \Delta$ and $L \subset D$ there is an index ν_0 such that $\varphi_{j_\nu}(K) \cap L = \emptyset$ if $\nu \ge \nu_0$ (see [KR])).

To prove that

$$\limsup_{\lambda \to 0} (k_{q,D}(w, X, \lambda) - \log|\lambda|) = \log\left(\frac{A_D(w, X)}{\tilde{\Gamma}_q(w, X)}\right),$$

it is sufficient to use the following property of Azukawa [A1]: *the upper limit* $\limsup_{\lambda\to 0}(g_D(w,\varphi(\lambda)) - p\log|\lambda|) = \log(A_D(w,X)|t|)$ *is independent of the choice of the map* $\varphi \in \mathcal{F}_p(w,X)$ *with* $\frac{1}{p!}\varphi^{(p)}(0) = tX$, *where* $t \in \mathbb{C}$. The last inequalities are a direct consequence of what precedes and the proof is complete. □

In some domains, the last inequalities of this lemma are equalities.

THEOREM 1.5. *If D is a convex or strictly lineally convex domain in $\mathbb{C}^n$, then for every integer $q \geq 1$ we have $k_{q,X}(\lambda) = \log|\lambda|$ on Δ and*

$$\tilde{\Gamma}_q(w,X) = A_D(w,X) \text{ on } D\times\mathbb{C}^n.$$

Proof. As in Theorem 1.2, this theorem is again an easy consequence of a strong result of Lempert [L1; L2]. Indeed, if D is as described here then, for all $X \in \mathbb{C}^n$, there exists a unique extremal disc $\varphi_X \in \mathcal{F}_1(w,X)$ such that $\varphi_X'(0) = \tilde{\Gamma}_1(w,X)^{-1}X$ and $g_D \circ \varphi_X(\lambda) = \log|\lambda|$ for all $\lambda \in \Delta$. Consequently, $k_{1,X}(\lambda) = \log|\lambda|$ for all $\lambda \in \Delta$ and $k_{q,X}(\lambda) = \log|\lambda|$ for all $\lambda \in \Delta$ and all $q \geq 1$, according to Lemma 1.4. For the equality of the pseudometrics $\tilde{\Gamma}_q(w,X)$ and $A_D(w,X)$, the proof is almost the same. □

Let us study what happens with a concrete example of a domain in $\mathbb{C}^n$, not necessarily convex. Let D be a complete circular domain with center 0; that is, $\lambda D \subset D$ for any $\lambda \in \bar{\Delta}$. Let l_X be a complex line in direction $X \in \mathbb{C}^n$ passing through 0. We denote the radius of the disc $l_X \cap D$ by $R(X)$. Note that $R(\lambda X) = R(X)$ for any $\lambda \in \mathbb{C}^*$. We define the function $r\colon \mathbb{C}^n \to [0,+\infty]$ by $r(z) = \|z\|/R(z)$ if $z \neq 0$ and $r(0) = 0$. Then r is upper semicontinuous and D is represented by $D = \{z \in \mathbb{C}^n : r(z) < 1\}$. The domain D is pseudoconvex if and only if $\log r$ is psh on $\mathbb{C}^n$. In addition, D is hyperconvex if and only if r is continuous.

Let us suppose now that D is pseudoconvex. It is well known that $g_D(0,z) = \log r(z)$ on D. Consequently $A_D(0,X) = \|X\|/R(X) = r(X)$ on $\mathbb{C}^n$, and it is not difficult to prove that we also have, for every $q \in \mathbb{N}^*$, $\tilde{\Gamma}_q(0,X) = r(X)$ on $\mathbb{C}^n$. Indeed, this last property is a consequence of the following lemma, which is an improvement of Sadullaev's Schwarz lemma [S].

LEMMA 1.6. *Let D be a pseudoconvex complete circular domain with center* 0, *and let $\varphi \in \mathcal{O}(\Delta, D)$ be such that $\varphi^{(k)}(0) = 0$ for all $k \leq q-1$ (q is an* integer $\geq$ 1). *Then we have*:

(1) $\|\varphi(\lambda)\| \leq |\lambda|^q R(\varphi(\lambda))$ *on* Δ;
(2) *if* $\varphi^{(q)}(0)/q!$ *is not equal to* 0 *then* $\varphi^{(q)}(0)/q!$ *is a vector in* $\bar{D}$.

Proof. We fix $0 < r < 1$ and denote $\varphi_r(\lambda) = \varphi(r\lambda)/\lambda^q$ on $\bar{\Delta}$. For every $\lambda \in \partial\Delta$, $\lambda = e^{i\theta}$, we have $\varphi_r(\lambda) = e^{-iq\theta}\varphi(re^{i\theta}) \in D$ because D is circular. In addition, D is pseudoconvex, so according to the continuity principle $\varphi_r(\Delta) \subset D$ and $\|\varphi_r(\lambda)\| \leq R(\varphi_r(\lambda)) = R(\varphi(r\lambda)/\lambda^q) = R(\varphi(r\lambda))$ on Δ. Consequently, for arbitrary λ in Δ we have $\|\varphi(\lambda)\| \leq |\lambda|^q R(\varphi(\lambda))$.

Let φ_1 be the analytic disc in $\mathbb{C}^n$ defined on Δ by

$$\varphi_1(\lambda) = \frac{\varphi(\lambda)}{\lambda^q} \quad \text{if } \lambda \in \Delta^* \text{ and } \varphi_1(0) = \frac{\varphi^{(q)}(0)}{q!}.$$

According to the first property, $\|\varphi_1(\lambda)\| = \|\varphi(\lambda)\|/|\lambda|^q \leq R(\varphi(\lambda))$ on Δ^*. Then $r(\varphi_1(\lambda)) = \|\varphi_1(\lambda)\|/R(\varphi_1(\lambda)) \leq 1$ on Δ^*. Since r is a psh function on $\mathbb{C}^n$ and φ_1 is a holomorphic mapping on Δ valued in $\mathbb{C}^n$, it follows that $r \circ \varphi_1$ is a subharmonic function in Δ. By the mean value property, we deduce that $r(\varphi_1(0)) \leq 1$. Finally,

$$r\left(\frac{\varphi^{(q)}(0)}{q!}\right) = \frac{\left\|\frac{\varphi^{(q)}(0)}{q!}\right\|}{R\left(\frac{\varphi^{(q)}(0)}{q!}\right)} \leq 1 \quad \text{and} \quad \frac{\varphi^{(q)}(0)}{q!} \in \bar{D}. \qquad \square$$

Now we shall study the general case. Proving that the sequence $(\tilde{\Gamma}_q(w, X))_q$ converges to $A_D(w, X)$ is almost equivalent to proving that the sequence $(k_{q,X}(\lambda))_q$ converges to $\log|\lambda|$ on Δ. In fact, if $(\tilde{\Gamma}_q(w, X))_q$ converges to $A_D(w, X)$, then $(k_{q,X}(\lambda))$ converges quasi-everywhere on Δ to $\log|\lambda|$ (i.e., except for a polar set). Conversely, if $(k_{q,X}(\lambda))$ converges on Δ to $\log|\lambda|$ then $(\tilde{\Gamma}_q(w, X))_q$ converges to $A_D(w, X)$.

Observe what happens in the 1-dimensional case. If D is a Dirichlet domain in $\mathbb{C}$, then its complementary set in $\mathbb{C}$ contains at least two points and hence the universal covering π of D is Δ. Let $\pi : \Delta \to D$ be such that $\pi(0) = w$. According to Azukawa [A2], $g_D(w, \pi(\lambda)) = \sum_j g_\Delta(t_j, \lambda)$, where $\pi^{-1}(w) = \{t_0 = 0, t_1, \ldots\}$. In this 1-dimensional case we obtain Proposition 2 and deduce Corollary 3 in the n-dimensional case.

Proof of Proposition 2. In what follows, we use k_q to denote $k_{q,D}(w, 1, \cdot)$.

If D is simply connected, then π is a biholomorphism from Δ to D and it is easy to prove that $A_D(w) = \tilde{\Gamma}_q(w)$ for every q in $\mathbb{N}^*$. If π is not a biholomorphism (i.e., D is not simply connected), then $\pi^{-1}(w)$ contains at least two distinct points in Δ. Since $\pi : \Delta \to D$ is the universal covering of D, for every holomorphic function φ in Δ with values in D there exists a holomorphic function $\tilde{\varphi}$ in Δ with values in Δ such that $\pi \circ \tilde{\varphi} = \varphi$. Then $k_q = \sup\{g_D(w, \pi \circ \tilde{\varphi} \circ \theta_q) : \theta_q \in \Theta_q, \tilde{\varphi} \in \mathcal{O}(\Delta, \Delta) \text{ s.t. } \tilde{\varphi}(0) \in \pi^{-1}(w) \text{ with order } q\}$. We can also write $k_q = \sup\{g_D(w, \pi \circ \theta \circ \tilde{\varphi} \circ \theta_q) : \theta_q \in \Theta_q, \tilde{\varphi} \in \mathcal{F}_{q,\Delta}(0), \theta \in \mathrm{Aut}(\Delta) \text{ s.t. } \theta(0) \in \pi^{-1}(w)\}$.

First we remark that all functions k_q are equal on Δ. In fact, for all t in Δ^*, according to the Schwarz lemma we have that $D(0, |t|) \subset \{\varphi(t), \varphi \in \mathcal{F}_{1,\Delta}(0)\} \subset \{\varphi(\theta_q(t)), \varphi \in \mathcal{F}_{q,\Delta}(0), \theta_q \in \Theta_q\} \subset D(0, |t|)$. Then the sequences $(k_q)_q$ and $(\tilde{\Gamma}_q(w))_q$ are constant in all cases.

Let us study the function u defined on Δ by

$$u(\lambda) = \sup\{g_D(w, \pi \circ \theta(\lambda)) : \theta \in \mathrm{Aut}(\Delta) \text{ s.t. } \theta(0) \in \pi^{-1}(w)\}.$$

We have $u(\lambda) \leq \log|\lambda|$ on Δ and $u(\lambda) > -\infty$ on $\Delta \setminus \{0\}$. The function u is continuous and subharmonic on Δ, and in its definition the supremum is in fact a maximum. Consequently, $\tilde{u}(\lambda) := u(\lambda) - \log|\lambda|$ is continuous, subharmonic on Δ, negative on Δ, and not equal to 0 everywhere. Then, according to the maximum principle, $\tilde{u}(0) < 0$ and there exists an open nonempty disc $D(0, r)$ where $\tilde{u}$ verifies $3\tilde{u}(0)/2 \leq \tilde{u}(\lambda) \leq \tilde{u}(0)/2$. Then, according to the Schwarz lemma,

$$\begin{aligned} k_1(\lambda) &\leq \sup\{\log|\tilde{\varphi}(\lambda)| + \tilde{u}(0)/2 : \tilde{\varphi} \in \mathcal{F}_{1,\Delta}(0)\} \\ &\leq \log|\lambda| + \tilde{u}(0)/2 \quad \text{on } D(0, r). \end{aligned}$$

Thus $\tilde{\Gamma}_1(w) > A_D(w)$ and the proof is complete. □

Proof of Corollary 3. It is sufficient to prove this result in $\mathbb{C}^2$. Let D be a product domain $D_1 \times D_2$ in $\mathbb{C}^2$, where D_1 and D_2 are two bounded Dirichlet domains in $\mathbb{C}$. Then D is a bounded hyperconvex domain in $\mathbb{C}^2$. Let $w = (w_1, w_2)$ be a point in D.

Any analytic disc $\varphi = (\varphi_1, \varphi_2) \in \mathcal{O}(\Delta, D)$ is given by two holomorphic functions φ_1 and φ_2 on Δ with values in D_1 and D_2, respectively. Note that $\varphi_j = \pi_j \circ \tilde{\varphi}_j$, where $\tilde{\varphi}_j \in \mathcal{O}(\Delta, \Delta)$ and $\pi_j : \Delta \to D_j$ is the universal covering of D_j with $\pi_j(0) = w_j$ for $j = 1, 2$. Then, according to the product property (see [E2] and [Z]), we have $g_D((w_1, w_2), (z_1, z_2)) = \max(g_{D_1}(w_1, z_1), g_{D_2}(w_2, z_2))$ and consequently, for any $X = (X_1, X_2) \in \mathbb{C}^2$,

$$k_{q,D}(w, X, \cdot) = \max(k_{q,D_1}(w_1, X_1, \cdot), k_{q,D_2}(w_2, X_2, \cdot)).$$

If, in addition, we suppose that D_1 and D_2 are not simply connected, then (according to Proposition 2) we obtain that, for all $w \in D$, all $X \in \mathbb{C}^2$, and all $q \geq 1$,

$$\tilde{\Gamma}_1(w, X) = \tilde{\Gamma}_q(w, X) > A_D(w, X).$$ □

We remark that this corollary illustrates a well-known result of Znamenskii: *If a lineally convex domain in* $\mathbb{C}^n$ $(n > 1)$ *can be exhausted by strictly lineally convex domains, then it is necessarily* $\mathbb{C}$*-convex* (see [H] and [Zn]).

1.2

In [P1] and [P2] (see also [E1]), Poletsky proved that, for any domain D in $\mathbb{C}^n$, the pluricomplex Green function $g_D(w, \cdot)$ verifies

$$g_D(w, z) = \inf\left\{ \sum_{\lambda \in \varphi^{-1}(w)} \log|\lambda| : \varphi \in \mathcal{O}(\Delta, D),\ \varphi(0) = z \right\}$$

if $z \in D$ and $z \neq w$, where we use $\varphi^{-1}(w)$ to denote the subset of Δ defined by $\varphi^{-1}(w) = \{\lambda \in \Delta : \varphi(\lambda) = w\}$.

This result does not change if we replace $\mathcal{O}(\Delta, D)$ by $\mathcal{O}(\bar{\Delta}, D)$, the set of holomorphic mappings in a neighborhood of $\bar{\Delta}$ with values in D. We remark that $\sum_{\lambda \in \varphi^{-1}(w)} \log|\lambda| = \sum_{\lambda \in \varphi^{-1}(w)} g_\Delta(\lambda, 0) = g_\Delta(\varphi^{-1}(w), 0)$. Let us consider the other pseudometrics $\Gamma_p(w, X)$ defined in the introduction. It is easy to see that, for any p and q in $\mathbb{N}^*$, we always have

$$A_D(w, X) \leq \Gamma_{pq}(w, X) \leq \Gamma_p(w, X) \leq \tilde{\Gamma}_p(w, X) \leq K_D(w, X);$$

for any convex domains and any strictly lineally convex domains, we have

$$A_D(w, X) = \Gamma_p(w, X) = K_D(w, X).$$

Proof of Theorem 1(2). First we consider the case $n = 1$. Note that D is a bounded hyperconvex domain in $\mathbb{C}$ if and only if it is a Dirichlet domain. As we have already seen, if π is a universal covering of D such that $\pi(0) = w$ then, according to Azukawa [A2],

$$g_D(w, \pi(\lambda)) = \sum_j g_\Delta(t_j, \lambda) = g_\Delta(\pi^{-1}(w), \lambda) \text{ on } \Delta,$$

where $\pi^{-1}(w) = \{t_0 = 0, t_1, \dots\}$ and 0 is a zero of order 1 of $\pi - w$. We note $\pi'(0) = t$, and then $\lim_{\lambda\to 0}(g_D(w, \pi(\lambda)) - \log|\lambda|) = \log(A_D(w)|t|)$. On the other hand,

$$\begin{aligned}\lim_{\lambda\to 0}(g_\Delta(\pi^{-1}(w), \lambda) - \log|\lambda|) &= \lim_{\lambda\to 0}\sum_{j\geq 1} g_\Delta(t_j, \lambda)\\ &= \lim_{\lambda\to 0}\sum_{j\geq 1}\log\left|\frac{\lambda - t_j}{1-\bar{t}_j\lambda}\right|\\ &= \sum_{j\geq 1}\log|t_j|.\end{aligned}$$

Consequently,

$$A_D(w) = \Gamma_1(w) = \frac{\left|\prod_{\lambda\neq 0, \pi(\lambda)=w}\lambda\right|}{|\pi'(0)|}.$$

Thus, $\Gamma_p(w) = A_D(w)$ for all $p \geq 1$.

Now, if $n \geq 1$ then we fix a "direction" X in $\mathbb{C}^n$ such that $\|X\| = 1$. We know by definition that $\log A_D(w, X) = \limsup_{\lambda\to 0}(g_D(w, w+\lambda X) - \log|\lambda|)$. Hence there exist a sequence $(t_m)_m$ in $\mathbb{C}^*$ that converges to 0 and a sequence $(\varphi_m)_m$ in $\mathcal{O}(\bar{\Delta}, D)$ such that $\varphi_m(0) = w + t_m X$, $\varphi_m^{-1}(w) = \{\lambda \in \Delta : \varphi_m(\lambda) = w\} = \{\lambda_{jm} : 1 \leq j \leq N_m\} \neq \emptyset$ (the zeros of $\varphi_m - w$ are repeated if they are of order greater than 1) and that verify

$$A_D(w, X) = \lim_{m\to\infty}\exp(g_D(w, w + t_m X) - \log|t_m|) = \lim_{m\to\infty}\frac{\left|\prod_{\lambda\in\varphi_m^{-1}(w)}\lambda\right|}{|t_m|}.$$

For any $m \geq 1$,

$$\varphi_m(\lambda) = w + B_m(\lambda)\left(\frac{t_m}{B_m(0)}X + \psi_m(\lambda)\right) \text{ on } \Delta,$$

where $B_m(\lambda)$ is the Blaschke product $\prod_{j=1}^{N_m}(\lambda - \lambda_{jm})/(1 - \overline{\lambda_{jm}}\lambda)$ and ψ_m is a holomorphic mapping on Δ in $\mathbb{C}^n$ such that $\psi_m(0) = 0$. If we use $\tilde{\varphi}_m$ to denote the analytic disc defined on Δ by

$$\tilde{\varphi}_m(\lambda) = w + (B_m(\lambda) - B_m(0))\left(\frac{t_m}{B_m(0)}X + \psi_m(\lambda)\right),$$

then we have the sup norm on Δ of $\tilde{\varphi}_m - \varphi_m$, which verifies $\|\tilde{\varphi}_m - \varphi_m\|_\Delta \leq |B_m(0)|M$ (where M is a constant independent of m). By hypothesis, D is strictly hyperconvex. Thus there exist a bounded domain Ω and a function $\varrho \in \mathcal{C}(\Omega,]-\infty,1[) \cap \mathrm{PSH}(\Omega)$ such that $D = \{z \in \Omega : \varrho(z) < 0\}$, ϱ is exhaustive for Ω, and the open set $\{z \in \Omega : \varrho(z) < c\}$ is connected for all real numbers $c \in [0, 1]$. If we denote D_m the bounded hyperconvex domain defined by $\{z \in \Omega : \varrho(z) < 1/m\}$ for any integer $m \geq 1$, then $\tilde{\varphi}_m$ is an analytic disc in $D_{\nu(m)}$, where $(\nu(m))_m$ is a sequence of integers that tends to ∞ when m tends to ∞.

Even if it means changing (for any m) a zero of φ_m without changing all others (this is possible because $\varphi_m \in \mathcal{O}(\bar{\Delta}, D)$), we can suppose that $B_m'(0) \neq 0$. That is, 0 is a zero of order 1 of $\tilde{\varphi}_m$. In addition, for any m sufficiently large, $|B_m(0)| < 1$. Thus, according to Rouché's theorem, B_m and $B_m - B_m(0)$ have N_m zeros (counted with multiplicity) in Δ. We have

$$\{\lambda \in \Delta : \lambda \neq 0,\ \tilde{\varphi}_m(\lambda) = w\} = \{\lambda \in \Delta : \lambda \neq 0,\ B_m(\lambda) - B_m(0) = 0\}$$
$$= \{\tilde{\lambda}_{jm}, 1 \leq j \leq N_m - 1\}.$$

We can write

$$B_m(\lambda) - B_m(0) = \tilde{B}_m(\lambda) = \lambda \prod_{j=1}^{N_m-1} \frac{\lambda - \tilde{\lambda}_{jm}}{1 - \overline{\tilde{\lambda}_{jm}}\lambda} \beta_m(\lambda) \text{ on } \Delta,$$

where β_m is a holomorphic function on Δ, never equal to zero, such that $\beta_m(0) = c_m$ is a complex number not equal to zero. Since $\tilde{B}_m'(0) = \left(\prod_{j=1}^{N_m-1} -\tilde{\lambda}_{jm}\right) c_m$, it follows that

$$\tilde{\varphi}_m'(0) = \left(\prod_{j=1}^{N_m-1} -\tilde{\lambda}_{jm}\right) c_m \frac{t_m}{B_m(0)} X.$$

Note that $|\tilde{B}_m(\lambda)| \leq 1 + |B_m(0)|$ on Δ. According to the maximum principle, we may deduce that $|\beta_m(\lambda)| \leq 1 + |B_m(0)|$ on Δ. In addition, on $\partial\Delta$, $|\beta_m(\lambda)| \geq 1 - |B_m(0)|$. Because β_m is never equal to zero on $\bar{\Delta}$, we deduce that $1/\beta_m \in \mathcal{O}(\Delta)$ and, according to the maximum principle, $|\beta_m(\lambda)| \geq 1 - |B_m(0)|$ on Δ. Consequently, $1 - |B_m(0)| \leq |c_m| \leq 1 + |B_m(0)|$, and $|c_m|$ converges to 1 when m tends to ∞.

We deduce that, for m large enough,

$$A_{D_{\nu(m)}}(w, X) \leq \Gamma_{1, D_{\nu(m)}}(w, X) \leq \frac{|B_m(0)|}{|c_m t_m|}.$$

It is simple to verify that $(A_{D_{\nu(m)}}(w, X))_m$ converges to $A_D(w, X)$ when m tends to ∞ (see [N2]).

Now, we just need a last lemma to conclude the proof. This result takes its inspiration from a result [Yu] concerning $\tilde{\Gamma}_p(w, X)$ pseudometrics.

LEMMA 1.7. *With the previous notation, we have*

$$\lim_{m\to\infty} \Gamma_{1, D_{\nu(m)}}(w, X) = \Gamma_{1, D}(w, X).$$

Proof. Assume, by way of contradiction, that $\Gamma_{1,D_{\nu(m)}}(w,X)$ does not converge to $\Gamma_{1,D}(w,X)$. Then there exist $\varepsilon_0 > 0$ and a subsequence $(\mu(m))_m$ of the sequence $(\nu(m))_m$ such that

$$|\Gamma_{1,D_{\mu(m)}}(w,X) - \Gamma_{1,D}(w,X)| > \varepsilon_0.$$

By definition, for any $\eta \in]0,1[$, there exist $\theta_m \in \mathcal{O}(\Delta, D_{\mu(m)})$ such that $\theta_m(0) = w$, $\theta_m'(0) = t_m X$, $t_m > 0$, and

$$\Gamma_{1,D_{\mu(m)}}(w,X) + \eta \geq \frac{\prod_{\lambda\neq 0,\theta_m(\lambda)=w}|\lambda|}{t_m}.$$

Claim. Every subsequence of the $(\theta_m)_m$ has itself a subsequence converging to some element $\theta \in \mathcal{O}(\Delta, D)$ such that $\theta(0) = w$ and $\theta'(0) = tX$ for some $t > 0$. As a result of the claim, we will obtain

$$\liminf_{m\to\infty} \Gamma_{1,D_{\mu(m)}}(w,X) \geq \Gamma_{1,D}(w,X). \tag{$*$}$$

First we check the claim. By hypothesis, $\theta_m \in \mathcal{O}(\Delta, D_1)$ for all m. By the tautness of D_1, every subsequence of $(\theta_m)_m$ has itself a subsequence either converging to some element $\theta \in \mathcal{O}(\Delta, D_1)$ or compactly divergent. Since $\theta_m(0) = w \in D$, only the first possibility could occur. Moreover, because $\theta_m(\Delta) \subset D_{\mu(m)} \to D$ (in the sense that $\lim_{m\to\infty} \operatorname{dist}(\partial D_{\mu(m)}, \partial D) = 0$), it follows that $\theta(\Delta) \subset \bar{D}$. But D is taut and $\theta(0) = w \in D$, so we must have $\theta(\Delta) \subset D$. Indeed, if there exists a point $\lambda_0 \in \Delta$ such that $\theta(\lambda_0) \in \partial D$, then the set $\theta(\Delta)$ is contained in ∂D. This verifies the claim.

It follows from the claim that

$$\liminf_{m\to\infty} \Gamma_{1,D_{\mu(m)}}(w,X) + \eta \geq \frac{\prod_{\lambda\neq 0,\theta(\lambda)=w}|\lambda|}{t} \geq \Gamma_{1,D}(w,X).$$

When η tends to 0, this implies $(*)$.

Now we seek a contradiction. By the tautness of D, there exists an extremal disc $\theta \in \mathcal{O}(\Delta, D)$ for $\Gamma_{1,D}(w,X)$. Namely, $\theta(0) = w$, $\theta'(0) = tX$ with $t > 0$, and $\left(\prod_{\lambda\neq 0,\theta(\lambda)=w}|\lambda|\right)/t = \Gamma_{1,D}(w,X)$. Since $\theta \in \mathcal{O}(\Delta, D_{\mu(m)})$ for all m, it follows that $\Gamma_{1,D_{\mu(m)}}(w,X) \leq \left(\prod_{\lambda\neq 0,\theta(\lambda)=w}|\lambda|\right)/t$. Taking lim sup with respect to m, we obtain

$$\limsup_{m\to\infty} \Gamma_{1,D_{\mu(m)}}(w,X) \leq \Gamma_{1,D}(w,X). \tag{$**$}$$

Obviously $(*)$ together with $(**)$ contradict the first assumption. This finishes the proof of Lemma 1.7. □

Consequently, it follows from this last lemma that $A_D(w,X) = \Gamma_1(w,X) = \Gamma_p(w,X)$ for all $p \geq 1$. The proof is complete. □

2. Boundary Behavior of a Pluricomplex Green Function with Several Poles

Let D be a strictly lineally convex domain in $\mathbb{C}^n$, and let $g_D(w,\cdot)$ be the pluricomplex Green function on D with logarithmic pole at w. Lempert [L2] proved that also the sublevel sets $\{z \in D : g_D(w, z) < c\}$ of $g_D(w, \cdot)$ are strictly lineally convex for any nonpositive real number c. If D is a bounded convex domain, then the sublevel sets of $g_D(w, \cdot)$ are again convex for any $c \le 0$ (see [L1] and the appendix in [Mo]). In addition, if D is strictly convex with smooth boundary ($\mathcal{C}^\infty$ or real analytic), then $g_D(w, \cdot)$ has the same regularity on $D \setminus \{w\}$ as the boundary.

In fact, this situation is quite special. Indeed, Bedford and Demailly proved in [BD] that there exist strongly pseudoconvex domains D in $\mathbb{C}^2$ with $\mathcal{C}^2$ boundary such that $g_D(w, \cdot)$ is not $\mathcal{C}^2$ in all $\bar{D} \setminus \{w\}$. Here, we consider instead the situation where the domain D is convex or even strictly convex but where the pluricomplex Green function has several logarithmic poles in D. For this case, we will prove that pluricomplex Green functions with several poles have no more these regularity properties. In Section 2.1, we show that there exist convex domains in $\mathbb{C}^2$ (respectively in $\mathbb{C}^n$) with pluricomplex Green functions g with two (or more) logarithmic poles such that any connected sublevel set of g is not lineally convex. In Section 2.2, where we obtain explicitly the pluricomplex Green function $g_{B_2(0,1)}$ in $B_2(0, 1)$ with two logarithmic poles of weight 1, we prove again (now we are in the situation of a strictly convex domain) that any connected sublevel set of $g_{B_2(0,1)}$ is not lineally convex and that this function is not $\mathcal{C}^2$ on $B_2(0, 1) \setminus \{\text{poles}\}$.

Before starting, observe what happens in the 1-dimensional case. The complex Monge–Ampère operator is the same as the Laplace operator and hence is linear. For any bounded Dirichlet domain in $\mathbb{C}$, the Green function $g_D(P, \cdot)$, $P := \{(p_1, c_1), \dots, (p_m, c_m)\}$, is a set of m distinct poles p_j in D, and the c_j ($j = 1, \dots, m$) are strictly positive reals, verifies $g_D(P, z) = \sum_{j=1}^m c_j g_D(p_j, z)$ on D. The function $g_D(P, \cdot)$ is continuous on $\bar{D}$ and harmonic, then $\mathcal{C}^\infty$ on D. Thus, by regularity, if D is strictly convex then the subsevel sets of $g_D(P, \cdot)$ are again strictly convex for any negative real c near 0. On the other hand, if D is only convex (and not strictly convex), the subsevel sets of $g_D(P, \cdot)$ can not be convex for any real $c < 0$. To understand this easily it is sufficient to consider the different sublevel sets of the Green function on Δ with two logarithmic poles in β and $-\beta$, where β is a real number in $]0, 1[$. Note that any domain in $\mathbb{C}$ is obviously lineally convex, so in this case there is nothing to say about the lineal convexity of sublevel sets.

The situation is quite different in the multidimensional case because, in particular, the Monge–Ampère operator $(dd^c)^n$ is no longer linear [Le1; Le2]. For any domain D in $\mathbb{C}^n$, we have $\sum_{j=1}^m c_j g_D(p_j, z) \le g_D(P, z) \le \inf_j c_j g_D(p_j, z)$ on D. And these inequalities cannot be replaced by equalities if $m > 1$ and $n > 1$.

2.1. A Counterexample of a Recent Result

First we briefly recall some properties of Möbius transformations of $B_n(0, 1)$ and of pluricomplex Green functions with one logarithmic pole in $B_n(0, 1)$. Let $a \in B_n(0, 1) \setminus \{0\}$. Denote by P_a the orthogonal projection onto the subspace of $\mathbb{C}^n$ generated by the vector a. Then $P_a(z) = (\langle z, a\rangle / \langle a, a\rangle) a$, where $\langle \cdot, \cdot \rangle$ is the standard complex scalar product and $\|.\|$ stands the Euclidean norm in $\mathbb{C}^n$. Let $Q_a(z) = z - P_a(z)$ denote the projection onto the orthogonal complement of the subspace generated by a. The Möbius transformation associated with a is the mapping $T_a(z) = (a - P_a(z) - s_a Q_a(z))/(1 - \langle z, a\rangle)$, where $s_a = (1 - \|a\|^2)^{1/2}$ and $\langle z, a\rangle \neq 0$. Note that, by the Cauchy–Schwarz inequality, $|\langle z, a\rangle| < 1$ if $\|z\| \leq 1$. Observe that $T_a(a) = 0$ and $T_a(0) = a$. We also define T_0 as the identity mapping. The Möbius transformation T_a is a homeomorphism of $\bar{B}_n(0,1)$ onto $\bar{B}_n(0,1)$, and it maps $B_n(0,1)$ onto $B_n(0,1)$ biholomorphically. The inverse of $T_a|_{B_n(0,1)}$ is $T_a|_{B_n(0,1)}$ itself. According to the properties of the Möbius transformations and the fact that the pluricomplex Green function is invariant by biholomorphism, the explicit formula for the pluricomplex Green function with one logarithmic pole w with weight 1 in $B_n(0,1)$ is

$$g_{B_n(0,1)}(w, z) = \log\|T_w(z)\|.$$

Now we shall construct a pluricomplex Green function with two logarithmic poles in a complex convex ellipsoid in $\mathbb{C}^2$. Let F be the mapping in $\mathbb{C}^2$ defined by $F\colon \mathbb{C}^2 \to \mathbb{C}^2$, $(z_1, z_2) \mapsto (z_1^2, z_2)$. It is a polynomial holomorphic proper function in $\mathbb{C}^2$ such that $F(D) = B_2(0,1)$, where D is the complex convex ellipsoid defined by $D = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^4 + |z_2|^2 < 1\}$; D is convex, but not strictly convex.

LEMMA 2.1. *Let D and F be defined as before. Then we have*

$$g_{B_2(0,1)}(a; F(z)) = g_D(p, q; z) \ \text{ on } D,$$

where $a = (\beta^2, 0) \in B_2(0,1)$, $p = (\beta, 0) \in D$, $q = (-\beta, 0) \in D$ (with $\beta \in\,]0,1[$), and $g_D(p, q; \cdot)$ is the pluricomplex Green function on D with the two logarithmic poles p and q of weight 1.

Proof. We can prove this lemma by using the fact that F is proper and $F(D) = B_2(0,1)$. But here let us prove this lemma directly. First we verify that $T_a(z) = (1/(1 - \beta^2 z_1))\big(\beta^2 - z_1, -\sqrt{1-\beta^4}\,z_2\big)$. Let the function

$$u(z_1, z_2) = \log\|T_a(F(z_1, z_2))\|;$$

u is plurisubharmonic on D (because it is the composition of a plurisubharmonic function and of a holomorphic one), continuous on $\bar{D}$, strictly negative on D $(F(D) = B_2(0,1))$, tends to zero when we approach the boundary of D, and has two logarithmic poles with weight 1 in p and q. In addition, u is of class $\mathcal{C}^2$ on $D \setminus \{p, q\}$, since $v(z) = g_{B_2(0,1)}(a, z) = \log\|T_a(z_1, z_2)\|$ is of class $\mathcal{C}^2$ on $B_2(0,1) \setminus \{a\}$. Because v is maximal on $B_2(0,1) \setminus \{a\}$ and $u = v \circ F$, u is also maximal on $D \setminus \{p, q\}$. In fact,

$$\det\left[\frac{\partial^2(v \circ F)}{\partial z_j \partial \bar{z}_k}(z)\right] = |\det \partial_z F|^2 \det\left[\frac{\partial^2 v}{\partial z_j \partial \bar{z}_k}(F(z))\right]$$

and then, on $D \setminus \{p, q\}$, $(dd^c u)^2(z) = |\det \partial_z F|^2 (dd^c v)^2(F(z))$. Consequently, u is the unique solution of the Dirichlet problem defined in the introduction with $m = 2$ and $c_1 = c_2 = 1$—that is, the pluricomplex Green function with logarithmic poles p and q with weight 1 on the convex complex ellipsoid D. The proof is complete. □

Let $c \in]-\infty,0]$ and denote $D_c = \{z \in D : g_D(p,q;z) < c\}$ the sublevel set of the pluricomplex Green function $g_D(p,q;\cdot)$. We remark that this open set is "very symmetric": if $(z_1,z_2) \in D_c$, then also $(-z_1,z_2)$, $(z_1,-z_2)$, $(\bar z_1,z_2)$, and $(z_1,\bar z_2)$ are in D_c. Note that $z \in D_c$ if and only if $|\beta^2 - z_1^2|^2 + (1-\beta^4)|z_2|^2 < e^{2c}|1-\beta^2 z_1^2|^2$; D_c is connected if and only if $\log\beta^2 < c \le 0$. In the case where $c \le \log\beta^2$, D_c has two connected components.

PROPOSITION 2.2. *If $\log\beta^2 < c < 0$, then D_c is connected and not lineally convex.*

Proof. Let z be a point in ∂D_c such that $z_1 = 0$. Then $|z_2|^2 = (e^{2c}-\beta^4)/(1-\beta^4)$, where $e^{2c} > \beta^4$. We see if this is possible for a complex line l through the point z such that $l \subset \mathbb{C}^2 \setminus D_c$. Denote by $l = \{(0,z_2)+\lambda(w_1,w_2),\ \lambda \in \mathbb{C}\}$ a complex line in $\mathbb{C}^2$ with direction $(w_1,w_2) \neq (0,0)$ through the point z.

If $w_1 = 0$, then $w_2 \neq 0$ and we can choose $w_2 = 1$. We have that $(0,z_2) + \lambda(0,1) \in \mathbb{C}^2 \setminus D_c$ if and only if $|z_2+\lambda|^2 \ge |z_2|^2$. Of course, there exist complex numbers λ that do not verify this inequality. If $w_2 = 0$, then $w_1 \neq 0$ and we can choose $w_1 = 1$. Now $(0,z_2)+\lambda(1,0) \in \mathbb{C}^2 \setminus D_c$ if and only if

$$\left|\lambda^2 - \beta^2\frac{1-e^{2c}}{1-\beta^4 e^{2c}}\right| \ge \beta^2\frac{1-e^{2c}}{1-\beta^4 e^{2c}}.$$

Again, there exist complex numbers λ that do not verify this inequality.

If neither w_1 nor w_2 equals zero then we can choose $w_2 = 1$. In this case, $(0,z_2)+\lambda(w_1,1) \in \mathbb{C}^2 \setminus D_c$ if and only if

$$\begin{aligned} f(\lambda) = (1-\beta^4)(z_2\bar\lambda + \bar z_2\lambda) + (1-\beta^4)|\lambda|^2 - \beta^2(1-e^{2c})((\lambda w_1)^2 + (\bar\lambda\bar w_1)^2) \\ + (1-\beta^4 e^{2c})|w_1|^4|\lambda|^4 \ge 0. \end{aligned}$$

But for $\lambda = -z_2\varepsilon$ (with small $\varepsilon > 0$),

$$\begin{aligned} f(\lambda) &= -2(1-\beta^4)|z_2|^2\varepsilon + (1-\beta^4)|z_2|^2\varepsilon^2 \\ &\quad - \beta^2(1-e^{2c})((z_2w_1)^2 + (\bar z_2\bar w_1)^2)\varepsilon^2 + |w_1|^4|z_2|^4(1-\beta^4e^{2c})\varepsilon^4 < 0. \end{aligned}$$

Finally, for any complex line through z, $l \cap D_c$ contains points other than z and D_c is not lineally convex. □

Now our question is as follows: Is it possible to obtain the same result for a strictly convex domain D in $\mathbb{C}^2$? In Section 2.2 we answer in the affirmative by finding an explicit formula for the pluricomplex Green function with two logarithmic poles of weight 1 in the open unit ball in $\mathbb{C}^2$.

2.2. The Pluricomplex Green Function of the Unit Ball in $\mathbb{C}^2$ with Two Logarithmic Poles of Weight 1

In proving Theorem 5, we shall first express the pluricomplex Green function $g_{B_2(0,1)}(p,q;\cdot)$ with the help of a pluricomplex Green function with one pole in a well-chosen convex complex ellipsoid, similarly as in the proof of Lemma 2.1. We then apply Lempert's method in convex domains, which is based on a study of extremal discs for the Kobayashi metric, to express this pluricomplex Green function with one pole. This is possible because Jarnicki, Pflug, and Zeinstra [JPZ] gave a complete description of all geodesics of any convex complex ellipsoid in $\mathbb{C}^n$.

Let D be the following complex convex ellipsoid in $\mathbb{C}^2$: $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| + |z_2|^2 < 1\}$. Let F be the following polynomial holomorphic proper function in $\mathbb{C}^2$: $\mathbb{C}^2 \to \mathbb{C}^2$, $(z_1, z_2) \mapsto (z_1^2, z_2)$. Let $F(B_2(0,1)) = D$ and $F(bB_2(0,1)) = bD$; D is convex, but not strictly convex.

LEMMA 2.3. *Let D and F be defined as before. Then we have*

$$g_{B_2(0,1)}(p, q; z) = g_D(a; F(z)) \text{ on } B_2(0,1),$$

where $a = (\beta^2, 0) \in D$ and where both $p = (\beta, 0)$ and $q = (-\beta, 0)$ are in $B_2(0,1)$ with $\beta \in]0,1[$.

Proof. This proof is similar to the one of Lemma 2.1. The only thing to remark is that we have the following general property. *Let $f \in H(\Omega, \Omega')$, where Ω and Ω' are two open sets in $\mathbb{C}^n$, and let $u \in \mathrm{PSH}(\Omega') \cap L^\infty_{\mathrm{loc}}(\Omega')$. Then, if we note $v = u \circ f \in \mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$, we have*

$$(dd^c v)^n(z) = |\det \partial_z f|^2 (dd^c u)^2(f(z)).$$

First, it is easy to verify this formula when $u \in \mathbb{C}^2(\Omega')$. To prove this property in the general case it is sufficient to see that, for any $u \in \mathrm{PSH}(\Omega') \cap L^\infty_{\mathrm{loc}}(\Omega')$ and for any exhaustive sequence of relatively compact open sets Ω'_j in Ω', there exists a sequence $(u_j)_j$ of psh and $\mathcal{C}^\infty$ functions on Ω'_j respectively, such that $(u_j)_j$ decreases to u on Ω'. By Bedford–Taylor's convergence property of the Monge–Ampère operator, we deduce that $\lim_j (dd^c u_j)^2 = (dd^c u)^2$ on Ω', in the sense of weak*-convergence of currents of order zero. Since the formula is valid for any u_j, it is also valid for u. □

Recall that a holomorphic mapping $\varphi \colon \Delta \to D$ is a complex geodesic if $k_D(\varphi(\lambda), \varphi(\lambda')) = k_\Delta(\lambda, \lambda')$ for λ and $\lambda' \in \Delta$, where k_Δ is the Kobayashi distance. In [JPZ] are described all complex geodesics of any convex complex ellipsoids $\mathcal{E}(p) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^{2p_j} < 1\}$ $\left(\text{with } p_1, \dots, p_n \geq \frac{1}{2}\right)$ in $\mathbb{C}^n$. Here we are in $\mathbb{C}^2$, $p_1 = 1/2$, and $p_2 = 1$. Let (φ_1, φ_2) be a complex geodesic of $D = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| + |z_2|^2 < 1\}$. If $\varphi_1 \equiv 0$ (resp. $\varphi_2 \equiv 0$), then φ_2 (resp. φ_1) is an automorphism of the unit disc Δ. This means that, without loss of generality, we can assume that φ_1 and φ_2 are not identically zero on Δ. Then any complex geodesic $\varphi = (\varphi_1, \varphi_2)$ of D has the following form:

$$\varphi_j(\lambda) = \begin{cases} a_j \dfrac{\lambda - \alpha_j}{1 - \bar{\alpha}_j \lambda} \left(\dfrac{1 - \bar{\alpha}_j \lambda}{1 - \bar{\alpha}_0 \lambda} \right)^{1/p_j} & \text{if the } \varphi_j \text{ have a zero in } \Delta, \\ a_j \left(\dfrac{1 - \bar{\alpha}_j \lambda}{1 - \bar{\alpha}_0 \lambda} \right)^{1/p_j} & \text{if the } \varphi_j \text{ have no zero in } \Delta, \end{cases} \tag{1}$$

where $a_1, a_2 \in \mathbb{C} \setminus \{0\}$, $\alpha_0 \in \Delta$, $\alpha_j \in \Delta$ if the φ_j have a zero in Δ, $\alpha_j \in \bar{\Delta}$ if the φ_j have no zero in Δ, $\alpha_0 = \sum_{j=1}^2 |a_j|^{2p_j} \alpha_j$, and $1 + |\alpha_0|^2 = \sum_{j=1}^2 |a_j|^{2p_j}(1 + |\alpha_j|^2)$.

The complex geodesics in D are uniquely determined mod $\mathrm{Aut}(\Delta)$. Let us remark, in addition, that any complex geodesic of D extends holomorphically to a neighborhood of $\bar{\Delta}$. We now recall Lempert's theorem.

THEOREM 2.4 [L1]. *Let D be a strictly convex, analytically bounded domain in $\mathbb{C}^n$, and let w be a point in D. For every $z \in D \setminus \{w\}$, there is a holomorphic imbedding $\varphi_z = \varphi \colon \bar{\Delta} \to \bar{D}$ such that $z, w \in \varphi(\bar{\Delta})$ and $g_D(w, \cdot)$ is harmonic restricted to $\varphi(\Delta) \setminus \{w\}$. This φ is unique up to automorphisms of Δ. Consequently, the corresponding analytic discs $\varphi_z(\bar{\Delta})$ yield a foliation of $\bar{D}$ that is singular at w.*

For the same reasons described after Theorem 1.2, this theorem is again true for any convex domain in $\mathbb{C}^n$. In particular, Theorem 2.4 states that $g_D(w, z) = \inf\{\log|\lambda| : \varphi \in H(\Delta, D),\ \varphi(0) = z,\ \varphi(\lambda) = w\}$, where the infimum is a minimum and is achieved for a complex geodesic of D.

To prove Theorem 5 we need the following theorem, which expresses the pluricomplex Green function on D with one logarithmic pole of weight 1. Hereafter, when we write $\lambda^{1/2}$ for $\lambda \in \mathbb{C} \setminus \{0\}$ we mean t such that $t^2 = \lambda$ and $\mathrm{Arg}(t) \in [-\pi/2, \pi/2[$ ($0^{1/2} = 0$).

THEOREM 2.5. *Let $\beta \in]0, 1[$. Let $a = (\beta^2, 0) \in D$ and let $p, q \in B_2(0, 1)$, where $p = (\beta, 0)$ and $q = (-\beta, 0)$. Then the following formula holds for the pluricomplex Green function on D with the logarithmic pole a of weight 1.*

$$g_D(a, w) = \begin{cases} \frac{1}{2}\log\left(\frac{|\beta - w_1^{1/2}|^2 + (1-\beta^2)|w_2|^2}{|1-\beta w_1^{1/2}|^2}\right) & \text{if } (w_1^{1/2}, w_2) \in \bar{\mathcal{C}}_p \cap D, \\ \frac{1}{2}\log\left(\frac{|\beta^2 - w_1|^2 + \beta^4|w_2|^4 + 2(1-\beta^4)|w_2|^2 + \sqrt{\Delta(w)}}{2|1-\beta^2 w_1|^2}\right) & \text{if } (w_1^{1/2}, w_2) \in D \setminus \mathcal{C}_p, \end{cases}$$

where

$$\Delta(w) = (\beta^4|w_2|^4 - |\beta^2 - w_1|^2)^2 + 4(1-\beta^4)|w_2|^2|\beta^2|w_2|^2 - (\beta^2 - w_1)|^2.$$

Proof. Let $w = (w_1, w_2) \in D$ be such that $w \neq a$. Because of (1), any complex geodesic $\varphi = (\varphi_1, \varphi_2)$ passing through a and w is such that either $\varphi^{-1}(0) = \emptyset$ or $\varphi^{-1}(0)$ contains exactly one point. Our aim is to decide for which pairs of points the complex geodesic joining these points is of the first type and for which it is of the second type.

We shall first compute $g_D(a, w)$ for all the points $w \in D$ such that $w_1 = 0$ or $w_2 = 0$. If $w_2 \in]0, 1[$, then the unique geodesic $\varphi = (\varphi_1, \varphi_2)$ of D that passes through both points a and $(0, w_2)$ and that verifies $\varphi(0) = (0, w_2)$ is defined on Δ by

$$\varphi_1(\lambda) = \frac{a_1\lambda}{(1-\bar{\alpha}_0\lambda)^2} \quad \text{and} \quad \varphi_2(\lambda) = a_2\frac{\lambda - \alpha_2}{1-\bar{\alpha}_0\lambda}, \tag{†}$$

where $\alpha_0 = |a_2|^2\alpha_2$, $1 + |\alpha_0|^2 = |a_1| + |a_2|^2(1 + |\alpha_2|^2)$, $-a_2\alpha_2 = w_2$, and $a_1\alpha_2/(1 - \bar{\alpha}_0\alpha_2)^2 = \beta^2$. Of course, $\alpha_2 \neq 0$. Then we have $a_2 = -w_2/\alpha_2$,

$\alpha_0 = w_2^2/\bar{\alpha}_2$, $a_1 = \beta^2(1-w_2^2)^2/\alpha_2$, and $|\alpha_2|$ verifies the following quadratic equation: $|\alpha_2|^2 - |\alpha_2|\beta^2(1-w_2^2) - w_2^2 = 0$. The unique solution for $|\alpha_2|$ is $\frac{1}{2}\big[\beta^2(1-w_2^2) + \sqrt{\beta^4(1-w_2^2)^2 + 4w_2^2}\big]$. If $w = (0, w_2) \in D$ with $w_2 \neq 0$, then $g_D(a, (0, w_2)) = g_D(a, (0, |w_2|))$, because of the invariance of $g_D(a, \cdot)$ under the mapping $(z_1, z_2) \mapsto (z_1, e^{i\theta}z_2)$ of $\mathbb{C}^2$, where θ is any real number. If $w_2 = 0$, then the unique geodesic φ of D which passes through points a and 0 and that verifies $\varphi(0) = 0$ is defined on Δ by $\varphi_1(\lambda) = \lambda$ and $\varphi_2(\lambda) = 0$. Thus, consequently, for all $w_2 \in \Delta$,

$$g_D(a, (0, w_2)) = \log\Big(\frac{1}{2}\big[\beta^2(1-|w_2|^2) + \sqrt{\beta^4(1-|w_2|^2)^2 + 4|w_2|^2}\big]\Big).$$

If $w_1 \in \Delta$ then, according to (1), the unique geodesic φ (mod Aut(Δ)) of D passing through a and $(w_1, 0)$ and such that $\varphi(0) = (w_1, 0)$ is defined on Δ by $\varphi_1(\lambda) = (\lambda + w_1)/(1 + \bar{w}_1\lambda)$ and $\varphi_2(\lambda) = 0$. Hence, for all $w_1 \in \Delta$, $g_D(a, (w_1, 0)) = \log(|\beta^2 - w_1|/|1 - \beta^2 w_1|)$.

Now observe what happens if $w = (w_1, w_2) \in D$ with $w_1 \neq 0$ and $w_2 \neq 0$. We can suppose that $w_2 > 0$ for the same reason as before: the invariance of $g_D(a, \cdot)$ under the mapping $(z_1, z_2) \mapsto (z_1, e^{i\theta}z_2)$ of $\mathbb{C}^2$, where θ is any real number. Suppose that $\varphi_1^{-1}(0) = \emptyset$. Then φ verifies on Δ

$$\varphi_1(\lambda) = a_1\left(\frac{1-\bar{\alpha}_1\lambda}{1-\bar{\alpha}_0\lambda}\right)^2 \quad \text{and} \quad \varphi_2(\lambda) = a_2\left(\frac{\lambda-\alpha_2}{1-\bar{\alpha}_0\lambda}\right),$$

where $\alpha_0 = |a_1|\alpha_1 + |a_2|^2\alpha_2$, $1 + |\alpha_0|^2 = |a_1|(1+|\alpha_1|^2) + |a_2|^2(1+|\alpha_2|^2)$, and $\varphi(0) = w$; that is, $a_1 = w_1$, $-a_2\alpha_2 = w_2$, and $\varphi(\alpha_2) = a$. In other words,

$$a_1\left(\frac{1-\bar{\alpha}_1\alpha_2}{1-\bar{\alpha}_0\alpha_2}\right)^2 = \beta^2.$$

We have immediately that

$$w_1^{1/2}\left(\frac{1-\bar{\alpha}_1\alpha_2}{1-\bar{\alpha}_0\alpha_2}\right) = +\beta \text{ or } -\beta.$$

After some calculations, we obtain in the first case

$$\alpha_1 = \frac{\beta(1-w_2^2) - \bar{w}_1^{1/2}}{\bar{\alpha}_2\bar{w}_1^{1/2}(-1+\beta w_1^{1/2})}, \qquad \alpha_0 = \frac{-\beta w_1^{1/2} + |w_1| + w_2^2}{\bar{\alpha}_2(1-\beta w_1^{1/2})},$$
$$|\alpha_2|^2 = \frac{|\beta - w_1^{1/2}|^2 + (1-\beta^2)w_2^2}{|1-\beta w_1^{1/2}|^2};$$

in the second case,

$$\alpha_1 = \frac{\beta(1-w_2^2) + \bar{w}_1^{1/2}}{\bar{\alpha}_2\bar{w}_1^{1/2}(1+\beta w_1^{1/2})}, \qquad \alpha_0 = \frac{\beta w_1^{1/2} + |w_1| + w_2^2}{\bar{\alpha}_2(1+\beta w_1^{1/2})},$$
$$|\alpha_2|^2 = \frac{|\beta + w_1^{1/2}|^2 + (1-\beta^2)w_2^2}{|1+\beta w_1^{1/2}|^2}.$$

We now use $\tilde{\varphi}$ to denote the complex geodesic of $B_2(0,1)$ defined on Δ by

$$\tilde{\varphi}_1(\lambda) = w_1^{1/2}\left(\frac{1-\bar{\alpha}_1\lambda}{1-\bar{\alpha}_0\lambda}\right) \quad \text{and} \quad \tilde{\varphi}_2(\lambda) = a_2\left(\frac{\lambda-\alpha_2}{1-\bar{\alpha}_0\lambda}\right).$$

In the first case, $\tilde{\varphi}(\Delta)$ is a part of the complex line of the equation $(Z_1-\beta)w_2 = Z_2(w_1^{1/2}-\beta)$, passing through p; in the second case, $\tilde{\varphi}(\Delta)$ is a part of the complex line of the equation $(Z_1+\beta)w_2 = Z_2(w_1^{1/2}+\beta)$, passing through q. By hypothesis, $\varphi_1^{-1}(0) = \emptyset$ and $\operatorname{Arg}(w_1^{1/2}) \in [-\pi/2, \pi/2[$, so the first case is the only one that can occur and necessarily $(w_1^{1/2}, w_2) \in \bar{\mathcal{C}}_p \cap D$. Conversely, if $(w_1^{1/2}, w_2) \in \bar{\mathcal{C}}_p \cap D$ then there exists a complex geodesic $\tilde{\varphi}$ of $B_2(0,1)$ that passes through $(w_1^{1/2}, w_2)$ and p. In fact, it is a part of a complex line. Then $\varphi = (\tilde{\varphi}_1^2, \tilde{\varphi}_2)$ is a complex geodesic of D that passes through w and a; it is such that $\varphi_1^{-1}(0) = \emptyset$. Consequently, for all $w = (w_1, w_2) \in D$ such that $(w_1^{1/2}, w_2) \in \bar{\mathcal{C}}_p \cap D$,

$$g_D(a,w) = \frac{1}{2}\log\left(\frac{|\beta - w_1^{1/2}|^2 + (1-\beta^2)|w_2|^2}{|1-\beta w_1^{1/2}|^2}\right).$$

Finally, let $w \in D$ be such that $(w_1^{1/2}, w_2) \in D \setminus \mathcal{C}_p$. Without restriction, we can suppose that $w_2 > 0$. Then, according to what is previous, the unique geodesic $\tilde{\varphi}$ (mod $\operatorname{Aut}(\Delta)$) of D passing through w and a is such that $\tilde{\varphi}_1^{-1}(0)$ contains exactly one element. In addition, there exists a unique point $w' = (0, w_2') \in D \cap \tilde{\varphi}(\Delta)$ and a unique geodesic φ of D of type (†) such that $\varphi(0) = w'$ and $\varphi(\alpha_2) = a$. We can suppose without restriction (with the help of well-chosen rotation) that $\alpha_2 > 0$. Now φ verifies on Δ

$$\varphi_1(\lambda) = \frac{a_1\lambda}{(1-\bar{\alpha}_0\lambda)^2} \quad \text{and} \quad \varphi_2(\lambda) = a_2\frac{\lambda-\alpha_2}{1-\bar{\alpha}_0\lambda},$$

where $\alpha_2 = \frac{1}{2}\left[\beta^2(1-|w_2'|^2)+\sqrt{\beta^4(1-|w_2'|^2)^2+4|w_2'|^2}\right]$, $a_1 = \beta^2(1-w_2'^2)^2/\alpha_2$, $a_2 = -w_2'/\alpha_2$, and $\alpha_0 = w_2'^2/\bar{\alpha}_2$. We remark that $\varphi(\Delta) \subset \{z = (z_1, z_2) \in D : w_2' z_1 = \beta^2(w_2' - z_2)(1-\bar{w}_2' z_2)\}$. Now $w_2' \in \Delta$ and verifies $w_2' w_1 = \beta^2(w_2' - w_2)(1-\bar{w}_2' w_2)$. This permits us to express w_2' with w_1 and w_2:

$$w_2' = \frac{1-\sqrt{1-4|\delta(w)|^2}}{2|\delta(w)|^2}\delta(w), \quad \text{where } \delta(w) = \frac{\beta^2 w_2(\beta^2 w_2^2 - \beta^2 + \bar{w}_1)}{\beta^4 w_2^4 - |\beta^2 - w_1|^2}.$$

Let $\lambda_0 \in \Delta$ be such that $\varphi(\lambda_0) = w$. From the equation $\varphi_2(\lambda_0) = w_2$, it is easy to obtain $\lambda_0 = (w_2 + a_2\alpha_2)/(a_2 + \bar{\alpha}_0 w_2)$. Since a_2, α_0, α_2 are expressed according to w', λ_0 also can be expressed according to w' and thence according to w. The relation between φ and $\tilde{\varphi}$ is $\tilde{\varphi} = \varphi \circ \theta$, where θ is the automorphism of Δ defined by $\theta(\lambda) = (\lambda + \lambda_0)/(1+\bar{\lambda}_0\lambda)$. Finally, $\tilde{\varphi}(\theta^{-1}(\alpha_2)) = a$,

$$g_D(a,w) = \log|\theta^{-1}(\alpha_2)| = \log\left|\frac{\alpha_2 - \lambda_0}{1-\bar{\lambda}_0\alpha_2}\right|,$$

and, for all $w \in D$ such that $(w_1^{1/2}, w_2) \in D \setminus \mathcal{C}_p$, we have

$$g_D(a,w)=\frac{1}{2}\log\left(\frac{|\beta^2-w_1|^2+\beta^4|w_2|^4+2(1-\beta^4)|w_2|^2+\sqrt{\Delta(w)}}{2|1-\beta^2w_1|^2}\right),$$

where

$$\Delta(w)=(\beta^4|w_2|^4-|\beta^2-w_1|^2)^2+4(1-\beta^4)|w_2|^2|\beta^2|w_2|^2-(\beta^2-w_1)|^2.$$

The proof is complete. □

Proof of Theorem 5. The theorem is a direct consequence of Lemma 2.3 and Theorem 2.5. Let us remark that, if $z\in\bar{\mathcal{C}}_p\cap B_2(0,1)$, then $\mathrm{Arg}(z_1)\in[-\pi/2,\pi/2[$ and $(z_1^2)^{1/2}=z_1$. Moreover, if $z\in\bar{\mathcal{C}}_q\cap B_2(0,1)$, then $\mathrm{Arg}(z_1)\in[-\pi,-\pi/2[\,\cup\,[\pi/2,\pi[$ and $(z_1^2)^{1/2}=-z_1$. □

Proof of Proposition 6. This proof is very similar to that for Proposition 2.2. The calculations are only a little bit more off-putting. Let z be a point in ∂B_c such that $z_1=0$ and $z_2>0$. Then $2e^c=\beta^2(1-z_2^2)+\sqrt{\beta^4(1-z_2^2)^2+4z_2^2}$, where $2e^c>2\beta^2$. We check for the possibility of a complex line l through the point z such that $l\subset\mathbb{C}^2\setminus B_c$. Denote by $l=\{(0,z_2)+\lambda(w_1,w_2),\ \lambda\in\mathbb{C}\}$ a complex line in $\mathbb{C}^2$ with direction $(w_1,w_2)\neq(0,0)$ through the point z. Then

$$g_{B_2(0,1)}(p,q,(\lambda w_1,z_2+\lambda w_2))=\tfrac{1}{2}\log f_1(\lambda),$$

where $f_1(\lambda)=g_1(\lambda)/(2|1-\beta^2w_1^2\lambda^2|^2)$ and

$$\begin{aligned}g_1(\lambda)=&\,|\beta^2-w_1^2\lambda^2|^2+\beta^4|z_2+w_2\lambda|^4+2(1-\beta^4)|z_2+w_2\lambda|^2\\&+\sqrt{\Delta(w_1^2\lambda^2,z_2+w_2\lambda)}\end{aligned}$$

if $\lambda\in\mathbb{C}$ is near 0.

If $w_2\neq0$, then we can choose $w_2=1$. We obtain that

$$f_1(\lambda)=f_1(0)+\tau_1(\lambda+\bar{\lambda})+\phi(|\lambda|),$$

where $\phi(|\lambda|)/|\lambda|$ tends to 0 when λ tends to 0 and

$$\tau_1=z_2(1-\beta^4(1-z_2^2))+\frac{\beta^2z_2(-\beta^4(1-z_2^2)^2+1-3z_2^2)}{\sqrt{\beta^4(1-z_2^2)^2+4z_2^2}}.$$

We remark that $\tau_1\neq0$ because $\beta\in]0,1[$. Since $f_1(0)=e^{2c}$, there exists $\lambda\in\mathbb{R}$ as near to zero as we want and such that $f_1(\lambda)<e^{2c}$; that means that $g_{B_2(0,1)}(p,q,(0,z_2+\lambda))<c$ for these λ and that $l\cap B_c\neq\emptyset$.

If $w_2=0$, then $w_1\neq0$ and we can choose $w_1=1$. We obtain that $f_1(\lambda)=f_1(0)+\tau_2(\lambda^2+\bar{\lambda}^2)+\phi(|\lambda|^2)$, where $\phi(|\lambda|^2)/|\lambda|^2$ tends to 0 when λ tends to 0 and

$$\begin{aligned}\tau_2=\frac{\beta^2}{2}\Bigg[&\beta^4(1-z_2^2)^2+2z_2^2-1\\&+\frac{(z_2^2-1)[-\beta^8(1-z_2^2)^3+\beta^4(1-z_2^2)(1-4z_2^2)+2z_2^2]}{\sqrt{\Delta(0,z_2)}}\Bigg].\end{aligned}$$

In fact, $\tau_2 = (\beta^2/4\sqrt{\Delta(0,z_2)}(\tau_3^2-1)$, where $\tau_3 = \beta^4(1-z_2^2)^2 + 2z_2^2 - 1 + \sqrt{\Delta(0,z_2)}$. Clearly, $\tau_3 + 1 > 0$; moreover, $\tau_3 < 1$. Indeed, this is equivalent to $\beta^4(1-z_2^2)^2 + \sqrt{\Delta(0,z_2)} < 2(1-z_2^2)$; after a simplification of both sides with $(1-z_2^2)$, this becomes $\beta^4(1-z_2^2) + \beta^2(\beta^4(1-z_2^2)^2 + 4z_2^2)^{1/2} < 2$. Note that, by the choice of z_2 in the beginning of the proof, the left-hand side of the last inequality is just $2\beta^2 e^c < 2$. Hence $\tau_2 < 0$, and one chooses $\lambda \in \mathbb{R}$ small to get $f_1(\lambda) < e^{2c}$; that means that $g_{B_2(0,1)}(p,q,(\lambda,z_2)) < c$ for these λ and that $l \cap B_c \neq \emptyset$.

Finally, for any complex line through z, $l \cap B_c$ contains others points than z, and B_c is not lineally convex. □

To conclude, we make a remark on the regularity of pluricomplex Green functions with one or several poles. Coman proved [Co] that $g_{B_2(0,1)}(p,q,\cdot)$ (p and q always are equal to $(\beta,0)$ and $(-\beta,0)$, resp., where $\beta \in]0,1[$) is: (a) of class $\mathcal{C}^{1,1}$ on $B_2(0,1) \setminus \{p,q\}$; (b) real analytic on $B_2(0,1) \setminus ((\bar{\mathcal{C}}_p - \mathcal{C}_p) \cup (\bar{\mathcal{C}}_q - \mathcal{C}_q))$; but (c) is not of class $\mathcal{C}^2$ on $B_2(0,1) \setminus \{p,q\}$.

If $D = \{(z_1,z_2) \in \mathbb{C}^2 : |z_1|^4 + |z_2|^2 < 1\}$, we have proved that $g_D(p,q,\cdot)$ is real analytic on $D \setminus \{p,q\}$. Note that D is convex, but not strictly convex like $B_2(0,1)$. Consequently, there is no relationship between the regularity of the boundary of the domain and the regularity of the pluricomplex Green function with two poles in this domain.

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