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Dimension of Julia Sets of Polynomial Automorphisms of $\mathbf{C}^2$

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1. Introduction

Let g be a polynomial automorphism of $\mathbf{C}^2$. In a similar way as is done for polynomials in $\mathbf{C}$, we denote by $K^{\pm}$ the set of points in $\mathbf{C}^2$ with bounded forward/backward orbit under g. We write $J^{\pm} = \partial K^{\pm}$ and $J = J^+ \cap J^-$. We refer to $J^{\pm}$ as the positive/negative Julia set and to J as the Julia set of g. The set $J^{\pm}$ is unbounded, closed, and connected, while J is compact (see [BS2; BS3; FM; HO] for more details).

The purpose of the main part of this paper is to show that, under the assumption that g is a hyperbolic mapping (i.e., the Julia set J is a hyperbolic set for g), the complete information about the Hausdorff dimensions of J^+ and J^- is already contained in the Julia set J itself. In particular, the results of Theorem 4.1–4.4 can be summarized by the following result.

THEOREM 1.1. *Let g be a hyperbolic polynomial automorphism of $\mathbf{C}^2$ and let $p \in J$. Then*

(i) $\dim_H J^{\pm} = \dim_H W_{\varepsilon}^{u/s}(p) \cap J + 2$;
(ii) $2 < \dim_H J^{\pm} < 4$;
(iii) $\dim_H J = \dim_H J^+ + \dim_H J^- - 4$.

The main idea in the proof of Theorem 1.1(i) is to construct locally a lamination of $\mathbf{C}^2$ such that the intersection of its leaves with $J^{\pm}$ can be represented as the image of $W_{\varepsilon}^{u/s}(p) \cap J$ under a particular holomorphic motion. It is then possible to verify that locally the Hausdorff dimension of $J^{\pm}$ is arbitrarily close to that of $W_{\varepsilon}^{u/s}(p) \cap J + 2$.

Only partial results are known about the Hausdorff dimensions of J^+ and J^- (see [FoS; Wo]). One difficulty for a direct calculation is that both J^+ and J^- are unbounded sets, and every restriction of g to a sufficiently large (in the sense of Hausdorff dimension) compact subset leads—either under forward or under backward iteration—out of the set. On the other hand, a result of Verjovsky and Wu [VW] shows that the Hausdorff dimension of $W_{\varepsilon}^{u/s}(p) \cap J$ can be calculated in terms of Bowen's formula. Therefore, Theorem 1.1(i) relates the Hausdorff dimension of $J^{\pm}$ to Bowen's formula.

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The (un)stable set of a hyperbolic set for a C^2-diffeomorphism has Lebesgue measure zero, except in the case of an attractor or repeller (see [Bo]). Theorem 1.1(ii) thus provides an even stronger result for Julia sets of polynomial automorphisms of $\mathbf{C}^2$.

Part (iii) of Theorem 1.1 is the main result of this paper and represents an intersection formula for the Hausdorff dimension of J. It turns out that the intersection between J^+ and J^- is "nice" in the sense of Hausdorff dimension.

In the second part of this paper we study dependence on the parameters. It is shown in [VW] that the Hausdorff dimension of J is a real-analytic function of the parameter of the mapping. This result can be easily extended by Theorem 1.1 to the positive/negative Julia set $J^{\pm}$.

For an analytic family of hyperbolic rational mappings on the Riemann sphere, it is shown by Ransford [Ra] that the Hausdorff dimension of the Julia set depends subharmonically on the parameter of the mapping. We show the higher-dimensional counterpart for polynomial automorphisms of $\mathbf{C}^2$.

COROLLARY 5.5. *The Hausdorff dimensions of $J^{\pm}$ and J depend plurisubharmonically* (*psh*) *on the parameter of the mapping.*

By proving this, we also obtain a new non–potential-theoretical proof for the fact that the Lyapunov exponent of the equilibrium measure depends pluriharmonically on the parameter of the mapping.

In the last part of this paper, we apply our results to polynomial automorphisms of $\mathbf{C}^2$ that are (in a particular sense) close to a hyperbolic polynomial in $\mathbf{C}$. Our results are essentially based on the work of Fornæss and Sibony [FoS], who showed the existence of a holomorphic motion that moves the Julia set of the polynomial holomorphically to a slice of J^+. We obtain that the Hausdorff dimension of J is close to that of the 1-dimensional Julia set (see Corollary 6.5). In addition, each value in $(3, 4)$ can occur for the Hausdorff dimension of $J^{\pm}$. This result is related to a result of Shishikura [Sh] about the Hausdorff dimension of Julia sets in hyperbolic components of the Mandelbrot set.

The results of [RR] and [BS2] imply that every basin of attraction of a nontrivial polynomial automorphism of $\mathbf{C}^2$ is biholomorphically equivalent to $\mathbf{C}^2$ and nondense in $\mathbf{C}^2$. Domains with that property are called Fatou–Bieberbach domains and are a subject of classical complex analysis. By the work of Stensønes [St], there exists a Fatou–Bieberbach domain in $\mathbf{C}^2$ with smooth boundary. As a counterpart to this remarkable result, we show in Corollary 6.8 that for all $s \in (0, 1)$ there exists a Fatou–Bieberbach domain in $\mathbf{C}^2$ whose boundary has Hausdorff dimension $3 + s$. These Fatou–Bieberbach domains are obtained as basins of attraction of hyperbolic quadratic polynomial automorphisms of $\mathbf{C}^2$. In view of Theorem 1.1(ii), our method cannot be applied to obtain a Fatou–Bieberbach domain in $\mathbf{C}^2$ whose boundary has maximal Hausdorff dimension equal to 4.

This paper is organized as follows. In Section 2 we present the basic definitions and notations. In Section 3 we show how holomorphic motions can be used to obtain estimates for the Hausdorff dimension of particular subsets of $\mathbf{C}^2$. Section 4 is devoted to the proof of Theorem 1.1 and represents the main part of this paper. In Section 5 we study the dependence on parameters for the Hausdorff dimensions of

the Julia sets; the main part of Section 5 is devoted to proving the facts that imply Corollary 5.5. The results of Section 4 and 5 are applied in Section 6 to polynomial automorphisms of $\mathbf{C}^2$ that are small perturbations of polynomials in $\mathbf{C}$.

2. Notation and Preliminaries

In this paper we consider polynomial automorphisms of $\mathbf{C}^2$ of the form

$$g = g_1 \circ \cdots \circ g_m. \tag{2.1}$$

Each mapping g_i is a generalized complex Hénon mapping, that is, a mapping of the form

$$g_i(z, w) = (w, P_i(w) + a_i z), \tag{2.2}$$

where P_i is a complex polynomial of degree $d_i \geq 2$ and a_i is a nonzero complex number. For $d = (d_1, \ldots, d_m)$ we denote by $\mathcal{H}_d$ the space of mappings of the form (2.1). Note that the degree of $g \in \mathcal{H}_d$ is equal to $\prod_{i=1}^m d_i$. Each $g \in \mathcal{H}_d$ depends on k complex and therefore on $2k$ real variables for some positive integer k. We can therefore identify $\mathcal{H}_d$ as a subspace of $\mathbf{R}^{2k}$.

It is a result due to Friedland and Milnor [FM] that every polynomial automorphism of $\mathbf{C}^2$ is conjugate either to a finite composition of elementary mappings (with trivial dynamics) or to a finite composition of generalized Hénon mappings (with nontrivial dynamics). Since dynamical properties are invariant under conjugation, each polynomial automorphism of $\mathbf{C}^2$ with nontrivial dynamics is represented in $\mathcal{H}_d$ for some d.

The function $\det Dg$ is constant in $\mathbf{C}^2$. We can thus restrict our considerations to the volume-decreasing case ($|\det Dg| < 1$) and the volume-preserving case ($|\det Dg| = 1$), because otherwise we can consider g^{-1}.

As pointed out in the introduction, a mapping $g \in \mathcal{H}_d$ is called *hyperbolic* if J is a hyperbolic set of g (see [BS2] for the details). We denote by Hyp_d the subspace of all hyperbolic mappings in $\mathcal{H}_d$. The most important feature of hyperbolic sets is that we can associate with each point p its local stable/unstable manifold $W_\varepsilon^{s/u}(p)$. We denote by $W^{s/u}(p)$ the (global) stable/unstable manifold of p. If $g \in \mathrm{Hyp}_d$, then the (local) stable/unstable manifolds are in fact complex manifolds (see [BS2]).

It is shown in [BS2] that $g \in \mathrm{Hyp}_d$ is an Axiom A diffeomorphism and that J is a basic set of g. Furthermore J has index 1; that is, $\dim_{\mathbf{C}} E_p^{s/u} = 1$ for all $p \in J$. Here $E_p^{s/u}$ denotes the stable/unstable subspace of p induced by the hyperbolic splitting. It follows that $Dg(p)\big|_{E_p^s}$ and $Dg(p)\big|_{E_p^u}$ can be identified as $\mathbf{C}$-linear mappings from $\mathbf{C}$ to $\mathbf{C}$. Therefore, $g\big|_J$ is a stable and unstable conformal diffeomorphism. For the definition of stable and unstable conformality and further details, see [P] and [Wo].

Finally, we recall the definition of the Hausdorff dimension. Assume (X, δ) is a metric space and $A \subset X$. For $s \geq 0$ we define the *s-dimensional outer Hausdorff measure* of A to be

$$H^s(A) = \sup_{\varepsilon > 0} \inf \left\{ \sum_{k=1}^{\infty} \operatorname{diam}(U_k)^s : A \subset \bigcup_{k=1}^{\infty} U_k,\ \operatorname{diam}(U_k) \leq \varepsilon \right\},$$

where "diam" denotes the diameter with respect to the metric δ. Then

$$\dim_H A = \inf\{s : H^s(A) = 0\} = \sup\{s : H^s(A) = \infty\}$$

is called the *Hausdorff dimension* of A.

3. Holomorphic Motions

In this section we introduce the concept of holomorphic motions, which has become a valuable tool for the analysis of dynamics of rational mappings on the Riemann sphere. In particular, Julia sets of rational mappings are moved holomorphically in hyperbolic parameter space (see [MSS]). Usually holomorphic motions are defined for subsets of the Riemann sphere $\bar{\mathbf{C}}$. Here we restrict our considerations to subsets of $\mathbf{C}$.

DEFINITION 3.1. Let $r > 0$, $X \subset \mathbf{C}$, and $T = D(0, r)$. A *holomorphic motion* of X is a mapping $h\colon T \times X \to \mathbf{C}$ such that

(i) $h(0, \cdot) = \mathrm{id}_X$;
(ii) $h(t, \cdot)$ is one-to-one for all $t \in T$; and
(iii) $h(\cdot, x)$ is holomorphic for all $x \in X$.

We consider t as a complex time parameter. Note that no continuity of $h(t, \cdot)$ is required in the definition.

Let X, Y be metric spaces. We call a bijective mapping $f : X \to Y$ an *α-Hölder homeomorphism* if both f and f^{-1} are Hölder-continuous with Hölder exponent α.

In general there exists no Fubini theorem for Hausdorff measures. It is therefore in general not possible to obtain an upper bound for the Hausdorff dimension of a set from the Hausdorff dimension of its level sets (see [Ma] for further details). However, if the level sets are moved holomorphically into each other, we obtain also an upper bound for the Hausdorff dimension of the set.

THEOREM 3.2. *Let $\delta > 0$ and let $h\colon T \times X \to \mathbf{C}$ be a holomorphic motion of X. Assume $\bigcup_{t \in D(0,r)} \{t\} \times h(t, X) \subset \mathbf{C}^2$ is bounded. Then there exists $r_0 > 0$ such that for all $0 < r_1 < r_0$ we have*

$$\dim_H \bigcup_{t \in D(0, r_1)} \{t\} \times h(t, X) \in [\dim_H X + 2,\ \dim_H X + 2 + \delta). \tag{3.3}$$

Proof. Let $\delta > 0$. The holomorphic motion h can be extended to a holomorphic motion of $\mathbf{C}$ (see [Sl]). On the other hand, the λ-lemma [MSS] implies that $h(t, \cdot)$ is a $K(|t|)$-quasiconformal homeomorphism. For $|t| < \frac{1}{3}$, a result of [BeR] implies

$$1 \le K(|t|) \le \frac{1 + 3|t|}{1 - 3|t|}. \tag{3.4}$$

By the Mori inequality (see [A]), we deduce that the mapping $h(t, \cdot)$ is a $K(|t|)^{-1}$-Hölder homeomorphism. This implies that $\dim_H h(t, X)$ is close to $\dim_H X$ when $|t|$ is small. Therefore, we conclude by [Ma, Thm. 7.7] that

$$\dim_H \bigcup_{t\in D(0,r_1)} \{t\}\times h(t,X) \geq \dim_H X + 2$$

for all $0 < r_1 \leq r$.

For $0 < r_1 \leq r$ we define $A_{r_1} = \bigcup_{t\in D(0,r_1)}\{t\}\times h(t,X) \subset \mathbf{C}^2$ and a mapping

$$h_{r_1}\colon D(0,r_1)\times X \to A_{r_1}, \quad (t,x)\mapsto (t,h(t,x)).$$

It follows from the definition that h_{r_1} is onto. Let ρ denote the spherical metric on $\bar{\mathbf{C}}$. By [BeR, Cor. 2], there exist $C_1, C_2 > 0$ and $\alpha(r_1) \leq 1$ with $\alpha(r_1) \to 1$ for $r_1 \to 0$ such that

$$\rho(h(t_1,x_1),h(t_2,x_2)) \leq C_1\rho(x_1,x_2)^{\alpha(r_1)} + C_2|t_1-t_2|$$

for all $x_1, x_2 \in X$ and all $t_1, t_2 \in D(0,r_1)$. The set A_r is bounded. Using that the spherical metric restricted to a bounded set is equivalent to the Euclidean metric, we deduce that the mapping h_{r_1} is Hölder-continuous with Hölder exponent $\alpha(r_1)$. We have $\dim_H D(0,r_1)\times X = 2+\dim_H X$. Hence we can choose $r_0 > 0$ such that $\dim_H A_{r_1} < \dim_H X + 2 + \delta$ for all $0 < r_1 < r_0$. This completes the proof. □

REMARK. It is possible to show that Theorem 3.2 also holds if the set A_r is unbounded. Since we do not use this fact in the sequel, we leave the proof for the reader.

4. The Intersection Formula

In this section we present the intersection formula for the Hausdorff dimension of Julia sets of polynomial automorphisms of $\mathbf{C}^2$. This result, Theorem 4.3, is the main result of this paper. For the proof we construct locally a lamination of $\mathbf{C}^2$ whose leaves intersected with $J^{\pm}$ are images of $W^{u/s}_{\varepsilon}(p)\cap J$ under a particular holomorphic motion. Throughout this section, $\varepsilon > 0$ is sufficiently small that the stable manifold theorem holds for $W^{s/u}_{\varepsilon}(p)$.

THEOREM 4.1. *Let* $g \in \mathrm{Hyp}_d$ *and* $p \in J$. *Then*

$$\dim_H J^+ = \dim_H W^u_{\varepsilon}(p)\cap J + 2.$$

Proof. The result of [VW] implies that $t^{u/s} = \dim_H W^{u/s}_{\varepsilon}(p)\cap J$ does not depend on p and ε. Let us now consider a fixed $p \in J$.

Assertion 1. For all $\delta > 0$ *there exists an* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon \leq \varepsilon_0$

$$\dim_H \bigcup_{q\in W^u_{\varepsilon}(p)\cap J} W^s_{\varepsilon}(q) \in [t^u+2, t^u+2+\delta). \tag{4.5}$$

Proof of Assertion 1. Let $\delta > 0$. If ε is small then there exists a domain $D \subset \mathbf{C}$ containing 0 and a biholomorphic mapping φ from a neighborhood $V \subset \mathbf{C}^2$ of p to a neighborhood $U \subset \mathbf{C}^2$ of 0 such that $\varphi(W^u_{\varepsilon}(p)) \subset D\times\{0\} \subset \mathbf{C}^2$ and $\varphi(p) = 0$. The stable manifold theorem implies that the local stable and unstable manifolds are uniformly transverse (see [KHa]). This property is invariant under a biholomorphic change of coordinates. Hence we can conclude that the sets $\varphi(W^s_{\varepsilon}(q))$ with $q \in W^u_{\varepsilon}(p)\cap J$ are uniformly transverse to $D\times\{0\}$. We

define $X = \varphi(W^u_\varepsilon(p) \cap J)$; for convenience, we use also the notation $W^s_\varepsilon(x) = \varphi(W^s_\varepsilon(\varphi^{-1}(x)))$ for $x \in X$. Let r be a small positive real number and $T = D(0, r)$. We define a mapping

$$h\colon T \times X \to \mathbf{C}, \quad (t, x) \mapsto \mathrm{Pr}_1(W^s_\varepsilon(x) \cap \mathbf{C} \times \{t\}).$$

Here Pr_1 denotes the projection to the first coordinate. It is well known that the property of transverse intersection between two submanifolds remains invariant if one of the manifolds makes a small change in C^1-topology. This shows that h is well-defined if r is small enough. Obviously $h(0, \cdot) = \mathrm{id}_X$. For all $q_1, q_2 \in W^u_\varepsilon(p) \cap J$ with $q_1 \neq q_2$, we have $W^s_\varepsilon(q_1) \neq W^s_\varepsilon(q_2)$. This follows because $g\big|_J$ is an expansive mapping and so $h(t, \cdot)$ is one-to-one for all $t \in T$. The local stable manifolds $W^s_\varepsilon(q)$ depend continuously on q in C^∞-topology, and this property is invariant under a biholomorphic change of coordinates. Making ε and r smaller if necessary, we may conclude that $W^s_\varepsilon(x)$ is transverse to $\mathbf{C} \times \{0\}$ for all $x \in X$; note that $W^s_\varepsilon(x)$ is a complex 1-dimensional submanifold of $\mathbf{C}^2$. Therefore, $h(\cdot, x)$ is holomorphic for all $x \in X$. Thus we have shown that h is a holomorphic motion. We now apply Theorem 3.2 to the holomorphic motion h. Note that Hausdorff dimension is invariant under a biholomorphic change of coordinates. This implies assertion 1.

Assertion 2. For all $\delta > 0$, there exist a neighborhood $U \subset J$ of p and an $\varepsilon > 0$ such that

$$\dim_H \bigcup_{q \in U} W^s_\varepsilon(q) \in [t^u + 2, t^u + 2 + \delta).$$

Proof of Assertion 2. Let $\delta > 0$, and assume that ε_0 is chosen as in assertion 1. We define $\varepsilon = \varepsilon_0/2$. The Julia set J has a local product structure (see [BS2]); hence there exists a neighborhood $U \subset J$ of p such that the mapping

$$H\colon U \to W^s_\varepsilon(p) \cap J \times W^u_\varepsilon(p) \cap J,$$
$$q \mapsto (W^s_\varepsilon(p) \cap W^u_\varepsilon(q), W^u_\varepsilon(p) \cap W^s_\varepsilon(q))$$

is a well-defined homeomorphism. Applying the triangle inequality yields

$$\bigcup_{q \in U} W^s_\varepsilon(q) \subset \bigcup_{q \in W^u_{\varepsilon_0}(p) \cap J} W^s_{\varepsilon_0}(q).$$

Therefore, assertion 2 follows from assertion 1.

Proof of the Theorem. Let $\delta > 0$. Assertion 2 implies that there exist $p_1, \ldots, p_n \in J$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$ with the property (4.5) such that for

$$\varepsilon = \min\left\{\frac{\varepsilon_1}{2}, \ldots, \frac{\varepsilon_n}{2}\right\}$$

and for all $p \in J$ the local stable manifold $W^s_\varepsilon(p)$ is contained in $W^s_{\varepsilon_k}(q)$ for some $q \in W^u_{\varepsilon_k}(p_k) \cap J$ and some $k \in \{1, \ldots, n\}$. This implies

$$\dim_H \bigcup_{p \in J} W^s_\varepsilon(p) \in [t^u + 2, t^u + 2 + \delta).$$

It is a result of Bedford and Smillie [BS2] that $W^s(J) = J^+$. We may thus conclude by [Bo, Prop. 3.10] that

$$\bigcup_{p\in J} W^s(p) = J^+.$$

On the other hand, we have

$$\bigcup_{n\in\mathbf{N}} g^{-n}\left(\bigcup_{p\in J} W^s_\varepsilon(p)\right) = \bigcup_{p\in J} W^s(p).$$

Hence

$$\dim_H J^+ \in [2 + t^u, 2 + t^u + \delta).$$

Since δ was arbitrary, the proof is complete. □

We obtain the analogous result for the Hausdorff dimension of J^- by applying Theorem 4.1 to the mapping g^{-1}.

THEOREM 4.2. *Let $g \in \mathrm{Hyp}_d$ and $p \in J$. Then*

$$\dim_H J^- = \dim_H W^s_\varepsilon(p) \cap J + 2.$$

Let f be an Axiom A diffeomorphism of a real surface and let Λ be a basic set for f. It is a result of [T] that the Hausdorff dimension and the upper box dimension of $W^{s/u}_\varepsilon(x) \cap \Lambda$ coincide. This result is generalized in [Ba] even to asymptotically conformal Axiom A homeomorphisms, so it holds in particular for $g \in \mathrm{Hyp}_d$. On the other hand, it follows by a result of [Ha] (see also [KHa]) that the holonomy mapping of $g \in \mathrm{Hyp}_d$ is Lipschitz-continuous. Combining these results yields

$$\dim_H J = \dim_H W^u_\varepsilon(p) \cap J + \dim_H W^s_\varepsilon(p) \cap J = t^u + t^s; \qquad (4.6)$$

see [P] and [Wo]. Note that (4.6) was already applied in [VW] and [FO].

The next theorem is the intersection formula for the Julia set of g. It turns out that the intersection between J^+ and J^- is "nice" in the sense of Hausdorff dimension.

THEOREM 4.3 (Intersection Formula). *Let $g \in \mathrm{Hyp}_d$. Then*

$$\dim_H J = \dim_H J^+ + \dim_H J^- - 4.$$

Proof. The result is a direct consequence of Theorem 4.1, Theorem 4.2, and (4.6). □

It is even for basic sets of Axiom A diffeomorphisms of real surfaces not known if an analogous intersection formula holds.

Theorem 4.11 of [Bo] implies that the stable/unstable set of J has Lebesgue measure zero (see Section 1). The next theorem provides an even stronger result: the Hausdorff dimension of $J^\pm$ is strictly less than 4.

THEOREM 4.4. *Let $g \in \mathrm{Hyp}_d$ and $p \in J$. Then*

(i) $0 < \dim_H W^{s/u}_\varepsilon(p) \cap J < 2$;
(ii) $2 < \dim_H J^\pm < 4$;
(iii) $0 < \dim_H J < 4$.

Proof. It is sufficient to show (i), because (ii) and (iii) follow immediately from (i) and Theorem 4.1, Theorem 4.2, and (4.6).

Proof of (i). Without loss of generality, we consider only the unstable manifold. That $\dim_H W^u_\varepsilon(p) \cap J > 0$ is well-known; see [VW] and [Wo]. Let $p \in J$ and $\varepsilon > 0$ small. We write $t^u = \dim_H W^u_\varepsilon(p) \cap J$. Let us assume $t^u = 2$. The mapping g is a stable and unstable conformal diffeomorphism. By [P, Thm. 22.1] we thus obtain $H^2(W^u_\varepsilon(p) \cap J) > 0$. Note that H^2 denotes the 2-dimensional Hausdorff measure defined in Section 2. Analogously to the proof of Theorem 4.1, there exist a set $X \subset \mathbf{C}$ and a holomorphic motion $h\colon T \times X \to \mathbf{C}$ such that $\bigcup_{t\in T}\{t\} \times h(t, X)$ is mapped diffeomorphically to $\bigcup_{q\in W^u_\varepsilon(p)\cap J} W^s_\varepsilon(q)$. We have $H^2(X) > 0$. Observe that the mapping $h(t, \cdot)$ is a quasi-conformal homeomorphism. Therefore, [As, Thm. 1.1] implies that there exists a $C > 0$ such that $H^2(h(t, X)) > C$ if $|t|$ is small enough. Thus, by Fubini's theorem, we conclude that $H^4\big(\bigcup_{t\in T}\{t\} \times h(t, X)\big) > 0$. In that case the Lebesgue measure of J^+ would be positive, which is a contradiction to Theorem 4.11 of [Bo]. □

REMARK. The statement $\dim_H J^\pm > 2$ holds true even without the assumption of hyperbolicity. This was derived in [FoS] by showing that the Green function $G^\pm$ is Hölder-continuous. In the volume-decreasing case we also have $\dim_H J^- < 4$ without the assumption of hyperbolicity (see [Wo]).

In [VW] the authors claim that $\dim_H W^{s/u}_\varepsilon(p) \cap J < 1$. Using the local product structure of J, this would imply that J is a Cantor set. Counterexamples to this statement are mappings in Hyp_d with a connected Julia set (considered in [BS5]) and mappings in Hyp_d with an attracting periodic orbit (see [Wo]). In the proof of [VW] there is a confusion related to the difference between real and complex Jacobian determinants. However, if the proof in [VW] is corrected, it also provides $\dim_H W^{s/u}_\varepsilon(p) \cap J < 2$.

5. Dependence on Parameters

Let A denote an open subset of $\mathbf{C}^k$. We identify $g = g_a$ for $a \in A$ and denote by $J^\pm_a$ and J_a its Julia sets respectively. The cases of interest are that either A is Hyp_d or a disk in $\mathbf{C}$. Note that we have also used a_i as a specific parameter, the Jacobian determinant of g_i, but there should be no confusion when we also use a as a general parameter.

As mentioned earlier, $t^{u/s} = \dim_H W^{u/s}_\varepsilon(p) \cap J$ is independent of $p \in J$ and $\varepsilon > 0$ for $g \in \mathrm{Hyp}_d$. Moreover, it is shown in [VW] that $t^{u/s}$ is given by the unique solution of

$$P\big(g\big|_J, \mp t\phi^{u/s}\big) = 0. \tag{5.7}$$

Here $P\big(g\big|_J, \cdot\big)$ denotes the topological pressure of $g\big|_J$ (see [Wa] for the definition) and $\phi^{u/s} \in C(J, \mathbf{R})$ is defined by $\phi^{u/s}(p) = \log\big\|Dg(p)\big|_{E^{u/s}_p}\big\|$. Equation (5.7) is usually called *Bowen's formula*. The mapping $a \mapsto t^{u/s}_a$ is real-analytic in Hyp_d (see [VW]). Hence Theorem 4.1 and 4.2 immediately imply the following.

COROLLARY 5.1. *The mapping $a \mapsto \dim_H J^\pm_a$ is real-analytic in* Hyp_d.

For $g \in \mathrm{Hyp}_d$ we denote by $M\big(J, g\big|_J\big)$ the space of all g-invariant Borel ergodic probability measures supported on J, and for each $\mu \in M\big(J, g\big|_J\big)$ the corresponding positive Lyapunov exponent $\Lambda(\mu)$ is defined by

$$\Lambda(\mu) = \lim_{n\to\infty} \frac{1}{n} \int \log\|Dg^n\|\, d\mu. \tag{5.8}$$

The multiplicative ergodic theorem of Oseledets [O] and the submultiplicativity of the operator norm guarantee the existence of the limit defining $\Lambda(\mu)$. Note that $\Lambda(\mu)$ is in fact positive, since J is a hyperbolic set for g of index 1. This implies that every $\mu \in M\big(J, g\big|_J\big)$ is a hyperbolic measure.

The next result provides information about the dependence of $M\big(J, g\big|_J\big)$ and $\Lambda(\mu)$ on the parameter of the mapping.

PROPOSITION 5.2. *Let $(g_a)_{a\in D}$ be a holomorphic family in Hyp_d, where D is a disk in $\mathbf{C}$ with center 0. Then there exist $r > 0$ and a family of mappings $(T_a)_{a\in D(0,r)}$, where each T_a is a bijection from $M\big(J_0, g_0\big|_{J_0}\big)$ to $M\big(J_a, g_a\big|_{J_a}\big)$, such that*

(1) *$\big(g_0\big|_{J_0}, \mu_0\big)$ and $\big(g_a\big|_{J_a}, T_a(\mu_0)\big)$ are measure-theoretically isomorphic for all $\mu_0 \in M\big(J_0, g_0\big|_{J_0}\big)$ and all $a \in D(0, r)$; and*
(2) *for all $\mu_0 \in M\big(J_0, g_0\big|_{J_0}\big)$, the mapping $a \mapsto \Lambda(T_a(\mu_0))$ is harmonic.*

Proof. The result of Jonsson [J] implies that there exist $r > 0$ and a holomorphic motion $h\colon D(0, r) \times J_0 \to \mathbf{C}^2$ that preserves the dynamics of $g_a\big|_{J_a}$. More precisely, we have the following statements:

(i) $h(0, \cdot) = \mathrm{id}_{J_0}$;
(ii) for all $a \in D(0, r)$, the mapping $h(a, \cdot)$ is a homeomorphism from J_0 to J_a such that $g_a\big|_{J_a} \circ h(a, \cdot) = h(a, \cdot) \circ g_0\big|_{J_0}$;
(iii) $h(\cdot, p)$ is holomorphic for all $p \in J_0$.

Note that here h denotes a holomorphic motion in complex dimension 2 (unlike our previous considerations). For all $a \in D(0, r)$ we define T_a by

$$T_a(\mu_0) = h(a, \cdot)_*\mu_0, \tag{5.9}$$

where $h(a, \cdot)_*\mu_0(A) = \mu_0(h(a, \cdot)^{-1}(A))$ for all Borel sets $A \subset J_a$. It is easy to see that T_a is a bijection between $M\big(J_0, g_0\big|_{J_0}\big)$ and $M\big(J_a, g_a\big|_{J_a}\big)$. It follows directly from the definition of T_a that $\big(g_0\big|_{J_0}, \mu_0\big)$ and $\big(g_a\big|_{J_a}, T_a(\mu_0)\big)$ are measure-theoretically isomorphic (property (1)).

It remains to show property (2). Consider a fixed $\mu_0 \in M\big(J_0, g_0\big|_{J_0}\big)$. For $a \in D(0, r)$, we have

$$\Lambda(T_a(\mu_0)) = \lim_{n\to\infty} \frac{1}{n} \int \log\|Dg_a^n \circ h(a, \cdot)\|\, d\mu_0 \tag{5.10}$$

(see e.g. [Ma, Thm. 1.19]). For $a \in D(0, r)$ and $n \in \mathbf{N}$, we define

$$\Lambda_n(a) = \frac{1}{n} \int \log\|Dg_a^n \circ h(a, \cdot)\|\, d\mu_0. \tag{5.11}$$

Property (iii) of the holomorphic motion h implies that, for a fixed $p \in J_0$, the mapping $a \mapsto Dg_a^n(h(a, p))$ is holomorphic. Therefore $a \mapsto \Lambda_n(a)$ is harmonic for all $n \in \mathbf{N}$. The operator norm is submultiplicative. Thus

$$(n+m)\Lambda_{n+m}(a) \le n\Lambda_n(a) + m\Lambda_m(a) \tag{5.12}$$

for all $n, m \in \mathbf{N}$ and all $a \in D(0, r)$. This implies that $(\Lambda_{2^n}(a))_{n\in\mathbf{N}}$ is a decreasing sequence of harmonic mappings; hence the mapping $a \mapsto \Lambda(T_a(\mu_0))$ is also harmonic. This completes the proof. □

For $g \in \mathrm{Hyp}_d$ we denote by $\mu(g)$ the equilibrium measure of g (see [BS2] and [BS4] for the definition) and by $\Lambda(g)$ the positive Lyapunov exponent of $\mu(g)$. It is shown in [BS4] via potential-theoretical arguments that $\mu(g)$ is the unique measure of maximal entropy for g and that $\Lambda(g)$ depends pluriharmonically on the parameter of g. We obtain a new proof for the latter result.

COROLLARY 5.3. *The mapping $a \mapsto \Lambda(g_a)$ is pluriharmonic in* Hyp_d.

Proof. We use the notation of Proposition 5.2. Consider a fixed mapping $g_0 \in \mathrm{Hyp}_d$, and assume that L is a complex line in parameter space containing g_0. By Proposition 5.2, the mapping $a \mapsto \Lambda(T_a(\mu(g_0)))$ is harmonic in a neighborhood of 0 in L. The equilibrium measure is the unique measure of maximal entropy, which implies that $T_a(\mu(g_0)) = \mu(g_a)$. This completes the proof. □

We now present the main result of this section.

THEOREM 5.4. *The mapping $a \mapsto t_a^{u/s}$ is plurisubharmonic in* Hyp_d.

Proof. Without loss of generality, we show only the result for t^u. First we consider the situation for a single mapping $g \in \mathrm{Hyp}_d$. The variational principle (see [Wa]) implies

$$P\big(g\big|_J, -t\phi^u\big) = \sup_{\mu\in M(J,g|_J)} \left(h_\mu(g) - t\int \phi^u \, d\mu \right), \tag{5.13}$$

where $h_\mu(g)$ denotes the measure-theoretic entropy of g with respect to μ. Since g is hyperbolic on J, there exists a $C_1 > 0$ such that

$$C_1 < \int \phi^u \, d\mu \tag{5.14}$$

for all $\mu \in M\big(J, g\big|_J\big)$. Hence (5.7), (5.13), and (5.14) imply that

$$t^u = \sup_{\mu\in M(J,g|_J)} \left(\frac{h_\mu(g)}{\int \log\big\| Dg\big|_{E^u}\big\| \, d\mu} \right). \tag{5.15}$$

Since E_p^u is of complex dimension 1, we obtain

$$\big\| Dg^n(p)\big|_{E_p^u}\big\| = \prod_{k=0}^{n-1} \big\| Dg(g^k(p))\big|_{E^u_{g^k(p)}}\big\| \tag{5.16}$$

for all $n \in \mathbf{N}$ and all $p \in J$. Thus

$$\int \log\big\| Dg\big|_{E^u}\big\| \, d\mu = \frac{1}{n}\int \log\big\| Dg^n\big|_{E^u}\big\| \, d\mu \tag{5.17}$$

for all $n \in \mathbf{N}$ and all $\mu \in M\big(J, g\big|_J\big)$. Since g is hyperbolic on J, there exists a $C_2 > 0$ such that

$$C_2 \|Dg^n(p)\| \le \big\|Dg^n(p)\big|_{E^u_p}\big\| \le \|Dg^n(p)\| \tag{5.18}$$

for all $p \in J$ and all $n \in \mathbf{N}$. Hence (5.8), (5.17), and (5.18) imply

$$\Lambda(\mu) = \int \log\big\|Dg\big|_{E^u}\big\|\, d\mu \tag{5.19}$$

for all $\mu \in M\big(J, g\big|_J\big)$. By (5.15), we conclude that

$$t^u = \sup_{\mu \in M(J, g|_J)} \left(\frac{h_\mu(g)}{\Lambda(\mu)} \right). \tag{5.20}$$

Let $g_0 \in \mathrm{Hyp}_d$ and let L be a complex line in parameter space containing g_0. Then there exists a holomorphic family $(g_a)_{a \in D} \subset \mathrm{Hyp}_d$, where D is a disk with center 0 in $\mathbf{C}$ such that $\{g_a : a \in D\}$ is a neighborhood of g_0 in L. We now apply Proposition 5.2 to the family $(g_a)_{a \in D}$. Equation (5.20) implies

$$t^u_a = \sup_{\mu_0 \in M(J_0, g_0|_{J_0})} \left(\frac{h_{T_a(\mu_0)}(g_a)}{\Lambda(T_a(\mu_0))} \right) = \sup_{\mu_0 \in M(J_0, g_0|_{J_0})} \left(\frac{h_{\mu_0}(g_0)}{\Lambda(T_a(\mu_0))} \right). \tag{5.21}$$

The mapping $a \mapsto \Lambda(T_a(\mu_0))$ is harmonic by Proposition 5.2. Note that $x \mapsto x^{-1}$ is a convex function on $\mathbf{R}^+$. This implies that the function $a \mapsto \Lambda(T_a(\mu_0))^{-1}$ is subharmonic (see [Kli, Thm. 2.6.6]). Therefore, t^u_a is given by the supremum over a family of subharmonic functions of a. The mapping $a \mapsto t^u_a$ is real-analytic and thus, in particular, continuous. We conclude that the mapping $a \mapsto t^u_a$ is subharmonic. This completes the proof. □

REMARK. Ransford [Ra] showed that the Hausdorff dimension of the Julia set of an analytic family of hyperbolic rational mappings on the Riemann sphere depends subharmonically on the parameter. Hence Theorem 5.4 can be considered as the higher-dimensional counterpart (for polynomial automorphisms of $\mathbf{C}^2$) to Ransford's result. It should be mentioned that some of the ideas in [Ra] are used in the proof of Theorem 5.4.

COROLLARY 5.5. *The mappings* $a \mapsto \dim_H J_a$ *and* $a \mapsto \dim_H J^\pm_a$ *are plurisubharmonic in* Hyp_d.

Proof. For $g_a \in \mathrm{Hyp}_d$ we have $\dim_H J_a = t^u_a + t^s_a$; see (4.6). On the other hand, it is shown in Theorems 4.1 and 4.2 that $\dim_H J^\pm_a = t^{u/s}_a + 2$. Therefore, the result follows immediately from Theorem 5.4. □

6. Small Perturbations of Polynomials in C

In this section we show that the Hausdorff dimensions of J^+ and J are related to the Hausdorff dimension of the Julia set of a hyperbolic quadratic polynomial in $\mathbf{C}$, provided the corresponding complex Hénon mapping is a small perturbation of the polynomial.

Let us recall some definitions for quadratic polynomials (see e.g. [CG]). For $c \in \mathbf{C}$ we consider the complex polynomial $P_c(z) = z^2 + c$. We will sometimes identify the map P_c with the complex number c. We use J_c to denote the Julia set of P_c; let $\mathcal{M}$ denote the Mandelbrot set. We call $C \subset \mathcal{M}$ a *hyperbolic component* of $\mathcal{M}$ if it is a connected component of the set of all $c \in \mathcal{M}$ such that P_c is hyperbolic. In particular, if C is the set of all $c \in \mathbf{C}$ such that P_c has an attracting fixed point, then we call C the *main cardioid*.

In the following we will consider a slightly different normal form for the complex Hénon mapping as in (2.2). For $(a, c) \in \mathbf{C}^2$ we consider the mapping $g_{a,c}\colon \mathbf{C}^2 \to \mathbf{C}^2$ defined by

$$g_{a,c}(z, w) = (P_c(z) + aw, az).$$

If $a \neq 0$ then $g_{a,c}$ is conjugate to a complex Hénon mapping in the usual normal form (2.2). For small $|a|$ we consider $g_{a,c}$ to be a small perturbation of the polynomial P_c. We will also use the notation $J_{a,c}$, $J^{\pm}_{a,c}$, and $K^{\pm}_{a,c}$ for the sets corresponding to the mapping $g_{a,c}$. In addition we define $J^{\pm}_{a,c,w} = J^{\pm}_{a,c} \cap \mathbf{C} \times \{w\}$ and $K^{\pm}_{a,c,w} = K^{\pm}_{a,c} \cap \mathbf{C} \times \{w\}$.

The dynamics of $g_{a,c}$ has been observed to be related to the dynamics of P_c for small values of $|a|$. The following result compiles some known results from [FoS] and [HO].

THEOREM 6.1. *Let C be a hyperbolic component of the Mandelbrot set, $c \in C$ and $R \gg 1$. Assume that P_c has an attracting cycle of period k. Then there exists an $a_0(c, R) > 0$ such that for all $0 < |a| < a_0(c, R)$ the following statements hold.*

(i) *$g_{a,c}$ has an attracting cycle of period k, $\{p_1, \ldots, p_k\}$; the interior of $K^+_{a,c}$ consists of k connected components, each of which is the immediate basin of attraction of one of $p_1, \ldots, p_k$.*

(ii) *If $w \in \mathbf{C}$ with $|w| \leq R$, then $K^+_{a,c,w}$ is a connected compact set.*

(iii) *$g_{a,c}$ is hyperbolic.*

(iv) *There exists a holomorphic motion $h_1\colon D(0, a_0(c, R)) \times J_c \to \mathbf{C}$ such that $h_1(a, J_c) = \mathrm{Pr}_1(J^+_{a,c,0})$, and there exists a holomorphic motion $h_2\colon D(0, R) \times \mathrm{Pr}_1(J^+_{a,c,0}) \to \mathbf{C}$ such that $h_2(w, \mathrm{Pr}_1(J^+_{a,c,0})) = \mathrm{Pr}_1(J^+_{a,c,w})$.*

(v) *The Hausdorff dimension of $J^-_{a,c}$ satisfies the inequality*

$$2 < \dim_H J^-_{a,c} \leq 2 - \frac{\log 2}{\log|a|}.$$

REMARK. The result of Theorem 6.1(iii) is extended in [BS4] to finite compositions of generalized Hénon mappings. In particular, one can deduce from [BS4] that there exists an $r > 0$ such that $g_{a,c}$ is hyperbolic for all $(a, c) \in P((0, c), r) \setminus \{0\} \times D(z, r)$, where $P((0, c), r)$ denotes the polydisk with center $(0, c)$ and radius r.

LEMMA 6.2. *Let $c \in \mathbf{C}$ and let $\alpha \in \mathbf{C}$ be a repelling periodic point of P_c with period $k \in \mathbf{N}$. Then there exist an $a_0(c, \alpha) > 0$ and a holomorphic mapping $h\colon D(0, a_0(c, \alpha)) \to \mathbf{C}^2$ such that, if $a \neq 0$, then $h(a)$ is a saddle point of $g_{a,c}$ with period k and $h(0) = (\alpha, 0)$.*

Proof. The proof for the existence of a mapping h with the property that $h(a)$ is a periodic point with period k is similar to that given for an attracting periodic point in [FoS, Lemma 3.10]. Analogous to [FoS], we have

$$g_{a,c}^k(z, w) = (P_c^k(z) + P(z, w), Q(z, w)),$$

where all the coefficients of P and Q contain positive powers of a. Note that α is a repelling fixed point of P_c^k. Thus $|(P_c^k)'(\alpha)| > 1$. This implies that if $a_0(c, \alpha)$ is small enough then $Dg_{a,c}^k(h(a))$ has at least one eigenvalue of modulus larger than 1. The mapping $g_{a,c}$ is volume-decreasing for $|a| < 1$. This implies that the modulus of the other eigenvalue is smaller than 1. Therefore, $h(a)$ is a saddle point of $g_{a,c}$. □

LEMMA 6.3. *Let $g \in \mathrm{Hyp}_d$ and $p \in J$, and let $U \subset \mathbf{C}^2$ be a neighborhood of p. Then* $\dim_H J^+ \cap U = \dim_H J^+$.

Proof. We choose $\varepsilon > 0$ such that

$$W^s = \bigcup_{q \in W_\varepsilon^u(p) \cap J} W_\varepsilon^s(q) \subset U.$$

Analogous to the proof of Theorem 4.1, we conclude that $\dim_H W^s = \dim_H J^+$. This completes the proof. □

We will now show that the Hausdorff dimension of $J_{a,c}^+$ is related to the Hausdorff dimension of J_c, if $|a|$ is small.

THEOREM 6.4. *Let C be a hyperbolic component of the Mandelbrot set, $c \in C$ and $\delta > 0$. Then there exists an $a_0(c, \delta) > 0$ such that for all $0 < |a| < a_0(c, \delta)$ we have*

$$\dim_H J_{a,c}^+ \in (\dim_H J_c + 2 - \delta,\ \dim_H J_c + 2 + \delta).$$

Proof. Let R and $a_0(c, R)$ be as in Theorem 6.1. Applying Theorem 6.1(iv) and making $a_0(c, R)$ smaller if necessary, we conclude (similarly to the proof of Theorem 3.2) that

$$\dim_H J_{a,c,0}^+ \in (\dim_H J_c - \delta/2,\ \dim_H J_c + \delta/2) \tag{6.22}$$

for all $0 < |a| < a_0(c, R)$. Let h_2 be the holomorphic motion in Theorem 6.1(iv). By Theorem 3.2, there exists an $r > 0$ such that

$$\dim_H \left(\bigcup_{|w|<r} J_{a,c,w}^+ \right) \in [\dim_H J_{a,c,0}^+ + 2,\ \dim_H J_{a,c,0}^+ + 2 + \delta/2)$$

for all $0 < |a| < a_0(c, R)$. Making again $a_0(c, R)$ smaller if necessary, we can assure by Lemma 6.2 that $\bigcup_{|w|<r} J_{a,c,w}^+$ contains a saddle orbit for all $0 < |a| < a_0(c, R)$. Thus Lemma 6.3 implies

$$\dim_H J_{a,c}^+ \in [\dim_H J_{a,c,0}^+ + 2,\ \dim_H J_{a,c,0}^+ + 2 + \delta/2).$$

Setting $a_0(c, \delta) = a_0(c, R)$ and applying (6.22) completes the proof. □

COROLLARY 6.5. *Let C be a hyperbolic component of the Mandelbrot set and let $c \in C$. Then*

$$\lim_{|a| \to 0} \dim_H J_{a,c} = \dim_H J_c.$$

Proof. For small $|a|$ we deduce from Theorem 6.4 that $\dim_H J^+_{a,c}$ is close to $\dim_H J_c + 2$. On the other hand, we conclude by Theorem 6.1(v) that $\dim_H J^-_{a,c}$ is close to 2. Therefore, the result follows immediately from Theorem 4.3. □

REMARK. Even if the Hausdorff dimensions of the Julia set $J_{a,c}$ and the 1-dimensional Julia set J_c are very close, their topological structures are completely different. If, for instance, c lies in the main cardioid, then J_c is a quasi-circle while $J_{a,c}$ is locally the product of a curve and a Cantor set.

Let C be a hyperbolic component of the Mandelbrot set. Theorem 6.1 implies that there exists a connected component C_H of Hyp_2 that contains C in its closure. If C is the main cardioid then we will refer to the corresponding component C_H as the main cardioid for complex Hénon mappings.

LEMMA 6.6. *The cardinality of attracting periodic points is constant in every connected component* C_H *of* Hyp_d.

Proof. By [BS2, Thm. 5.6], each $g \in C_H$ has finitely many attracting periodic points. Since hyperbolicity is an open property and since attracting periodic points are hyperbolic, we can conclude that the mapping $C_H \ni g \mapsto \#\{\alpha : \alpha \text{ is an attracting periodic point of } g\}$ is locally constant; because C_H is connected, it is constant in C_H. □

We recall that $t^{u/s}_{a,c}$ denotes the Hausdorff dimension of the unstable/stable slice of $g_{a,c}$.

COROLLARY 6.7. *Let* C_H *be the main cardioid for complex Hénon mappings. Then*

(i) $\inf\{t^u_{a,c} : (a,c) \in C_H\} = 1$;
(ii) $\sup\{t^u_{a,c} : (a,c) \in C_H\} = 2$;
(iii) $\inf\{t^s_{a,c} : (a,c) \in C_H\} = 0$.

Proof. (i) By Theorem 6.1(i), we know there exists a $g_{a_0,c_0} \in C_H$ with an attracting fixed point. Therefore Lemma 6.6 implies that all $g_{a,c}$ in C_H have an attracting fixed point. We can thus apply a result of [Wo] that implies that the topological dimension of $W^u_\varepsilon(p) \cap J_{a,c}$ is equal to 1 for all $(a,c) \in C_H$. This gives the required lower bound for $t^u_{a,c}$. On the other hand, we conclude by Theorem 4.1 and Theorem 6.4 that $t^u_{a,0}$ is close to 1 if $|a|$ is small.

(ii) There exists a sequence $(c_i)_{i\in\mathbf{N}}$ in the main cardioid such that

$$\lim_{i\to\infty} \dim_H J_{c_i} = 2$$

(see [Sh]). Hence the result follows from Theorem 4.1 and Theorem 6.4.

(iii) The result follows immediately from Theorem 4.2 and Theorem 6.1(v). □

REMARK. We do not have a nontrivial upper bound for $t^s_{a,c}$. However, Theorem 4.2 and a result of [Wo] imply that, for $t^s_{a,c}$ close to 2, it would be necessary that $|a|$ be close to 1.

A domain $\Omega \subset \mathbf{C}^n$ ($n \geq 2$) is called *Fatou–Bieberbach domain* if it is biholomorphically equivalent to $\mathbf{C}^n$ and $\bar{\Omega} \neq \mathbf{C}^n$. It is a well-known fact that basins of attraction of automorphisms of $\mathbf{C}^n$ are biholomorphically equivalent to $\mathbf{C}^n$ (see [RR]). This implies that basins of attraction of mappings in $\mathcal{H}_d$ are Fatou–Bieberbach domains. See [BS1; RR] for further information about Fatou–Bieberbach domains.

We obtain the following result about the Hausdorff dimension of boundaries of Fatou–Bieberbach domains in $\mathbf{C}^2$.

COROLLARY 6.8. *For all* $s \in [0, 1)$ *there exists a Fatou–Bieberbach domain* $\Omega \subset \mathbf{C}^2$ *such that* $\dim_H \partial\Omega = 3 + s$.

Proof. By a result of Stensønes [St], there exists a Fatou–Bieberbach domain in $\mathbf{C}^2$ with smooth boundary. This implies the result when $s = 0$. Suppose now that $0 < s < 1$. We conclude by Theorem 4.1 and Corollary 6.7 that there exist (a_1, c_1), (a_2, c_2) in the main cardioid for complex Hénon mappings C_H such that

$$\dim_H J^+_{a_1,c_1} < 3 + s < \dim_H J^+_{a_2,c_2}.$$

According to Corollary 5.1, there exists $(a, c) \in C_H$ such that $\dim_H J^+_{a,c} = 3 + s$. Theorem 6.1 and Lemma 6.6 imply that $g_{a,c}$ has an attracting fixed point $\alpha \in \mathbf{C}^2$. By [BS3, Thm. 2] we have $\partial W^s(\alpha) = J^+_{a,c}$. This completes the proof. □

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