Michigan Math. J. 48 (2000)

Height Formulas for Homogeneous Varieties

Harry Tamvakis

Dedicated to my teacher, William Fulton

In this paper we use classical Schubert calculus to evaluate the integral formula of Kaiser and Köhler [KK] for the Faltings height of certain homogeneous varieties in terms of combinatorial data, and we verify their conjecture for the size of the denominators.

1. Introduction

Consider a system of diophantine equations with integral coefficients which defines an arithmetic variety X in projective space $\mathbb{P}^n_{\mathbb{Z}}$. The Faltings height $h(X)$ of X is a measure of the arithmetic complexity of the system; it is an arithmetic analog of the geometric notion of the degree of a projective variety. The height $h(X)$ generalizes the classical height of a rational point of projective space, used by Siegel [S], Northcott [N] and Weil [W] to study questions of diophantine approximation. Faltings [F] defined $h(X)$ using the arithmetic intersection theory of Gillet and Soulé [GS2]; if $\overline{\mathcal{O}}(1)$ denotes the canonical hermitian line bundle on $\mathbb{P}^n$, then the height

$$h(X) = h_{\overline{\mathcal{O}}(1)}(X) = \widehat{\deg}(\hat{c}_1(\overline{\mathcal{O}}(1))^{\dim(X)} \mid X)$$

is the arithmetic degree of $X \subset \mathbb{P}^n$ with respect to $\overline{\mathcal{O}}(1)$. More generally, one has a notion of height of algebraic cycles with respect to hermitian line bundles; see [BGS, Sec. 3]. Our interest here is in explicit computations for these heights when $X = G/P$ is a homogeneous space of a Chevalley group G.

There are several alternative ways to identify the Faltings height $h(X)$. Although not as intrinsic as the above definition, they involve a more direct use of the equations in the system defining X. The approach by Philippon [Ph] uses an "alternative Mahler measure" of the Chow form of X. When X is a hypersurface defined by a homogeneous polynomial $f \in \mathbb{Z}[z_0, \dots, z_n]$, this gives

$$h(X) = \deg(f)h(\mathbb{P}^n) + \int_{S^{2n+1}} \log|f(z)|\,d\sigma, \tag{1}$$

where $d\sigma$ denotes the $U(n+1)$-invariant probability measure on the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$; the Faltings height of projective space is given by

$$h(\mathbb{P}^n) = \frac{1}{2}\sum_{k=1}^{n} \mathcal{H}_k \tag{2}$$

(see also [BGS, Sec. 3.3.1]). Here $\mathcal{H}_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$ is a *harmonic number.*

Received January 4, 2000. Revision received April 13, 2000.

The author was supported in part by a National Science Foundation postdoctoral research fellowship.

Cassaigne and Maillot [CM] used (1) to compute the height of certain toric hypersurfaces and even-dimensional quadrics. Maillot [Ma] later computed arithmetic intersections in the Arakelov Chow ring of general SL_N-Grassmannians $G(m,n)$ and arrived at an algorithm for calculating the Faltings height of $G(m,n)$ under its Plücker embedding. In [T1]–[T4] the author used *arithmetic Schubert calculus* to obtain simple formulas for the heights of SL_N- and Lagrangian Grassmannians as well as an algorithm to compute the height of flag varieties SL_N/P with respect to their natural geometric embeddings in projective space.

A third general approach to computing the height $h(X)$ is via an arithmetic analog of the classical Hilbert–Samuel formula. The latter identifies the degree of $X(\mathbb{C})$ with respect to an ample line bundle $L(\mathbb{C})$ in the leading term of the Hilbert polynomial of L. The *arithmetic Hilbert–Samuel formula* states that

$$\widehat{\deg}(H^0(X, L^{\otimes n}), \|\cdot\|_2) = \frac{n^{d+1}}{(d+1)!} h(X) + O(n^d \log n). \tag{3}$$

Here $L = \mathcal{O}(1)\big|_X$ is the very ample line bundle inducing the projective embedding of X, and $d = \dim_{\mathbb{C}} X(\mathbb{C})$ is the dimension of X relative to $\operatorname{Spec}\mathbb{Z}$. The left-hand side of (3) is defined as follows: for each $n \geq 0$ the lattice $V = H^0(X, L^{\otimes n})$ is a torsion-free abelian group. Choose a Kähler metric on $X(\mathbb{C})$ with volume form dx, and equip $L(\mathbb{C})$ with its standard hermitian metric and the real vector space $V_{\mathbb{R}} = V \otimes_{\mathbb{Z}} \mathbb{R}$ with the L_2 norm $\|s\|^2 = \int_{X(\mathbb{C})} |s(x)|^2\, dx$. If we provide $V_{\mathbb{R}}$ with the Haar measure that gives volume 1 to the unit ball, then $-\widehat{\deg}(H^0(X, L^{\otimes n}), \|\cdot\|_2)$ is the logarithm of the covolume (that is, the measure of a fundamental domain) of the lattice V in $V_{\mathbb{R}}$. The asymptotic formula (3) was first shown by Gillet and Soulé [GS1] using, among other things, a weak form of their arithmetic Riemann–Roch theorem; Abbès and Bouche [AB] later gave a simpler direct proof.

Recently, Kaiser and Köhler [KK] used (3) to produce a formula for the height of generalized flag varieties X with respect to natural very ample hermitian line bundles. They compute the covolume on the left-hand side of (3) by using the Jantzen sum formula [J, Sec. 8.16] for integral representations of Chevalley schemes over $\mathbb{Z}$, which is identified in [KK] with an analog of the Weyl character formula in Arakelov geometry. The asymptotics as $n \to +\infty$ are evaluated by applying the Riemann–Roch theorem, and the result is a fascinating integral formula for the height $h(X)$.

To describe their formula, let G be a semisimple Chevalley group over $\operatorname{Spec}\mathbb{Z}$, let $T \subset G$ be a maximal split torus with set of roots R, and fix an ordering $R = R^+ \cup R^-$ with basis Δ. Parabolic subgroups of G correspond to subsets $I \subset \Delta$; for each such I let $X = G/P$ denote the smooth projective scheme over $\mathbb{Z}$ that represents the fpqc- or étale-sheafification of the functor $S \mapsto G(S)/P(S)$ for any parabolic $P \subset G$ of type I (see [DG, XXVI, Sec. 3.3] and [KK, Sec. 2]).

Let $\mathfrak{g}$ and $\mathfrak{t}$ be the Lie algebras of G and T, respectively, and consider a standard parabolic subgroup P containing T whose Lie algebra $\mathfrak{p}$ decomposes into root spaces of G:

$$\mathfrak{p} = \mathfrak{t} + \sum_{\alpha \in R_P} \mathfrak{g}_\alpha$$

for some R_P with $R^- \subset R_P \subset R$. Following Snow [Sn], we define the set of *roots of* X by $R_X = R \setminus R_P$. For any weight λ and $\alpha \in R$, set $\langle \lambda, \alpha \rangle = 2(\lambda, \alpha)/(\alpha, \alpha)$, where $(\cdot, \cdot)$ is the pairing induced by the Killing form on $\mathfrak{g}$. The very ample line bundles L_λ on G/P are given by P-representations with weights λ such that $\langle \lambda, \alpha \rangle = 0$ if $\alpha \in R^+ \setminus R_X$ and $\langle \lambda, \alpha \rangle > 0$ for $\alpha \in R_X$. The bundle L_λ comes with an equivariant hermitian metric, which is normalized by setting the length of the generator of the corresponding P-module equal to 1.

THEOREM 1 [KK]. *The height of $X = G/P$ with respect to the hermitian line bundle $\overline{L}_\lambda$ is given by*

$$h_{\overline{L}_\lambda}(X) = \frac{1}{2} \sum_{k=0}^{d} \frac{(-1)^k}{k+1} \binom{d+1}{k+1} \sum_{j>0} j^{k+1} \int_X p_k(E_j) c_1(L_\lambda)^{d-k}. \tag{4}$$

Here E_j is the homogeneous vector bundle over X associated to the virtual P-representation with character

$$\chi_j = \sum_{\alpha : \langle \lambda, \alpha \rangle = j} e^{2\pi i \alpha}, \tag{5}$$

and $p_k(E)$ is the kth *power sum* of E, that is, the characteristic class associated to the symmetric function $p_k(x) = \sum_i x_i^k$.

One of the merits of (4) is that it is a purely cohomological formula, whereas general arithmetic intersections on flag varieties involve nonclosed currents (see [T2]). One may readily evaluate (4) using standard localization techniques, as in [KK, Sec. 8]; the resulting explicit but rather complicated expressions give rational numbers for the height. An interesting feature of the formulas in [KK] is that the size of the denominators seems larger than expected. More precisely, let $m(G)$ be the largest exponent of G; note that $c(G) = m(G) + 1$ is the Coxeter number of G (see e.g. [OV, p. 289]). It is shown in [KK] that the largest prime power occurring in the denominator of $2h_{\overline{L}_\lambda}(G/P)$ is no greater than $2m(G)$. Based on computer calculations and the results of [T3; T4], Kaiser and Köhler formulate the following.

CONJECTURE 1. *The height $h_{\overline{L}_\lambda}(G/P)$ is a number in $\frac{1}{2}\sum_{k=1}^{m(G)}\left(\frac{1}{k}\mathbb{Z}\right)$.*

This paper grew out of the author's attempts to understand (4) and compare it with the formulas in [T3] and [T4]. We use Schubert calculus to evaluate the integrals in (4) directly in several examples that include some of those studied in [T2]–[T4]. Specifically, we consider the complete flag variety (Section 2) and Grassmannian (Section 3) for SL_N, as well as the Grassmannians parametrizing maximal isotropic subspaces in the symplectic and even orthogonal cases (Section 4). This leads to formulas for the height similar to the ones in [T3] and [T4] but which are qualitatively quite different, as they come from classical rather than arithmetic Schubert calculus. It turns out that the formulas derived from (4) giving the heights of the Lagrangian and even orthogonal Grassmannians are very similar. We combine them with the height calculation in [T4] and arrive at an analog of [T4, Thm. 3] in the orthogonal case (Theorem 6 of the present paper).

We are able to prove that Conjecture 1 holds in all these examples. In the SL_N case we do this directly, without using the results of [T2; T3]. Our explanation for the cancellation of the denominators is surprisingly subtle; we could not show this without using techniques from classical Schubert calculus and combinatorics of symmetric functions.

I wish to thank Kai Köhler and Damian Roessler for discussing their results with me. A special thank-you goes to Bill Fulton for many years of encouragement and enlightenment—in conversations and through his exemplary teaching and research.

2. Height of the Complete SL_n-Flag Variety

The homogeneous spaces $X = G/P$ considered in this paper are all smooth over $\operatorname{Spec}\mathbb{Z}$ and have cellular decompositions in the sense of [Fu, Ex. 1.9.1]. It follows that the Chow rings $\mathrm{CH}(X)$ may be defined with $\mathbb{Z}$-coefficients (following [Fu, Secs. 1–8 and 20]) and are isomorphic to the integral cohomology rings $H^*(X(\mathbb{C}),\mathbb{Z})$. Throughout this paper we will identify the two and use $\int_X \colon \mathrm{CH}(X) \to \mathbb{Z}$ to denote the classical degree map.

In this section $F = \mathrm{SL}_n/B$ will denote the complete SL_n-flag variety, which parametrizes, over any base field k, the complete flags in a k-vector space of dimension n. There is a tautological filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = E$$

of the trivial rank-n vector bundle E over F. The dimension $d = \dim_{\mathbb{C}} F(\mathbb{C}) = \binom{n}{2}$.

Let $\mathfrak{t} = \{\mathrm{diag}(x_1,\dots,x_n) \mid \sum x_i = 0\}$ be the Cartan subalgebra of diagonal matrices in $\mathfrak{sl}_n$. The set of positive roots R^+ can be identified with the linear functionals $x_i - x_j$ for $1 \le i < j \le n$ and $R_F = R^+$. Each x_i maps to $-c_1(E_i/E_{i-1})$ under the Borel characteristic map

$$\mathrm{Sym}(\mathrm{Char}(B)) \longrightarrow \mathrm{CH}(F)$$

and will be identified with its image in the Gysin computations that follow.

PROPOSITION 1. *For any $\alpha = (\alpha_1,\dots,\alpha_n) \in (\mathbb{Z}_+)^n$ we have*

$$\int_F x_1^{\alpha_1}\cdots x_n^{\alpha_n} = \begin{cases} \mathrm{sgn}(w) & \text{if } \alpha = (w(n-1),\dots,w(1),w(0)) \text{ for } w \in S_n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Recall from [BGG] and [D] that the degree map

$$\int_F \colon \mathrm{CH}(F) \to \mathbb{Z}$$

can be identified with the divided difference operator ∂_{w_0}, where w_0 denotes the element of longest length in the symmetric group S_n. Moreover, the operator

$$\partial = \partial_{w_0} \colon \mathbb{Z}[x_1,\dots,x_n] \to \mathbb{Z}[x_1,\dots,x_n]^{S_n}$$

coincides with the Jacobi symmetrizer, whose value at a polynomial f is

$$\partial(f) = \frac{1}{V} \sum_{w \in S_n} \operatorname{sgn}(w) w(f),$$

where $V = \prod_{1 \le i < j \le n}(x_i - x_j)$ is the Vandermonde determinant.

Because $\int_F \equiv \partial$, the result is implied by the following three facts: (i) $x_i^n = 0$ in $\mathrm{CH}(F)$ for each i; (ii) the image of the antisymmetrizing operator $\sum \operatorname{sgn}(w) w$ consists of the skew-symmetric polynomials; and (iii) we have

$$\int_F x_1^{n-1} x_2^{n-2} \cdots x_{n-1} = 1,$$

as $x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$ is dual to the class of a point in $\mathrm{CH}(F)$. □

The very ample line bundles L_λ on F correspond to weights λ of the form $\lambda(a) = \sum_{i=1}^n a_i x_i$ with $a = (a_i)$ integers such that $a_1 > \cdots > a_n \ge 0$. If $a = \rho := (n-1, \ldots, 1, 0)$ then $\lambda = \lambda(\rho)$ is the half sum of the positive roots and the corresponding embedding of F in projective space is the pluri-Plücker embedding, considered in [T2, Sec. 9].

In the statement of the next result we use multiindex notation: $\mathbb{Z}_+$ denotes the nonnegative integers and, for n-tuples $a, \nu \in (\mathbb{Z}_+)^n$, $a^\nu := a_1^{\nu_1} \cdots a_n^{\nu_n}$ with the convention that $0^0 = 1$. Moreover, $|\nu| = \sum \nu_i$, $\binom{|\nu|}{\nu}$ is a multinomial coefficient, and $e_1, \ldots, e_n$ are the standard basis elements for the lattice $\mathbb{Z}^n$.

THEOREM 2. (a) *For the very ample metrized line bundle $\overline{L}_{\lambda(a)} \to F$, we have*

$$h_{\overline{L}_{\lambda(a)}}(F) = \frac{1}{2} \sum_{r,s,i,\nu} \frac{(-1)^i \operatorname{sgn}(w)}{k+1} \binom{k}{i} \binom{d+1}{k+1} \binom{d-k}{\nu} (a_r - a_s)^{k+1} a^\nu, \quad (6)$$

the sum over all $r, s, i \in \mathbb{Z}_+$ and $\nu \in (\mathbb{Z}_+)^n$ with $1 \le r < s \le n$ and $k = d - |\nu| \ge 0$ and such that $\nu + ie_r + (k-i)e_s = w(\rho)$ for a (unique) permutation $w \in S_n$.

(b) *Conjecture 1 is true for F.*

Proof. (a) A direct application of Theorem 1 gives

$$h_{\overline{L}_{\lambda(a)}}(F) = \frac{1}{2} \sum_{k=0}^{d} \frac{(-1)^k}{k+1} \binom{d+1}{k+1} \sum_{r<s} (a_r - a_s)^{k+1} \int_F (x_r - x_s)^k \Big(\sum a_i x_i \Big)^{d-k}. \quad (7)$$

Now (6) follows from (7) by expanding the factors in the integrand and applying Proposition 1.

(b) Note that the condition $\nu + ie_r + (k-i)e_s = w(\rho)$ with $0 \le i \le k = d - |\nu|$ is quite restrictive on ν; in particular, $k \le 2n-3$. Suppose that $k+1 = p^r$ is a prime power greater than $m(\mathrm{SL}_n) = n-1$. The key number theoretic result that is used to simplify the denominators in the examples we consider is the following.

LEMMA 1. *If $k+1 = p^r$ is a prime power, then $\binom{k}{i} \equiv (-1)^i \bmod p$ for all i.*

Proof.

$$\binom{k}{0} = 1 \quad \text{and} \quad \binom{k}{i+1} = \frac{k-i}{i+1}\binom{k}{i} \equiv -\binom{k}{i} \bmod p. \qquad \square$$

Observe now that any fixed r, s, and ν with $|\nu| = d-k$ contribute either zero or two terms to the sum (6). The latter case is determined by the relations

$$(\nu_r + i, \nu_s + k - i) = (b, c) \quad \text{and} \quad (\nu_r + i', \nu_s + k - i') = (c, b)$$

for some $i, i' \geq 0$; the corresponding permutations w, w' differ by a transposition and hence have opposite signs. Lemma 1 now implies that the numerator of the sum of the two terms is divisible by p. Since $k+1 \leq 2m(\mathrm{SL}_n)$, it follows that the highest power of p in the denominator of $2h(F)$ is at most $m(\mathrm{SL}_n)$. $\square$

3. Height of the SL_N-Grassmannian

In this section we will adopt the notational conventions of [T3, Secs. 2, 4, 5]. Let $N = m+n$ and $G = G(m,n) = \mathrm{SL}_N/P_{m,n}$ denote the Grassmannian over $\mathrm{Spec}\,\mathbb{Z}$ that parametrizes m-planes in k^N for any field k. The universal exact sequence of vector bundles over G is

$$0 \longrightarrow S \longrightarrow E \longrightarrow Q \longrightarrow 0. \tag{8}$$

These become hermitian vector bundles by giving the trivial rank-N bundle $E(\mathbb{C})$ the trivial hermitian metric and the tautological rank-m subbundle $S(\mathbb{C})$ and quotient bundle $Q(\mathbb{C})$ the induced metrics.

We have a geometric basis of Schubert classes for the Chow ring $\mathrm{CH}(G)$; these coincide with the characteristic classes $\{s_\lambda(Q)\}_\lambda$. Here the indexing set consists of partitions λ whose Young diagrams are contained in the $n \times m$ rectangle (m^n)—in this section we consider only such diagrams—and s_λ is the Schur polynomial corresponding to λ. If we rotate the complement of λ in (m^n) by 180° then we obtain the *dual diagram* $\widehat{\lambda}$ (see Figure 1); this corresponds to the Poincaré dual of $s_\lambda(Q)$ in geometry. For each box x in λ, the *hook length* h_x equals the number of boxes directly to the right and below x, including x itself. Define

$$h_\lambda = \prod_{x \in \lambda} h_x \quad \text{and} \quad f^\lambda = \frac{|\lambda|!}{h_\lambda},$$

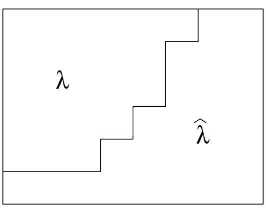


Figure 1 Dual Young diagrams in (8^6)

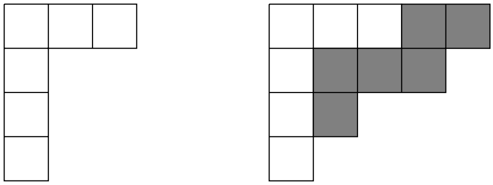


Figure 2 A hook and a double hook

and recall that f^{λ} counts the number of *standard Young tableaux* on λ, that is, the number of fillings of the boxes of λ with the integers $1, \ldots, |\lambda|$ such that the entries are strictly increasing along each row and column.

A partition λ is a *hook* if $\lambda = (a, 1^b)$ for some $a > 0$ and $b \geq 0$. We define a *double hook* to be a pair $(\mu \subset \lambda)$ of partitions such that μ is a hook and the skew diagram λ/μ is a rim hook. Figure 2 shows a hook μ and a double hook $\mu \subset \lambda$ (with λ/μ shaded). We assume that $0 < |\mu| < |\lambda|$ and call the pair $(|\mu|, |\lambda| - |\mu|)$ the *weight* of the double hook. Define the sign of a hook and double hook by

$$\operatorname{sgn}(\lambda) = (-1)^{ht(\lambda)} \quad \text{and} \quad \operatorname{sgn}(\mu \subset \lambda) = (-1)^{ht(\mu)+ht(\lambda/\mu)}$$

respectively, where as usual the height $ht(\gamma)$ of a rim hook γ is one less than the number of rows it occupies.

THEOREM 3. (a) *The Faltings height of the Grassmannian $G = G(m, n)$ under its Plücker embedding in projective space is given by*

$$h(G) = \binom{mn+1}{2} f^{(m^n)} + \frac{1}{2} \sum_{\lambda} \operatorname{sgn}(\lambda) \frac{(-1)^{|\lambda|} m - n}{|\lambda| + 1} \binom{mn+1}{|\lambda|+1} f^{\hat{\lambda}} \tag{9}$$

$$- \frac{1}{2} \sum_{\mu \subset \lambda} \operatorname{sgn}(\mu \subset \lambda) \frac{(-1)^{|\lambda|+|\mu|}}{|\lambda|+1} \binom{|\lambda|}{|\mu|} \binom{mn+1}{|\lambda|+1} f^{\hat{\lambda}} \tag{10}$$

where the first sum is over all hooks λ and the second over all double hooks $\mu \subset \lambda$ (with λ contained in (m^n)).

(b) *Conjecture 1 is true for G.*

REMARKS. (1) Note that the sums in Theorem 3 may be indexed by simple integral parameters. For instance, the hook partitions $\lambda = (a, 1^b)$ in (9) have a and b in the ranges $1 \leq a \leq m$ and $0 \leq b \leq n-1$, and the sum may be written as

$$\frac{1}{2} \sum_{a,b} \frac{(-1)^a m - (-1)^b n}{a+b+1} \binom{mn+1}{a+b+1} f^{(a,1^b)^{\wedge}}.$$

However, the second sum (10) requires four integral parameters (cf. [T3, Thm. 2]).

(2) All the diagrams that occur in Theorem 3 are contained in the $n \times m$ rectangle (m^n), in contrast to the corresponding formula of [T3, Thm. 2], where the diagrams have weight $mn + 1$. This stems from the fact that the former comes

from classical intersection theory and Schubert calculus whereas the latter from their arithmetic analogs.

(3) Part (b) follows immediately from the formula for $h(G)$ in [T3, Thm. 2] coming from arithmetic Schubert calculus. The point here is to check the conjecture directly using part (a). Note that, a priori, the denominators in (10) are as large as $2m(\mathrm{SL}_N)-1$.

Proof of Theorem 3. (a) For the homogeneous spaces X such that $X(\mathbb{C})$ is a hermitian symmetric space, there is a unique positive primitive hermitian line bundle $\overline{L}_\nu$ on X (given by a fundamental weight ν). One can check (see [KK, p. 28]) that $\langle \nu, \alpha\rangle \in \{1,2\}$ for any $\alpha \in R_X$.

In our case $X = G(m,n)$, the line bundle is $\det(\overline{Q})$ and the corresponding embedding is the Plücker embedding. Moreover, only one homogeneous vector bundle E_1 occurs in (4), and the sum (5) is over all the roots of X; thus $E_1 = TG$. Since $\dim_{\mathbb{C}} G(\mathbb{C}) = mn$, formula (4) becomes

$$h(G) = \frac{1}{2}\sum_{k=0}^{mn} \frac{(-1)^k}{k+1}\binom{mn+1}{k+1}\int_G p_k(TG)c_1(Q)^{mn-k} \tag{11}$$

(see also [KR, Sec. 1]). To evaluate the integrals in (11), recall that the tangent bundle $TG \cong S^*\otimes Q$ and

$$p_k(S)+p_k(Q) = p_k(S\oplus Q) = 0$$

for positive k, as E is trivial in (8). It follows that

$$p_k(TG) = \sum_i \binom{k}{i} p_i(S^*)p_{k-i}(Q) \tag{12}$$

$$= (m-(-1)^k n)p_k(Q) + \sum_{0<i<k}(-1)^{i+1}\binom{k}{i}p_i(Q)p_{k-i}(Q) \tag{13}$$

for positive k, while $p_0(TG) = \mathrm{rk}(TG) = mn$.

Let $p_k = p_k(Q)$ and $s_\lambda = s_\lambda(Q)$ for the remainder of this section. The power sums and their products are related to the Schur functions s_λ as follows:

$$p_k = \sum_\lambda \mathrm{sgn}(\lambda)s_\lambda, \tag{14}$$

$$p_k p_l = \sum_{\mu\subset\lambda}\mathrm{sgn}(\mu\subset\lambda)s_\lambda \tag{15}$$

with (14) summed over all hooks λ of weight k and (15) summed over all double hooks $\mu\subset\lambda$ of weight (k,l) (see [M, Ex. I.3.11]). Moreover, the degree map satisfies

$$\int_G s_\lambda s_1^{mn-|\lambda|} = f^{\widehat{\lambda}} \tag{16}$$

for any $\lambda \subset (m^n)$. Indeed, by iterating the Pieri rule for a product $s_\lambda s_1$, we see that

$$s_\lambda s_1^{mn-|\lambda|} = N_\lambda s_{(m^n)}$$

in $\mathrm{CH}(G)$, where N_λ is the number of ways of filling in the boxes of $(m^n)\setminus\lambda$ with the numbers $1, 2, \ldots, mn-|\lambda|$ so that the entries are strictly increasing along rows and columns. By rotating this picture 180°, one sees that N_λ equals the number $f^{\widehat{\lambda}}$ of standard tableaux on $\widehat{\lambda}$. Equation (16) now follows, as $s_{(m^n)}$ is dual to the class of a point in $\mathrm{CH}(G)$.

The proof of part (a) is completed by using (14) and (15) in (13), substituting the result in (11), and applying (16). Note that the initial term in (9) comes from the $k=0$ term in (11), which is treated separately.

(b) Observe that it suffices to verify this statement for the unique primitive line bundle, that is, for $h(G)$ as given in part (a). This follows from the basic properties of heights [BGS, Sec. 3.2.1].

Suppose that $k+1=p^k$ is a prime power which is at least $m(\mathrm{SL}_N)+1=m+n$; we must check that the numerator of the sum of terms in (9) and (10) with $|\lambda|=k$ is divisible by p. For this it is convenient to visualize the partitions that occur in these sums using β-sequences, as in [T3, pp. 430–431]. The β-sequence of a partition $\lambda\subset(m^n)$ is the n-tuple

$$\beta(\lambda)=(\lambda_1+n-1,\ \lambda_2+n-2,\ \ldots,\ \lambda_n+n-n)$$

of distinct integers between 0 and $m+n-1$. We picture each such sequence as a collection of n checkers on the Young diagram of $(n+m)$ (that is, $n+m$ squares in a row). The checker positions correspond to the numbers β_i, ordered as on the real line. Let us agree to identify a Young diagram with its β-sequence; for example, the empty diagram corresponds to the picture in Figure 3.

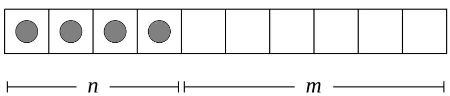


Figure 3 The β-sequence of the empty diagram

The β-sequences are convenient when working with the power sums p_k. A checker C makes a *move* of length k when it moves to an empty square located k squares to the right of its initial position. The *sign* of the move is $+1$ (resp. -1) if the number of checkers C "jumped over" is even (resp. odd). We then have the following multiplication rule:

$$p_k s_{\beta(\lambda)}=\sum(\pm1)s_{\beta(\mu)},$$

the sum over all $\beta(\mu)$ obtained from $\beta(\lambda)$ by a move of length k, with the sign equal to the sign of the move. In other words (recalling (14) and (15)), each move of length k starting from $\beta(\lambda)$ corresponds to adding a rim k-hook γ to λ, and the parity mod 2 of $ht(\gamma)$ determines the sign of the move.

Let $\beta(\lambda)\to\beta(\mu)$ denote a single move from $\beta(\lambda)$ to $\beta(\mu)$. Observe that the sum (9) is over all moves $\beta(\emptyset)\to\beta(\lambda)$, while (10) is summed over pairs $\beta(\emptyset)\to\beta(\mu)\to\beta(\lambda)$ of consecutive moves, starting from the β-sequence of the empty diagram. We are now ready to study the numerators in (9) and (10) mod p and distinguish two cases.

Case 1. Summing the terms with $|\lambda| = k > m+n-1$. Observe that there are no such terms coming from (9), since the longest possible hook $(m, 1^{n-1})$ has weight $m+n-1$. Let us consider those summands $\beta(\emptyset) \to \beta(\mu) \to \beta(\lambda)$ with a *fixed final position* $\beta(\lambda)$. Note that each of these is a sequence of two moves involving distinct checkers (i.e., the same checker cannot have moved twice). It is easy to see that there are exactly four such sequences, corresponding to the two choices for the first checker and the two possible destination squares (all determined by $\beta(\lambda)$). By counting the total number of checkers jumped over for each move, one sees that two of the four sequences have sign $+1$ and the other two sign -1. Figure 4 illustrates the smallest example, the four pairs of moves from $\beta(\emptyset) = (1,0)$ to $\beta(2,2) = (3,2)$ when $m = n = 2$. The first move in each pair is shown by the arrow *above* the diagram, and the corresponding double hooks $\mu \subset \lambda$ are illustrated above these.

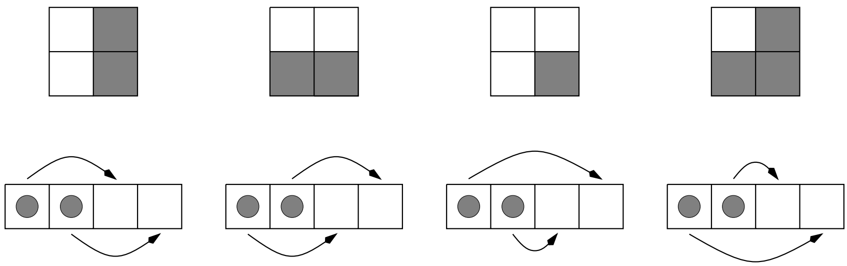


Figure 4 Four move pairs with signs $+1$, $+1$, -1, -1

Now Lemma 1 shows that the numerator of the sum of these four terms has residue, mod p, of

$$(-1)^{|\lambda|+1}\binom{mn+1}{|\lambda|+1} f^{\widehat{\lambda}} \sum_{\mu\subset\lambda} \mathrm{sgn}(\beta(\emptyset) \to \beta(\mu) \to \beta(\lambda)) = 0,$$

and we are done with this case.

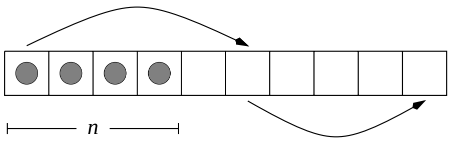
Case 2. Summing the terms with $|\lambda| = m+n-1 = p^r - 1$. The sum of all such terms that come from pairs of moves $\beta(\emptyset) \to \beta(\mu) \to \beta(\lambda)$ using two distinct checkers *and with* λ *not a hook* is handled exactly as in Case 1. The remaining extra terms have λ equal to the hook $(m, 1^{n-1})$, and are analyzed as follows.

(i) There is a single such term in the sum (9); the residue of the numerator mod p for this term is

$$(-1)^{n-1}((-1)^{m+n-1}m - n) \cdot F = ((-1)^m m + (-1)^n n) \cdot F, \tag{17}$$

where F is the fixed factor $F = \binom{mn+1}{m+n} f^{(m,1^{n-1})^\wedge}$.

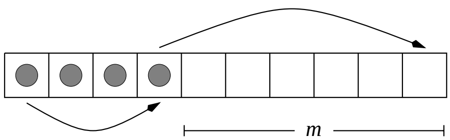
(ii) There are $m-1$ terms of (10) of the form



contributing total numerator residue

$$(-1)^{m+n-1}(-1)^{n-1}(m-1)\cdot F=(-1)^{m}(m-1)\cdot F, \tag{18}$$

and $n-1$ terms of (10) of the form



contributing total numerator residue

$$(-1)^{m+n-1}(-1)^{n-2}(n-1)\cdot F=(-1)^{m-1}(n-1)\cdot F. \tag{19}$$

The total contribution of the extra terms to the numerator is therefore

$$(17)-(18)-(19)=[(-1)^m+(-1)^n]\cdot nF \mod p.$$

Recall that $m+n=p^r$ is a prime power. If p is odd, then m and n have different parity mod 2 and hence $(-1)^m+(-1)^n=0$. Otherwise $p=2$ while clearly $(-1)^m+(-1)^n$ is even. The proof is complete. $\square$

EXAMPLE 1. When applied to projective space $\mathbb{P}^n=G(n,1)$, Theorem 3 gives

$$h(\mathbb{P}^n)=\binom{n+1}{2}+\frac{1}{2}\sum_{k=1}^{n}\frac{(-1)^k n-1}{k+1}\binom{n+1}{k+1}-\frac{1}{2}\sum_{\substack{i,k\\0<i<k}}\frac{(-1)^{i+k}}{k+1}\binom{k}{i}\binom{n+1}{k+1}.$$

We leave it as an exercise for the reader to check that this agrees with the Stoll number from (2). Another method of evaluating (4) for $X=\mathbb{P}^n$ is given in [KK, Sec. 8].

4. Heights of Isotropic Grassmannians

In this section we adopt the notational conventions from [T4] unless otherwise indicated. All constructions and results will be type-C and type-D analogs of those of the previous section. Our aim is to study the height formula (4) for the

Lagrangian and even orthogonal Grassmannians, whose complex points are hermitian symmetric spaces. We begin with the former in order to draw from the analysis in [T4], although the final formulas are simpler in the orthogonal case (this is to be expected, as the same is true for the degrees in geometry).

Let $\mathrm{LG} = \mathrm{LG}(n, 2n) = \mathrm{Sp}_n/P_n$ denote the Lagrangian Grassmannian over $\operatorname{Spec}\mathbb{Z}$ which parametrizes maximal isotropic subspaces in k^{2n}, with respect to the standard symplectic form, for any field k. We have a universal exact sequence of vector bundles over LG,

$$0 \longrightarrow S \longrightarrow E \longrightarrow Q \longrightarrow 0, \tag{20}$$

with E the trivial bundle of rank $2n$. We can use the symplectic form to identify the quotient bundle Q with S^*. The complex points of these bundles can be metrized as in the previous section (see [T4]). We let $d = \dim_{\mathbb{C}}(\mathrm{LG}(\mathbb{C})) = \binom{n+1}{2}$.

We have a geometric basis of Schubert classes for the Chow ring $\mathrm{CH}(\mathrm{LG})$, given by the characteristic classes $\{\sigma_\lambda(Q)\}_\lambda$. Here the indexing set consists of strict partitions λ whose Young diagrams are contained in the triangular partition $\rho(n) = (n, n-1, \ldots, 1)$. Furthermore, σ_λ denotes the $\tilde{Q}$-polynomial indexed by λ; these symmetric polynomials were defined and studied by Pragacz and Ratajski and are type-C analogs of Schur polynomials (see [PR, Thm. 2.1]).

The Poincaré dual $\sigma_{\widehat{\lambda}}(Q)$ of $\sigma_\lambda(Q)$ is indexed by the *dual diagram* $\widehat{\lambda}$, whose parts complement the parts of λ in the set $\{1, \ldots, n\}$. The relation between the two diagrams can be visualized as follows: the shifted diagram $\mathcal{S}(\widehat{\lambda})$ is obtained by rotating the complement of $\mathcal{S}(\lambda)$ in $\mathcal{S}(\rho(n))$ by 90° and then reflecting it about the vertical axis (see [P, Cor. 6.9] and Figure 5).

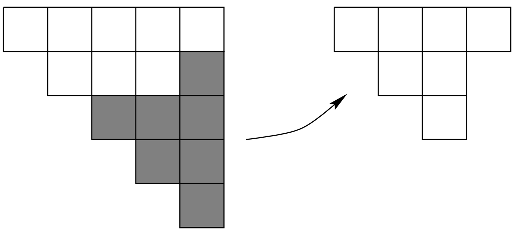


Figure 5 Shifted diagrams for $\lambda = (5, 3) \subset \rho(5)$ and $\widehat{\lambda} = (4, 2, 1)$

Let g^λ denote the number of standard tableaux on the shifted diagram $\mathcal{S}(\lambda)$. Then one has $g^\lambda = |\lambda|!/h^s_\lambda$, where h^s_λ is the product of the hook lengths at boxes of $\mathcal{S}(\lambda)$ (these hook lengths are taken with respect to the double diagram of λ; see [M, Ex. III.8.12] for details). In [T4, Sec. 3] we extended the definition of the numbers g^λ to arbitrary Young diagrams λ with $\lambda_1 \leq n$. The g^λ count the number of *proper standard tableaux* on λ (as defined in [T4]). It would be interesting to find an analog of the hook length formula for these more general g-numbers. The (geometric) degree of $\mathrm{LG}(\mathbb{C})$ is given by

$$\deg(\mathrm{LG}(\mathbb{C})) = rg^{\rho(n)}, \quad \text{where } r := 2^{d-n} = 2^{n(n-1)/2}.$$

Following [T4, Sec. 4.2] and [M, Ex. III.8.11] we define, in the context of shifted diagrams, a *double rim* to be the skew diagram formed by the union of two rim hooks that both end on the main diagonal $D = \{(i,i) \mid i > 0\}$. Each double rim $\delta = \alpha \cup \beta$ is a union of two non-empty connected pieces; α consists of the diagonals of length 2 in δ (which are parallel to D) and $\beta = \delta \setminus \alpha$ is a rim hook (two double rims appear in Figure 6). For any such double rim δ and for any rim hook γ, let

$$\varepsilon(\delta) = (-1)^{|\alpha|/2 + ht(\beta)}2 \quad \text{and} \quad \varepsilon(\gamma) = (-1)^{ht(\gamma)}.$$

We define a *shape* to be the Young diagram of a partition $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 + \lambda_2$ odd. Note that if λ is a shape then the shifted diagram $\mathcal{S}(\lambda)$ is a (rim) hook or a double rim. Define a *double shape* to be a pair $(\mu \subset \lambda)$ of strict partitions, with $|\lambda|$ even and $|\mu|$ odd, such that μ is a shape and $\mathcal{S}(\lambda/\mu)$ is a rim hook or a double rim; the *weight* of $\mu \subset \lambda$ is the pair $(|\mu|, |\lambda| - |\mu|)$. An example in $\mathcal{S}(\rho(7))$ is illustrated in Figure 6. Finally, to any shape λ and double shape $\mu \subset \lambda$ we associate the integers

$$\varepsilon(\lambda) = \varepsilon(\mathcal{S}(\lambda)) \quad \text{and} \quad \varepsilon(\mu \subset \lambda) = \varepsilon(\mathcal{S}(\mu))\varepsilon(\mathcal{S}(\lambda/\mu)).$$

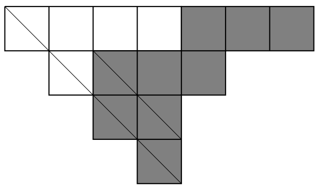


Figure 6 The (shifted) double shape $(4,1) \subset (7,4,2,1)$

THEOREM 4. (a) *The Faltings height* $h(\mathrm{LG})$ *of the Lagrangian Grassmannian* $\mathrm{LG} = \mathrm{LG}(n, 2n)$ *under its fundamental embedding in projective space satisfies*

$$h(\mathrm{LG}) = \frac{rn^2}{2}(d+1)g^{\rho(n)} - r\sum_{\lambda} \varepsilon(\lambda)\frac{2n+1-2^{|\lambda|}}{|\lambda|+1}\binom{d+1}{|\lambda|+1}g^{\widehat{\lambda}} \tag{21}$$

$$+\, 2r\sum_{\mu\subset\lambda} \frac{\varepsilon(\mu\subset\lambda)}{|\lambda|+1}\binom{|\lambda|}{|\mu|}\binom{d+1}{|\lambda|+1}g^{\widehat{\lambda}}, \tag{22}$$

where the first sum is over all shapes λ *and the second is over all double shapes* $\mu \subset \lambda$ *(with* λ *contained in* $\rho(n)$*), and* $r = 2^{d-n}$.

(b) *Conjecture 1 is true for* LG.

Proof. (a) The line bundle giving the embedding in this case is $\det(\overline{Q})$. Let $x_1, \ldots, x_n$ be the Chern roots of Q. One may check, using for instance [BH], that in this case two vector bundles occur in formula (4): a bundle E_1 with roots $\{2x_i\}$ and another $E_2 = \wedge^2(Q)$ with roots $\{x_i + x_j \mid i < j\}$. Hence Theorem 1 gives

$$h(\mathrm{LG}) = \frac{1}{2}\sum_{k=0}^{d} \frac{(-1)^k}{k+1}\binom{d+1}{k+1}\int_{\mathrm{LG}}(p_k(E_1)+2^{k+1}p_k(E_2))c_1(Q)^{d-k}. \tag{23}$$

Observe that

$$\sum_{i<j} e^{x_i+x_j} = \frac{1}{2}\Big[-\sum_i e^{2x_i} + \Big(\sum_i e^{x_i}\Big)^2\Big];$$

hence

$$\mathrm{ch}(\wedge^2(Q)) = \frac{1}{2}\Big(-\sum_k 2^k\,\mathrm{ch}_k(Q) + \mathrm{ch}(Q)^2\Big), \tag{24}$$

where ch_k is the kth homogeneous component of the Chern character. Now substitute $p_k = k!\,\mathrm{ch}_k$ in (24) and equate the degree-k components to obtain

$$p_k(\wedge^2(Q)) = -2^{k-1}p_k(Q) + \frac{1}{2}\sum_i \binom{k}{i} p_i(Q)p_{k-i}(Q). \tag{25}$$

It follows that

$$p_k(E_1)+2^{k+1}p_k(E_2) = 2^k\Big[(2n+1-2^k)p_k(Q)+\sum_{0<i<k}\binom{k}{i}p_i(Q)p_{k-i}(Q)\Big] \tag{26}$$

for positive k, while $p_0(E_1)+2p_0(E_2)=n^2$.

For the remainder of this proof let p_k and σ_λ denote $p_k(Q)$ and $\sigma_\lambda(Q)$, respectively. The *length* of a partition λ (i.e., the number of nonzero parts) is denoted by $\ell(\lambda)$. In [T4, eq. (19)] we showed that, for k odd,

$$p_k\sigma_\mu = \sum_\lambda \varepsilon(\mathcal{S}(\lambda/\mu))2^{\ell(\mu)-\ell(\lambda)+1}\sigma_\lambda, \tag{27}$$

the sum over all strict $\lambda\supset\mu$ with $|\lambda| = |\mu|+k$ such that $\mathcal{S}(\lambda/\mu)$ is a rim hook or a double rim. Equation (27) implies the relations

$$p_k = 2\sum_\lambda \varepsilon(\lambda)2^{-\ell(\lambda)}\sigma_\lambda \tag{28}$$

and

$$p_k p_l = 4\sum_{\mu\subset\lambda}\varepsilon(\mu\subset\lambda)2^{-\ell(\lambda)}\sigma_\lambda, \tag{29}$$

where (28) is summed over all shapes λ of weight k and (29) is summed over all double shapes $\mu\subset\lambda$ of weight (k,l). Moreover, the degree map satisfies

$$\int_{\mathrm{LG}}\sigma_\lambda\sigma_1^{d-|\lambda|} = 2^{\ell(\lambda)-|\lambda|}rg^{\widehat{\lambda}} \tag{30}$$

for any $\lambda\subset\rho(n)$. Equation (30) is justified in the same way as (16), by iterating the Pieri rule of [BoH] for a product $\sigma_\lambda\sigma_1$, and recalling the picture of the dual diagram $\widehat{\lambda}$ from Figure 5. We complete the argument by using the ingredients (26), (28), (29), and (30) in (23), noting that for any shape λ we have $(-1)^{|\lambda|} = -1$ whereas, for a double shape $\mu\subset\lambda$, $(-1)^{|\lambda|} = 1$.

(b) This follows immediately from the formula for $h(\mathrm{LG})$ in [T4] (reproduced here in equation (34)), which comes from arithmetic Schubert calculus. It is likely that one can show the cancellation of the denominators directly (as in Theorem 3(b)) using the analogs of β-sequences in this setting. □

EXAMPLE 2. We compute the height of the quadric LG(2, 4) using Theorem 4. The constants are $n = 2$, $d = 3$, $r = 2$, and the g-numbers

$$g^{\emptyset} = g^1 = g^2 = g^{(2,1)} = 1.$$

The shapes contained in $\rho(2) = (2, 1)$ are (1) and (2, 1), while there is a single double shape $(1) \subset (2)$. Moreover, their associated integers are

$$\varepsilon((1)) = 1, \quad \varepsilon((2, 1)) = -2, \quad \varepsilon((1) \subset (2)) = 1.$$

Substituting these into (21) yields

$$h(\mathrm{LG}(2,4)) = 16 - 2\left(\frac{3}{2}\binom{4}{2} + \frac{3}{2}\binom{4}{4}\right) + \frac{4}{3}\binom{2}{1}\binom{4}{3} = \frac{17}{3},$$

in agreement with [CM] and [T4].

We turn now to the even orthogonal Grassmannian $\mathrm{OG} = \mathrm{OG}(n + 1, 2n + 2) = \mathrm{SO}_{2n+2}/P_{n+1}$, where P_{n+1} is the maximal parabolic subgroup corresponding to a "right end root" in the Dynkin diagram. Here we can immediately state our theorem.

THEOREM 5. (a) *The Faltings height $h(\mathrm{OG})$ of the even orthogonal Grassmannian $\mathrm{OG} = \mathrm{OG}(n + 1, 2n + 2)$ under its fundamental embedding in projective space satisfies*

$$h(\mathrm{OG}) = \binom{d+1}{2} g^{\rho(n)} - \sum_{\lambda} \varepsilon(\lambda) \frac{n + 1 - 2^{|\lambda|-1}}{|\lambda| + 1} \binom{d+1}{|\lambda|+1} g^{\widehat{\lambda}} \tag{31}$$

$$+ \sum_{\mu \subset \lambda} \frac{\varepsilon(\mu \subset \lambda)}{|\lambda| + 1} \binom{|\lambda|}{|\mu|} \binom{d+1}{|\lambda|+1} g^{\widehat{\lambda}}, \tag{32}$$

where the first sum is over all shapes λ and the second is over all double shapes $\mu \subset \lambda$ (with λ contained in $\rho(n)$).

(b) *Conjecture 1 is true for* OG.

Proof. (a) The argument is similar (and in fact simpler) than that for Theorem 4. We will confine ourselves to pointing out the differences in the orthogonal case. Note that the space of maximal isotropic subspaces in $\mathbb{C}^{2n+2}$ (with respect to a nondegenerate symmetric form) has two connected components, and $\mathrm{OG}(\mathbb{C})$ parametrizes one of them. We have a universal subbundle S and quotient bundle Q that fit into an exact sequence (20) as before.

The dimension $d = \dim_{\mathbb{C}}(\mathrm{OG}(\mathbb{C})) = n(n + 1)/2$ and the Chow ring $\mathrm{CH}(\mathrm{OG})$ has a basis of Schubert classes $\{\tau_\lambda(Q)\}$ for strict partitions λ with $\lambda \subset \rho(n)$. Here τ_λ is a $\tilde{P}$-polynomial, related to the $\tilde{Q}$-polynomial σ_λ by

$$\tau_\lambda = 2^{-\ell(\lambda)} \sigma_\lambda$$

(we refer again to [PR, Thm. 2.1] for details).

The generator L_ν of Pic(OG) giving the fundamental embedding has $c_1(L_\nu) = \tau_1(Q)$. Moreover, we have $\langle \nu, \alpha\rangle = 1$ for all $\alpha \in R_{\mathrm{OG}}$, and the vector bundle $E_1 = T(\mathrm{OG}) = \wedge^2(Q)$. We can thus use (25) in the computation of the integrals, noting that (28) and (29) have the simpler form

$$p_k = 2\sum_\lambda \varepsilon(\lambda)\tau_\lambda,$$

$$p_k p_l = 4\sum_{\mu\subset\lambda} \varepsilon(\mu\subset\lambda)\tau_\lambda$$

in this case. In addition, the degree map for OG satisfies

$$\int_{\mathrm{OG}} \tau_\lambda \tau_1^{d-|\lambda|} = g^{\widehat{\lambda}}.$$

(b) We argue indirectly by comparing the formula in part (a) with that for $h(\mathrm{LG})$, for which we know the result is true. Since $m(\mathrm{SO}_{2n+2}) = 2n+1$ and the largest denominators in the first sum (31) have order $2n$, we are left with checking the denominators in the second sum (32). It is clear by comparing (32) with (22) that we need only check what happens when $|\lambda|+1 = 2^k$ is a large power of 2. But this never occurs, because for each double shape $\mu\subset\lambda$ the weight $|\lambda|$ is *even*. □

EXAMPLE 3. The case $n = 2$ is similar to Example 2, since all the constants are the same. Substituting into (31) gives

$$h(\mathrm{OG}(3,6)) = 6 - \left(\frac{2}{2}\binom{4}{2} + \frac{1}{2}\binom{4}{4}\right) + \frac{1}{3}\binom{2}{1}\binom{4}{3} = \frac{13}{6} = h(\mathbb{P}^3),$$

as expected.

Combining the two preceding theorems with [T4, Thm. 3] leads to an analog of the latter result for OG. It is clear from the formulas in Theorems 4 and 5 that

$$h(\mathrm{OG}) = (2r)^{-1}h(\mathrm{LG}) + \frac{n(d+1)}{4}g^{\rho(n)} - \frac{1}{2}\sum_\lambda \frac{\varepsilon(\lambda)}{|\lambda|+1}\binom{d+1}{|\lambda|+1}g^{\widehat{\lambda}}, \quad (33)$$

the sum over all shapes λ (with $\lambda_1 \le n$). We now recall the situation in [T4, Sec. 5]: for each shape $\lambda\subset\rho(n)$ there is a unique Young diagram $[\lambda]$ of weight $d+1$ such that (i) there is a shifted hook operation (as defined in [T4]) from $[\lambda]$ to $\rho(n)$ and (ii) the diagram $\bar{\lambda}$, which is obtained from $[\lambda]$ by deleting two equal parts, is the dual diagram of λ. Observe that all shapes $\lambda\subset\rho(n)$ are of the form $\lambda = (a+2b+1, a)$ for unique $a, b\in\mathbb{Z}_+$ with $a+2b<n$, and we have $[\lambda] = [a,b]_n$ in the notation of [T4, Sec. 5]. Note also the equality

$$\varepsilon(\lambda) = \varepsilon((a+2b+1,a)) = (-1)^a 2^{1-\delta_{a0}},$$

where δ_{a0} is the Kronecker delta. It follows that Theorem 3 of [T4] may be stated in the following form:

$$h(\mathrm{LG}) = \frac{r}{2}\sum_\lambda (-1)^{(|\lambda|-1)/2}\varepsilon(\lambda)\mathcal{H}_{|\lambda|}g^{[\lambda]}, \quad (34)$$

the sum over all shapes λ with $\lambda_1 \le n$. We see that *the parameter space* $\mathcal{E}(n)$ *for the sum in* [T4, Thm. 3] *is in bijection with the set of shapes* λ *with* $\lambda_1 \le n$. This is a pleasant surprise, as the formula in [T4] and (21) were shown using different methods (arithmetic and classical Schubert calculus, respectively). Putting (33) and (34) together produces the following theorem.

THEOREM 6. *The Faltings height of* $\mathrm{OG} = \mathrm{OG}(n+1, 2n+2)$ *under its fundamental embedding in projective space is given by*

$$\begin{aligned} h(\mathrm{OG}) = {} & \frac{n(d+1)}{4} g^{\rho(n)} \\ & + \frac{1}{4} \sum_{\lambda} \left((-1)^{(|\lambda|-1)/2} \varepsilon(\lambda) \mathcal{H}_{|\lambda|} g^{[\lambda]} - \frac{2\varepsilon(\lambda)}{|\lambda|+1} \binom{d+1}{|\lambda|+1} g^{\widehat{\lambda}} \right), \end{aligned}$$

the sum over all shapes λ *with* $\lambda_1 \le n$. *This may be written using the parameters* $a, b \in \mathbb{Z}_+$ *as follows*:

$$\begin{aligned} h(\mathrm{OG}) = {} & \frac{n(d+1)}{4} g^{\rho(n)} + \frac{1}{2} \sum_{0 \le a+2b < n} (-1)^b 2^{-\delta_{a0}} \mathcal{H}_{2a+2b+1} g^{[a,b]_n} \\ & - \frac{1}{2} \sum_{0 \le a+2b < n} \frac{(-1)^a 2^{-\delta_{a0}}}{a+b+1} \binom{d+1}{2a+2b+2} g^{(a+2b+1,a)^\wedge}. \end{aligned}$$

References

[AB] A. Abbès and T. Bouche, *Théorème de Hilbert–Samuel "arithmetique"*, Ann. Inst. Fourier (Grenoble) 45 (1995), 375–401.

[BGG] I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand, *Schubert cells and the cohomology of the spaces G/P*, Uspekhi Mat. Nauk 28 (1973), 1–26.

[BoH] B. Boe and H. Hiller, *Pieri formula for* SO_{2n+1}/U_n *and* Sp_n/U_n, Adv. Math. 62 (1986), 49–67.

[BH] A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces. I,* Amer. J. Math. 80 (1958), 458–538.

[BGS] J.-B. Bost, H. Gillet, and C. Soulé, *Heights of projective varieties and positive Green forms,* J. Amer. Math. Soc. 7 (1994), 903–1027.

[CM] J. Cassaigne and V. Maillot, *Hauteurs des hypersurfaces et fonctions zêta d'Igusa*, preprint, 1994.

[D] M. Demazure, *Invariants symétriques entiers des groupes de Weyl et torsion,* Invent. Math. 21 (1973), 287–301.

[DG] M. Demazure and A. Grothendieck, *Schémas en groupes. I–III,* Lecture Notes in Math., 151–153, Springer-Verlag, New York, 1970.

[F] G. Faltings, *Diophantine approximation on abelian varieties,* Ann. of Math. (2) 133 (1991), 549–576.

[Fu] W. Fulton, *Intersection theory,* Ergeb. Math. Grenzgeb. (3), 2, Springer-Verlag, Berlin, 1998.

[GS1] H. Gillet and C. Soulé, *Amplitude arithmétique*, C.R. Acad. Sci. Paris Sér. I Math. 307 (1988), 887–890.

[GS2] ———, *Arithmetic intersection theory,* Inst. Hautes Études Sci. Publ. Math. 72 (1990), 94–174.

[J] J. C. Jantzen, *Representations of algebraic groups,* Pure Appl. Math., 131, Academic Press, Boston, 1987.

[KK] C. Kaiser and K. Köhler, *A fixed point formula of Lefschetz type in Arakelov geometry III: Representations of Chevalley schemes and heights of flag varieties,* preprint, 1999.

[KR] K. Köhler and D. Roessler, *A fixed point formula of Lefschetz type in Arakelov geometry I: Statement and proof,* preprint, 1999.

[M] I. G. Macdonald, *Symmetric functions and Hall polynomials,* Oxford Math. Monographs, Clarendon Press, New York, 1995.

[Ma] V. Maillot, *Un calcul de Schubert arithmétique,* Duke Math. J. 80 (1995), 195–221.

[N] D. G. Northcott, *An inequality in the theory of arithmetic on algebraic varieties,* Proc. Cambridge Philos. Soc. 45 (1949), 502–509.

[OV] A. L. Onishchik and E. B. Vinberg, *Lie groups and algebraic groups,* Springer Ser. Soviet Math., Springer-Verlag, Berlin, 1990.

[Ph] P. Philippon, *Sur des hauteurs alternatives. I,* Math. Ann. 289 (1991), 255–283; *II,* Ann. Inst. Fourier (Grenoble) 44 (1994), 1043–1065; *III,* J. Math. Pures Appl. (9) 74 (1995), 345–365.

[P] P. Pragacz, *Algebro-geometric applications of Schur S- and Q-polynomials,* Lecture Notes in Math., 1478, pp. 130–191, Springer-Verlag, Berlin, 1991.

[PR] P. Pragacz and J. Ratajski, *Formulas for Lagrangian and orthogonal degeneracy loci; $\tilde{Q}$-polynomial approach*, Compositio Math. 107 (1997), 11–87.

[S] C. L. Siegel, *Über einige Andwendungen diophantischer Approximationen*, Abh. Preuss. Akad. Wiss. (1929), 41–69.

[Sn] D. M. Snow, *The nef value and defect of homogeneous line bundles,* Trans. Amer. Math. Soc. 340 (1993), 227–241.

[T1] H. Tamvakis, *Bott–Chern forms and arithmetic intersections,* Enseign. Math. (2) 43 (1997), 33–54.

[T2] ———, *Arithmetic intersection theory on flag varieties,* Math. Ann. 314 (1999), 641–665.

[T3] ———, *Schubert calculus on the arithmetic Grassmannian*, Duke Math. J. 98 (1999), 421–443.

[T4] ———, *Arakelov theory of the Lagrangian Grassmannian*, J. Reine Angew. Math. 516 (1999), 207–223.

[W] A. Weil, *Arithmetic on algebraic varieties,* Ann. of Math. (2) 53 (1951), 412–444.

Department of Mathematics
University of Pennsylvania
Philadelphia, PA 19104-6395

harryt@math.upenn.edu