Michigan Math. J. 48 (2000)

The Algebra of Jets

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To W. Fulton on the occasion of his 60th birthday

0. Introduction

0.1. Let A be a not necessarily commutative algebra over a commutative ring k, and let $d\colon A \to M$ be a derivation into an A-A-module. In this article we construct noncommutative k-algebras $J^0 = A, J^1, \dots,$ depending on d, that are A-A-modules, with natural inclusions $M \to J^n$ for $n \geq 1$ and with natural k-algebra homomorphisms $r^n\colon J^n \to J^{n-1}$. The algebras J^n fit into exact sequences $M^{\otimes_A n} \xrightarrow{i^n} J^n \xrightarrow{r^n} J^{n-1} \to 0$, where i^n is induced by the multiplication on J^n; the sequence is short exact when M is flat over A.

When M is free we show that J^n has a second natural structure as an algebra under a multiplication, the *shuffle* product. Denote J^n with this product by J^n_{shuff}. We show that J^n_{shuff} has a subalgebra J^n_{sym} of symmetric elements, depending on the choice of basis of M, and we have natural exact sequences $0 \to (M^{\otimes_A n})^{S_n} \xrightarrow{i^n} J^n_{\text{sym}} \xrightarrow{r^n} J^{n-1}_{\text{sym}} \to 0$.

When A is commutative we show that J^n always has a shuffle product, which coincides with the aforementioned product when M is free. In the commutative case we define, for every k-linear map $\varphi\colon M \to \bigwedge^2 M$ such that $\varphi d = 0$, a subalgebra J^n_φ of J^n_{shuff}; we also give natural conditions for J^n_φ to be equal to J^n_{sym} when M is a free A-module.

Finally, we show that the theory globalizes.

0.2. Let X be a Riemann surface. For each integer $n \geq 0$ there is a natural locally free $\mathcal{O}_X$-module $\mathcal{J}^n_X$ of rank $n+1$, called the *bundle of n-jets,* with a natural $\mathbb{C}$-linear map

$$\delta\colon \mathcal{O}_X \to \mathcal{J}^n_X.$$

The bundle, with the map δ, is characterized by the following property: Associated with each parameter x on an open subset U of X is an $\mathcal{O}_U$-basis $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n$ of $\mathcal{J}^n_X\big|_U$, in terms of which $\delta\big|_U$ is given by the *Taylor expansion,*

$$\delta(f) = f\varepsilon_0 + (df/dx)\varepsilon_1 + \cdots + (d^n f/dx^n)\varepsilon_n. \tag{0.2.1}$$

More classically, the coefficients are taken with denominators $(1/i!)(d^i f/dx^i)$.

Received January 31, 2000. Revision received March 31, 2000.
The second author was supported in part by the Danish Natural Science Research Council.

The jet bundle $\mathcal{J}_X^n$ may be constructed by gluing local pieces as follows. Let y be a second parameter on U. Clearly, then, for $f \in \mathcal{O}_X(U)$ there is an equation

$$(f, df/dy, \dots, d^n f/dy^n) = (f, df/dx, \dots, d^n f/dx^n)T(x, y)$$

with a matrix $T(x, y)$ whose coefficients are polynomials in the derivatives $d^i x/dy^i$ for $i = 1, \dots, n$. It is easily verified that the matrices $T(x, y)$ satisfy the cocycle conditions: $T(x, x) = 1$ and, if z is a third parameter on U, then $T(x, z) = T(x, y)T(y, z)$. These properties suffice to define $\mathcal{J}_X^n$ and the map δ, first locally on trivial open subsets and then globally by gluing the local pieces.

It is easy to see that the jet bundles $\mathcal{J}_X^n$ come with a sequence of surjections,

$$\cdots \to \mathcal{J}_X^{n+1} \to \mathcal{J}_X^n \to \cdots \to \mathcal{J}_X^1 \to \mathcal{J}_X^0 = \mathcal{O}_X.$$

For $n = 1$, we have the decomposition

$$\mathcal{J}_X^1 = \mathcal{O}_X \oplus \Omega_X^1, \qquad \delta(f) = f + df,$$

where Ω_X^1 is module of differentials and d is the global differential,

$$d\colon \mathcal{O}_X \to \Omega_X^1. \tag{0.2.2}$$

The construction is well known and is easily generalized to other types of manifolds.

There are several other constructions of jets—for example, the principal parts $\mathcal{P}_X^n$ defined for any scheme or any analytic space X. In analytic geometry it is more usual to start with the module of nth-order differential operators, $\mathcal{D}iff_X^n$, related to the principal parts by $\mathcal{D}iff_X^n = \mathcal{H}om(\mathcal{P}_X^n, \mathcal{O}_X)$, and define jets to be the module $\mathcal{H}om(\mathcal{D}iff_X^n, \mathcal{O}_X)$. When X is smooth, the two definitions coincide.

0.3. It is a fundamental observation, made by Gatto [G1], that the construction of jets depends only on the derivation d of (0.2.2) in the following sense. Let X be any variety, and let

$$d\colon \mathcal{O}_X \to \mathcal{M} \tag{0.3.1}$$

be any derivation into an invertible $\mathcal{O}_X$-module $\mathcal{M}$. Then there is an analogous construction leading to a globally defined locally free rank-$(n+1)$ module $\mathcal{J}^n = \mathcal{J}_{\mathcal{M},d}^n$ and a $\mathbb{C}$-linear map

$$\delta\colon \mathcal{O}_X \to \mathcal{J}_{\mathcal{M},d}^n.$$

To construct δ we associate with each basis ε for $\mathcal{M}\big|_U$ over an open set U the derivation $D_\varepsilon\colon \mathcal{O}_U \to \mathcal{O}_U$ defined by the equation

$$df = (D_\varepsilon f)\varepsilon.$$

Define $\mathcal{J}_{\mathcal{M},d}^n\big|_U$ as the free $\mathcal{O}_U$-module on a basis $(\varepsilon_0, \dots, \varepsilon_n)$ and $\delta\colon \mathcal{O}_U \to \mathcal{J}_{\mathcal{M},d}^n\big|_U$ by the expansion, analogous to (0.2.1),

$$\delta(f) = f\varepsilon_0 + (D_\varepsilon f)\varepsilon_1 + \cdots + (D_\varepsilon^n f)\varepsilon_n. \tag{0.3.2}$$

Let η be a second basis for $\mathcal{M}\big|_U$, say $\varepsilon = u\eta$ for a unit u of $\mathcal{O}_X(U)$. Then there is an equation

$$(f, D_\eta f, \ldots, D_\eta^n f) = (f, D_\varepsilon f, \ldots, D_\varepsilon^n f) T(\varepsilon, \eta)$$

with a matrix $T(\varepsilon, \eta)$ whose coefficients are polynomials in the derivatives $D_\varepsilon^i(u)$ for $i = 0, \ldots, n-1$. Again the cocyle conditions are verified, and the bundle $\mathcal{J}^n_{\mathcal{M},d}$ together with δ is obtained by gluing the local pieces.

0.4. The theory presented in Section 0.1, when applied to the case when X is a scheme and $d\colon \mathcal{O}_X \to \mathcal{M}$ is a derivation into an arbitrary $\mathcal{O}_X$-module, will give an intrinsic definition of the modules $\mathcal{J}^n = \mathcal{J}^n_{\mathcal{M},d}$. A technical advantage of the approach (as compared to the constructions sketched in Sections 0.2 and 0.3) is that the intrinsic definition avoids the tedious verification of the cocycle relations. Our main construction in Section 1.6 is conceptual and simple, and it allows natural proofs of the properties of the sheaves $\mathcal{J}^n$. However, the most important feature of the intrinsic definition is that it throws additional light on the properties of $\mathcal{J}^n$. For instance, it becomes clear that:

(1) $\mathcal{J}^n$ is, in a natural way, a noncommutative algebra over the ground field;
(2) $\mathcal{J}^n$ contains $\mathcal{O}_X$ as a subalgebra;
(3) the map $\delta\colon \mathcal{O}_X \to \mathcal{J}^n$ is a map of algebras;
(4) there is a natural imbedding $\mathcal{M} \to \mathcal{J}^n$, and when $\mathcal{M}$ is locally free then the $\mathcal{O}_X$-modules $\mathcal{J}^n$ are free. If $\mathcal{M}$ is invertible with basis ε over an open set U, then the base elements ε_i considered in (0.3.2) are the powers ε^i. In particular, in (0.2.1) we have that $\varepsilon_i = (dx)^i$.

Let us emphasize that the algebra $\mathcal{J}^n$ is necessarily noncommutative: in a commutative algebra, the equation (0.3.2), with $\varepsilon_i = \varepsilon^i$ and $\delta(fg) = \delta(f)\delta(g)$, would contradict the Leibniz rule for the higher derivatives of a product.

0.5. The construction of jet bundles from a derivation (0.3.1) arose in the study of Weierstrass points for families of curves. In earlier work [LT1], [LT2], [LT3] we have shown how Weierstrass points for a family $X \to S$ can be defined in a natural way from a *Wronski system* on the family. Such a system consists of maps between locally free $\mathcal{O}_X$-modules. For smooth families the relative principal parts give rise to a Wronski system. In constructing a Wronski system, Gatto was led to the construction of jet bundles as just described for a smooth curve; this was extended to smooth families by Gatto and Ponza [GP]. In both cases, the construction is performed for the global relative differentials analogous to (0.2.2) and in characteristic zero. In these cases, as indicated above and proved below, the module of jets is isomorphic to the module of principal parts.

The importance of the construction of Gatto and Ponza comes from its use to families that are not smooth but where there is a natural derivation $d\colon \mathcal{O}_X \to \omega_{X/S}$ into an invertible $\mathcal{O}_X$-module. In such cases the principal parts are not locally free, but the construction gives locally free jets. Gatto [G2] used the construction for a Gorenstein curve with the composite $\mathcal{O}_X \to \omega_{X/S}$ of the global differential with the natural map $\Omega^1_{X/S} \to \omega_{X/S}$ into the *dualizing sheaf*, sketching in [G3] such a construction in the case of families of stable curves. Our construction allows us to construct Wronski systems whenever we have a derivation $d\colon \mathcal{O}_X \to \mathcal{M}$ into a locally free $\mathcal{O}_X$-module. It therefore makes possible the definition and study of Weierstrass points under very general conditions.

A natural homomorphism $\Omega^1_{X/S} \to \omega_{X/S}$ was constructed by Buchweitz and Greuel [BG, Thm. (4.2.4)] for a flat family of germs of reduced curves over the complex numbers. It also exists for flat families that are locally complete intersections. We do not know whether such a map exists for any flat family of curves in any characteristic.

In arbitrary characteristic, Esteves [E1] constructed Wronski systems for flat families that are locally complete intersections. For families of Gorenstein curves with a simultaneous resolution by a smooth family, yet another construction of Wronski systems was given in [LT2].

The jets constructed here give a Wronski system for a family X/S of Gorenstein curves and a map $\Omega^1_{X/S} \to \omega_{X/S}$ into the invertible dualizing sheaf $\omega_{X/S}$. However, in positive characteristic, the construction is not satisfactory in the sense that it does not give the principal parts even in the case when X is a single smooth curve. It would be interesting to know the kind of extra information needed, in addition to the map $\Omega^1_{X/S} \to \omega_{X/S}$, for a satisfactory construction of Wronski systems.

1. Jets

In this section we present the construction of the algebra of jets J associated to a derivation $d\colon A \to M$ from a not necessarily commutative algebra A over a commutative ring k, into an A-A-module M. By truncation we obtain the algebras of n-truncated jets $J^n = J^n_{M,d}$ that are the main objects of study in the article. We establish the main properties of these jets. In particular, we prove that they fit into exact sequences similar to the well-known exact sequences for principal parts.

1.1. SETUP. Let A be a not necessarily commutative algebra over a commutative ground ring k. Moreover, let M be an A-A-module—that is, a k-module on which A acts from the left and from the right, so that the two actions commute and extend the given action of k. We fix a derivation

$$d\colon A \to M,$$

a k-linear map such that $d(fg) = f(dg) + (df)g$. Throughout, linearity is with respect to the ground ring k. The dependence on k is often omitted in the notation. We use the convention that an *algebra*, with no further specification, is a k-algebra with unit.

1.2. DEFINITION. An *A-A-algebra E* is an A-A-module with a k-bilinear product, $x \otimes y \mapsto x \cdot y$, which is associative and left A-linear in the first factor and right A-linear in the second:

$$(fx) \cdot y = f(x \cdot y), \qquad x \cdot (yf) = (x \cdot y)f. \tag{1.2.1}$$

Note that, in contrast to the convention used for k-algebras, we do not require an A-A-algebra to have a unit, nor do we require the product to be *A-balanced*—that is, we do not require that $x \cdot (fy) = (xf) \cdot y$. Let E be an A-A-algebra. A derivation $D\colon A \to E$ is called *balanced* if

$$x \cdot (fy) = (xf) \cdot y + x \cdot Df \cdot y. \tag{1.2.2}$$

1.3. NOTE. Let E be an A-A-algebra with a unit 1_E. Then E is a k-algebra. Moreover, the equations $\delta(f) := f1_E$ and $\iota(f) := 1_E f$ define maps of algebras $\delta, \iota\colon A \to E$. There is a unique balanced derivation $d_E\colon A \to E$ determined by the equation $d_E(f) = f1_E - 1_E f = \delta(f) - \iota(f)$. The structure of E as an A-A-module is determined by the maps δ and ι by the equations $fx = \delta(f) \cdot x$ and $xf = x \cdot \iota(f)$, respectively.

On the other hand, take a k-algebra E with a pair of algebra homomorphisms $\delta, \iota\colon A \to E$; then, with the structure of an A-A-module determined by the equations $fx = \delta(f) \cdot x$ and $xf = x \cdot \iota(f)$, we have that E is an A-A-algebra. The unique balanced derivation d_E is given by $d_E(f) = \delta(f) - \iota(f)$.

1.4. ACTIONS ON A-A-ALGEBRAS. Let E be an A-A-algebra and $D\colon A \to E$ a balanced derivation. We define new left and right actions of A on E as follows:

$$\iota(f) \cdot x := fx - Df \cdot x,$$

$$x \cdot \delta(f) := xf + x \cdot Df.$$

We will indicate with the notation ${}_\iota E$ (resp. E_δ) that E is considered as an A-A-module via the left action ι and the original right action (resp., with the original left action and the right action δ). The new left action of A on ${}_\iota E$ will also be written $f \cdot x := \iota(f) \cdot x$.

The product ${}_\iota E \otimes {}_\iota E \to {}_\iota E$ is left A-linear in the first factor and right A-linear in the second. It is also balanced; that is, $x \cdot (f \cdot y) = (xf) \cdot y$.

1.5. ADJUNCTION OF A UNIT. Let E be an A-A-algebra and $D\colon A \to E$ a balanced derivation. Since the product on ${}_\iota E$ is balanced, the direct sum

$$\tilde{E} := A \oplus {}_\iota E, \tag{1.5.1}$$

becomes a k-algebra under the product $(f, x) \cdot (g, y) = (fg,\ f \cdot y + xg + x \cdot y)$. The maps $\iota(f) := (f, 0)$ and $\delta(f) := (f, Df)$ are both maps of algebras $A \to \tilde{E}$. We shall identify $f \in A$ with $\iota(f) \in \tilde{E}$.

We have that $\tilde{E}$ is *augmented* over A by the algebra map $(f, x) \mapsto f$. Identify the augmentation ideal with E via the map $x \to (0, x)$. The augmentation map $\tilde{E} \to A$ is split by the map $\iota\colon A \to \tilde{E}$, and the resulting decomposition $\tilde{E} = \iota(A) \oplus E$ corresponds to the decomposition (1.5.1). Similarly, there is a decomposition

$$\tilde{E} = \delta(A) \oplus E. \tag{1.5.2}$$

1.6. THE ALGEBRA OF JETS. We shall construct an algebra $J = J_d = J_{M,d}$ associated to the derivation d.

Consider the tensor algebra $T = T_k M$ of M over k. Let $\mathcal{R}$ be the ideal generated by the differences for $\omega', \omega'' \in M$ and $f \in A$,

$$\omega' \otimes f\omega'' - \omega' f \otimes \omega'' - \omega' \otimes df \otimes \omega''. \tag{1.6.1}$$

The terms in (1.6.1) are of degrees 2, 2, and 3. Hence $\mathcal{R}$ is a nonhomogeneous ideal contained in $T_{\geq 2}$ and thus, in particular, is contained in the positive part T_+ of T. Clearly, the positive part T_+ is an A-A-algebra and the ideal $\mathcal{R}$ is an A-A-submodule. Let

$$J_+ := T_+/\mathcal{R}$$

be the residue class algebra of T_+ by the ideal $\mathcal{R}$. Then J_+ is an A-A-algebra, and the natural map $T_+ \to J_+$ is a homomorphism of A-A-algebras. Clearly, this map restricts to an A-A-linear embedding of the degree-1 part,

$$M \to J_+.$$

We will identify an element $\omega \in M$ with its image in J_+. Under this identification, the following equation holds in J_+ for $f \in A$ and $x, y \in J_+$:

$$x \cdot (fy) = (xf) \cdot y + x \cdot df \cdot y.$$

In other words, the given derivation d induces a balanced derivation $d\colon A \to J_+$.

We let $J = J_d = J_{M,d}$ be the algebra obtained from J_+ by adjoining a unit as in Section 1.5. The k-algebra J is called the *algebra of jets associated to* d. We have two injections of algebras $\iota, \delta\colon A \to J$, and we shall identify $f \in A$ with its image $\iota(f)$ in J. The algebra J comes with an augmentation that is split by ι and δ, and the splitting $J = \iota(A) \oplus J_+$ corresponds to the splitting $J = A \oplus {}_\iota J_+$ used in the adjunction of a unit. Hence, every jet φ has a unique decomposition $\varphi = \varphi_0 + \varphi_+$, with $\varphi_0 \in A$ and $\varphi_+ \in J_+$. The term φ_0 is called the *constant term* of φ. Under the embedding of M into J, we have

$$\delta(f) = f + df.$$

The algebra J has two structures as a left A-module, via ι and via δ; likewise, it has two structures as a right A-module. On the ideal J_+, the structures are given by the equations

$$\iota(f) \cdot x = f \cdot x = fx - df \cdot x, \quad x \cdot \iota(f) = xf,$$

$$\delta(f) \cdot x = fx, \quad x \cdot \delta(f) = xf + x \cdot df.$$

Note the distinction between fx and $f \cdot x$. Note also that the decompositions $J = A \oplus J_+$ and $J = \delta(A) \oplus J_+$ are decompositions of A-A-modules,

$${}_\iota J = A \oplus {}_\iota(J_+), \qquad J_\delta = \delta(A) \oplus (J_+)_\delta. \tag{1.6.2}$$

1.7. THE UNIVERSAL PROPERTY. It follows from the construction of jets in Section 1.6 and Note 1.3 that the A-A-algebra J_+, as well as the algebra $J = J_{M,d}$, have the following universal properties.

Let E be an A-A-algebra. Then any A-A-linear map $\chi\colon M \to E$ such that $\chi d\colon A \to E$ is a balanced derivation factors uniquely through a map of A-A-algebras $J_+ \to E$.

Let E be a k-algebra with a pair of maps of algebras $\delta_E, \iota_E\colon A \to E$, and let E have the A-A-module structure given by δ_E to the left and ι_E to the right. Then any A-A-linear map $\chi : M \to E$ such that $\chi d = \delta_E - \iota_E$ factors uniquely through a map of k-algebras $J \to E$ commuting with the maps δ and ι.

1.8. PROPOSITION. *Multiplication in the A-A-algebra J_+ induces a J-A-linear injection $(J_+)_\delta \otimes_A M \to J_+$ that gives rise to a decomposition of J_+ into A-A-submodules,*

$$J_+ = M \oplus [(J_+)_\delta \otimes_A M]. \tag{1.8.1}$$

In addition, multiplication in J induces a J-A-isomorphism $J_\delta \otimes_A M = J_+$.

Proof. The A-A-module $T_+ = T_+ M$ is the direct sum of the degree-1 piece $T_1 = M$ and $T_{\geq 2} = T_+ \otimes M$. As noted previously, the ideal $\mathcal{R}$ defining J_+ is contained in $T_{\geq 2}$. Hence, if $K := T_{\geq 2}/\mathcal{R}$ is the residue class module of the A-A-module $T_{\geq 2}$ modulo $\mathcal{R}$, we obtain the decomposition

$$J_+ = M \oplus K.$$

Multiplication in J_+ induces a surjection $J_+ \otimes M \to K$, which factors through $(J_+)_\delta \otimes_A M$ since d is balanced. The composition

$$T_{\geq 2} = T_+ \otimes M \to J_+ \otimes M \to (J_+)_\delta \otimes_A M$$

vanishes on the ideal $\mathcal{R}$. Hence we obtain an inverse $K = T_{\geq 2}/\mathcal{R} \to (J_+)_\delta \otimes_A M$ to the map $(J_+)_\delta \otimes_A M \to K$, which is therefore injective, and we have obtained the splitting (1.8.1).

To see that the multiplication $J_\delta \otimes_A M \to J_+$ is an isomorphism, we use the decomposition (1.6.2) to obtain a decomposition $J_\delta = \delta(A) \oplus (J_+)_\delta$ of A-A-modules; then we use the decomposition (1.8.1) on the target. □

1.9. TRUNCATED JETS. For each integer $n \geq 0$ we define the *k-algebra of n-truncated jets, $J^n = J^n_{M,d}$*, as the residue algebra of J modulo the $(n+1)$th power of the augmentation ideal J_+,

$$J^n := J/(J_+)^{n+1}.$$

The algebra J^n becomes an A-augmented k-algebra with augmentation ideal $J^n_+ = J_+/(J_+)^{n+1}$ and with maps of algebras $\iota, \delta \colon A \to J^n$ obtained as the compositions of $\iota, \delta \colon A \to J$ with the residue class map $J \to J^n$.

For $n \geq 1$, we have surjective maps of k-algebras,

$$J^n \xrightarrow{r^n} J^{n-1},$$

commuting with the maps ι and δ, and we have A-A-linear inclusions,

$$M \to J^n,$$

commuting with the surjections r^n.

The algebra J^n is universal in the following sense. Let E be a k-algebra with a pair of maps of algebras $\delta_E, \iota_E \colon A \to E$, and let E have the A-A-module structure given by δ_E to the left and ι_E to the right. Then any A-A-linear map $\chi : M \to E$

such that $\chi d = \delta_E - \iota_E$ and $\chi(M)^{n+1} = 0$ factors uniquely through a map of k-algebras $J^n \to E$ commuting with the maps δ and ι.

We have that $\delta, \iota\colon A \to J^n$ split the augmentation on J^n. Identifying $f \in A$ with $\iota(f) \in J^n$, we obtain the decompositions of A-A-modules ${}_\iota J^n = A \oplus {}_\iota(J^n_+)$ and $(J^n)_\delta = \delta(A) \oplus (J^n_+)_\delta$.

We have that $J^0 = A$, and the map $M \to J^0$ is the zero map. In the following, we fix $n \geq 1$.

1.10. PROPOSITION. *Multiplication in the A-A-algebra J^n_+ induces a J^n-A-linear injection $(J^{n-1}_+)_\delta \otimes_A M \to J^n_+$, and there is a decomposition of J^n_+ into A-A-submodules,*

$$J^n_+ = M \oplus [(J^{n-1}_+)_\delta \otimes_A M].$$

In addition, multiplication in J^n induces a J^n-A-isomorphism $J^{n-1}_\delta \otimes_A M = J^n_+$.

Proof. The assertions follow easily from (1.8). □

1.11. COROLLARY. *Multiplication in J^n induces a natural map $i^n\colon M^{\otimes_A n} \to J^n$. The map i^n is left A-linear with respect to both left structures on J^n and right A-linear with respect to both right structures on J^n. The following sequence is exact*:

$$M^{\otimes_A n} \xrightarrow{i^n} J^n \xrightarrow{r^n} J^{n-1} \to 0, \tag{1.11.1}$$

and i^n is injective if M is left A-flat.

Proof. The kernel of the surjection $r^n\colon J^n \to J^{n-1}$ is additively generated by all products of n elements of M. It follows that there is an exact sequence,

$$M^{\otimes n} \to J^n \to J^{n-1} \to 0,$$

where the tensor product $M^{\otimes n}$ is over k. We have that $\omega' \cdot (f\omega'') = (\omega' f) \cdot \omega'' + \omega' \cdot df \cdot \omega''$ in J^n, and the product of $n+1$ elements of M vanishes in J^n. It follows that the map $M^{\otimes n} \to J$ factors through $M^{\otimes_A n}$ and that the resulting map $i^n\colon M^{\otimes_A n} \to J^n$ is left A-linear when the target is considered as the A-module ${}_\iota J^n$. The map i^n is trivially left A-linear for the original structure on J^n. Similarly, the map i^n is right A-linear for both structures on J^n.

To prove the last part of the corollary, we note that the map i^{n+1} is the composition

$$M^{\otimes_A n} \otimes_A M \xrightarrow{i^n \otimes_A 1} J^n_\delta \otimes_A M \to J^{n+1}.$$

It follows from Proposition 1.10 that the right map is injective. When M is left A-flat, it follows by induction on n that the left map is injective. Hence the last assertion of the corollary holds. □

1.12. NOTE. It follows from Corollary 1.11 that when $\mathcal{M}$ is a finitely generated A-A-module the same is true for J^n, and when $\mathcal{M}$ is a left flat A-module the same is true for J^n.

1.13. NOTE. Let A^{op} be the opposite algebra and let M^{op} be the A^{op}-A^{op}-module obtained by viewing the given left and right A-module structures on M as a right and left A^{op}-module structures. View d as a derivation $A^{\mathrm{op}} \to M^{\mathrm{op}}$, denoted d^{op}. Let $\iota^{\mathrm{op}}, \delta^{\mathrm{op}} \colon A^{\mathrm{op}} \to J^{\mathrm{op}}$ be the maps of algebras defined by $\iota^{\mathrm{op}} := \delta$ and $\delta^{\mathrm{op}} := \iota$. Finally, let $\chi \colon M \to J$ be the map $\chi(\omega) = -\omega$. Then $\chi(df) = -df = \iota(f) - \delta(f) = \delta^{\mathrm{op}}(f) - \iota^{\mathrm{op}}(f)$, and $\chi \colon M^{\mathrm{op}} \to J^{\mathrm{op}}$ has the linearity properties required for applying the universal property. As a consequence, we obtain the isomorphism

$$J_{M^{\mathrm{op}}, d^{\mathrm{op}}} \simeq (J_{M,d})^{\mathrm{op}} \tag{1.13.1}$$

and a similar isomorphism for the n-truncated jets. In the commutative case we have that $d^{\mathrm{op}} = d$, so the isomorphism may be viewed as an anti-involution of the algebra of jets.

In particular, assume that M is right A-flat. Then M^{op} is left A^{op}-flat. Hence $(J^n)^{\mathrm{op}}$ is left A^{op}-flat with respect to its two left A^{op}-structures. Hence J^n is right A-flat with respect to its two right A-structures.

1.14. Definition. Let L be a left A-module. Define the module of *L-twisted* jets $J^n(L) = J^n_{M,d}(L)$ as the tensor product

$$J^n(L) := J^n_\delta \otimes_A L.$$

1.15. Note. Note that $J^n(L)$ is a left J^n-module and in particular, via the inclusion $\iota \colon A \to J^n$, a left A-module. From the algebra homomorphism $\delta \colon A \to J$, we obtain a homomorphism

$$\delta_L = \delta \otimes 1_L \colon L \to J^n(L).$$

Clearly,

$$\delta_L(fx) = \delta(f) \cdot \delta_L(x).$$

From the exact sequence in Corollary 1.11 we obtain the exact sequence

$$M^{\otimes_A n} \otimes_A L \to J^n(L) \to J^{n-1}(L) \to 0;$$

it is left exact if M is right A-flat.

1.16. Note. The formation of jets commutes with scalar extensions as follows. Let $k \to k'$ be a homomorphism of commutative rings, and form the k'-algebra $A' = A \otimes_k k'$. Then $M' := M \otimes_k k'$ is an A'-A'-module and $d' := d \otimes 1$ is a derivation,

$$d' \colon A' \to M'.$$

From the universal property, we obtain isomorphisms of noncommutative algebras:

$$J^n_{M',d'} \simeq J^n_{M,d} \otimes k'.$$

2. Jets on Free Modules

In this section we study the jets $J^n = J^n_{M,d}$ when M is free and of finite rank over the not necessarily commutative algebra A. We describe explicitly a basis for J^n in terms of a given basis $\varepsilon_1, \dots, \varepsilon_r$ of M. Relative to the basis $\varepsilon_1, \dots, \varepsilon_r$ we define a shuffle product on J^n that makes J^n into a commutative algebra J^n_{shuff}, and we define symmetric elements with respect to the basis and show they form a subalgebra J^n_{sym} of J^n_{shuff}. We show that the shuffle product is independent of the choice of basis, whereas J^n_{sym} depends on the basis. The main tool used in the study of the shuffle product and symmetric elements will be the partial derivatives introduced in Section 2.3.

2.1. SETUP. Keep the setup of Section 1. We shall call the A-A-module M *free of rank* r if it has a *basis* $\varepsilon_1, \dots, \varepsilon_r$; that is, the elements ε_i are central (i.e., $\varepsilon_i f = f\varepsilon_i$ for all $f \in A$) and every element ω of M has an expansion

$$\omega = \varepsilon_1 f_1 + \cdots + \varepsilon_r f_r$$

with uniquely determined coefficients $f_i \in A$. With respect to the basis, let $D_i \colon A \to A$ be the derivation induced by d and the ith coordinate projection; that is, the D_i are defined by the equation in M,

$$df = \varepsilon_1 D_1 f + \cdots + \varepsilon_r D_r f.$$

2.2. PROPOSITION. *Assume that $\varepsilon_1, \dots, \varepsilon_r$ is a basis for M as an A-A-module. For any sequence $\alpha = (\alpha_1, \dots, \alpha_t)$ of indices with $1 \le \alpha_j \le r$, form the following product in J^n and the composition of derivations on A:*

$$\varepsilon_\alpha := \varepsilon_{\alpha_1} \cdots \varepsilon_{\alpha_t}, \qquad D_\alpha := D_{\alpha_1} \cdots D_{\alpha_t}.$$

Then, for all sequences $\alpha = (\alpha_1, \dots, \alpha_t)$ of length $t = \ell(\alpha)$ at most n, the set of products ε_α is a free A-basis with respect to any of the four A-actions on J^n. Moreover, for $f \in A$, we have the following equation in J^n:

$$\delta(f) = \sum_\alpha D_\alpha(f) \cdot \varepsilon_\alpha. \tag{2.2.1}$$

Proof. The assertions are proved by induction on n. We have that the products ε_α in J^{n-1}, for $\ell(\alpha) < n$, form a basis of J^{n-1} by the induction hypothesis, and the natural basis of $M \otimes_A \cdots \otimes_A M$ is mapped by the map i^n of Corollary 1.11 to the products ε_α in J^n for $\ell(\alpha) = n$. The first assertion thus follows from Corollary 1.11.

To prove equation (2.2.1) we note that, under the identification $J^n_+ = J^{n-1}_\delta \otimes_A M$ of Proposition 1.10, we have $\varepsilon_i D_i f = (D_i f)\varepsilon_i = \delta(D_i f) \otimes \varepsilon_i$. Hence,

$$\delta(f) = f + df = f + \sum_i \varepsilon_i D_i f = f + \sum_i \delta(D_i f) \otimes \varepsilon_i.$$

Obviously, the formula now follows by applying the induction hypothesis to the elements $\delta(D_i f) \in J^{n-1}$. □

2.3. PARTIAL DERIVATION. Assume that the A-A-module M is free with a given basis $\varepsilon_1, \dots, \varepsilon_r$. For $i = 1, \dots, r$, define the *partial derivation* $\partial_i = \partial_i^n$ as the composition

$$\partial_i \colon J^{n+1} \to J^{n+1}_+ = J^n_\delta \otimes_A M \to J^n,$$

where the first map is the projection corresponding to the decomposition $J^{n+1} = A \oplus J_+^{n+1}$, the identification is that of Proposition 1.10, and the last map is induced by the ith coordinate projection. In other words, under the identifications $J^{n+1} = A \oplus J_+^{n+1}$ and $J_+^{n+1} = J_\delta^n \otimes_A M$, we have

$$\varphi = \varphi_0 + \sum_i \partial_i \varphi \otimes \varepsilon_i,$$

where $\varphi_0 \in A$ is the constant term of φ.

Note the following equation for φ, ψ in J^{n+1}:

$$\partial_i(\varphi \cdot \psi) = \partial_i \varphi \cdot \delta(\psi_0) + r\varphi \cdot \partial_i \psi, \tag{2.3.1}$$

where $r\varphi = r^{n+1}\varphi \in J^n$ is the restriction of $\varphi \in J^{n+1}$. Indeed,

$$\varphi \cdot \psi_0 = \varphi_0 \psi_0 + \sum_i (\partial_i \varphi \cdot \delta(\psi_0)) \otimes \varepsilon_i$$

and

$$\varphi \cdot \sum_i \partial_i \psi \otimes \varepsilon_i = \sum_i (r\varphi \cdot \partial_i \psi) \otimes \varepsilon_i,$$

so (2.3.1) follows from the resulting expression for $\varphi \cdot \psi = \varphi \cdot \psi_0 + \varphi \cdot \sum_i \partial_i \psi \otimes \varepsilon_i$.

For $\omega = \sum_i \varepsilon_i f_i$ in M, we have that $\omega = \sum_i \delta(f_i) \otimes \varepsilon_i$ and thus $\partial_i \omega = \delta(f_i)$. It follows from (2.3.1) that

$$\partial_i(\varphi \cdot \omega) = r\varphi \cdot \delta(f_i). \tag{2.3.2}$$

In particular,

$$\partial_i(\delta(f)) = \partial_i(df) = \delta(D_i f). \tag{2.3.3}$$

2.4. Definition. Let $\varepsilon_1, \dots, \varepsilon_r$ be a basis for the A-A-module M. The products ε_α for $\ell(\alpha) \leq n$ form a basis for ${}_\iota J^n$ as a left A-module. Define, relative to the basis $\varepsilon_1, \dots, \varepsilon_r$, a *shuffle multiplication* on the basis elements of J^n by

$$\varepsilon_\alpha * \varepsilon_\beta := \sum_\gamma \varepsilon_\gamma,$$

where the sum is over all shuffles γ of the two sets of indices α and β. Extend the product to all of J^n so that the elements of A commute with the base elements ε_α, that is, $f * \varepsilon_\alpha = \varepsilon_\alpha * f = f \cdot \varepsilon_\alpha$. In particular, then, $f * g = fg$ for $f, g \in A$.

Note that the base elements commute with respect to the shuffle product and commute with the elements of A, but the shuffle product is only commutative when A is commutative. Note also that the νth power of ε_i with respect to the shuffle product is given by the equation

$$\varepsilon_i^{*\nu} = \nu!\, \varepsilon_i^\nu,$$

where ε_i^ν is the power with respect to the original multiplication in J^n.

Clearly, with the shuffle product, J^n is an algebra (denoted J^n_{shuff}) with unit 1 corresponding to the empty product of base elements. The restrictions r^{n+1} are maps of algebras $J^{n+1}_{\mathrm{shuff}} \to J^n_{\mathrm{shuff}}$. Moreover, ι is a map of algebras $A \to J^n_{\mathrm{shuff}}$.

The maps ∂_i are derivations with respect to the shuffle product, that is,

$$\partial_i(\varphi * \psi) = \partial_i\varphi * r\psi + r\varphi * \partial_i\psi, \tag{2.4.1}$$

where $r\varphi = r^{n+1}\varphi \in J^n_{\text{shuff}}$ is the restriction of $\varphi \in J^{n+1}_{\text{shuff}}$. Indeed, it follows from (2.3.1) that $\partial_i(f \cdot \psi) = f \cdot \partial_i(\psi)$. Hence ∂_i is a left A-linear map ${}_\iota J^{n+1} \to {}_\iota J^n$. It therefore suffices to prove (2.4.1) when φ and ψ are the base elements $\varepsilon_\alpha = \varepsilon_{\alpha_1} \cdots \varepsilon_{\alpha_s}$ and $\varepsilon_\beta = \varepsilon_{\beta_1} \cdots \varepsilon_{\beta_t}$, respectively. Then the formula (2.4.1) is easily checked by dividing into classes according to whether the α_s or β_t are equal to i.

We also have the equation

$$\varphi \cdot \delta(f) = \varphi * \delta(f). \tag{2.4.2}$$

Indeed, the two sides have the same constant term, $\varphi_0 \cdot f$. Hence it suffices to prove that the two sides become equal under ∂_i. We proceed by induction on n. By (2.3.1) and (2.4.1), we have

$$\begin{aligned} \partial_i(\varphi \cdot \delta(f)) &= \partial_i(\varphi) \cdot \delta(f) + r\varphi \cdot \partial_i(\delta f), \\ \partial_i(\varphi * \delta(f)) &= \partial_i(\varphi) * \delta(f) + r\varphi * \partial_i(\delta f). \end{aligned} \tag{2.4.3}$$

By (2.3.3), we have that $\partial_i(\delta(f)) = \delta(D_i f)$. It follows from the induction hypothesis that (2.4.2) holds for $n-1$. Hence $\partial_i(\varphi) \cdot \delta(f) = \partial_i(\varphi) * \delta(f)$ and $r\varphi \cdot \partial_i(D_i f) = r\varphi * \partial(D_i f)$. Consequently, it follows from (2.4.3) that $\partial_i(\varphi \cdot \delta(f)) = \partial_i(\varphi * \delta(f))$, as we wanted to prove.

Since $\delta\colon A \to J^n$ is a map of algebras, it follows from (2.4.2) that δ is a map of algebras $\delta\colon A \to J^n_{\text{shuff}}$.

2.5. PROPOSITION. *The shuffle product in J^n is independent of the choice of basis.*

Proof. Consider a second basis $\tilde\varepsilon_1, \ldots, \tilde\varepsilon_r$ of M and the corresponding derivations $\tilde D_p$ and $\tilde\partial_p$. If (u_{ip}) is the transition matrix, then we have the equations

$$\tilde\varepsilon_p = \sum_i \varepsilon_i u_{ip}, \qquad D_i = \sum_p u_{ip}\tilde D_p.$$

Note that the u_{ip} belong to the center of A, as is seen by expanding the equation $f\tilde\varepsilon_p = \tilde\varepsilon_p f$ in the basis $\varepsilon_1, \ldots, \varepsilon_r$. Thus $u = u_{ip}$ as an element of J^n is in the center of J^n_{shuff}. We obtain that $\delta(u)$ is in the center of J^n_{shuff}. Indeed, by (2.2.1) it suffices to prove that $D_i u$ is in the center of A. However, apply D_i to the equation $fu = fu$, and use $u(D_i f) = (D_i f)u$ to deduce that $(D_i u)f = f(D_i u)$.

The equation in J^{n+1},

$$\varphi \mathbin{\tilde *} \psi = \varphi * \psi, \tag{2.5.1}$$

where the left term is the product with respect to the second basis, will be proved by induction on n.

Note first that, for $\varphi \in J^{n+1}$,

$$\varphi - \varphi_0 = \sum_p \tilde\partial_p(\varphi) \otimes \tilde\varepsilon_p = \sum_{i,p} (\tilde\partial_p(\varphi) \cdot \delta(u_{ip})) \otimes \varepsilon_i.$$

Hence,

$$\partial_i \varphi = \sum_p \tilde{\partial}_p(\varphi) \cdot \delta(u_{ip}) = \sum_p \tilde{\partial}_p(\varphi) \mathbin{\tilde{*}} \delta(u_{ip}).$$

Since $\delta(u_{ip})$ is in the center of J^n_{shuff}, we have that $\partial_i = \sum_p \delta(u_{ip}) \mathbin{\tilde{*}} \tilde{\partial}_p$. Therefore, since $\tilde{\partial}_p$ is a derivation with respect to the operator $\tilde{*}$, it follows that ∂_i is a derivation with respect to $\tilde{*}$.

In equation (2.5.1) the two sides have the same constant term. Hence, it suffices to prove that the two sides of (2.5.1) yield the same result under ∂_i. Since ∂_i is a derivation with respect to the products $*$ and $\tilde{*}$ both, the equality follows from the induction hypothesis. □

2.6. DEFINITION. The base elements ε_α, defined from a given basis of M, may be grouped according to the length $l = \ell(\alpha)$, and the symmetric group S_l acts on the base elements ε_α with $\ell(\alpha) = l$ by permuting the indices $(\alpha_1, \ldots, \alpha_l)$. Call a jet in J^n *symmetric* with respect to the given basis if its degree-l part is symmetric under the action of S_l for all l.

Note that the different permutations of $(1, \ldots, 1, \ldots, r, \ldots, r)$, where the number i appears ν_i times and where $\nu_1 + \cdots + \nu_r = l$, correspond bijectively to the terms in the shuffle product $\varepsilon_1^{\nu_1} * \cdots * \varepsilon_r^{\nu_r}$.

Note that if $\varepsilon_1, \ldots, \varepsilon_r$ is basis for the A-A-module M, then the tensors $\varepsilon_{\alpha_1} \otimes \cdots \otimes \varepsilon_{\alpha_n}$ form a basis for the A-A-module $M^{\otimes_A n}$. Accordingly, the symmetric group S_n acts linearly on $M^{\otimes_A n}$. It is easily seen that the action is independent of the choice of basis; if A is commutative then the action is the usual action obtained by permuting the factors of the tensor product. We denote by $(M^{\otimes_A n})^{S_n}$ the A-A-submodule of invariant tensors.

2.7. PROPOSITION. *The subset J^n_{sym} of jets that are symmetric with respect to a given basis $\varepsilon_1, \ldots, \varepsilon_r$ for M is a free A-module having as basis the set of all shuffle products,*

$$\varepsilon^{(\nu)} := \varepsilon_1^{\nu_1} * \cdots * \varepsilon_r^{\nu_r}, \tag{2.7.1}$$

for exponents $\nu = (\nu_1, \ldots, \nu_r)$ with $|\nu| \le n$. In particular, J^n_{sym} is the subalgebra of J^n_{shuff} generated by the elements $\varepsilon_j^{\nu_j}$.

Moreover, we have an exact sequence

$$0 \to (M^{\otimes_A n})^{S_n} \xrightarrow{i^n} J^n_{\text{sym}} \xrightarrow{r^n} J^{n-1}_{\text{sym}} \to 0. \tag{2.7.2}$$

Proof. An element $\sum_\alpha f_\alpha \cdot \varepsilon_\alpha$ is symmetric if and only if, for any α and any permutation β of α, we have $f_\alpha = f_\beta$. In other words, the sums $\sum_\beta \varepsilon_\beta$, where the sum is over all different permutations of a given sequence, form a free A-basis for J^n_{sym}. The latter sums are exactly the elements of the form (2.7.1). It follows from the equation $\varepsilon_j^\nu * \varepsilon_j^\mu = \binom{\nu+\mu}{\nu}\varepsilon_j^{\nu+\mu}$ that J^n_{sym} is a subalgebra of J^n_{shuff}.

The sums $\sum_\beta \varepsilon_{\beta_1} \otimes \cdots \otimes \varepsilon_{\beta_n}$, where the sum is over all different permutations $\beta = (\beta_1, \ldots, \beta_n)$ of a given sequence $\alpha = (\alpha_1, \ldots, \alpha_n)$, form a basis for the A-module $(M^{\otimes_A n})^{S_n}$. We have that

$$i^n\Big(\sum_\beta \varepsilon_{\beta_1}\otimes\cdots\otimes\varepsilon_{\beta_n}\Big)=\sum_\beta \varepsilon_\beta.$$

The exact sequence (2.7.2) thus follows from Proposition 1.11. □

2.8. COROLLARY. *An element λ in J^n is symmetric if and only if $\partial_i\lambda$ is symmetric for all i and $\partial_i\partial_j\lambda=\partial_j\partial_i\lambda$ for all i, j.*

Proof. We use the description of J^n_{sym} from Proposition 2.7, along with the obvious equation

$$\partial_i(\varepsilon_j^\nu)=\begin{cases}\varepsilon_j^{\nu-1} & \text{if } i=j \text{ and } \nu>0,\\ 0 & \text{otherwise.}\end{cases}$$

Since ∂_i is an A-derivation with respect to the shuffle product, we obtain from this equation an obvious formula for the value of ∂_i on an element $\varepsilon^{(\nu)}$ of the form (2.7.1). It follows that ∂_i maps J^n_{sym} surjectively onto J^{n-1}_{sym} and that $\partial_i\partial_j\lambda=\partial_j\partial_i\lambda$ when λ is symmetric.

Assume conversely that $\partial_i\lambda\in J^{n-1}_{\mathrm{sym}}$ and that $\partial_i\partial_j\lambda=\partial_j\partial_i\lambda$. As $\partial_r\lambda$ is symmetric, there is an element $\mu\in J^n_{\mathrm{sym}}$ such that $\partial_r\lambda=\partial_r\mu$. By subtracting μ from λ, we may assume that $\partial_r\lambda=0$. Again, $\partial_{r-1}\lambda$ is symmetric and, by commutation, $\partial_r\partial_{r-1}\lambda=0$. Therefore, no ε_r^ν occurs in $\partial_{r-1}\lambda$. Hence there is a symmetric element μ, in which no ε_r^ν occurs, such that $\partial_{r-1}\lambda=\partial_{r-1}\mu$. By subtracting μ from λ, we may assume that $\partial_i\lambda=0$ for $i=r-1,r$. Proceed by descending induction on i to show, by subtracting symmetric elements from λ, that we may assume $\partial_i\lambda=0$ for all i. Then λ is a constant and hence λ is symmetric. □

2.9. NOTE. We note that the symmetric part J^n_{sym} depends on the choice of basis $\varepsilon_1,\dots,\varepsilon_r$. Obviously, for $n=1$ we have that $J^1_{\mathrm{sym}}=J^1$. In general, for $n\geq 2$ we do not have that M, as a submodule of J^n, is contained in J^n_{sym}. For instance, it follows from (2.2.1) that df is symmetric if and only if $D_iD_jf=D_jD_if$.

In Section 4 we consider conditions under which the algebra J^n_{sym} is independent of the choice of basis.

When $r=1$, we have that $J^n_{\mathrm{sym}}=J^n$.

3. The Shuffle Product

In this section we study the case when A is commutative and M is an A-module. We show that, in this case, the shuffle product on the tensor algebra of M induces a shuffle product on J^n that makes it into a commutative algebra J^n_{shuff}. When M is free, this shuffle product is the same as the shuffle product defined in Section 2 for free modules over not necessarily commutative rings. We show that the formation of J^n_{shuff} commutes with localization in A. This is a quite delicate property. We also show how our constructions and results of the first three sections globalize.

3.1. SETUP. In the rest of the paper we consider the commutative case exclusively; that is, A is assumed to be a commutative algebra and M is an A-module.

Denote by $*$ the shuffle product in the tensor algebra $T = T_k M$. The shuffle product of two tensor products is the sum of all tensor products obtained as shuffles of the two sets of factors. The algebra T with the shuffle product is a commutative algebra with unit.

3.2. LEMMA. *Modulo the ideal $\mathcal{R}$ in $T = T_k M$ defining J_+, the following three expressions, for $f \in A$ and $x, y \in T_+$ and $z \in T$, are congruent to zero*:

$$\begin{aligned} e_0 &:= fx - xf + df * x - df \cdot x - x \cdot df, \\ e_1 &:= [yf \cdot x - y \cdot fx + y \cdot df \cdot x] * z, \\ e_2 &:= f(x * z) - df \cdot (x * z) - (fx) * z + (df \cdot x) * z. \end{aligned}$$

Proof. Clearly, we may assume that the elements x, y, z are tensor products of elements of M, and in particular homogeneous elements of T.

Let us first observe that, for the shuffle product $u * v$ of two tensors of the form $u = \omega \cdot u'$ and $v = \pi \cdot v'$ where $\omega, \pi \in M$, we have

$$u * v = \omega \cdot (u' * v) + \pi \cdot (u * v'). \tag{3.2.1}$$

Indeed, (3.2.1) follows by separating the terms in the shuffle product according as ω or π is the first factor.

The assertion for e_0 is proved by induction on the degree of x. The expression vanishes if the degree is 1. If the degree is larger than 1, then $x = \omega \cdot x'$ with $x' \in T_+$. Apply the observation to the shuffle product $df * x = (df \cdot 1) * (\omega \cdot x')$. It follows that

$$e_0 = f\omega \cdot x' - \omega \cdot x'f + \omega \cdot (df * x') - \omega \cdot x' \cdot df.$$

Since $f\omega = \omega f$, it follows that $f\omega \cdot x' \equiv \omega \cdot fx' - \omega \cdot df \cdot x'$. Hence $e_0 \equiv \omega \cdot e_0'$, where e_0' is the expression obtained by replacing x by x' in e_0. Thus $e_0 \equiv 0$ by induction.

The assertion for e_1 is proved by induction on the sum of the degrees of y and z. With $w := yf \cdot x - y \cdot fx + y \cdot df \cdot x$, we have that $e_1 = w \cdot z$. The assertion holds if the degree of z is zero, since $w \in \mathcal{R}$. Assume that the degree of z is positive, and write $z = \pi \cdot z'$ with $\pi \in M$. If the degree of y is greater than 1, write $y = \omega \cdot y'$ with $y' \in T_+$. Then $w = \omega \cdot w'$, where w' is the expression obtained from w by replacing y by y'. By the observation,

$$e_1 = (\omega \cdot w') * (\pi \cdot z') = \omega \cdot (w' * z) + \pi \cdot (w * z').$$

By the induction hypothesis, we have that $w' * z \equiv 0$ and $w * z' \equiv 0$, hence $e_1 \equiv 0$. If y is of degree 1, say $y = \omega \in M$, then ω is not a common factor of the three terms in w. However, applying the observation to each of the three terms, it follows that

$$e_1 = \omega \cdot (x * z) - \omega \cdot (fx * z) + \omega \cdot ((df \cdot x) * z) + \pi \cdot (w * z').$$

The last term is congruent to 0 by the induction hypothesis. The sum of the remaining terms is of the form $\omega \cdot e_2$ and therefore (as we shall prove) is congruent to zero. Thus $e_1 \equiv 0$.

Finally, the assertion for the expression e_2 is proved by induction on the degree of z. Clearly, the expression vanishes when the degree is 0, that is, when $z = 1$. If the degree is positive, then $z = \pi \cdot z'$ with $\pi \in M$. Write $x = \omega \cdot x'$, and apply the observation to the shuffle products $f(x * z)$ and $fx * x$. It follows that

$$f(x * z) - fx * z = f\omega \cdot (x * z') - \omega \cdot (fx * z').$$

Again, by the observation, it follows that

$$(df \cdot x) * z = df \cdot (x * z) + \pi \cdot ((df \cdot x) * z').$$

Consequently,

$$e_2 = f\pi \cdot (x * z') - \pi \cdot (fx * z') + \pi \cdot [(df \cdot x) * z'].$$

In the first term on the right, use that $f\pi = \pi f$ and $\pi f \cdot y \equiv \pi \cdot fy - \pi \cdot df \cdot y$. Hence $e_2 \equiv \pi \cdot e_2'$, where e_2' is the expression obtained by replacing z by z' in e_2. Thus $e_2 \equiv 0$ by induction. □

3.3. PROPOSITION. *The shuffle product $*$ in T_+ induces a shuffle product on the quotient J_+ under which J_+ is a commutative algebra without unit. The shuffle product in J_+ extends uniquely to a shuffle product in J such that J becomes a commutative algebra, denoted J_{shuff}, and the two maps $\iota, \delta\colon A \to J$ become maps of commutative algebras $A \to J_{\text{shuff}}$. For $f \in A$ and $\varphi \in J$, the extension is determined by*

$$f * \varphi = \iota(f) \cdot \varphi = f \cdot \varphi, \tag{3.3.1}$$

and we have that $\delta(f) * \varphi = \varphi \cdot \delta(f)$.

Proof. Since the expressions of the form e_1 in Lemma (3.2) are congruent to zero, it follows that the ideal $\mathcal{R}$ of T defining J_+ is also an ideal with respect to the shuffle product on T. Hence the quotient $J_+ = T_+/\mathcal{R}$, with the induced shuffle product, is a commutative algebra without unit.

Because expressions of the form e_0 vanish in J_+, we have the following equation in J_+ for $f \in A$ and $\varphi \in J_+$:

$$f\varphi - df \cdot \varphi + df * \varphi = \varphi f + \varphi \cdot df.$$

Since $f\varphi - df \cdot \varphi = f \cdot \varphi$ and $\varphi f + \varphi \cdot df = \varphi \cdot \delta(f)$, it follows that

$$f \cdot \varphi + df * \varphi = \varphi \cdot \delta(f). \tag{3.3.2}$$

Similarly, since expressions of the form e_2 vanish in J_+, for $y \in J_+$ we obtain

$$(f \cdot \varphi) * y = f \cdot (\varphi * y). \tag{3.3.3}$$

It follows from (3.3.3) that shuffle multiplication in J_+ is A-bilinear when J_+ is considered as the A-module ${}_\iota J_+$. Hence the sum $J = A \oplus {}_\iota J_+$ becomes a commutative A-algebra with unit under the product $(f, \varphi) * (g, y) = (fg, f \cdot y + g \cdot \varphi + \varphi * y)$. Clearly, (3.3.1) holds with this definition of the shuffle product in J. Moreover, it follows from (3.3.2) that, for $\varphi \in J_+$,

$$\delta(f) * \varphi = \varphi \cdot \delta(f). \tag{3.3.4}$$

Hence the last equation of Proposition 3.3 holds when φ belongs to J_+, and it holds trivially when φ is in A. Hence it holds for all $\varphi \in J$. In particular, since the equation holds for $\varphi = \delta(g)$, it follows that δ is a homomorphism of commutative algebras $\delta\colon A \to J_{\text{shuff}}$. □

3.4. NOTE. Obviously, the ideal $(J_+)^{n+1}$ defining the truncated jets J^n is an ideal with respect to the shuffle product. Hence J^n has the structure of a commutative algebra, denoted J^n_{shuff}, and the restrictions are maps of algebras $J^n_{\text{shuff}} \to J^{n-1}_{\text{shuff}}$. It should be clear that, if M is a free A-module of finite rank, then the product in J^n_{shuff} is equal to the shuffle product defined relative to a basis in (2.4). In particular, in this case it follows that the partial derivatives define derivations $\partial_i\colon J^n_{\text{shuff}} \to J^{n-1}_{\text{shuff}}$.

3.5. LOCALIZATION. Let S be a multiplicative subset S of the algebra A. Consider the induced derivation

$$d_S\colon S^{-1}A \to S^{-1}M.$$

Indicate with the subscript S the algebras of jets and corresponding maps associated to the derivation d_S. In particular, then, $J^n_S = J^n_{S^{-1}M,d_S}$ is an algebra and we have two maps of algebras, $\iota_S, \delta_S\colon S^{-1}A \to J^n_S$. By the universal property of J, we obtain a map of algebras

$$J^n \to J^n_S \tag{3.5.1}$$

that commutes in the obvious sense with the maps $A \to S^{-1}A$ and $M \to S^{-1}M$. View the target of (3.5.1) as a left $S^{-1}A$-module via $\iota_S\colon S^{-1}A \to J^n_S$. Then we obtain an induced left $S^{-1}A$-linear homomorphism,

$$\sigma^n\colon S^{-1}A \otimes_A {}_\iota J^n \to {}_\iota J^n_S; \tag{3.5.2}$$

obviously, σ^n is right A-linear when the source and the target are considered as right A-modules via δ.

3.6. PROPOSITION. *The map σ^n of* (3.5.2) *is an isomorphism of $S^{-1}A$-modules. With respect to the shuffle product, it is an isomorphism of commutative algebras*

$$S^{-1}A \otimes_A {}_\iota J^n_{\text{shuff}} \xrightarrow{\sim} (J^n_S)_{\text{shuff}}.$$

Proof. Obviously, when the jets are considered with the shuffle product, the map (3.5.1) and the map σ^n are maps of commutative algebras. We prove, by induction on the degree n, that σ^n is an isomorphism.

In degree $n+1$, we have the exact sequence given by the decomposition in Proposition 1.10,

$$0 \to J^n_\delta \otimes_A M \xrightarrow{j} J^{n+1} \xrightarrow{r} A \to 0. \tag{3.6.1}$$

Here j is the inclusion obtained from the isomorphism $(J^n_+)_\delta \otimes_A M = J^{n+1}_+$ of Proposition 1.10, and r is the augmentation map. Note that the maps j and r are left A-linear when J^n and J^{n+1} are considered as left A-modules via ι.

Consider the following diagram:

$$\begin{CD}
0 @>>> S^{-1}A\otimes_{A}{}_{\iota}J^n_\delta\otimes_A M @>{j}>> S^{-1}A\otimes_{A}{}_{\iota}J^{n+1} @>{r}>> S^{-1}A @>>> 0\\
@. @V{\sigma^n\otimes 1_M}VV @V{\sigma^{n+1}}VV @| \\
0 @>>> (J^n_S)_\delta\otimes_A M @>{j'}>> J^{n+1}_S @>{r_S}>> S^{-1}A @>>> 0.
\end{CD}$$

The top row is obtain by tensoring (3.6.1) from the left by $S^{-1}A$. In particular, the top row is exact. The map j' is induced by the multiplication in J_S^{n+1}. Clearly, the diagram is commutative.

In the source of the map j', the factor $(J_S^n)_\delta$ is a right A-module via δ and $\delta\colon A \to (J_S^n)$ is the restriction to A of $\delta_S\colon S^{-1}A \to J_S^n$. Hence, for the source of j' we have a canonical identification

$$(J_S^n)_\delta \otimes_A M = (J_S^n)_{\delta_S} \otimes_A M = (J_S^n)_{\delta_S} \otimes_{S^{-1}A} S^{-1}A \otimes_A M = (J_S^n)_{\delta_S} \otimes_{S^{-1}A} S^{-1}M.$$

Under this identification, the bottom row of the diagram becomes the sequence (3.6.1) for the derivation d_S. In particular, the bottom row is exact. Consequently, since we may assume by induction that $\sigma^n \otimes 1_M$ is an isomorphism, it follows that σ^{n+1} is an isomorphism. Hence we have proved the proposition. □

3.7. NOTE. Clearly the isomorphism (1.13.1) is an isomorphism of commutative algebras when the jets are considered with the shuffle product. Similarly, the formation of twisted jets commutes with the scalar extension.

3.8. GLOBALIZATION. Given the localization property in Proposition 3.6, it is immediate to globalize the construction as follows. Let X be a Y-scheme. Fix a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ and an $\mathcal{O}_Y$-linear derivation of sheaves,

$$d\colon \mathcal{O}_X \to \mathcal{M}.$$

Then, associated with d, there are sheaves $\mathcal{J}^0 = \mathcal{O}_X, \mathcal{J}^1, \dots$ of $\mathcal{O}_Y$-algebras on X and maps $r^n\colon \mathcal{J}^n \to \mathcal{J}^{n-1}$ of algebras. There are two maps of algebras $\iota, \delta\colon \mathcal{O}_X \to \mathcal{J}^n$ commuting with the maps r^n. The maps ι and δ give rise to four $\mathcal{O}_X$-module structures on $\mathcal{J}^n$, two structures ${}_\iota\mathcal{J}^n$ and ${}_\delta\mathcal{J}^n$ obtained by left multiplication via ι and δ plus two right structures $\mathcal{J}^n_\iota$ and $\mathcal{J}^n_\delta$ obtained by right multiplication by ι and δ.

For $n \geq 1$, there is a natural inclusion of sheaves $\mathcal{M} \to \mathcal{J}^n$ that is $\mathcal{O}_X$-linear as a map $\mathcal{M} \to {}_\delta\mathcal{J}^n$ and as a map $\mathcal{M} \to \mathcal{J}^n_\iota$. Under any of the four structures $\mathcal{J}^n$ is a quasi-coherent $\mathcal{O}_X$-module, and it is locally of finite type if $\mathcal{M}$ is.

Multiplication in $\mathcal{J}^n$ induces a natural map $i^n\colon \mathcal{M}^{\otimes_{\mathcal{O}_X} n} \to \mathcal{J}^n$, which is left $\mathcal{O}_X$-linear with respect to both left structures and right linear with respect to both right structures on $\mathcal{J}^n$. The sequence

$$\mathcal{M}^{\otimes_{\mathcal{O}_X} n} \xrightarrow{i^n} \mathcal{J}^n \xrightarrow{r^n} \mathcal{J}^{n-1} \to 0 \tag{3.8.1}$$

is exact and i^n is injective if $\mathcal{M}$ is flat over $\mathcal{O}_X$.

For each $\mathcal{O}_X$-module $\mathcal{L}$ there is a module of $\mathcal{L}$-twisted jets $\mathcal{J}^n(\mathcal{L}) = \mathcal{J}^n_\delta \otimes_{\mathcal{O}_X} \mathcal{L}$. Then $\mathcal{J}^n(\mathcal{L})$ is a left $\mathcal{J}^n$-module and in particular, via the inclusion $\iota\colon \mathcal{O}_X \to \mathcal{J}^n$, a left $\mathcal{O}_X$-module. From the algebra homomorphism $\delta\colon \mathcal{O}_X \to \mathcal{J}$ we obtain a homomorphism $\delta_{\mathcal{L}}\colon \mathcal{L} \to \mathcal{J}^n(\mathcal{L})$, and from the exact sequence (3.8.1) we obtain an exact sequence

$$\mathcal{M}^{\otimes_{\mathcal{O}_X} n} \otimes_{\mathcal{O}_X} \mathcal{L} \to \mathcal{J}^n(\mathcal{L}) \to \mathcal{J}^{n-1}(\mathcal{L}) \to 0,$$

which is left exact when $\mathcal{M}$ is flat.

The noncommutative algebra $\mathcal{J}^n$ has a second natural product called the *shuffle product*, under which it is a commutative algebra (denoted $\mathcal{J}^n_{\text{shuff}}$). For the shuffle product the maps ι and δ are maps of sheaves of commutative algebras, and the structures of an $\mathcal{O}_X$-module on $\mathcal{J}^n_{\text{shuff}}$ defined via the maps ι and δ are equal to ${}_\iota\mathcal{J}^n$ and $\mathcal{J}_\delta^n$, respectively.

The formation of jets commutes in the obvious sense with base change $Y' \to Y$.

4. Symmetric Jets

In this section we continue the study of jets in the commutative case. A key role is played by the notion of a flat extension of the given derivation $d\colon A \to M$; see Section 4.1. Given a flat extension $\varphi\colon M \to \bigwedge^2 M$ of d, we define a subalgebra J^n_φ of J^n_{shuff}. When M is free there is a canonical extension φ_ε associated to a basis $\varepsilon_1, \dots, \varepsilon_r$ of M.

Assume in particular that the basis has the form $dx_1, \dots, dx_r$ and that the associated partial derivatives $\partial/\partial x_1, \dots, \partial/\partial_r$ commute. We prove in this case that the extension φ_ε is the unique flat extension of d. Moreover, we prove that the subalgebra $J^n_{\varphi_\varepsilon}$ is equal to the subalgebra J^n_{sym} (considered in Definition 2.6 and Proposition 2.7) of jets that are symmetric with respect to the basis. In particular, the subalgebra J^n_{sym} is independent of the choice of basis.

We also show under suitable hypotheses on M that the symmetric jets fit into exact sequences similar to those obtained for the truncated jets.

4.1. Setup. Keep the setup of Section 3.1. Form the exterior square $\bigwedge^2 M$ of M as an A-module. We will say that a k-linear map $\varphi\colon M \to \bigwedge^2 M$ is an *extension* of the given derivation $d\colon A \to M$ if, for $f \in A$ and $\omega \in M$,

$$\varphi(f\omega) = f\varphi(\omega) + df \wedge \omega. \tag{4.1.1}$$

The extension will be called *flat* if $\varphi d = 0$. Equivalently, φ is flat if

$$\varphi(fdg) = df \wedge dg. \tag{4.1.2}$$

In particular, if M is generated as an A-module by elements of the form dg, then d has at most one flat extension.

4.2. Note. Let φ be an extension of d. Then, as is well known, there is a unique sequence of k-linear maps,

$$A \xrightarrow{d^0} M \xrightarrow{d^1} \textstyle\bigwedge^2 M \xrightarrow{d^2} \bigwedge^3 M \to \cdots, \tag{4.2.1}$$

such that $d^0 = d$, $d^1 = \varphi$, and

$$d^{p+q}(\pi \wedge \nu) = d^p\pi \wedge \nu + (-1)^p \pi \wedge d^q \nu$$

for $\pi \in \bigwedge^p M$ and $\nu \in \bigwedge^q M$. The map d^n is determined by the equation

$$d^n(\omega_1 \wedge \cdots \wedge \omega_n) = \sum_{i=1}^{n} (-1)^{i-1} \omega_1 \wedge \cdots \wedge \varphi(\omega_i) \wedge \cdots \wedge \omega_n.$$

Moreover, φ is flat if and only if the sequence is a complex.

4.3. REMARK. The universal derivation $d = d_{A/k} \colon A \to \Omega^1_{A/k}$ has a unique flat extension. Indeed, the k-linear map $A \otimes_k A \to \Omega^1_{A/k}$ given by $f \otimes g \mapsto fdg$ is surjective, and its kernel is the additive subgroup generated by elements of the special form $f \otimes gh - fg \otimes h - fh \otimes g$. Hence it suffices to note that the map $A \otimes A \to \bigwedge^2 \Omega^1_{A/k}$ given by the right side of (4.1.2) (i.e., $f \otimes g \mapsto df \wedge dg$) vanishes on elements of the special form.

4.4. LEMMA. *Assume that M is a free A-module with basis $\varepsilon_1, \dots, \varepsilon_r$. Let $\varphi_\varepsilon \colon M \to \bigwedge^2 M$ be the additive map determined by the equations*

$$\varphi_\varepsilon(f\varepsilon_i) = df \wedge \varepsilon_i \quad \text{for } i = 1, \dots, r.$$

Then φ_ε is an extension of d as well as the unique extension with the elements ε_i in the kernel. Moreover, the extension φ_ε is flat if and only if the derivations D_i associated with the basis $\varepsilon_1, \dots, \varepsilon_r$ commute.

Proof. The assertions are easily proved. □

4.5. PROPOSITION. *Consider a second basis $\tilde\varepsilon_1, \dots, \tilde\varepsilon_r$ given by a transition matrix (u_{ip})*:

$$\tilde\varepsilon_p = \sum_i \varepsilon_i u_{ip}.$$

Then the following conditions are equivalent:

(i) $\varphi_{\tilde\varepsilon} = \varphi_\varepsilon$;
(ii) $\varphi_\varepsilon(\tilde\varepsilon_p) = 0$ *for all* p;
(iii) $D_j(u_{ip}) = D_i(u_{jp})$ *for all* i, j, p.

Proof. By Lemma 4.4, $\varphi_{\tilde\varepsilon}$ is the unique extension with $\tilde\varepsilon_p$ in the kernel. Hence assertion (i) is equivalent to assertion (ii). Obviously, assertion (ii) is equivalent to assertion (iii). □

4.6. COROLLARY. (A) *Assume that condition* (iii) *of Proposition 4.5 holds. Then the corresponding condition holds for the derivations $\tilde D_p$ and the inverse of the matrix (u_{ip}). Moreover, the D_i commute if and only if the $\tilde D_p$ commute.*

(B) *Assume that M is generated as an A-module by elements of the form dg and that the derivations D_i associated with the basis ε_i commute. Then the derivations $\tilde D_p$ associated with the basis $\tilde\varepsilon_p$ commute if and only if condition* (iii) *of Proposition 4.5 holds.*

(C) *Assume that M has a basis of the form $dx_1, \ldots, dx_r$ for elements $x_1, \ldots, x_r$ in A. Then the following conditions are equivalent*:

(i) *there exists a flat extension of d*;

(ii) *the derivations $\partial/\partial x_1, \ldots, \partial/\partial x_r$ associated with the basis $dx_1, \ldots, dx_r$ commute.*

In particular, if M has a basis $\varepsilon_1, \ldots, \varepsilon_r$ such that the corresponding D_i commute, then (ii) *holds for any basis of the form $dx_1, \ldots, dx_r$.*

Proof. Assume the condition in (A). Then condition (i) of Proposition 4.5 holds. Hence condition (iii) of Proposition 4.5 holds for the $\tilde{D}_p$ and the inverse of the matrix (u_{ip}), since condition (i) of Proposition 4.5 is symmetric in ε and $\tilde{\varepsilon}$. By the same argument, since the D_i commute if and only if φ_ε is flat (by Lemma 4.4), it follows that the D_i commute if and only if the $\tilde{D}_p$ commute.

Assume the first condition in (B). Then, as noted in Section 4.1, there is at most one flat extension of d. Assume in addition the second condition in (B). Then it follows from Lemma 4.4 that φ_ε is the unique flat extension. Hence, by Lemma 4.4, the $\tilde{D}_p$ commute if and only if condition (i) of Proposition 4.5 holds—that is, if and only if condition (iii) of Proposition 4.5 holds.

Assume the condition in (C), and let φ_{dx} be the extension determined by the basis dx_i. If there is a flat extension φ of d (i.e., if $\varphi d = 0$), it follows that the dx_i belong to the kernel of φ. Hence, by the uniqueness in Lemma 4.4, $\varphi = \varphi_{dx}$. Thus φ_{dx} is flat and, again by Lemma 4.4, the $\partial/\partial x_i$ commute.

Conversely, if the $\partial/\partial x_i$ commute, then it follows that φ_{dx} is flat. Hence the conditions (i) and (ii) in (C) are equivalent. The last assertion in (C) follows from the equivalence of (i) and (ii) because, by Lemma 4.4, we have that φ_ε is flat. □

4.7. Definition. Let φ be a flat extension of d. Denote by J^n_φ the A-subalgebra of the shuffle algebra J^n_{shuff} generated by all powers ω^ν, where ω^ν is the power with respect to the original multiplication in J^n for all ω in M that are in the kernel of φ.

Note that $df \in J^n_\varphi$, because φ is a flat extension. Hence $\delta(f) = f + df$ belongs to J^n_φ. Thus we may view ι and δ as maps of commutative algebras,

$$\iota, \delta \colon A \to J^n_\varphi.$$

The elements of J^n_φ will be called *symmetric* with respect to φ. If M is generated as an A-module by elements of the form df, then we observed in Section 4.1 that there is at most one flat extension and hence at most one algebra of symmetric jets. It follows from Corollary 4.6(C) that if M has a basis of the form $dx_1, \ldots, dx_r$ such that the corresponding derivations $\partial/\partial x_i$ commute, then there is a unique subalgebra of symmetric jets, denoted $J^n_{M,d,\text{sym}}$. Similarly, by Remark 4.3, when the given derivation is the universal Kähler derivation $d \colon A \to \Omega^1_{A/k}$, there is a unique subalgebra of symmetric jets, denoted $J^n_{A/k,\text{sym}}$.

4.8. Proposition. *Assume that M is free with a basis $\varepsilon_1, \ldots, \varepsilon_r$ such that the corresponding derivations D_i commute. Then the subalgebra of jets that are symmetric with respect to the corresponding extension φ_ε is equal to the subalgebra*

J^n_{sym} of jets, defined in Definition 2.6, that are symmetric with respect to the basis $\varepsilon_1, \dots, \varepsilon_r$. In particular, the subalgebra has as free A-basis the set of all products,

$$\varepsilon^{(\nu)} := \varepsilon_1^{\nu_1} * \cdots * \varepsilon_r^{\nu_r} \tag{4.8.1}$$

for all multi-indices $\nu = (\nu_1, \dots, \nu_r)$ with $|\nu| \le n$.

Proof. The products form an A-basis for J^n_{sym} by Proposition 2.7, and $J^n_{\text{sym}} \subseteq J^n_{\varphi_\varepsilon}$ because ε_i is in the kernel of φ_ε. As the D_i commute, it follows from (2.2.1) that $\delta(f)$ belongs to J^n_{sym}.

To show equality we must prove that, for any ω in the kernel of φ_ε, the power ω^ν belongs to J^n_{sym}. Write $\omega = \varepsilon_1 f_1 + \cdots + \varepsilon_r f_r$. Then $D_i(f_j) = D_j(f_i)$, since $\varphi_\varepsilon(\omega) = 0$.

The proof is by induction on ν. The assertion is obvious for $\nu = 0$. In the inductive step we assume that $\omega^{\nu-1} \in J^n_{\text{sym}}$ for all n. As observed in Note 3.4, we have the partial derivations $\partial_i \colon J^n_{\text{shuff}} \to J^{n-1}_{\text{shuff}}$ and, by Corollary 2.8, it suffices to prove that $\partial_i(\omega^\nu) \in J^{n-1}_{\text{sym}}$ and that $\partial_j\partial_i(\omega^\nu) = \partial_i\partial_j(\omega^\nu)$. By (2.3.1), (2.3.2), and (2.4.2), we have

$$\partial_i(\omega^\nu) = \omega^{\nu-1} \cdot \partial_i(\omega) = \omega^{\nu-1} \cdot \delta(f_i) = \omega^{\nu-1} * \delta(f_i). \tag{4.8.2}$$

Since $\delta(f) \in J^{n-1}_{\text{sym}}$ for any $f \in A$ and since $\omega^{\nu-1} \in J^{n-1}_{\text{sym}}$ by the induction hypothesis, it follows that $\partial_i(\omega^\nu) \in J^{n-1}_{\text{sym}}$. Moreover, since ∂_j is a derivation with respect to the shuffle product and $\partial_j(\delta(f)) = \delta(D_j f)$ by (2.3.3), it follows from (4.8.2) that

$$\partial_j\partial_i(\omega^\nu) = \partial_j(\omega^{\nu-1} * \delta(f_i)) = \omega^{\nu-2} * \delta(f_j) * \delta(f_i) + \omega^{\nu-1} * \delta(D_j f_i).$$

Since $D_i(f_j) = D_j(f_i)$ we have that $\partial_j\partial_i(\omega^\nu) = \partial_i\partial_j(\omega^\nu)$, and the proof is complete. □

4.9. COROLLARY. *Under the conditions of Proposition 4.8, there is an exact sequence*

$$0 \to (M^{\otimes_A n})^{\mathrm{S}_n} \to J^n_{\text{sym}} \to J^{n-1}_{\text{sym}} \to 0. \tag{4.9.1}$$

Proof. Clearly, the restriction $J^n_{\text{sym}} \to J^{n-1}_{\text{sym}}$ is surjective. It follows from the exact sequence (1.9.1) that the kernel is the intersection of $M^{\otimes_A n}$ and J^n_{sym}. Hence Proposition 4.8 implies that the intersection is the free A-module generated by all products $\varepsilon^{(\nu)}$, where $|\nu| = n$. The intersection is thus the part of $M^{\otimes_A n}$ that is invariant under the symmetric group S_n . □

4.10. NOTE. The corollary applies in particular when it is possible to choose for M a basis of form $dx_1, \dots, dx_r$ such that the corresponding derivations $\partial/\partial x_i$ commute. In this case there is a unique algebra $J^n_{M,d,\text{sym}}$ of symmetric jets.

Note also that the zero map may by taken as an extension of d if either $d = 0$ (the trivial case) or $\bigwedge^2 M = 0$ (e.g., if M is free and of rank 1). In the first case, we have that $J = T_A M$ and J^n is the truncated tensor algebra $J^n = T_A M_{\le n}$; moreover, if M is free then J^n_{sym} is the subalgebra of symmetric tensors in $T_A M_{\le n}$. In the second case, all jets are symmetric: $J^n_{\text{sym}} = J^n$.

4.11. Comparison with Principal Parts. Consider the A-A-linear map $A \otimes A \to {}_\iota J_\delta$ defined by

$$f \otimes g \mapsto f \cdot \delta(g) = f * \delta(g);$$

the equality $f \cdot \delta(g) = f * \delta(g)$ holds by (3.3.1). For $f \in A$, let $\Delta(f) := 1 \otimes f - f \otimes 1$. Then

$$\Delta(f) \mapsto \delta(f) - f = df.$$

Now, with respect to the shuffle product, J^n_{shuff} is a commutative algebra. Hence the map $f \otimes g \mapsto f * \delta(g)$ is a homomorphism of algebras $A \otimes A \to J^n_{\mathrm{shuff}}$ that vanishes on any product on $n+1$ elements of the form $\Delta(f)$. Therefore, it induces a map of algebras from the algebra of nth-order principal parts $P^n_{A/k}$ to the algebra $J^n_{M,d}$ with the shuffle product

$$P^n_{A/k} \to J^n_{\mathrm{shuff}}. \tag{4.11.1}$$

Note that the power $(\Delta f)^\nu$ in $P^n_{A/k}$ is mapped to the shuffle power $(df)^{*\nu} = \nu!(df)^\nu$. If φ is any flat extension of d, then δ maps into the corresponding algebra J^n_φ; hence (4.11.1) maps into J^n_φ. In particular, when d is the universal derivation $d = d_{A/k}$ into the module of Kähler differentials $\Omega^1_{A/k}$, we obtain a canonical map of algebras:

$$P^n_{A/k} \to J^n_{A/k,\mathrm{sym}}. \tag{4.11.2}$$

Assume that M is a free A-module of finite rank and that A contains the field of rational numbers. Then the canonical homomorphism $(M^{\otimes_A n})^{S_n} \to \mathrm{Sym}^n_A M$ is an isomorphism. Assume that M has a basis of the form $dx_1, \dots, dx_r$. Then $(dx_i)^\nu$ is the image of $(1/\nu!)(\Delta x_i)^\nu$. Therefore, since the $(dx_i)^\nu$ generate the algebra J^n_{sym} of symmetric jets by Proposition 2.7, it follows that the image of (4.11.1) contains J^n_{sym}. Conversely, if the partial derivatives $\partial/\partial x_i$ commute, then $\delta(f)$ is symmetric and it follows that (4.11.1) maps into J^n_{sym}; hence the image of (4.11.1) is equal to J^n_{sym}.

Assume in particular that $\Omega^1_{A/k}$ has a free basis of the form $dx_1, \dots, dx_r$. Then the map (4.11.2) induces an isomorphism $P^n_{A/k} \to J^n_{A/k,\mathrm{sym}}$. In fact, there is a commutative diagram

$$\begin{tikzcd}
 & \mathrm{Sym}^n \Omega^1_{A/k} \arrow[r] \arrow[d, equal] & P^n_{A/k} \arrow[r] \arrow[d] & P^{n-1}_{A/k} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Sym}^n \Omega^1_{A/k} \arrow[r] & J^n_{A/k,\mathrm{sym}} \arrow[r] & J^{n-1}_{A/k,\mathrm{sym}} \arrow[r] & 0,
\end{tikzcd}$$

where the top row is the usual exact sequence of principal parts (cf. [Gr]) and the bottom row is obtained from (2.7.2); the two vertical right maps are obtained from (4.11.2). It follows by induction on n that the map (4.11.2) is an isomorphism in this case. In particular, we obtain the well-known result that, over the rationals, A is differentially smooth over k when $\Omega^1_{A/k}$ is free.

Note that Δx_i^ν is mapped to $\nu!(dx_i)^\nu$. In particular, in positive characteristic p, the image vanishes when $\nu \geq p$. In positive characteristic, then, the map (4.11.2) is in general neither injective nor surjective.

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