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Good Representations and Solvable Groups

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Dedicated to William Fulton on his 60th birthday

1. Introduction

The purpose of this paper is to provide a characterization of solvable linear algebraic groups in terms of a geometric property of representations. Representations with a related property played an important role in the proof of the equivariant Riemann–Roch theorem [EG2]. In that paper, we constructed representations with that property (which we call freely good) for the group of upper triangular matrices in GL_n. We noted that it seemed unlikely that such representations exist for arbitrary groups; the main result of this paper implies that they do not.

To state our results, we need some definitions. A representation V of a linear algebraic group G is said to be *good* (resp. *freely good*) if there exists a nonempty G-invariant open subset $U \subset V$ such that

(i) G acts properly (resp. freely) on U.
(ii) $V \setminus U$ is the union of a finite number of G-invariant linear subspaces.

Note that freely good representations were called "good" in [EG2].

The main result of the paper is the following theorem.

Theorem 1.1. *Let G be a connected algebraic group over a field k of characteristic not equal to 2. Then G is solvable if and only if G has a good representation. Moreover, if G is solvable and k is perfect then G has a freely good representation.*

In characteristic 2, a solvable group still has good representations, and a partial converse holds (Corollary 4.1). A key step in the proof of the main result is Theorem 4.1, which is inspired by an example of Mumford [MFK, Ex. 0.4].

In characteristic 0, solvable groups are characterized by a weaker property that does not require the action to be proper. (In general, if G acts properly on X then G acts with finite stabilizers on X, but the converse need not hold.)

Theorem 1.2. *Let G be a connected algebraic group over a field of characteristic 0. Suppose that G has a representation V that contains a nonempty open set U such that*:

(1) *the complement of U is a finite union of invariant linear subspaces*; *and*
(2) *G acts with finite stabilizers on U.*

Then G is solvable.

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Examples (see Section 6) show that this weaker property does not characterize solvability in positive characteristic.

2. Preliminaries

GROUPS AND REPRESENTATIONS. We let k denote a field with algebraic closure $\bar{k}$ and separable closure k_s. If Z is a k-variety and $k' \supset k$ is any extension of k, then $Z(k')$ denotes the k'-valued points of Z and $Z_{k'}$ denotes the k'-variety $Z \times_k k'$.

All groups in this paper are assumed to be linear algebraic groups over a field k. We assume that such a group G is geometrically reduced (i.e., that $G_{\bar{k}}$ is reduced). The identity component of a group G is denoted G^0.

Unless otherwise stated, a representation V of a group G is assumed to be k-rational; that is, V is a k-vector space and the action map $G \times V \to V$ is a morphism of k-varieties.

If $k' \supset k$ is a field extension then we call a k'-rational representation V of $G_{k'}$ a k'-representation of G. We say that V is defined over k if it is obtained by base change from a k-rational representation.

If $k' \supset k$ is a Galois field extension, then $\mathrm{Gal}(k'/k)$ acts on k'-representations of G. Indeed, let V be a k'-representation of G corresponding to a k'-morphism $\rho\colon G_{k'} \to \mathrm{GL}(V)$. For $g \in G(k_s)$ and σ in $\mathrm{Gal}(k'/k)$, we define ${}^{\sigma}\rho(g)$ as follows (cf. [B, AG14.3, 24.5]). Because ρ is defined over k', for any $\tau \in \mathrm{Gal}(k_s/k')$ and any $g \in G(k_s)$ we have $\tau(\rho(\tau^{-1}(g))) = \rho(g)$. Thus, if $\sigma \in \mathrm{Gal}(k'/k) = \mathrm{Gal}(k_s/k)/\mathrm{Gal}(k'/k)$, then

$$\sigma'(\rho((\sigma')^{-1}g)) \in \mathrm{GL}(V)(k_s)$$

is independent of the lift of σ to an element $\sigma' \in \mathrm{Gal}(k_s/k)$. We will call this point ${}^{\sigma}\rho(g)$ and set ${}^{\sigma}\rho(g) = \sigma(\rho(\sigma^{-1}g))$.

The k'-representation V is obtained by base change from a representation defined over k if and only if ${}^{\sigma}\rho = \rho$ for all $\sigma \in \mathrm{Gal}(k'/k)$.

FREE AND PROPER ACTIONS. The action of a group G on a scheme X is said to be *free* if the action map $G \times X \to X \times X$ is a closed embedding. The action is said to be *proper* if the map $G \times X \to X \times X$ is proper. If the action is proper then the stabilizer of every point is finite. If the stabilizer of every geometric point is a trivial group scheme then we say that the action is *set-theoretically free.* An action that is set-theoretically free and proper is free [EG1].

Let $H \to G$ be a finite morphism of algebraic groups. If G acts properly on a scheme X then H also acts properly on X. Thus, if V is a good representation of G then V is also a good representation of H via the action induced by the map $H \to G$. Moreover, if H is a closed subgroup and V is a freely good representation of G, then V is a freely good representation of H.

EXAMPLE 2.1. Let B be the group of upper triangular matrices in $\mathrm{GL}(n)$. The group B acts by left multiplication on the vector space V of upper triangular matrices; it acts with trivial stabilizers on the open subset U of invertible upper triangular matrices. Since the matrices are upper triangular, $V \setminus U$ is the union of the invariant subspaces $L_i = \{A \in V \mid A_{ii} = 0\}$. This representation is freely good because the action of B on U is identified with B acting itself by left multiplication. The map $B \times B \to B \times B$ given by $(A, A') \mapsto (A, AA')$ is an isomorphism, so the action of B on U is free.

By contrast, the action of $\mathrm{GL}(n)$ by left multiplication on the vector space M_n of $n \times n$ matrices is not good.

3. Existence of Good Representations

In this section we show that every connected solvable group G has good representations and, if k is perfect, freely good representations.

By the Lie–Kolchin theorem, $G_{\bar{k}}$ is trigonalizable; that is, it can be embedded in the group $B_{\bar{k}} \subset \mathrm{GL}_n$ of upper triangular matrices.

Let $V_{\bar{k}}$ be the vector space of upper triangular $n \times n$ matrices. The group $B_{\bar{k}}$ acts on $V_{\bar{k}}$ by left multiplication, and we have seen that this representation is freely good. By restriction, $V_{\bar{k}}$ is a good representation of $G_{\bar{k}}$. Consider the morphism $\rho\colon G_{\bar{k}} \to \mathrm{GL}(V_{\bar{k}})$ corresponding to the action of $G_{\bar{k}}$ on $V_{\bar{k}}$.

Since ρ is a morphism of schemes of finite type, it is defined over a field extension $k' \supset k$ of finite degree. Write $V = V_{k'}$ for the corresponding k'-representation; then we have $\rho\colon G_{k'} \to \mathrm{GL}(V)$.

Case I: k' is Separable over k

(This will occur when k is perfect.) In this case we will use Galois descent to construct a freely good representation of G.

Replacing k' by a possibly bigger field extension, we may assume that $k' \supset k$ is Galois. Enumerate the elements of $\mathrm{Gal}(k'/k)$ as $\{1 = \sigma_1, \sigma_2, \dots, \sigma_d\}$ and consider the representation $\Phi\colon G_{k'} \to \mathrm{GL}(V^{\oplus d})$, where $G_{k'}$ acts on the jth factor by the representation ${}^{\sigma_j}\rho\colon G_{k'} \to \mathrm{GL}(V)$.

We define $U_d \subset V^{\oplus d}$ to be the open set whose k_s-rational points are the d-tuples $(A_1, \dots, A_d)$, where some A_i is invertible. We realize U_d as a complement of $G_{k'}$-invariant linear subspaces as follows. Let $L_j = \{A \in V \mid A_{jj} = 0\}$, a $G_{k'}$-invariant subspace of V. Given a d-tuple $(j_1, \dots, j_d)$, define

$$L_{(j_1,\dots,j_d)} = L_{j_1} \oplus \cdots \oplus L_{j_d}.$$

This is a $G_{k'}$-invariant subspace of $V^{\oplus d}$ and $U_d = V_d \setminus \bigcup L_{(j_1,\dots,j_d)}$.

Lemma 3.1 (cf. [EG2, Thm. 2.2]). *$G_{k'}$ acts freely on U_d.*

Proof. Since $G_{k'}$ is a closed subgroup of $B_{k'}$ and the open set U_d is $B_{k'}$ invariant, it suffices to show that $B_{k'}$ acts freely on U_d. To do this, we must show that the map $B_{k'} \times U_d \to U_d \times U_d$ given on k_s-points by

$$(A, A_1, \dots, A_d) \mapsto (AA_1, \sigma_2(A\sigma_2^{-1}(A_2)), \dots, \sigma_d(A\sigma_d^{-1}(A_d)))$$

is a closed embedding.

First we show that the image Z of $B_{k'} \times U_d$ is closed in $U_d \times U_d$. Let $(A_1, A_2, \ldots, A_d, C_1, \ldots, C_d)$ be matrix coordinates on $U_d \times U_d$. Expanding the inverse out in terms of the adjoint, we see that the image is contained in the subvariety defined by the matrix equations

$$\sigma_j\sigma_i^{-1}(\det A_i)C_j = \sigma_j\sigma_i^{-1}(C_i \operatorname{Adj} A_i)A_j.$$

Suppose that a $2d$-tuple of matrices $(A_1, A_2, \ldots, A_d, C_1, C_2, \ldots, C_d) \in U_d \times U_d$ satisfies the matrix equations above. At least one of the A_i and one of the C_j is invertible because we are in $U_d \times U_d$. Let $A = \sigma_i^{-1}(C_i A_i^{-1})$. Substituting into our equations we see that $C_l = \sigma_l(A\sigma_l^{-1}(A_l))$ for all l. Moreover, A is invertible since C_j is invertible and $C_j = \sigma_j(A\sigma_j^{-1}(A_j))$. Hence every point satisfying the matrix equations is in the image Z of $B_{k'} \times U_d$, so Z is closed.

The variety Z is covered by open sets of the form

$$\{(A_1, A_2, \ldots, A_j, \ldots, A_d, AA_1, \ldots, AA_j, \ldots, AA_d) \mid \det A_j \neq 0\}.$$

These open sets are isomorphic to $V^{d-1} \times B_{k'}$, where V^{d-1} is the $(d-1)$-fold Cartesian product of V. Hence the image is smooth—in particular, normal. The action of $G_{k'}$ on U_d is set-theoretically free, so $G_{k'} \times U_d \to Z$ is a birational bijection. By Zariski's main theorem (cf. [B, AG18]), a birational bijection of normal varieties is an isomorphism; hence $G_{k'} \times U_d \to Z$ is an isomorphism. Therefore, $G_{k'} \times U_d \to U_d \times U_d$ is a closed embedding. □

REMARK. The proof of [EG2, Thm. 2.2] is incomplete; the last paragraph of the preceding argument is needed.

For any basis of V, there is a natural choice of basis so that, with respect to this basis, if $g \in G(k_s)$ then $\Phi(g)$ is represented by the block diagonal matrix

$$\begin{bmatrix} \rho(g) & & & \\ & {}^{\sigma_2}\rho(g) & & \\ & & \ddots & \\ & & & {}^{\sigma_d}\rho(g) \end{bmatrix}.$$

This representation is not defined over k because the Galois group acts by permuting the blocks. More precisely, we have the following. Given a $d \times d$ matrix M, let $M[n]$ denote the $nd \times nd$ matrix whose ij block is $M_{ij} \cdot I_n$, where I_n is the $n \times n$ identity matrix. If $\sigma \in \operatorname{Gal}(k'/k)$, let J_σ denote the permutation matrix corresponding to the permutation $\sigma_i \mapsto \sigma\sigma_i$. In matrix form, for $g \in G(k_s)$ we have

$${}^{\sigma}\Phi(g) = J_\sigma[n]^{-1}\Phi(g)J_\sigma[n].$$

We will show that Φ is k'-isomorphic to a freely good representation defined over k. Choose a primitive element α for the extension $k' \supset k$, and let A be the $d \times d$ matrix with $A_{ij} = \sigma_j(\alpha^i)$. The matrix A is invertible since $\alpha, \sigma_2(\alpha), \ldots, \sigma_d(\alpha)$ are exactly the roots of the irreducible polynomial $f \in k[x]$ of α over k, so $\det A = \prod_{i<j}(-1)^d(\sigma_i(\alpha) - \sigma_j(\alpha)) \neq 0$. The Galois group acts by $\sigma(A) = AJ_\sigma$. Consider the morphism $\Psi\colon G_{k'} \to \mathrm{GL}(V^{\oplus d})$ defined by $\Psi(g) = A[n]\Phi(g)A[n]^{-1}$ for $g \in k_s$. Then ${}^{\sigma}\Psi(g) = \Psi(g)$ for any $\sigma \in \operatorname{Gal}(k'/k)$; hence Ψ is defined over k.

Each of the subspaces $L_{(j_1,\dots,j_d)}$ is $G_{k'}$-invariant under the action Ψ. Moreover, because each $L_{(j_1,\dots,j_d)}$ is a vector subspace of $V^{\oplus d}$ defined over k, the corresponding subrepresentations $G_{k'} \to \mathrm{GL}(L_{(j_1,\dots,j_d)})$ are also defined over k. Therefore Ψ is obtained by base change from a freely good representation of G.

Case II: The General Case

Here we may assume that there is a freely good k'-rational representation $\rho\colon G_{k'} \to \mathrm{GL}(V)$ defined over a finite normal extension k' of k. Then $k' \supset k$ factors as $k' \supset k'' \supset k$, with k'/k'' purely inseparable of degree p^n and k''/k Galois. The Frobenius endomorphism on V induces a group homomorphism of $\mathrm{GL}(V)$. Composing ρ with the nth power of Frobenius on $\mathrm{GL}(V)$, we obtain a representation defined over k''. Because the Frobenius has finite kernel, this representation will no longer be faithful. However, the action of Frobenius is trivial on geometric points, so G will act properly on an open set whose complement is a union of linear subspaces. We can now use the Galois descent argument of Case I to obtain a good k-rational representation of G.

4. Characterization of Solvable Groups by Good Representations

In this section we show that if $\operatorname{char} k \neq 2$ then every group with a good representation is solvable. However, many of the results of this section are valid in arbitrary characteristic. In particular, we prove that every reductive group with a good representation is a torus. We only need that $\operatorname{char} k \neq 2$ in part of the proof of Theorem 1.1. We will explicitly say when we start assuming this; until then, $\operatorname{char} k$ is arbitrary.

Let T be the diagonal torus in SL_2 and let $N(T)$ be the normalizer of T. We will first show that $N(T)$ has no good representations. We begin by recalling some facts about $N(T)$. First, let

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix};$$

we will also write

$$H(t) = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}.$$

The group $N(T)$ is generated by T and J; it has two components, T and $J(T)$. The action of SL_2 on its 2-dimensional standard representation V induces an action on $S(V^*) \cong k[x, y]$ given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : y \mapsto ay - cx, \qquad x \mapsto -by + dx.$$

Let W_i denote the subspace of $S(V^*)$ spanned by x^i and y^i; this is an irreducible representation of $N(T)$ of dimension 2 (if $i > 0$). Let W_0' denote the 1-dimensional irreducible representation of $N(T)$ on which T acts trivially and J acts by multiplication by -1.

If $\operatorname{char} k \neq 2$, the group $N(T)$ is linearly reductive; that is, its action on any representation is completely reducible [MFK, p. 191]. The next lemma shows that much of this survives in arbitrary characteristic.

LEMMA 4.1. *Let V be a representation of $N(T)$.*

(1) *As a representation of $N(T)$, V splits as a direct sum of $N(T)$-submodules*:

$$V = V_0 \oplus \bigoplus_{i>0} V_{\pm i}.$$

Here $V_{\pm i}$ is the sum of the i and $-i$ weight spaces of T on V, and V_j is the j-weight space.

(2) *The action of $N(T)$ on $V_{\pm i}$ $(i > 0)$ is completely reducible, and $V_{\pm i}$ is isomorphic as $N(T)$-module to a direct sum of copies of W_i.*

(3) *If $\operatorname{char} k \neq 2$, then V_0 is isomorphic to a direct sum of copies of W_0 and W_0'.*

Proof. (1) Because the action of T on V is completely reducible, we can decompose $V = \bigoplus V_i$ as a T-module. As $JH(t)J^{-1} = H(t^{-1})$, we have $JV_i = V_{-i}$. Hence $V_{\pm i}$ is an $N(T)$-submodule and so we have the desired direct sum decomposition of V.

(2) Let $v_1, \ldots, v_d$ be a basis for V_i $(i > 0)$. The map $v_r \mapsto y$, $Jv_r \mapsto -x$ defines an isomorphism of the span of v_r, Jv_r (denoted $\langle v_r, Jv_r \rangle$) with W_i, and the map $W_i^{\oplus d} \to V_{\pm i}$, taking the rth component to $\langle v_r, Jv_r \rangle$, is an $N(T)$-module isomorphism.

(3) Decompose the 0-weight space of V into the $+1$ and -1 eigenspaces of J; these are isomorphic to sums of copies of W_0 and W_0', respectively. □

The proof of the following result was motivated by [MFK, Ex. 0.4].

THEOREM 4.1. *The group $N(T)$ has no good representations.*

Proof. If a group G has good representations, then so does $G_{\bar{k}}$, so we may assume that k is algebraically closed. Suppose that V is a representation of $N(T)$, and let $U \subset V$ be the complement of a finite set of invariant linear subspaces $\mathcal{S}$. We will show that $N(T)$ does not act properly on $U = V - \bigcup_{L \in \mathcal{S}} L$. The strategy of the proof is as follows. Consider the action map $\Phi\colon N(T) \times U \to U \times U$. We will find a closed subvariety Z of $N(T) \times U$ whose closed points are of the form

$$\left(\begin{bmatrix} 0 & -\lambda^{-1} \\ \lambda & 0 \end{bmatrix}, v_\lambda \right)$$

and whose image is not closed in $U \times U$. Hence Φ is not proper, so the representation is not good.

We now carry out the proof. Decompose $V = \bigoplus V_i$, where V_i is the i-weight space of V for T. Pick $u \in U$ and write $u = \sum u_i$, where $u_i \in V$. Some of the u_i may be 0; let d be the dimension of the space spanned by the nonzero u_i.

Step 1. If $a_i \neq 0$ for all i with $u_i \neq 0$, then $w = \sum a_i u_i \in U$. Indeed, suppose not; then $w \in L$ for some $L \in \mathcal{S}$. For almost all choices $t_1, \ldots, t_d$ of d elements of k^*, the vectors

$$H(t_q)w = \sum_p t_q^{i_p} a_{i_p} u_{i_p} \in L$$

are linearly independent. (Here $i_1, \ldots, i_d$ are the indices i_p with $u_{i_p} \neq 0$.) This follows because the $d \times d$ matrix A with entries

$$A_{pq} = t_q^{i_p}$$

is nonsingular for almost all $t_1, \ldots, t_d$. (This is because det A is a sum of monomials, where each monomial is a product of one term from each row and each column; each monomial has different multi-degree, so det A is not the zero polynomial.) Therefore, the vectors $H(t_q)w$ span the same space as the u_i and so $u \in L$, contradicting our assumption that $u \in U$. We conclude that $w \in U$, as claimed.

A similar argument shows that $Ju_0 + \sum_{i \neq 0} u_i \in U$.

Step 2. There exists an element $u' = \sum u'_i \in U$ with $Ju'_i = u'_{-i}$ for all $i > 0$. To see this, suppose $u_{-j} \neq Ju_j$ for some $j > 0$. Let $W_j \subset V_j \oplus V_{-j}$ be the subspace of vectors of the form $v_j + Jv_j$ $(v_j \in V_j)$. Note that W_j generates $V_j \oplus V_{-j}$ as an $N(T)$-module. Consider the affine linear subspace

$$B = \sum_{i \neq \pm j} u_i + W_j.$$

We claim that $B \cap U$ is nonempty. If it is empty then, because B is affine linear and is contained in a finite union of the subspaces in $\mathcal{S}$, we see that $B \subset L$ for some $L \in \mathcal{S}$. But then the span of B is contained in L, so $\sum_{i \neq \pm j} u_i \in L$ and $W_j \subset L$. Because L is $N(T)$-stable, $V_j \oplus V_{-j} \subset L$ as well; hence

$$u \in \sum_{i \neq \pm j} u_i + (V_j \oplus V_{-j}) \subset L,$$

contradicting $u \in U$. We conclude that $B \cap U$ is nonempty. Replacing u by an element of $B \cap U$, which we again call u, we do not change u_i for $i \neq \pm j$ but we do obtain $u_{-j} = Ju_j$. Iterating this process, we obtain u' of the desired form.

Replacing u by u', we will assume that $Ju_i = u_{-i}$ for all $i > 0$. From the $N(T)$-module isomorphism of $\langle u_i, Ju_i \rangle$ with $\langle x^i, y^i \rangle$ we see that, for $i > 0$,

$$\begin{bmatrix} 0 & -\lambda^{-1} \\ \lambda & 0 \end{bmatrix} : u_i \mapsto \lambda^i u_{-i}, \quad u_{-i} \mapsto (-1)^i \lambda^{-i} u_i.$$

Note also that

$$\begin{bmatrix} 0 & -\lambda^{-1} \\ \lambda & 0 \end{bmatrix} u_0 = Ju_0.$$

Step 3. Define

$$v_\lambda = u_0 + \sum_{i>0} (u_{-i} + \lambda^{-i} u_i),$$

$$v'_\lambda = Ju_0 + \sum_{i>0} (u_{-i} + (-1)^i \lambda^{-i} u_i).$$

For all $\lambda \neq 0$, both v_λ and v'_λ are in U (by Step 1). Define Z to be the closed subvariety of $N(T) \times U$ whose points are the pairs

$$\left(\begin{bmatrix} 0 & -\lambda^{-1} \\ \lambda & 0 \end{bmatrix}, v_\lambda\right).$$

Then

$$\Phi(Z) = \{(v_\lambda, v'_\lambda)\}_{\lambda \neq 0} \subset U \times U.$$

Consider the point

$$(v, v') = \left(u_0 + \sum_{i>0} u_{-i},\ Ju_0 + \sum_{i>0} u_{-i}\right).$$

Reasoning as in Step 1 shows that u is in the $N(T)$-module generated by v or v', so if either v or v' were in L then u would be; but this is impossible since $u \in U$. Hence v and v' are in U, so $(v, v') \in U \times U$. Also, (v, v') is not in $\Phi(Z)$ but is in the closure of $\Phi(Z)$ in $U \times U$. We conclude that Φ is not proper, so the representation is not good. □

Proof of Theorem 1.1. Let G be a connected nonsolvable linear algebraic group. Consider the surjective map $\pi : G \to G_1 = G/\mathcal{R}_u G$, where $\mathcal{R}_u G$ is the unipotent radical of G and where G_1 is reductive. Because G is not solvable, G_1 is neither trivial nor a torus. Let T be a maximal torus of G. Then $T_1 = \pi(T)$ is a maximal torus of G_1, and π induces an isomorphism of Weyl groups $W(T, G) \to W(T_1, G_1)$ [B, 11.20]. (Here $W(T, G) = N_G(T)/Z_G(T)$, where N_G and Z_G denote normalizer and centralizer of T in G, and similarly for G_1.) Because $\ker \pi$ is a unipotent group, $\pi\big|_T : T \to T_1$ is an isomorphism. Note that $Z_G(T) = T \cdot (\mathcal{R}_u G)^T$ [B, 13.17]. Because G_1 is reductive, this fact (applied to G_1) implies that $Z_{G_1}(T_1) = T_1$. Moreover, any $g_1 \in N_{G_1}(T_1)$ can be lifted to $g \in N_G(T)$. This follows because the isomorphism of Weyl groups just described, together with the structure of the centralizers, implies that each component of $N_{G_1}(T_1)$ is the image of a surjective map of a component of $N_G(T)$.

Since G_1 is not a torus, there is a root α and a homomorphism $\phi_\alpha : \mathrm{SL}_2 \to G_1$ with kernel either trivial or the set of matrices $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ with $a^2 = 1$. Moreover (using the subscript SL_2 to denote terms for SL_2 defined in the previous subsection), $\phi_\alpha(T_{\mathrm{SL}_2}) \subset T$ and $J_1 := \phi_\alpha(J_{\mathrm{SL}_2}) \in N_{G_1}(T_1)$. (See [J, p. 176] for these facts.) Let $H_1 = \phi_\alpha(N(T_{\mathrm{SL}_2}))$; its identity component $H_1^0 = \phi_\alpha(T_{\mathrm{SL}_2}) \subset T_1$. Because H_1 is a finite image of $N(T_{\mathrm{SL}_2})$, it has no good representations (and hence neither does G_1).

Up to this point, $\mathrm{char}\, k$ *has been arbitrary*; *now we assume that* $\mathrm{char}\, k \neq 2$.

Because $\pi\big|_T$ is an isomorphism, there is a unique subgroup $H^0 \subset T$ projecting isomorphically to H_1^0. As noted previously, we can choose a lift $J \in N_G(T)$ of $J_1 \in N_{G_1}(T_1)$. Write $J = J_s J_u$ for the Jordan decomposition of J. Because $\mathrm{char}\, k \neq 2$, J_1 is semisimple and so $\pi(J_u) = 1$. Therefore we can replace J by J_s and assume that J is semisimple. Now J^2 corresponds to the identity element in the Weyl group (as J_1^2 does), so $J^2 \in Z_G(T) = T \cdot (\mathcal{R}_u G)^T$. Since J is semisimple, we conclude that $J^2 \in T$. Because J_1^2 is in the subgroup H_1^0 of T_1 and T maps isomorphically to T_1, we conclude that $J^2 \in H^0$. Therefore, the group H generated by H^0 and J maps isomorphically to H_1 and thus has no good representations; hence G has no good representations. This proves Theorem 1.1. □

The proof of Theorem 1.1 yields the following weaker statement in characteristic 2. Note that Levi decompositions need not exist in positive characteristic [B, 11.22].

COROLLARY 4.1. *Suppose* $\operatorname{char} k = 2$. *If the connected algebraic group G has a Levi decomposition and if G has a good representation, then G is solvable. In particular, any connected reductive group with a good representation is diagonalizable.*

Proof. Suppose that $G = LN$, where L is reductive and N unipotent. If G has a good representation then so does L. As we have shown, this implies that L is a torus, so G is solvable. □

5. Proof of Theorem 1.2

If G has a representation V that contains an open set U whose complement is a finite union of invariant subspaces such that G acts with finite stabilizers on U, then $G_{\bar{k}}$ also has such a representation. Thus we can assume that k is algebraically closed.

Assume that G is not solvable, and let V be a representation of G. Since the characteristic is 0, G has a Levi subgroup L. Since G is assumed to be nonsolvable, L contains a Borel subgroup that is not a torus. Hence L contains a nontrivial unipotent subgroup N.

Since the characteristic is 0 and L is reductive, V decomposes as a direct sum $V = V_1 \oplus V_2 \oplus \cdots \oplus V_p$ of irreducible L-modules. Every vector in the subspace $V^N = V_1^N \oplus V_2^N \oplus \cdots \oplus V_p^N$ has a positive dimensional stabilizer. Since N is unipotent, $V_i^N \neq 0$ for each i and so $L(V_i^N)$ spans all of V_i. Hence the subset $LV^N = L(V_1^N \oplus \cdots \oplus V_p^N)$ that consists of vectors with positive dimensional stabilizers cannot be contained in any proper L-invariant subspace. Since L is a subgroup of G, this means that LV^N is not contained in any proper G-invariant subspace. Hence V does not have properties (1) and (2). □

6. Examples and Complements

In this section we discuss set-theoretic versions of the conditions "freely good" and "good". We will say a representation V is *set-theoretically freely good* (resp. *set-theoretically good*) if it contains a nonempty open subset U whose complement is a union of invariant subspaces such that G acts with trivial stabilizers (resp. finite stabilizers) on U (cf. Theorem 1.2). Surprisingly, these conditions are not enough to characterize solvability in arbitrary characteristic.

EXAMPLE 6.1. Let V be the standard representation of SL_2 and let $V_d = S(V^*)$ be the vector space of homogeneous forms of degree d. As in Section 4, SL_2 acts on V_d. If $p = \operatorname{char} k$ is an odd prime then $W_p = V_{2p-2} \oplus V_1$ is a set-theoretically freely good representation of SL_2. The reason is as follows. The stabilizer of any pair of forms $(f(x, y), l(x, y))$ is trivial as long as $l(x, y) \neq 0$ and the coefficient of $x^{p-1}y^{p-1}$ in f is nonzero. Since the characteristic is p, the subspace

$L_{2p-2} \subset V_{2p-2}$ of forms with no $x^{p-1}y^{p-1}$ term is an SL_2 invariant subspace (cf. [J, II2.16]). Thus, SL_2 acts with trivial stabilizers on the open set $U_p = W_p \setminus (W_{2p-2} \oplus V_1 \cup V_{2p-2} \oplus 0)$.

In characteristic 2, the representation $W_2 = V_2 \oplus V_1$ is not set-theoretically freely good because the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ stabilizes the pair $(x^2y^2, x+y)$. However, W_2 is set-theoretically good.

In positive characteristic, we do not know if the group SL_n admits set-theoretically good representations for $n \geq 3$.

EXAMPLE 6.2. Assume that k is algebraically closed and that $\operatorname{char} k \neq 2$. Then $G = \mathrm{PGL}_2$ has no representation that is set-theoretically freely good. Indeed, let

$$g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad H = \{1, g\}.$$

If V is any representation of G, then V^H generates V as a representation of G. Indeed, this holds if V is irreducible because, for any vector v, the vector $v + gv$ is a nonzero H-invariant. Since $\operatorname{char} k \neq 2$, the action of H is completely reducible; hence, if

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

is an exact sequence of G-modules then the corresponding sequence of H-invariants is also exact. By induction, we may assume that V_1^H and V_3^H generate V_1 and V_3 as G-modules, and a diagram-chase then shows that V_2^H generates V_2 as a G-module.

It follows that if V is any representation then there is no proper invariant linear subspace of V containing V^H. Therefore, V is not set-theoretically freely good. A similar argument shows that PGL_n and GL_n do not have set-theoretically freely good representations.

We conclude with a proposition about the inductive construction of good representations.

PROPOSITION 6.1. *Let G be a connected linear algebraic group and H a normal subgroup. Assume that k is algebraically closed. If H and G/H have set-theoretically freely good representations, then so does G.*

Proof. For this proof only, we will use "good" to mean "set-theoretically freely good". Let W be a good representation of H, with M_i a finite set of proper invariant subspaces containing the vectors with nontrivial stabilizers. Because G/H is affine [B, Thm. 6.8], the vector bundle $G \times^H W$ is generated by a finite-dimensional space of global sections Γ. We will view sections of the vector bundle as regular functions $\gamma \colon G \to W$ satisfying $\gamma(gh) = h^{-1} \cdot \gamma(g)$, where on the right side we are using the action of H on W. The action of G on the space of sections of the vector bundle corresponds to the left action of G on regular functions: $(g \cdot \gamma)(g_0) = \gamma(g^{-1}g_0)$. Because the action of G on regular functions is locally finite, by enlarging the space Γ we may assume that Γ is stable under the G-action.

Define L_i to be the subspace of Γ consisting of those elements of Γ that are sections of $G \times^H M_i$. Each L_i is a G-stable subspace of Γ. Let Γ^0 denote the complement of the L_i in Γ.

Let V be a good representation of G/H, viewed as a representation of G via the map $G \to G/H$. We claim that $V \oplus \Gamma$ is a set-theoretically good representation of G. Indeed, let V_j be a finite set of invariant subspaces of V containing the vectors with nontrivial stabilizer. It suffices to show that the vectors with nontrivial stabilizer in $V \oplus \Gamma$ are contained in the union of the subspaces $V_j \oplus \Gamma$ and $V \oplus L_i$. To see this, let (v, γ) be in the complement of these subspaces; hence $v \notin V_j$ and $\gamma \notin L_i$ for any i, j. We must show that $\mathrm{stab}_G(v, \gamma)$ is trivial. First, $\mathrm{stab}_G(v, \gamma) \subset \mathrm{stab}_G(v) = H$. Let $h \in \mathrm{stab}_G(v, \gamma)$. As before, we will view γ as a function $G \to W$. Because γ is not in any L_i, we have that γ is not a section of $G \times^H M_i$ for any i. In other words, the open subsets $\gamma^{-1}(W \setminus M_i)$ of G are nonempty. Choose g_0 in the intersection of these sets, so $s(g_0) \notin M_i$ for any i. Our hypothesis implies that $h \cdot \gamma = \gamma$. By definition, we have

$$(h \cdot \gamma)(g_0) = \gamma(h^{-1} g_0) = \gamma(g_0(g_0^{-1} h^{-1} g_0)) = (g_0^{-1} h g_0)\gamma(g_0).$$

But $\mathrm{stab}_H\, \gamma(g_0) = \{1\}$, so we conclude that $h = 1$, as desired. □

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