

Descent for Shimura Varieties

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In his Corvallis article, Langlands [L, Sec. 6] stated a conjecture that identifies the conjugate of a Shimura variety by an automorphism of \mathbb{C} with the Shimura variety defined by different data, and he sketched a proof that his conjecture implies the existence of canonical models. However, as J. Wildeshaus and others have pointed out to me, it is not obvious that the descent maps defined by Langlands satisfy the continuity condition necessary for the descent to be effective. In this note, I prove that they do satisfy this condition and hence that Langlands's conjecture *does* imply the existence of canonical models—this is our only proof of the existence of these models for a general Shimura variety. The proof is quite short and elementary. I give it in Section 2 after reviewing some generalities on the descent of varieties in Section 1.

NOTATION AND CONVENTIONS. A *variety* over a field k is a geometrically reduced scheme of finite type over $\text{Spec } k$ (*not* necessarily irreducible). For a variety V over a field k and a homomorphism $\sigma : k \rightarrow k'$, σV is the variety over k' obtained by base change. The ring of finite adèles for \mathbb{Q} is denoted by \mathbb{A}_f .

1. Descent of Varieties

In this section, Ω is an algebraically closed field of characteristic zero. For a field $L \subset \Omega$, $A(\Omega/L)$ denotes the group of automorphisms of Ω fixing the elements of L .

Let V be a variety over Ω , and let k be a subfield of Ω . A family $(f_\sigma)_{\sigma \in A(\Omega/k)}$ of isomorphisms $f_\sigma : \sigma V \rightarrow V$ will be called a *descent system* if $f_{\sigma\tau} = f_\sigma \circ \sigma f_\tau$ for all $\sigma, \tau \in A(\Omega/k)$. We say that a model $(V_0, f : V_{0,\Omega} \rightarrow V)$ of V over k *splits* (f_σ) if $f_\sigma = f \circ (\sigma f)^{-1}$ for all $\sigma \in A(\Omega/k)$, and that a descent system is *effective* if it is split by some model over k . The next theorem restates results of Weil [W].

THEOREM 1.1. *Assume that Ω has infinite transcendence degree over k . A descent system $(f_\sigma)_{\sigma \in A(\Omega/k)}$ on a quasiprojective variety V over Ω is effective if, for some subfield L of Ω finitely generated over k , the descent system $(f_\sigma)_{\sigma \in A(\Omega/L)}$ is effective.*

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Proof. Let k' be the algebraic closure of k in L ; then k' is a finite extension of k and L is a regular extension of k' . Let $(V_{t'}, f': V_{t',\Omega} \rightarrow V)$ be the model of V over L splitting $(f_\sigma)_{\sigma \in A(\Omega/L)}$. Let $t: L \rightarrow k_t$ be a k' -isomorphism from L onto a subfield k_t of Ω linearly disjoint from L over k' , and let $V_t = V_{t'} \otimes_{L,t} k_t$. Zorn's lemma allows us to extend t to an automorphism τ of Ω over k' . The isomorphism

$$f_{t,t'}: V_{t',\Omega} \xrightarrow{f'} V \xrightarrow{f_\tau^{-1}} \tau V \xrightarrow{(\tau f')^{-1}} V_{t,\Omega}$$

is independent of the choice of τ , is defined over $L \cdot k_t$, and satisfies the hypothesis of [W, Thm. 6], which gives a model (W, f) of V over k' splitting $(f_\sigma)_{\sigma \in A(\Omega/k')}$.

For $\sigma \in A(\Omega/k)$, $g_\sigma \stackrel{\text{df}}{=} f_\sigma \circ \sigma f: \sigma W_\Omega \rightarrow V$ depends only on $\sigma|k'$. For k -homomorphisms $\sigma, \tau: k' \rightarrow \Omega$, define $f_{\tau,\sigma} = g_\tau^{-1} \circ g_\sigma: \sigma W \rightarrow \tau W$. Then $f_{\tau,\sigma}$ is defined over the Galois closure of k' in Ω , and the family $(f_{\tau,\sigma})$ satisfies the hypotheses of [W, Thm. 3], which gives a model of V over k splitting $(f_\sigma)_{\sigma \in A(\Omega/k)}$. □

COROLLARY 1.2. *Let Ω, k , and V be as in the theorem, and let $(f_\sigma)_{\sigma \in A(\Omega/k)}$ be a descent system on V . If there is a finite set Σ of points in $V(\Omega)$ such that*

- (a) *any automorphism of V fixing all $P \in \Sigma$ is the identity map and*
- (b) *there exists a subfield L of Ω finitely generated over k such that $f_\sigma(\sigma P) = P$ for all $P \in \Sigma$ and all $\sigma \in A(\Omega/L)$,*

then $(f_\sigma)_{\sigma \in A(\Omega/k)}$ is effective.

Proof. After possibly replacing the L in (b) with a larger finitely generated extension of k , we may suppose that V has a model (W, f) over L for which the points of Σ are rational—that is, such that, for each $P \in \Sigma$, $P = f(P')$ for some $P' \in W(L)$. Now, for each $\sigma \in A(\Omega/L)$, f_σ and $f \circ \sigma f^{-1}$ are both isomorphisms $\sigma V \rightarrow V$ sending σP to P , and so hypothesis (a) implies they are equal. Hence $(f_\sigma)_{\sigma \in A(\Omega/L)}$ is effective, and the theorem applies. □

REMARK 1.3. (a) It is easy to construct noneffective descent systems. For example, take Ω to be the algebraic closure of k , and let V be a variety k . A 1-cocycle $h: A(\Omega/k) \rightarrow \text{Aut}(V_\Omega)$ can be regarded as a descent system by identifying h_σ with a map $\sigma V_\Omega = V_\Omega \rightarrow V_\Omega$. If h is not continuous—for example, if it is a homomorphism into $\text{Aut}(V)$ whose kernel is not open—then the descent system will not be effective.

(b) An example [Di, p. 131] shows that the hypothesis that V be quasiprojective in (1.1) is necessary unless the model V_0 is allowed to be an algebraic space in the sense of M. Artin.

(c) Theorem 1.1 and its corollary replace Lemma 3.23 of [M3], which omits the continuity conditions.

APPLICATION TO MODULI PROBLEMS. Suppose we have (a) a contravariant functor \mathcal{M} from the category of algebraic varieties over Ω to the category of sets and (b) equivalence relations \sim on each of the sets $\mathcal{M}(T)$ compatible with morphisms. The pair (\mathcal{M}, \sim) is then called a *moduli problem* over Ω . A t in $T(\Omega)$ defines a map

$$m \mapsto m_t \stackrel{\text{df}}{=} t^*m : \mathcal{M}(T) \rightarrow \mathcal{M}(\Omega).$$

A *solution to the moduli problem* is a variety V over Ω , together with an isomorphism $\alpha : \mathcal{M}(\Omega)/\sim \rightarrow V(\Omega)$, such that:

- (a) for all varieties T over Ω and all $m \in \mathcal{M}(T)$, the map $t \mapsto \alpha(m_t) : T(\Omega) \rightarrow V(\Omega)$ is regular (i.e., defined by a morphism $T \rightarrow V$ of Ω -varieties); and
- (b) for any variety W over Ω and map $\beta : \mathcal{M}(\Omega)/\sim \rightarrow W(\Omega)$ satisfying condition (a), the map $P \mapsto \beta(\alpha^{-1}(P)) : V(\Omega) \rightarrow W(\Omega)$ is regular.

Clearly, a solution (when it exists) to a moduli problem is unique up to a unique isomorphism.

Let (\mathcal{M}, \sim) be a moduli problem over Ω , and let k be a subfield Ω . For $\sigma \in A(\Omega/k)$, define ${}^\sigma\mathcal{M}$ to be the functor sending an Ω -variety T to $\mathcal{M}(\sigma^{-1}T)$. We say that (\mathcal{M}, \sim) is *rational over k* if there is given a family $(g_\sigma)_{\sigma \in A(\Omega/k)}$ of isomorphisms $g_\sigma : {}^\sigma\mathcal{M} \rightarrow \mathcal{M}$, compatible with \sim , such that $g_{\sigma\tau} = g_\sigma \circ \sigma g_\tau$ for all $\sigma, \tau \in A(\Omega/k)$ and where the last equation means that $g_{\sigma\tau}(T) = g_\sigma(T) \circ g_\tau(\sigma^{-1}T)$ for all varieties T . Note that ${}^\sigma\mathcal{M}(\Omega) = \mathcal{M}(\Omega)$ and that the rule $\sigma m = g_\sigma(m)$ defines an action of $A(\Omega/k)$ on $\mathcal{M}(\Omega)$. A *solution to a moduli problem $(\mathcal{M}, \sim, (g_\sigma))$ rational over k* is a variety V_0 over k together with an isomorphism $\alpha : \mathcal{M}(\Omega)/\sim \rightarrow V_0(\Omega)$ such that

- (a) (V_0, Ω, α) is a solution to the moduli problem (\mathcal{M}, \sim) over Ω and
- (b) α commutes with the actions of $A(\Omega/k)$ on $\mathcal{M}(\Omega)$ and $V_0(\Omega)$.

Again, (V_0, α) is uniquely determined up to a unique isomorphism (over k) when it exists.

THEOREM 1.4. *Assume that Ω has infinite transcendence degree over k . Let $(\mathcal{M}, \sim, (g_\sigma))$ be a moduli problem rational over k for which (\mathcal{M}, \sim) has a solution (V, α) over Ω . Then $(\mathcal{M}, \sim, (g_\sigma))$ has a solution over k if there exists a finite subset $\Sigma \subset \mathcal{M}(\Omega)$ such that*

- (a) *any automorphism of V fixing $\alpha(P)$ for all $P \in \Sigma$ is the identity map and*
- (b) *there exists a subfield L of Ω finitely generated over k such that $g_\sigma(P) \sim P$ for all $P \in \Sigma$ and all $\sigma \in A(\Omega/L)$.*

Proof. The family (g_σ) defines a descent system on V , which Corollary 1.2 shows to be effective. □

2. Descent of Shimura Varieties

In this section, all fields will be subfields of \mathbb{C} . For a subfield E of \mathbb{C} , E^{ab} denotes the composite of all the finite abelian extensions of E in \mathbb{C} .

Let (G, X) be a pair satisfying the axioms (2.1.1.1)–(2.1.1.3) used in [D2] to define a Shimura variety, and let $\text{Sh}(G, X)$ be the corresponding Shimura variety over \mathbb{C} . We regard $\text{Sh}(G, X)$ as a pro-variety endowed with a continuous action of $G(\mathbb{A}_f)$; in particular [D2, 2.7.1], this means that $\text{Sh}(G, X)$ is a projective system of varieties $(\text{Sh}_K(G, X))$ indexed by the compact open subgroups K of

$G(\mathbb{A}_f)$. Let $[x, a] = ([x, a]_K)_K$ denote the point in $\text{Sh}(G, X)(\mathbb{C})$ defined by a pair $(x, a) \in X \times G(\mathbb{A}_f)$, and let $E(G, X)$ be the reflex field of (G, X) . For a special point $x \in X$, let $E(x) \supset E(G, X)$ be the reflex field for x , and let

$$r_x : \text{Gal}(E(x)^{\text{ab}}/E(x)) \rightarrow T(\mathbb{A}_f)/T(\mathbb{Q})^-$$

be the reciprocity map defined in [M2, p. 164] (inverse to that in [D2, 2.2.3]). Here T is a subtorus of G such that $\text{Im}(h_x) \subset T_{\mathbb{R}}$, and $T(\mathbb{Q})^-$ is the closure of $T(\mathbb{Q})$ in $T(\mathbb{A}_f)$. A model of $\text{Sh}(G, X)$ over a field k is a pro-variety S over k endowed with an action of $G(\mathbb{A}_f)$ and a $G(\mathbb{A}_f)$ -equivariant isomorphism $f : S_{\mathbb{C}} \rightarrow \text{Sh}(G, X)$. A model of $\text{Sh}(G, X)$ over $E(G, X)$ is *canonical* if, for each special point $x \in X$ and $a \in G(\mathbb{A}_f)$, $[x, a]$ is rational over $E(x)^{\text{ab}}$ and $\sigma \in \text{Gal}(E(x)^{\text{ab}}/E(x))$ acts on $[x, a]$ according to the rule

$$\sigma[x, a] = [x, r_x(\sigma) \cdot a].$$

(More precisely, the condition for (S, f) to be canonical is as follows: If $P \in S(\mathbb{C})$ corresponds under f to $[x, a]$, then σP corresponds under f to $[x, r_x(\sigma) \cdot a]$.)

Let k be a field containing $E(G, X)$. A *descent system* for $\text{Sh}(G, X)$ over k is a family of isomorphisms

$$(f_{\sigma} : \sigma \text{Sh}(G, X) \rightarrow \text{Sh}(G, X))_{\sigma \in A(\mathbb{C}/k)}$$

such that:

- (a) for all $\sigma, \tau \in A(\mathbb{C}/k)$, $f_{\sigma\tau} = f_{\sigma} \circ \sigma f_{\tau}$; and
- (b) for all $\sigma \in A(\mathbb{C}/k)$, f_{σ} is equivariant for the actions of $G(\mathbb{A}_f)$ on $\text{Sh}(G, X)$ and $\sigma \text{Sh}(G, X)$.

We say that a model (S, f) of $\text{Sh}(G, X)$ over k *splits* (f_{σ}) if $f_{\sigma} = f \circ \sigma f^{-1}$ for all $\sigma \in A(\mathbb{C}/k)$, and that a descent system is *effective* if it is split by some model over k . A descent system (f_{σ}) for $\text{Sh}(G, X)$ over $E(G, X)$ is *canonical* if

$$f_{\sigma}(\sigma[x, a]) = [x, r_x(\sigma|E(x)^{\text{ab}}) \cdot a]$$

whenever x is a special point of X , $\sigma \in A(\mathbb{C}/E(x))$, and $a \in G(\mathbb{A}_f)$.

REMARK 2.1. (a) For a Shimura variety $\text{Sh}(G, X)$, there exists at most one canonical descent system for $\text{Sh}(G, X)$ over $E(G, X)$. (Apply [D1, 5.1, 5.2].)

(b) Let (S, f) be a model of $\text{Sh}(G, X)$ over $E(G, X)$, and let $f_{\sigma} = f \circ (\sigma f)^{-1}$. Then $(f_{\sigma})_{\sigma \in A(\mathbb{C}/k)}$ is a descent system for $\text{Sh}(G, X)$, and (f_{σ}) is canonical if and only if (S, f) is canonical.

(c) Suppose $\text{Sh}(G, X)$ has a canonical descent system $(f_{\sigma})_{\sigma \in A(\mathbb{C}/E(G, X))}$; then $\text{Sh}(G, X)$ has a canonical model if and only if (f_{σ}) is effective. (This follows from (a) and (b).)

(d) A descent system $(f_{\sigma})_{\sigma \in A(\mathbb{C}/k)}$ on $\text{Sh}(G, X)$ defines, for each compact open subgroup K of $G(\mathbb{A}_f)$, a descent system $(f_{\sigma, K})_{\sigma \in A(\mathbb{C}/k)}$ on the variety $\text{Sh}_K(G, X)$ (in the sense of Section 1). If (f_{σ}) is effective, then so also is $(f_{\sigma, K})$ for all K ; conversely, if $(f_{\sigma, K})_{\sigma \in A(\mathbb{C}/k)}$ is effective (in the sense of Section 1) for all sufficiently small K , then $(f_{\sigma})_{\sigma \in A(\mathbb{C}/k)}$ is effective (in the sense of Section 2).

LEMMA 2.2. *The automorphism group of the quotient of a bounded symmetric domain by a neat arithmetic group is finite.*

Proof. According to [Mu, Prop. 4.2], such a quotient is an algebraic variety of logarithmic general type, which implies that its automorphism group is finite [I, 11.12].

Alternatively, one sees easily that the automorphism group of the quotient of a bounded symmetric domain D by a neat arithmetic subgroup Γ is N/Γ , where N is the normalizer of Γ in $\text{Aut}(D)$. Now N is countable and closed (because Γ is closed), and hence it is discrete (by the Baire category theorem). Because the quotient of $\text{Aut}(D)$ by Γ has finite measure, this implies that Γ has finite index in N (cf. [Ma, II.6.3]). \square

THEOREM 2.3. *Every canonical descent system on a Shimura variety is effective.*

Proof. Let $(f_\sigma)_{\sigma \in A(\mathbb{C}/E(G, X))}$ be a canonical descent system for the Shimura variety $\text{Sh}(G, X)$. Let K be a compact open subgroup of $G(\mathbb{A}_f)$, chosen so small that the connected components of $\text{Sh}_K(G, X)$ are quotients of bounded symmetric domains by neat arithmetic groups. Let x be a special point of X . According to [D1, 5.2], the set $\Sigma = \{[x, a]_K \mid a \in G(\mathbb{A}_f)\}$ is Zariski dense in $\text{Sh}_K(G, X)$. Because the automorphism group of $\text{Sh}_K(G, X)$ is finite, there is a finite subset Σ_f of Σ such that any automorphism α of $\text{Sh}_K(G, X)$ fixing each $P \in \Sigma_f$ is the identity map.

The rule

$$\sigma * [x, a]_K = [x, r_x(\sigma) \cdot a]_K$$

defines an action of $\text{Gal}(E(x)^{\text{ab}}/E(x))$ on Σ for which the stabilizer of each point of Σ is open. Therefore, there exists a finite abelian extension L of $E(x)$ such that $\sigma * P = P$ for all $P \in \Sigma_f$ and all $\sigma \in \text{Gal}(E(x)^{\text{ab}}/L)$.

Now, because $(f_\sigma)_{\sigma \in A(\Omega/E(G, X))}$ is canonical, it follows that $f_{\sigma, K}(\sigma P) = P$ for all $P \in \Sigma_f$ and all $\sigma \in A(\mathbb{C}/L)$. Hence we may apply Corollary 1.2 to conclude that $(f_{\sigma, K})_{\sigma \in A(\mathbb{C}/E(G, X))}$ is effective. As this holds for all sufficiently small K , $(f_\sigma)_{\sigma \in A(\mathbb{C}/E(G, X))}$ is effective. \square

REMARK 2.4. (a) If Langlands’s conjugacy conjecture [L, pp. 232–233] is true for a Shimura variety $\text{Sh}(G, X)$, then $\text{Sh}(G, X)$ has a canonical descent system ([L, Sec. 6]; see also [MS, Sec. 7]).

(b) Langlands’s conjugacy conjecture is true for all Shimura varieties [M1]. Hence canonical models exist for all Shimura varieties.

Another proof (based on different ideas) that the descent maps given by Langlands’s conjecture are effective can be found in [Mo]. (I thank the referee for this reference.)

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