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Gherardelli Linkage and Complete Intersections

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To Bill, with best wishes for many more years of fruitful endeavors

1. Introduction

Our main result is Theorem 3.2. It characterizes the complete intersections of codimension 2 in $\mathbf{P}^n$ ($n \geq 3$), over an algebraically closed field of characteristic 0, among the Cohen–Macaulay X as those that are subcanonical and self-linked. This characterization was formulated by Ellia (private comm.), who proved it in a joint work with Beorchia [BE, Thm. 5, p. 556] assuming X is smooth. In Remark 6.1 [BE, p. 557], Beorchia and Ellia said they don't know whether the smoothness "can be avoided." It can! Furthermore, X can be reducible and nonreduced.

More precisely, an X is said to be *a-subcanonical* if its dualizing sheaf ω_X is of the form $\omega_X = \mathcal{O}_X(a)$. An X is said to be *self-linked* by two hypersurfaces F_1 and F_2 if X is equal to its own residual scheme in the complete intersection of F_1 and F_2. For example, suppose X is the complete intersection of F_1 and F_3. Then X is self-linked by F_1 and F_2, where $F_2 := 2F_3$ or where F_2 is, more generally, any hypersurface such that $F_1 \cap F_2 = F_1 \cap 2F_3$. Furthermore, X is a-subcanonical where a is the following integer: denote the degree of F_i by m_i; then $a := m_1 + m_3 - n - 1$. Now, Theorem 3.2 says that this is, in fact, the only example!

This second formulation of Theorem 3.2 is more refined than the first. After all, the first says nothing much about the hypersurfaces F_i involved. In particular, the first does not suggest anything like the equation $m_2 = 2m_3$. Indeed, [BE, Cor. 4, p. 557] offers an alternative proof of the first formulation in the case where X is a curve and $m_3 \geq m_2 \geq m_1$. The proof is correct, but the case is vacuous!

Our proof of Theorem 3.2 follows, to a fair extent, the lines of Beorchia and Ellia's proof of their Theorem 5. In both proofs, a key step is to split the normal bundle of X in $\mathbf{P}^n$. At this stage, if $n \geq 4$ and X is smooth, then we're done simply because the normal bundle splits; indeed, Basili and Peskine [BP, p. 87] proved that then X is a complete intersection. However, in order to prove Theorem 3.2 in full generality, we must split the normal bundle with care. For example, consider the twisted cubic space curve X; its normal bundle is split because X is rational, and it is known that X is self-linked by a quadric cone and a cubic surface, but of course X is neither a complete intersection nor even subcanonical.

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To split the normal bundle, we'll use (the Gherardelli linkage) Theorem 2.5. It asserts that, when two hypersurfaces F_1 and F_2 of $\mathbf{P}^n$ intersect partially in an X, then X is subcanonical if and only if its residual scheme Y is, scheme-theoretically, of the form $Y = F_1 \cap F_2 \cap F_3$, where F_3 is a suitable hypersurface. (Such a Y is called a *quasi-complete intersection*.) In particular, if X is subcanonical and is self-linked by F_1 and F_2, then $X = F_1 \cap F_2 \cap F_3$. In this case, we'll form the conormal bundles of X in F_3 and in $\mathbf{P}^n$, and we'll split the natural map from the latter bundle onto the former.

We will then conclude that some multiple of X is numerically equivalent to a hypersurface section of F_3, at least after we've replaced F_3 by an integral component; we'll simply apply Braun's main theorem [Br, p. 403]. (Braun followed the lines of Ellingsrud, Gruson, Peskine, and Strømme's remarkable proof of the theorem in the case of a curve on a smooth connected surface. This case had been treated earlier, in a very different fashion, by Griffiths, Harris, and Hulek. See Braun's paper [Br, p. 411] for all the references.) Finally, to conclude that X is a complete intersection, Beorchia and Ellia used Gruson and Peskine's work on space curves. Instead, we'll make a direct geometric argument and so obtain our more refined statement of Theorem 3.2.

If $n \geq 6$ and X is smooth, then, since X is a quasi-complete intersection, it is, in fact, a complete intersection by Faltings' Korollar of Satz 3 [Fa, p. 398]. This line of proof is significant because it is valid in any characteristic, whereas Basili and Peskine work in characteristic 0—and we must too, although only to apply Braun's theorem. Beorchia and Ellia [BE, p. 556] suggested that there might be a problem in characteristic 2 by pointing out the following result, due in part to Rao [R2, p. 272] and in part to Migliore [Mi, p. 185]: a double line in $\mathbf{P}^3$ of arithmetic genus -2 or less is self-linked if and only if the characteristic is 2. We'll pursue this suggestion in Example 3.4. On the other hand, it would be nice to know whether Theorem 3.2 is valid except for certain X of small dimension in characteristic 2.

The Gherardelli linkage theorem holds in greater generality than that stated above. In Theorem 2.5, we'll replace $\mathbf{P}^n$ by any Gorenstein projective scheme P having pure dimension 2 or more and satisfying this vanishing condition: $H^q(\mathcal{O}_P(m)) = 0$ for three specific values of the pair (q, m). For example, P can be a complete intersection in $\mathbf{P}^n$. Thus we'll recover Theorem 2(i) of Fiorentini and Lascu [FL2, p. 170], where, in addition, X and Y are assumed to have no common components; in fact, our proof was inspired by theirs. Beorchia and Ellia [BE, p. 556] proved the existence of the hypersurface F_3 directly in the case at hand by using the mapping cone. Earlier, Rao [R1, pp. 209–10] proved (burying it among other things) a version of the Gherardelli linkage theorem in which the condition that X be subcanonical is replaced by the condition that X be the zero scheme of a section of a rank-2 vector bundle on $\mathbf{P}^n$; these two conditions are equivalent by a famous theorem of Serre's (see [Fe, Prop. 3, p. 346]). On the other hand, our Theorem 3.2 does not hold even if $\mathbf{P}^n$ is replaced by a smooth hypersurface P, as we'll see in Example 3.3.

To prove (the Gherardelli linkage) Theorem 2.5, we'll use the Noether linkage sequence (2.3.1), which presents the dualizing sheaf of a partial intersection in any Gorenstein ambient scheme P having pure dimension 2 or more. The case where P is a complete intersection in $\mathbf{P}^n$ was treated in [FL2, Lemma 1] and [PS, 1.6] and was used in [R2, p. 253]. The general case is, as we'll see, no more difficult to prove.

In short, in Section 2 we will review some basic linkage theory, including the Peskine–Szpiro linkage theorem (cf. [Ei, 21.23, p. 541; PS, 1.3, p. 274]), the Noether linkage sequence, and the Gherardelli linkage theorem. This theory is all more or less well known but has not always been developed exactly as here, and it is all essential for our work in Section 3. In Section 3, we'll prove our main theorem, our characterization of complete intersections of codimension 2 in $\mathbf{P}^n$. Finally, we'll discuss two examples: the first shows that the ambient projective space cannot be replaced even by a smooth hypersurface; the second shows that our characterization fails in characteristic 2.

2. Gherardelli Linkage

PROPOSITION 2.1 (Peskine–Szpiro Linkage Theorem). *Let Z be a Gorenstein scheme, $X \subset Z$ a proper closed subscheme, and Y the residual scheme of X. If X is Cohen–Macaulay of pure codimension 0, then so is Y; furthermore, X is then also the residual scheme of Y.*

Proof. Let $\mathcal{I}_{X/Z}$ and $\mathcal{I}_{Y/Z}$ denote the ideals. Then we have

$$\mathcal{I}_{Y/Z} := \operatorname{Ann}_{\mathcal{O}_Z} \mathcal{I}_{X/Z} \xleftarrow{\sim} \operatorname{Hom}_{\mathcal{O}_Z}(\mathcal{O}_X, \mathcal{O}_Z), \tag{2.1.1}$$

where the equation holds by definition and the isomorphism is given by evaluation at 1.

It is a basic fact (see [Ei, 21.21, p. 538]) that, on the category of maximal (dimensional) Cohen–Macaulay $\mathcal{O}_Z$-modules $\mathcal{M}$, the functor

$$\mathcal{D}(\mathcal{M}) := \operatorname{Hom}_{\mathcal{O}_Z}(\mathcal{M}, \mathcal{O}_Z)$$

is dualizing. Now, $\mathcal{D}$ interchanges the two basic exact sequences

$$0 \to \mathcal{I}_{X/Z} \to \mathcal{O}_Z \to \mathcal{O}_X \to 0 \quad \text{and} \quad 0 \to \mathcal{I}_{Y/Z} \to \mathcal{O}_Z \to \mathcal{O}_Y \to 0;$$

indeed, $\mathcal{D}$ carries the first sequence to the second thanks to (2.1.1), and so, as $\mathcal{D}$ is dualizing, $\mathcal{D}$ carries the second sequence back to the first. Thus, $\mathcal{O}_Y = \mathcal{D}(\mathcal{I}_{X/Z})$ and $\mathcal{I}_{X/Z} = \mathcal{D}(\mathcal{O}_Y)$. The latter equation implies that X is the residual scheme of Y. The former equation implies that $\mathcal{O}_Y$ is maximal Cohen–Macaulay because $\mathcal{I}_{X/Z}$ is so, since at any $x \in X$ we have

$$\operatorname{depth} \mathcal{I}_{X/Z,x} \geq \min(\operatorname{depth} \mathcal{O}_{Z,x}, 1 + \operatorname{depth} \mathcal{O}_{X,x})$$

(see [Ei, 18.6b, p. 451]). The proof is now complete. □

SETUP 2.2. Let P be a complete scheme defined over an algebraically closed field of arbitrary characteristic. Assume that P is Gorenstein of pure dimension at least 2, and equip P with an invertible sheaf $\mathcal{O}_P(1)$ that is not necessarily ample. For $i = 1, 2$, let $f_i \in H^0(\mathcal{O}_P(m_i))$ be a section and let $F_i : f_i = 0$ be its scheme of zeros. Set

$$Z := F_1 \cap F_2,$$

and assume that Z has pure codimension 2.

Let $X \subset Z$ be a proper closed subscheme, and assume that X is Cohen–Macaulay of pure codimension 2 in P. Let $Y \subset Z$ be the residual scheme of X. By the Peskine–Szpiro linkage theorem (Proposition 2.1), Y also is Cohen–Macaulay of pure codimension 2 in P, and X is also the residual scheme of Y.

PROPOSITION 2.3 (Noether Linkage Sequence). *In Setup 2.2, the dualizing sheaves and the ideals in P are related by the following short exact sequence*:

$$0 \to \mathcal{I}_{Z/P} \otimes \omega_P(m_1+m_2) \to \mathcal{I}_{Y/P} \otimes \omega_P(m_1+m_2) \to \omega_X \to 0. \quad (2.3.1)$$

Proof. First, note the following two equations:

$$\omega_Z = \omega_P(m_1+m_2)\big|_Z \quad \text{and} \quad \omega_X = \mathcal{I}_{Y/Z} \otimes \omega_Z. \quad (2.3.2)$$

The first equation is standard and results from basic duality theory (see e.g. [AK, Chap. 1]):

$$\omega_Z = \mathrm{Ext}^2_P(\mathcal{O}_Z, \omega_P) = \mathrm{Hom}_Z\big(\det(\mathcal{I}_{Z/P}/\mathcal{I}^2_{Z/P}), \omega_P\big|_Z\big).$$

The second equation in (2.3.2) results from a series of three other equations:

$$\omega_X = \mathrm{Hom}(\mathcal{O}_X, \omega_Z) = \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_Z) \otimes \omega_Z = \mathcal{I}_{Y/Z} \otimes \omega_Z.$$

These hold by elementary duality theory, by the invertiblity of ω_Z, and by (2.1.1).

Finally, the Noether linkage sequence (2.3.1) results from the basic sequence

$$0 \to \mathcal{I}_{Z/P} \to \mathcal{I}_{Y/P} \to \mathcal{I}_{Y/Z} \to 0$$

by tensoring it with $\omega_P(m_1+m_2)$ and then using the two equations in (2.3.2). □

REMARK 2.4. According to Enriques [EC, Vol. 3, p. 534], Noether obtained the preceding proposition in the special case where P is the projective 3-space. Noether stated it virtually as follows:

> *If the curve X is the partial intersection of two surfaces F_1 and F_2 of degrees m_1 and m_2, meeting further in a curve Y, then the surfaces of degree m_1+m_2-4 passing through Y cut on X the complete canonical series.*

To derive this statement, take (2.3.1), replace ω_P by $\mathcal{O}_P(-4)$, and extract cohomology, obtaining the following exact sequence:

$$H^0(\mathcal{I}_{Y/P}(m_1+m_2-4)) \to H^0(\omega_X) \to H^1(\mathcal{I}_{Z/P}(m_1+m_2-4)).$$

The third term vanishes because Z is a complete intersection, and Noether's statement follows.

THEOREM 2.5 (Gherardelli Linkage). *Preserve the assumptions of Setup 2.2. Let* $m_3 > 0$. *If there exists an* $f_3 \in H^0(\mathcal{O}_P(m_3))$ *such that* $Y = F_1 \cap F_2 \cap F_3$, *where* $F_3 : f_3 = 0$, *then*

$$\omega_X = \omega_P(m_1 + m_2 - m_3)\big|_X.$$

The converse holds if, in addition,

$$H^1(\mathcal{O}_P(m_3 - m_1)) = 0, \qquad H^1(\mathcal{O}_P(m_3 - m_2)) = 0$$

and

$$H^2(\mathcal{O}_P(m_3 - m_1 - m_2)) = 0.$$

Proof. Assume an f_3 exists. Then $Y = Z \cap F_3$. Hence, multiplication by f_3 gives a surjection $\mu \colon \mathcal{O}_Z(-m_3) \twoheadrightarrow \mathcal{I}_{Y/Z}$. Its kernel Ann $\mathcal{I}_{Y/Z}(-m_3)$ is equal to $\mathcal{I}_{X/Z}(-m_3)$ because X is also the residual scheme of Y, owing to (2.1). So μ induces an isomorphism $\mathcal{O}_X(-m_3) \xrightarrow{\sim} \mathcal{I}_{Y/Z}$. Hence, by (2.3.2), ω_X has the asserted form.

Conversely, assume that $\omega_X = \omega_P(m_1 + m_2 - m_3)\big|_X$. Twisting the Noether linkage sequence (2.3.1) then yields the following exact sequence:

$$0 \to \mathcal{I}_{Z/P}(m_3) \to \mathcal{I}_{Y/P}(m_3) \to \mathcal{O}_X \to 0. \tag{2.4.1}$$

Extracting cohomology yields the next exact sequence:

$$H^0(\mathcal{I}_{Y/P}(m_3)) \to H^0(\mathcal{O}_X) \to H^1(\mathcal{I}_{Z/P}(m_3)).$$

Assume the additional vanishing conditions. Then $H^1(\mathcal{I}_{Z/P}(m_3)) = 0$ thanks to the twisted Koszul resolution,

$$0 \to \mathcal{O}_P(m_3 - m_1 - m_2) \to \mathcal{O}_P(m_3 - m_1) \oplus \mathcal{O}_P(m_3 - m_2) \to \mathcal{I}_{Z/P}(m_3) \to 0.$$

Hence, we may lift $1 \in H^0(\mathcal{O}_X)$ to an $f_3 \in H^0(\mathcal{I}_{Y/P}(m_3))$. Set $F_3 : f_3 = 0$.

In (2.4.1), we may replace $\mathcal{O}_X$ by $\mathcal{I}_{Y/Z}(m_3)$. Hence $\mathcal{I}_{Y/Z}(m_3)$ is generated by the image of f_3 in $H^0(\mathcal{I}_{Y/Z}(m_3))$. Therefore, $Y = Z \cap F_3$ and the proof is complete. □

3. Complete Intersections

DEFINITION 3.1. Let P be a Gorenstein scheme and X a closed Cohen–Macaulay subscheme. We'll say that X is *subcanonical* in P if P is equipped with an invertible sheaf $\mathcal{O}_X(1)$ and if, for some integer α, we have

$$\omega_X = \omega_P(\alpha)\big|_X.$$

Assume that P has pure dimension at least 3 and that X has pure codimension 2. We'll say that X is *self-linked* in P by two effective Cartier divisors F_1 and F_2 if they meet properly in a subscheme Z containing X and if X is equal to the residual scheme Y of X in Z.

THEOREM 3.2. *Let P be a projective space of dimension $n \geq 3$ over an algebraically closed field of characteristic* 0. *Let $X \subset P$ be a closed subscheme that* *is Cohen–Macaulay of pure codimension* 2. *Assume that X is subcanonical and self-linked. Then X is a complete intersection.*

In fact, say X is self-linked by hypersurfaces F_1 and F_2 of degrees m_1 and m_2. Then, after F_1 and F_2 are switched if need be, m_2 is even and there is a hypersurface F_3 of degree $m_2/2$ such that $X = F_1 \cap F_3$ and $Z = F_1 \cap 2F_3$, where $Z := F_1 \cap F_2$.

Proof. Since P is smooth and X is subcanonical, X is Gorenstein. Hence, since X has pure codimension 2, it is locally a complete intersection in P by one of Serre's results [Ei, 21.10, p. 537]. Hence, on X, the conormal sheaf $\mathcal{I}_{X/P}/\mathcal{I}_{X/P}^2$ is locally free of rank 2.

By another celebrated theorem of Serre's, $H^i(\mathcal{O}_P(j)) = 0$ for $i = 1, 2$ and for any j, since $n \geq 3$. Hence, by Gherardelli linkage (Theorem 2.5) there is a hypersurface F_3 such that $X = Z \cap F_3$.

Let $x \in X$. For $i = 1, 2, 3$, let $\varphi_i \in \mathcal{O}_{P,x}$ generate the ideal of F_i. Then $\mathcal{I}_{X/P,x}$ is generated by φ_1, φ_2, and φ_3 but not by φ_1 and φ_2, since $X = Z \cap F_3$ but $X \neq Z$. Since $\mathcal{I}_{X/P,x}$ is generated by two elements, it must be generated either by φ_1 and φ_3 or by φ_2 and φ_3. Hence X is a Cartier divisor on F_3.

For $i = 1, 2$, set $Z_i := F_i \cap F_3$. Let $x \in X$. Then, by the preceding paragraph, $\mathcal{I}_{X/P,x}$ is equal either to $\mathcal{I}_{Z_1/P,x}$ or to $\mathcal{I}_{Z_2/P,x}$. Put geometrically, X is equal, in a neighborhood of x in P, either to Z_1 or to Z_2.

For $i = 1, 2, 3$, say $F_i : f_i = 0$. For $i = 1, 2$, form the greatest common divisor g_i of f_i and f_3, and set $G_i : g_i = 0$.

First, suppose that both G_1 and G_2 are nonempty and let x be a common point. Since G_1 is a component of both F_1 and F_3, their intersection Z_1 is not equal to X in a neighborhood of x. Similarly, Z_2 is not equal to X in a neighborhood of x. This conclusion stands in contradiction to our previous conclusion that X is equal, in a neighborhood of x in P, either to Z_1 or to Z_2. Therefore, G_1 and G_2 cannot both be nonempty; say G_2 is empty.

Then Z_2 has pure codimension 2 in P, and $Z_2 \supseteq X$. If $Z_2 = X$, then $X = F_2 \cap F_3$. So suppose not, and we'll prove that $X = F_1 \cap F_3$. Form the residual scheme X_2 of X in Z_2. By general principles, X_2 is a Cartier divisor on F_3 because X and Z_2 are so; moreover, $Z_2 = X + X_2$.

Suppose G_1 is nonempty, and set $C := G_1 \cap F_2$. Then C is a hypersurface section of F_2. Hence C has a point x in common with X_2, which also lies on F_2. Then $x \in X$, because $C \subset Z$ and Z has the same support as X. Since G_1 is a component of both F_1 and F_3, their intersection Z_1 is not equal to X in a neighborhood of x. Since x lies on both X_2 and X, also Z_2 is not equal to X in a neighborhood of x. As before, there is a contradiction. Therefore, G_1 is empty.

It follows that Z_1 has pure codimension 2 in P and that $Z_1 \supseteq X$. If $Z_1 = X$, then $X = F_1 \cap F_3$ as claimed. So suppose not, and form the residual scheme X_1 of X in Z_1. By general principles, X_1 too is a Cartier divisor on F_3. After a bit of work, we'll achieve a contradiction.

First, we'll construct a natural splitting of the natural surjection,

$$\mathcal{I}_{X/P}/\mathcal{I}_{X/P}^2 \twoheadrightarrow \mathcal{I}_{X/F_3}/\mathcal{I}_{X/F_3}^2. \tag{3.2.1}$$

To do so, form the image $\mathcal{L}$ of $\mathcal{I}_{Z/P}$ in $\mathcal{I}_{X/P}/\mathcal{I}^2_{X/P}$; we will show that $\mathcal{L}$ maps isomorphically onto $\mathcal{I}_{X/F_3}/\mathcal{I}^2_{X/F_3}$. Since $\mathcal{L}$ maps surjectively and since $\mathcal{I}_{X/F_3}/\mathcal{I}^2_{X/F_3}$ is invertible (because X is a Cartier divisor on F_3), we need only show that $\mathcal{L}$ is invertible.

Let $x \in X$. Say, as before, that $\mathcal{I}_{X/P,x} = \mathcal{I}_{Z_1/P,x}$. Set $W := F_1 \cap 2F_3$. Then $W \supseteq Z$; indeed, $\mathcal{I}^2_{F_3/P} \subset \mathcal{I}_{Z/P}$ because $\mathcal{I}_{X/Z} = \operatorname{Ann} \mathcal{I}_{X/Z}$, since X is self-linked. Since also $W \supseteq Z_1$, there is a natural commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_{Z_1/W} & \longrightarrow & \mathcal{O}_W & \longrightarrow & \mathcal{O}_{Z_1} & \longrightarrow & 0 \\ & & {\scriptstyle u}\big\downarrow & & {\scriptstyle v}\big\downarrow & & {\scriptstyle w}\big\downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_{X/Z} & \longrightarrow & \mathcal{O}_Z & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0. \end{array}$$

Clearly, $\mathcal{I}_{Z_1/W} = \mathcal{O}_P(-F_3)\big|_{Z_1}$. Moreover, $\mathcal{I}_{X/Z} = \omega_X \otimes \omega_P(m_1+m_2)^{-1}$ owing to (2.3.2) with $Y := X$. Thus the source of u is invertible on Z_1, and the target is invertible on X. Now, $\mathcal{I}_{X/P,x} = \mathcal{I}_{Z_1/P,x}$. Hence, w is an isomorphism at x; in other words, X and Z are the same scheme in a neighborhood of x. Also, u is surjective at x, and its source and target are invertible sheaves on the same scheme in a neighborhood of x; hence, u is an isomorphism at x. Therefore, v is an isomorphism at x and so $\mathcal{I}_{W/P,x} = \mathcal{I}_{Z/P,x}$.

Thus, in $\mathcal{I}_{X/P}/\mathcal{I}^2_{X/P}$, the images of $\mathcal{I}_{W/P}$ and $\mathcal{I}_{Z/P}$ are equal at x. The image of $\mathcal{I}_{W/P}$ is equal to $\mathcal{O}_P(-F_1)\big|_X$ at x; indeed, the latter sheaf maps naturally into the former, and this map is surjective (since $X \subset F_3$) and injective at x, since its natural image is a direct summand of $\mathcal{I}_{X/P}/\mathcal{I}^2_{X/P}$ at x (because $\mathcal{I}_{X/P,x} = \mathcal{I}_{Z_1/P,x}$). The image of $\mathcal{I}_{Z/P}$ is $\mathcal{L}$, by definition; thus, $\mathcal{L}$ is invertible at x. Since $x \in X$ is arbitrary, $\mathcal{L}$ is invertible. Hence $\mathcal{L} \xrightarrow{\sim} \mathcal{I}_{X/F_3}/\mathcal{I}^2_{X/F_3}$, and (3.2.1) splits.

Let F be any irreducible component of F_3, and equip F with its reduced structure. Since F is a hypersurface, F meets X. Set $V := X \cap F$. Then V is a Cartier divisor on F and hence V is locally a complete intersection in P. Consider the natural commutative diagram of sheaves on V,

$$\begin{array}{ccc} \mathcal{L}\big|_V & \longrightarrow & (\mathcal{I}_{X/F_3}/\mathcal{I}^2_{X/F_3})\big|_V \\ \big\downarrow & & \big\downarrow \\ \mathcal{I}_{V/P}/\mathcal{I}^2_{V/P} & \longrightarrow & \mathcal{I}_{V/F}/\mathcal{I}^2_{V/F}\,. \end{array}$$

The top horizontal map is an isomorphism because it is the restriction of an isomorphism. The right vertical map is an isomorphism because it is surjective and its source and target are invertible. Therefore, the lower horizontal map splits.

Because

(a) the lower map splits,
(b) V is a Cartier divisor on F and is locally a complete intersection in P,
(c) F is reduced, irreducible, and closed, and
(d) P is a projective space of dimension $n \geq 3$ over an algebraically closed field of characteristic 0,

Braun's main theorem [Br, p. 26] implies that some multiple of V is numerically equivalent to a hypersurface section of F.

Since F is a hypersurface, it follows that F meets both X_1 and X_2, which are supposedly nonempty. For $i = 1, 2$, set $V_i := X_i \cap F$. Then V_i is a Cartier divisor on F, and $V + V_i = F_i \cap F$. Hence some multiple of V_i, too, is numerically equivalent to a hypersurface section of F. Therefore, V_1 and V_2 have a common point x. Then x lies on both Z_1 and Z_2 and thus on their intersection, which is X. However, there is no neighborhood of x in which either Z_1 or Z_2 is equal to X, because x lies on both X_1 and X_2. Thus, we've achieved the desired contradiction and so $X = F_1 \cap F_3$.

Then $W = Z$ everywhere (by the previous reasoning); in other words, $Z = F_1 \cap 2F_3$. Finally, set $m_3 := \deg F_3$. Then $\deg Z = 2m_1m_3$. Now, $Z := F_1 \cap F_2$, so $\deg Z = m_1m_2$. Hence $2m_3 = m_2$. The proof is complete. □

Example 3.3. Most of the proof of Theorem 3.2 works without change in the relative case, where P is a smooth projectively Cohen–Macaulay variety of pure dimension at least 3. However, to apply Braun's theorem, we must know that the surjection (3.2.1) splits when P is replaced by the ambient projective space; the proof shows that (3.2.1) itself splits, but this splitting is insufficient. The theorem does not hold even when P is replaced by a smooth hypersurface, as the following paragraph shows.

Let P be a smooth quadric hypersurface in $\mathbf{P}^4$. Let F_1 be the section of P by a hyperplane H_1 that is tangent to P at a point x. Then F_1 is a cone in H_1 with vertex at x and with base a smooth (plane) conic C. Fix $y \in C$. Then y determines a generator X of the cone F_1. Let H_2 be a hyperplane in $\mathbf{P}^4$ that cuts H_1 in the plane spanned by x and by the tangent line to C at y. Then X is a line and thus is subcanonical in P. Moreover, X is self-linked in P by F_1 and F_2 with $F_2 := H_2 \cap P$. However, X is not the complete intersection of two hypersurface sections of P, since any such complete intersection has even degree in $\mathbf{P}^4$.

Example 3.4. Theorem 3.2 is not valid in positive characteristic without some further restriction on X. Indeed, we will see that, in characteristic 2, there exists an example of an irreducible, but nonreduced, Cohen–Macaulay space curve X that is subcanonical and self-linked yet is not a complete intersection.

Ferrand [Fe, p. 345] explained how to put a subcanonical double structure on a line (indeed, on any complete curve that is locally a complete intersection) in $\mathbf{P}^3$ in any characteristic; moreover, the double curve can have arbitrarily negative arithmetic genus. Migliore [Mi, p. 185] proved that, in characteristic 2, a double line X is self-linked if its arithmetic genus is -2 or less. Such an X is not a complete intersection, because every complete intersection Z has nonnegative arithmetic genus by (2.3.2).

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