Differential Polynomials That Share Three Finite Values with Their Generating Meromorphic Function

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1. Introduction

In this paper, “meromorphic function” means meromorphic in the whole plane $\mathbb{C}$. We shall assume that the reader is familiar with the notation and elementary aspects of Nevanlinna theory (cf. [3] or [4]).

We say that two meromorphic functions f and g share a value a “IM” (resp. CM) if $f - a$ and $g - a$ have the same zeros ignoring multiplicities (counting multiplicities). The subject on sharing values between meromorphic functions and their derivatives was first studied by Rubel and Yang [9].

THEOREM A. *Let f be a nonconstant entire function. If f and f' share two finite values CM, then $f = f'$.*

This result was improved independently by Gundersen [2], and Mues and Steinmetz [7].

THEOREM B. *Let f be meromorphic and nonconstant. If f and f' share three finite and distinct values b_1, b_2, b_3 IM, then $f = f'$.*

Frank and Schwick [1] generalized this to the kth derivative.

THEOREM C. *Let f be meromorphic and nonconstant, $k \in \mathbb{N}$. If f and $f^{(k)}$ share three finite and distinct values b_1, b_2, b_3 IM, then $f = f^{(k)}$.*

In the sequel, we set

$$L(f) := a_k f^{(k)} + a_{k-1} f^{(k-1)} + \cdots + a_0 f \quad (a_k \neq 0), \tag{1}$$

where $a_k, \dots, a_0$ are finite constants. Mues-Reinders [6] proved the following result.

THEOREM D. *Let f be meromorphic and nonconstant, $2 \le k \le 50$. If f and $L(f)$ share three finite and distinct values b_1, b_2, b_3 IM, then $f = L(f)$. Furthermore, if $a_{k-1} = a_{k-2} = 0$, then the restriction $k \le 50$ can be omitted.*

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The purpose of this paper is to cancel the restriction $k \leq 50$.

THEOREM 1. *Let $f(z)$ be nonconstant and meromorphic, $k \geq 2$. If f and $L(f)$ share three finite and distinct values b_1, b_2, b_3 IM, then $f = L(f)$.*

The following example will show that three finite values in our theorem are best possible.

EXAMPLE. Let

$$f(z) = 2\frac{e^{2\sqrt{2}iz} + 4e^{\sqrt{2}iz} + 1}{(e^{\sqrt{2}iz} - 1)^2}.$$

Then $2 - f \neq 0$, $f'' = f(2-f)$, and $f'' - 1 = -(f-1)^2$. Thus f and f'' share 0 and 1 IM, but $f \not\equiv f''$.

2. One Basic Lemma

For the sake of convenience, we define

$$\Psi(f) := \frac{f'(L(f))'(f - L(f))^2}{(f-b_1)(f-b_2)(f-b_3)(L(f)-b_1)(L(f)-b_2)(L(f)-b_3)}, \tag{2}$$

$$N_0\left(r, \frac{1}{f - L(f)}\right) := N\left(r, \frac{1}{f - L(f)}\right) - \sum_{j=1}^{3} \bar{N}\left(r, \frac{1}{f - b_j}\right),$$

$$N_0\left(r, \frac{1}{f'}\right) := N\left(r, \frac{1}{f'}\right) - \sum_{j=1}^{3} N_1\left(r, \frac{1}{f - b_j}\right),$$

$$N_0\left(r, \frac{1}{(L(f))'}\right) := N\left(r, \frac{1}{(L(f))'}\right) - \sum_{j=1}^{3} N_1\left(r, \frac{1}{L(f) - b_j}\right),$$

$$N_1(r, f) := N(r, f) - \bar{N}(r, f),$$

$$A := \frac{(L(f))'(f - L(f))}{(L(f)-b_1)(L(f)-b_2)(L(f)-b_3)}. \tag{3}$$

LEMMA 1. *Let f be a nonconstant meromorphic function, $k \in \mathbb{N}$. If f and $L(f)$ share three finite values b_1, b_2, b_3 IM, and if $f \not\equiv L(f)$, then the following conclusions hold*:

$$T(r, f) = T(r, L(f)) + S(r, f), \qquad T(r, L(f)) = T(r, f) + S(r, f); \tag{4}$$

$$2T(r, L(f)) = \bar{N}(r, f) + \sum_{j=1}^{3} \bar{N}\left(r, \frac{1}{L(f) - b_j}\right) + S(r, f); \tag{5}$$

$$N_1(r, f) = S(r, f); \tag{6}$$

$$T(r, \Psi(f)) = m(r, \Psi(f)) = S(r, f); \tag{7}$$

$$N_0\left(r, \frac{1}{f - L(f)}\right),\ N_0\left(r, \frac{1}{f'}\right),\ N_0\left(r, \frac{1}{(L(f))'}\right) = S(r, f); \tag{8}$$

$$m(r, L(f)) = S(r, f); \tag{9}$$

$$T(r, L(f)) = (k+1)\bar{N}(r, f) + S(r, f); \tag{10}$$

$$T(r, f) = (k+1)\bar{N}(r, f) + S(r, f); \tag{11}$$

$$m(r, f) = k\bar{N}(r, f) + S(r, f); \tag{12}$$

$$T(r, A) = m(r, A) = k\bar{N}(r, f) + S(r, f). \tag{13}$$

Proof. The proof is actually given in [2], [5], and [6]. We include it here for the sake of completeness. Take $c \in \mathbb{C} - \{b_1, b_2, b_3\}$ and let $\hat{b}_j = \frac{1}{b_j - c}$ $(j = 1, 2, 3)$, $\hat{b}_4 = 0$, $g_1 = \frac{1}{f-c}$, and $g_2 = \frac{1}{L(f)-c}$. Then g_1 and g_2 share the values $\hat{b}_j$ $(j = 1, \ldots, 4)$ IM. By the second fundamental theorem, we have

$$\begin{aligned} 2T(r, g_i) &\leq \sum_{j=1}^{4} \bar{N}\left(r, \frac{1}{g_i - \hat{b}_j}\right) + S(r, g_i) \\ &\leq N\left(r, \frac{1}{g_1 - g_2}\right) + S(r, g_i) \\ &\leq T(r, g_1) + T(r, g_2) + S(r, g_i) \quad (i = 1, 2). \end{aligned}$$

Equations (4) and (5) follow from this and the first fundamental theorem. Now, by

$$\begin{aligned} k\bar{N}(r, f) + N(r, f) &\leq T(r, L(f)) = T(r, f) + S(r, f) \\ &= \sum_{j=1}^{3} \bar{N}\left(r, \frac{1}{f - b_j}\right) + \bar{N}(r, f) - T(r, f) + S(r, f) \\ &\leq N\left(r, \frac{1}{f - L(f)}\right) + \bar{N}(r, f) - T(r, f) + S(r, f) \\ &\leq (k+1)\bar{N}(r, f) + S(r, f), \end{aligned}$$

we obtain (6), (10), (11), and (12). By the assumptions and (2), it is easy to see that Ψ is entire. Now (2) can be written in the form

$$\Psi(f) = \sum_{s,t=1}^{3} c_{st} \frac{f'}{f - b_s} \frac{(L(f))'}{L(f) - b_t},$$

where c_{st} $(s, t = 1, 2, 3)$ are constants depending only on b_j $(j = 1, 2, 3)$. Equation (7) follows from this and the theorem on the logarithmic derivative; (7) and (2) yield (8). Now, by (10),

$$\begin{aligned} m(r, L(f)) + (k+1)\bar{N}(r, L(f)) &\leq m(r, L(f)) + N(r, L(f)) \\ &\leq (k+1)\bar{N}(r, f) + S(r, f). \end{aligned}$$

This gives (9). From (3), we now have

$$m(r, A) \leq m(r, f) + S(r, f)$$

and

$$k\bar{N}(r, f) \leq N(r, 1/A) \leq T(r, A) + S(r, f). \tag{14}$$

Combining these two inequalities with (12), we obtain (13). This completes the proof of the lemma. □

3. Proof of Theorem 1

To prove our theorem, we follow some ideas in Mues and Reinders [6]. We suppose that $f \not\equiv L(f)$ and $k \geq 3$. Let z_0 be a simple pole of f, and let

$$f(z) = \frac{R}{z - z_0} + O(1)$$

near $z = z_0$. Then

$$L(f)(z) = \frac{a_k k!\,(-1)^k R}{(z - z_0)^{k+1}} + \frac{a_{k-1}(k-1)!\,(-1)^{k-1} R}{(z - z_0)^k} + \cdots .$$

Put

$$\phi := \frac{A'}{A}. \tag{15}$$

We then have

$$\phi(z) = \frac{k}{z - z_0} + \sigma + \frac{\tau^2}{3k}(z - z_0) + O((z - z_0)^2), \tag{16}$$

where σ and τ are constants depending only on the coefficients of $L(f)$ and k as follows:

$$\sigma := \sigma(f) := \frac{(k+2)}{k(k+1)} \frac{a_{k-1}}{a_k}, \tag{17}$$

$$\tau := \tau(f) := \left[3k \left(\frac{k^2 + 4k + 2}{k^2(k+1)^2} \left(\frac{a_{k-1}}{a_k} \right)^2 - \frac{2k+6}{k(k^2-1)} \frac{a_{k-2}}{a_k} \right) \right]^{1/2}. \tag{18}$$

Obviously, by (15),

$$m(r, \phi) = S(r, f) \tag{19}$$

and

$$N(r, \phi) = \bar{N}(r, 1/A) = \bar{N}(r, f) + S(r, f).$$

Let

$$H := k\phi' - \tau^2 + (\phi - \sigma)^2;$$

then, by (16), $H(z_0) = 0$ at the simple pole of f and so $N(r, H) = S(r, f)$, which results in $T(r, H) = S(r, f)$ by (19). If $H(z) \not\equiv 0$, then

$$N(r, f) \leq N(r, 1/H) + S(r, f) = S(r, f),$$

which contradicts (11). Thus $H(z) \equiv 0$; that is,

$$k\phi' = \tau^2 - (\phi - \sigma)^2.$$

If $\tau = 0$ then ϕ is fractional linear, and so f has at most one pole by (15) and (3), which contradicts (11). Thus $\tau \neq 0$. From the preceding equality we have

$$\phi(z) = \sigma + \tau \frac{a\exp(uz) - b\exp(-uz)}{a\exp(uz) + b\exp(-uz)},$$

where a, b are constants and

$$u = \frac{\tau}{k}. \tag{20}$$

If $ab = 0$ then $\phi(z)$ is constant. By (14) and (15),

$$k\bar{N}(r, f) \le N(r, 1/A) = 0,$$

which contradicts (11). Thus $ab \ne 0$. Take c satisfying $\exp(2uc) = -a/b$. Then ϕ has the form

$$\phi(z) = \sigma + \tau \frac{\exp(u(z+c)) + \exp(-u(z+c))}{\exp(u(z+c)) - \exp(-u(z+c))}.$$

Using the transformation $z \to z - c$ if necessary, we may let $c = 0$. Thus

$$\phi(z) = \sigma + \tau \coth(uz).$$

By (15), we have

$$A(z) = De^{\sigma z}\left(\frac{e^{uz} - e^{-uz}}{2}\right)^k \tag{21}$$

with a constant $D \ne 0$. This, together with (3), (6), and (13), imply that

$$\bar{N}(r, f) = \frac{2|u|r}{\pi} + O(1)$$

and so, by (11),

$$T(r, f) = \frac{2(k+1)|u|}{\pi} r + S(r, f).$$

This implies that the order $\rho(f)$ of f is less than or equal to 1. Thus

$$T(r, f) = O(r) \quad \text{for } r \to \infty.$$

It follows from (2) and [8] that

$$m(r, \Psi(f)) = \circ(\log r) \quad \text{for } r \to \infty.$$

Combining this with the fact that $\Psi(f)$ is entire, we obtain

$$\Psi(f) \equiv \text{constant}. \tag{22}$$

By (22) and (2), the functions f', $(L(f))'$, and $f - L(f)$ have only zeros at the zeros of $f - b_j$ $(j = 1, 2, 3)$; $f - L(f)$ has only simple zeros and f has only simple poles that coincide with zeros of A. Thus, the poles of f are

$$z_\nu = \nu \frac{\pi}{u} i \quad (\nu \in \mathbb{Z}),$$

which gives

$$\bar{N}(r, f) = N(r, f) = \frac{2|u|r}{\pi} + O(1). \tag{23}$$

Note that, since $\rho(f) \le 1$, it follows from [8], (9), and (12) that

$$m(r, L(f)) = \circ(\log r), \qquad m(r, f) = \frac{2k|u|r}{\pi} + \circ(\log r). \tag{24}$$

Let

$$f(z) = \frac{R_\nu}{z - z_\nu} + O(1). \tag{25}$$

Then

$$\Psi(f) = \frac{k+1}{R_\nu^2}, \tag{26}$$

by (2). Set

$$v := v(L(f)) := \frac{\sigma}{u} = \frac{\sigma k}{\tau}. \tag{27}$$

From (21) and (3) it follows that

$$Du^k e^{v\nu\pi i}(-1)^{k\nu} = \frac{(-1)^k(k+1)}{k!\, a_k R_\nu} \tag{28}$$

for all $\nu \in \mathbb{Z}$. Squaring (28) and combining with (22) and (26), we deduce that

$$e^{2v\nu\pi i} \equiv \text{constant} \quad \text{for all } \nu \in \mathbb{Z}.$$

Taking $\nu = 0$, we know that

$$e^{2v\nu\pi i} \equiv 1 \quad \text{for all } \nu \in \mathbb{Z}.$$

Thus $e^{2v\pi i} = 1$, which results in $v \in \mathbb{Z}$. By (28) we have

$$R_\nu = (-1)^{(k-v)\nu} B, \tag{29}$$

where

$$B = \left(-\frac{1}{u}\right)^k \frac{k+1}{Dk!\, a_k}. \tag{30}$$

Now, by (21),

$$T(r, A) = m(r, A) = \{k + \max(k, |v|)\}\frac{|u|r}{\pi} + O(1).$$

On the other hand, by (13) and (23),

$$T(r, A) = k\frac{2|u|r}{\pi} + \circ(\log r).$$

These two equations imply that $|v| \le k$, and so

$$v \in \mathbb{Z}, \quad -k \le v \le k. \tag{31}$$

We define:

$$G(w) := \begin{cases} 2Bu/(w^2-1) & \text{if } k-v \text{ is even,} \\ 2Buw/(w^2-1) & \text{if } k-v \text{ is odd;} \end{cases} \tag{32}$$

$$g(z) := G(e^{uz}); \tag{33}$$

$$h(z) := f(z) - g(z). \tag{34}$$

Then h is entire by (25), (29), and (34). Let

$$L(g) = \sum_{j=0}^{k} a_j g^{(j)}, \qquad L(h) = \sum_{j=0}^{k} a_j h^{(j)}. \tag{35}$$

It is easy to check that $m(r, g^{(j)}) = O(1)$ for $j = 0, \ldots, k$; (24) gives

$$m(r, L(h)) \leq m(r, L(f)) + m(r, L(g)) + O(1) = \circ(\log r)$$

for $r \to \infty$. Note that $L(h)$ is entire and so we have

$$L(h) = \text{constant}; \tag{36}$$

hence,

$$L(f)(z) = L(g)(z) + L(h)(z) = S(e^{uz}) \tag{37}$$

for a rational function $S(w)$. Note that, as $|\mathrm{Re}(uz)| \to \infty$,

$$g^{(j)}(z) = O(1) \quad (0 \leq j \leq k).$$

We deduce that

$$S(0) \neq \infty, \qquad S(\infty) \neq \infty. \tag{38}$$

From (21), (33), (34), (37), and (3), we see that h is a $(2\pi/u)i$-periodic entire function, and (24) and (34) yield

$$m(r, h) = m(r, f) + O(1) = \frac{2k|u|r}{\pi} + \circ(\log r)$$

for $r \to \infty$. Thus $h(z)$ is of the form

$$h(z) = \sum_{j=p}^{q} A_j e^{juz} \quad (p \leq q,\ A_j \in \mathbb{C},\ A_p A_q \neq 0), \tag{39}$$

with

$$\max\{q, 0\} - \min\{p, 0\} = 2k. \tag{40}$$

Therefore,

$$f(z) = R(e^{uz}) \tag{41}$$

with a rational function

$$R(w) = \sum_{j=p}^{q} A_j w^j + G(w). \tag{42}$$

By (21), (37), (41), and (3), we now have

$$\frac{uwS'(w)(R(w) - S(w))}{(S(w) - b_1)(S(w) - b_2)(S(w) - b_3)} = \frac{D}{2^k} \frac{(w^2 - 1)^k}{w^{k-v}}. \tag{43}$$

From (33), (37), and (40), we may suppose that

$$S(w) = \frac{P(w)}{(w^2 - 1)^{k+1}}, \tag{44}$$

where

$$P(w) = d_t w^t + \cdots + d_1 w + d \quad (d_t \neq 0,\ t \leq 2(k+1),\ P(\pm 1) \neq 0).$$

Substituting this into (43), we obtain

$$\frac{\begin{array}{c} w[(w^2-1)P'(w)-2(k+1)wP(w)] \\ \times[R(w)w^{k-v}(w^2-1)^{k+1}-P(w)w^{k-v}] \end{array}}{\prod_{j=1}^{3}[P(w)-b_j(w^2-1)^{k+1}]}=\frac{D}{u2^k}. \tag{45}$$

From (44) we see that there exists an integer m with

$$m \geq 2(k+1)-t+1 \tag{46}$$

such that

$$S'(w)=O(w^{-m}) \quad \text{for } w \to \infty.$$

Dividing both sides of (43) by w^{k+v} and letting $w \to \infty$, it follows from (40), (42), and (43) that

$$q=k+v+m-1 \geq k+v. \tag{47}$$

Similarly, by considering $w \to 0$ we obtain

$$p \leq v-k.$$

Combining this, (24), (40), and (47), we have

$$q=v+k, \qquad p=v-k.$$

Thus, (39) and (42) now read

$$h(z)=\sum_{j=v-k}^{v+k} A_j e^{juz} \quad (A_j \in \mathbb{C},\ A_p A_q \neq 0) \tag{48}$$

and

$$R(w)=\sum_{v-k}^{v+k} A_j w^j + G(w), \tag{49}$$

respectively. Furthermore, from $q=v+k$ and (47) we deduce that $m=1$; hence by (46), $1 \geq 2(k+1)-t+1$—that is, $t \geq 2(k+1)$. This and the condition $t \leq 2(k+1)$ imply that

$$t=2(k+1).$$

Thus by (44),

$$S(\infty)=d_t \neq 0,$$

which implies that

$$(w^2-1)P'(w)-2(k+1)wP(w)$$

is a polynomial of w with order $\leq 2k+2$. Therefore, the order of the numerator of (45) is at most $6k+5$. If

$$S(\infty)=d_t \neq b_1, b_2, b_3,$$

then $P(w)-b_j(w^2-1)^{k+1}$ is a polynomial of w with degree $2k+2$ for $j=1, 2, 3$, so that the order of the denominator of (45) is $6k+6$—a contradiction. Thus $d_t = b_j$ $(1 \leq j \leq 3)$. We may let

$$b_1=S(\infty)=d_t \neq 0. \tag{50}$$

This, (36), and (48) imply that

$$S(e^{uz}) = b_1 + L(g(z)) \tag{51}$$

and

$$d_t = b_1 = a_0 A_0 = L(h). \tag{52}$$

Without loss of generality, we may suppose that

$$a_0 \neq \frac{b_1 - b_2}{2Bu} \quad \text{and} \quad a_0 \neq \frac{b_1 - b_3}{2Bu}. \tag{53}$$

Otherwise, we consider $\tilde{f} = f(\alpha z)$ and

$$\tilde{L}(\tilde{f}) := \sum_{j=0}^{k} \tilde{a}_j \tilde{f}^{(j)}$$

for some suitable positive constant α, where

$$\tilde{a}_j = a_j \alpha^{-j} \quad (j = 0, 1, \ldots, k).$$

It is obvious that $\tilde{L}(\tilde{f}) = L(f)$ and that $\tilde{f}$ and $\tilde{L}(\tilde{f})$ share b_j IM for $j = 1, 2, 3$. Let $\tilde{A}$, $\tilde{B}$, $\tilde{u}$, $\tilde{v}$, $\tilde{\tau}$, $\tilde{\sigma}$, and $\tilde{D}$ correspond to A, B, u, v, τ, σ, and D, respectively. Then, by (3) and (21), $\tilde{D} = D$; by (17) and (18), $\tilde{\sigma} = \alpha\sigma$ and $\tilde{\tau} = \alpha\tau$, so that $\tilde{u} = \alpha u$ and $\tilde{v} = v$ by (20) and (27). Thus (31) still holds for $\tilde{v}$ and by (30), $\tilde{B} = B$. As a result,

$$\frac{b_1 - b_j}{2\tilde{B}\tilde{u}} = \alpha^{-1} \frac{b_1 - b_2}{2Bu} \quad (j = 2, 3).$$

We can therefore choose a suitable positive constant α such that

$$\frac{b_1 - b_j}{2\tilde{B}\tilde{u}} \neq \tilde{a}_0 = a_0 \quad (j = 2, 3).$$

Next we consider two cases.

Case 1: $k - v$ is even. Then, by (32),

$$G(w) = \frac{2Bu}{w^2 - 1} \quad \text{and} \quad g(z) = \frac{2Bu}{e^{2uz} - 1}.$$

These equalities imply that

$$g^{(j)}(z) = e^{2uz} \frac{\sum_{l=0}^{j-1} c_l e^{2luz}}{(e^{2uz} - 1)^{j+1}} \quad (j \geq 1),$$

where all c_l are constants. Thus, by (50) and (51), we may let

$$S(w) = b_1 + \frac{Q(w)}{(w^2 - 1)^{k+1}}, \tag{54}$$

where

$$Q(w) = 2Bua_0(w^2 - 1)^k + w^2 P_{k-1}(w^2); \tag{55}$$

here $P_{k-1}(w^2)$ is a polynomial of w^2 of degree less than or equal to $k - 1$. We rewrite $Q(w)$ in the form

$$Q(w) = e_m w^m + \cdots + e_1 w + e_0 \quad (e_m \neq 0,\ m \leq 2k).$$

Combining (55), (54), and (53), we obtain that $S(0)-b_2\neq 0$ and $S(0)-b_3\neq 0$. By (43), we thus have

$$\frac{uw(w^2-1)S'(w)(R(w)w^{k-v}-S(w)w^{k-v})}{Q(w)(S(w)-b_2)(S(w)-b_3)}=D2^{-k}.$$

Now the numerator is zero at $w=0$ and so $Q(0)=0$, which results in $a_0=0$ by (55). Hence $b_1=0$ by (52), which contradicts (50).

Case 2: $k-v$ is odd. Then $G(w)=2Buw/(w^2-1)$ and $g(z)=G(e^{uz})$. We can easily deduce that

$$g^{(j)}(z)=2B(-u)^{j+1}\frac{wQ_j(w^2)}{(w^2-1)^{j+1}}\circ e^{uz}\quad (j\geq 0),$$

where $Q_j(w^2)$ is a polynomial of w^2 with degree j. It follows from (33) and (35) that

$$L(g)=\frac{wQ(w^2)}{(w^2-1)^{k+1}}\circ e^{uz},$$

where $Q(\zeta)$ is a polynomial of ζ with degree $\leq k$. This and (51) imply

$$S(w)=b_1+\frac{U(w)}{(w^2-1)^{k+1}}, \tag{56}$$

where

$$U(w)=wQ(w^2). \tag{57}$$

Substituting (49) and (56) into (43), we have

$$\begin{aligned}&\frac{\begin{aligned}&w\{(w^2-1)U'(w)-2(k+1)wU(w)\}\\&\qquad\times\big\{\big(\textstyle\sum_{i=v-k}^{v+k}A_iw^i+G(w)-b_1\big)(w^2-1)^{k+1}-U(w)\big\}\end{aligned}}{\{(b_2-b_1)(w^2-1)^{k+1}+U(w)]U(w)[(b_3-b_1)(w^2-1)^{k+1}+U(w)\}}\\&\qquad=\frac{D}{2^k}w^{v-k}.\end{aligned}$$

We rewrite this in the form

$$\begin{aligned}\sum_{j=0}^{k}A_{v-k+2j}w^{2j}&+\sum_{j=0}^{k-1}A_{v-k+2j+1}w^{2j+1}-b_1w^{k-v}\\&+\frac{2Buw^{k-v+1}}{w^2-1}-\frac{w^{k-v+1}Q(w^2)}{(w^2-1)^{k+1}}\\=\frac{D}{2^k}&\cdot\frac{Q(w^2)}{(w^2-1)U'(w)-2(k+1)wU(w)}\\&\cdot\bigg\{(b_2-b_1)(b_3-b_1)(w^2-1)^{k+1}\\&\quad+\frac{[wQ(w^2)]^2}{(w^2-1)^{k+1}}+(2b_1-b_2-b_3)wQ(w^2)\bigg\},\end{aligned} \tag{58}$$

where we have replaced some of the $U(w)$ by (57). From (57) we now see that

$$(w^2-1)U'(w)-2(k+1)wU(w)$$

is a polynomial of w^2. By multiplying the factor

$$\{(w^2-1)U'(w)-2(k+1)wU(w)\}(w^2-1)^{k+1}$$

to both sides of (58) and then comparing all the terms with odd degree, we obtain

$$\sum_{j=0}^{k-1} A_{v-k+2j+1}w^{2j+1} - b_1 w^{k-v}$$
$$= \frac{(2b_1-b_2-b_3)D}{2^k}\frac{[Q(w^2)]^2}{(w^2-1)U'(w)-2(k+1)wU(w)}w. \quad (59)$$

It is easy to see that the right-hand side can not be a polynomial unless

$$2b_1-b_2-b_3=0. \quad (60)$$

Therefore, $3b_1=b_1+b_2+b_3$. From this and (50), we thus have the following lemma.

LEMMA 2. *Let f be nonconstant and meromorphic, and let $L(f)$ and v be defined* (*resp.*) *by* (1) *and* (27), $k\geq 2$. *Suppose that f and $L(f)$ share three finite values b_1, b_2, b_3 IM, where $f\not\equiv L(f)$. If $k-v$ is odd, then there exists some $b_j\neq 0$* $(1\leq j\leq 3)$ *such that*

$$3b_j=b_1+b_2+b_3.$$

Proof of the Theorem (cont.). From (60) we see that the left-hand side of (59) is identically zero. Thus, $b_1=A_0$. Together with (50) and (52), this implies that $a_0=1$ and so

$$L(f)=a_k f^{(k)}+\cdots+a_1 f'+f. \quad (61)$$

Let

$$\hat{f}=f-b_1$$

and

$$\hat{L}(\hat{f})=a_k\hat{f}^{(k)}+\cdots+a_1\hat{f}'+\hat{f}.$$

Then—from (60), (61), and the assumptions of the theorem—we deduce that $\hat{f}$ and $\hat{L}(\hat{f})$ share three values (0, x, and $-x$) IM, where x can be chosen as b_2-b_1 or b_3-b_1 and $x\neq 0$. On the other hand, since $\hat{L}(\hat{f})$ and $L(f)$ have the same coefficients, it follows from (27), (17), and (18) that $v(\hat{L}(\hat{f}))=v(L(f))$ and so $k-v(\hat{L}(\hat{f}))$ is also odd. Obviously, $\hat{L}(\hat{f})\not\equiv\hat{f}$ by the assumption $L(f)\not\equiv f$. Thus all the conditions of Lemma 2 are satisfied. By Lemma 2,

$$3x=x+0+(-x)=0$$

and so $x=0$, which is impossible.

This completes the proof of the theorem. □

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