On a Minimal Lagrangian Submanifold of C^n Foliated by Spheres

Ildefonso Castro & Francisco Urbano

1. Introduction

In general, not much is known about minimal submanifolds of Euclidean space of high codimension. In [1], Anderson studies complete minimal submanifolds of Euclidean space with finite total scalar curvature, trying to generalize classical results of minimal surfaces. More recently, Moore [10] continues the study of this kind of minimal submanifolds.

Harvey and Lawson [6] also study a particular family of minimal submanifolds of complex Euclidean space, the *special Lagrangian* submanifolds—that is, oriented minimal Lagrangian submanifolds. They have the property of being absolutely volume minimizing. Among other things, they construct important examples of the previously mentioned minimal Lagrangian submanifolds. Following their ideas, new examples of this kind of submanifolds are also obtained in [2]. This family is well known in the case of surfaces, because an orientable minimal surface of $\mathbb{C}^2$ is Lagrangian if and only if it is holomorphic with respect to some orthogonal complex structure on $\mathbb{R}^4$ (see [3]).

Among the examples constructed by Harvey and Lawson in [6], we emphasize the one given in Theorem 3.5. In this example, we emphasize one of its connected components, which is defined by

$$\begin{aligned} M_0 = \{(x, y) \in \mathbb{C}^n \equiv \mathbb{R}^n \times \mathbb{R}^n;\ |x|y = |y|x;\\ \operatorname{Im}(|x| + i|y|)^n = 1;\ |y| < |x| \tan(\pi/n)\}. \end{aligned}$$

Besides being a minimal Lagrangian submanifold of complex Euclidean space $\mathbb{C}^n$, M_0 is invariant under the diagonal action of $SO(n)$ on $\mathbb{C}^n \equiv \mathbb{R}^n \times \mathbb{R}^n$. This paper is inspired by this example. We start by showing that it is a very regular example with many similar properties to the classical catenoid. So, from now on we will refer to M_0 as the *Lagrangian catenoid*. Topologically it is $\mathbb{R} \times \mathbb{S}^{n-1}$. Geometrically it is foliated by $(n-1)$-dimensional round spheres of $\mathbb{C}^n$, and it has finite total scalar curvature (see Proposition 1). When $n = 2$ it has total curvature -4π, being one of the examples described by Hoffman and Osserman in [7].

Received November 20, 1997. Revision received July 22, 1998.
Research partially supported by a DGICYT grant no. PB97-0785.
Michigan Math. J. 46 (1999).

In this case, the Lagrangian catenoid can be written as the following holomorphic surface of $\mathbb{C}^2$:

$$M_0 = \{(z, \tfrac{1}{z}) \in \mathbb{C}^2;\ z \in \mathbb{C}^*\}.$$

In [4] for surfaces and in [8] for higher dimension, it was proved that the catenoid, the Riemann surfaces, and the generalized catenoid are the only (nonflat) minimal hypersurfaces of Euclidean space $\mathbb{R}^{n+1}$ foliated by pieces of $(n-1)$-dimensional round spheres of $\mathbb{R}^{n+1}$. In Theorem 1, the Lagrangian version of this result is obtained, showing that here there are no examples similar to the Riemann minimal surfaces.

THEOREM 1. *Let $\phi\colon M \to \mathbb{C}^n$ be a minimal (nonflat) Lagrangian immersion of an n-dimensional manifold M. Then M is foliated by pieces of round $(n-1)$-spheres of $\mathbb{C}^n$ if and only if ϕ is congruent (up to dilations) to an open subset of the Lagrangian catenoid.*

The proof of Theorem 1 is first given for $n \geq 3$. Theorem 1 in the case $n = 2$ is a consequence of our next, more general result.

THEOREM 2. *Let $\phi\colon M^n \to \mathbb{C}^m$ be a (nonflat) complex immersion of a complex n-dimensional Kähler manifold M. Then M is foliated by pieces of round $(2n-1)$-spheres of $\mathbb{C}^m$ if and only if $n = 1$ and ϕ is congruent (up to dilations) to an open subset of the Lagrangian catenoid.*

On the other hand, using results of Anderson [1] and Enomoto [5], we find that any complete minimal submanifold M (of dimension $n \geq 3$) of finite total scalar curvature and properly immersed in Euclidean space $\mathbb{R}^m$ has a compactification by an inversion. This means that there exists a compact n-dimensional submanifold $\bar{M}$ of $\mathbb{R}^m$ passing through the origin such that M is the image of $\bar{M} - \{0\}$ by the inversion $F\colon \mathbb{R}^m \cup \{\infty\} \to \mathbb{R}^m \cup \{\infty\}$ defined by $F(p) = p/|p|^2$. Following these ideas, we obtain a global characterization of the Lagrangian catenoid in the family of minimal submanifolds with finite total scalar curvature, which is given in the following result.

THEOREM 3. *Let M be an n-dimensional $(n \geq 3)$, complete minimal (nonflat) submanifold with finite total scalar curvature immersed in Euclidean space $\mathbb{R}^{2n}$. Then the compactification by the inversion of M is Lagrangian for a certain orthogonal complex structure on $\mathbb{R}^{2n}$ if and only if M is (up to dilations) the Lagrangian catenoid.*

We mention that Lawlor [9] generalizes the Lagrangian catenoid by constructing a family of complete Lagrangian minimal submanifolds of $\mathbb{C}^n$ with finite total scalar curvature.

When $n = 2$, and assuming that the minimal surface admits a compactification by the inversion, the proof also works and the result is true.

The method used here is different from the one used in the case of Euclidean space [4; 8] and, in some sense, it follows some ideas developed in [11]. Basically it consists of proving that the foliation of the submanifold $\phi\colon M \to \mathbb{C}^n$ is invariant under the action of a uniparametric group of conformal transformations of M, and that the associated conformal vector field X satisfies the vectorial equation $\phi = f\phi_* X$, where f is a smooth complex-valued function of constant length. Integrating this equation we obtain our uniqueness results.

2. The Lagrangian Catenoid

Let $\mathbb{C}^n$ be complex Euclidean space with complex coordinates z_i $(i = 1, \ldots, n)$, and let

$$\Omega = \frac{i}{2}\sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$$

be its symplectic structure. If $\langle \cdot, \cdot \rangle$ denotes the Euclidean metric and J the standard complex structure on $\mathbb{C}^n$, then $\Omega(u, v) = \langle Ju, v\rangle$ for vectors u, v. Let $\phi \colon M \to \mathbb{C}^n$ be an immersion of an n-dimensional manifold M. Then ϕ is called *Lagrangian* if $\phi^*\Omega \equiv 0$.

If NM denotes the normal bundle of ϕ, then J defines an isometry between TM and NM such that

$$J\nabla = \nabla^{\perp} J,$$

where ∇ (resp. $\nabla^{\perp}$) is the Levi–Civita connection (resp., the normal connection) of the induced metric, which will be denoted again by $\langle \cdot, \cdot \rangle$.

Moreover, if σ is the second fundamental form of ϕ and A_η the Weingarten endomorphism associated to a normal vector field η, then

$$\sigma(u, v) = JA_{Ju}v$$

for vectors u, v tangent to M. Denoting by H the mean curvature vector of ϕ, the 1-form γ on M defined by

$$\gamma(v) = \langle JH, v\rangle$$

is closed and, up to a constant, is the known *Maslov 1-form* on M.

Suppose now that ϕ is a minimal immersion, that is, $H = 0$. If τ is the scalar curvature of the submanifold M, then $\tau = -|\sigma|^2$. The total scalar curvature of M is defined by

$$\int_M |\sigma|^n \, dV$$

(see [1]), where dV is the measure associated to the metric $\langle \cdot, \cdot \rangle$.

Let $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$, and let $\phi_0 \colon \mathbb{R} \times \mathbb{S}^{n-1} \to \mathbb{C}^n \equiv \mathbb{R}^n \times \mathbb{R}^n$ be the map given by

$$\phi_0(t, p) = \cosh^{1/n}(nt)e^{i\beta(t)}(p, 0),$$

where $\beta(t) = \frac{\pi}{2n} - \frac{2}{n}\arctan\big(\tanh\frac{nt}{2}\big)$. We point out that $\beta(t) \in \big(0, \frac{\pi}{n}\big)$.

Proposition 1.

(a) *ϕ_0 is a minimal Lagrangian embedding such that $\phi_0(\mathbb{R} \times \mathbb{S}^{n-1}) = M_0$.*

(b) *When $n = 2$, M_0 can be identified with the holomorphic surface*

$$\big\{\big(z, \tfrac{1}{z}\big);\ z \in \mathbb{C}^*\big\}.$$

(c) *The induced metric by ϕ_0 on $\mathbb{R}\times\mathbb{S}^{n-1}$ is*

$$\langle\cdot,\cdot\rangle = \cosh^{2/n}(nt)(dt^2+g_0),$$

where g_0 is the standard metric on $\mathbb{S}^{n-1}$ of constant curvature 1. *In particular, $(M_0,\langle\cdot,\cdot\rangle)$ is complete.*

(d) *M_0 is foliated by round $(n-1)$-spheres of $\mathbb{C}^n$.*

(e) *M_0 has finite total scalar curvature. In fact, if σ is the second fundamental form of M_0, then*

$$\int_{M_0}|\sigma|^n\,dV = ((n+2)(n-1))^{n/2}c_{n-1}I_n,$$

where c_{n-1} is the volume of the $(n-1)$-dimensional unit sphere, $I_2=1$, $I_3=\pi/6$ and

$$I_n = \frac{(n-2)^2}{n(n-1)}I_{n-2}, \quad n\geq 4.$$

Proof. Since we know that M_0 is a minimal Lagrangian submanifold of $\mathbb{C}^n$, (a) is reduced to proving the equality $M_0=\phi_0(\mathbb{R}\times\mathbb{S}^{n-1})$ and that ϕ_0 is an embedding. It is straightforward that $\phi_0(\mathbb{R}\times\mathbb{S}^{n-1})\subset M_0$. Conversely, given $z=(x,y)\in M_0$, we take the only $t\in\mathbb{R}$ such that $\beta(t)=\arctan(|y|/|x|)$. Using that $|z|^n\sin(n\beta(t))=1$, it is easy to check that $\cosh^{1/n}(nt)=|z|$ and then $\phi_0(t,x/|x|=y/|y|)=z$. On the other hand, it is not difficult to prove that ϕ_0 is an embedding using that $\beta=\beta(t)\colon\mathbb{R}\to(0,\pi/n)$ is a 1:1 map.

When $n=2$, $(x,y)\in\mathbb{C}^2$ belongs to M_0 if and only if $x/|x|=y/|y|$ and $|x||y|=1/2$; so, if we define $F\colon\mathbb{C}^2\to\mathbb{C}^2$ by $F(z,w)=\sqrt{2}(z,\bar{w})$, it happens that $F(M_0)=\{(z,1/z);\ z\in\mathbb{C}^*\}$.

Computing $d\phi_0$ from the explicit expression of ϕ_0, (c) is an easy exercise. It is clear that $\{t\times\mathbb{S}^{n-1}\mid t\in\mathbb{R}\}$ defines a foliation on M_0 by round $(n-1)$-spheres of $\mathbb{C}^n$.

Finally, it is not difficult to prove that

$$|\sigma|^2(t,p)=\frac{(n+2)(n-1)}{\cosh^{2(n+1)/n}(nt)}.$$

Hence, using that $dV=\cosh(nt)(dt+dV_0)$, with dV_0 the measure of the unit sphere $\mathbb{S}^{n-1}$, and the Fubini theorem, the total scalar curvature of M_0 is computed as follows:

$$\int_{M_0}|\sigma|^n\,dV=((n+2)(n-1))^{n/2}c_{n-1}\int_{-\infty}^{\infty}\frac{dt}{\cosh^n(nt)},$$

where c_{n-1} is the volume of the unit $(n-1)$-sphere. The formula for $I_n=\int_{-\infty}^{\infty}(dt/\cosh^n(nt))=(1/n)\int_{-\infty}^{\infty}(dx/\cosh^n x)$ follows from the equality

$$\int\frac{dx}{\cosh^n x}=\frac{\sinh x}{(n-1)\cosh^{n-1}x}+\frac{n-2}{n-1}\int\frac{dx}{\cosh^{n-2}x}. \qquad \square$$

3. Proof of the Theorems

Proof of Theorem 1 with $n \geq 3$. Proposition 1(d) states that M_0 is foliated by round $(n-1)$-spheres of $\mathbb{C}^n$.

Conversely, suppose that M is foliated by pieces of round $(n-1)$-spheres of $\mathbb{C}^n$. Let $\mathcal{D}$ be the corresponding foliation on M and let X be an orthogonal vector field to $\mathcal{D}$, tangential to M, such that $|X_p| = r(p)$ for any $p \in M$, where $r(p)$ is the radius of the round sphere where the leaf of $\mathcal{D}$ containing p lies. Locally, we can parameterize M as $(-\delta, \delta) \times N$ in such a way that $X \equiv \partial/\partial t$ and, for any $t \in (-\delta, \delta)$, $\{t\} \times N$ is a leaf of the foliation $\mathcal{D}$.

Thus, $\phi\colon \{t\} \times N \to \mathbb{C}^n$ is an embedding and $\phi(\{t\} \times N)$ is an open subset of a round $(n-1)$-sphere. If σ^t is its second fundamental form, then

$$\sigma^t = -r^{-2}\langle \cdot, \cdot \rangle (\phi - c), \tag{1}$$

where $r(t)$ and $c(t)$ are the radius and the center of the sphere. But $\sigma^t = \sigma \mid (\{t\} \times N) + \theta^t$, where θ^t is the second fundamental form of the leaf $\{t\} \times N$ in M. So we get

$$-r^{-2}\langle v, w\rangle(\phi - c) = \sigma(v, w) - r^{-2}\langle \nabla_v X, w\rangle X \tag{2}$$

for any vectors v and w tangent to the leaf $\{t\} \times N$. Taking trace in the above expression and using the minimality of M, we obtain

$$(n-1)(\phi - c) = \sigma(X, X) + (n-1)aX \tag{3}$$

for a certain function a on M. Using (3) in (2), comparing tangent and normal components, and taking into account that $v(|X|^2) = 0$ for any vector v tangent to the leaf $\{t\} \times N$, we deduce that

$$\nabla_v X = av \quad \text{and} \quad (n-1)\sigma(v, w) = -r^{-2}\langle v, w\rangle \sigma(X, X) \tag{4}$$

for any vectors v, w tangent to the leaf $\{t\} \times N$.

On the other hand, we consider the 1-form α on $\{t\} \times N$ defined by

$$\alpha(v) = \langle J(\phi - c), v\rangle.$$

Then, since the immersion ϕ is Lagrangian, the differential of α is given by

$$d\alpha(v, w) = 2\langle J\phi_*(v), w\rangle = 0.$$

In addition, the codifferential of α is given by

$$\delta\alpha = \sum_{i=1}^{n-1} \langle J(\phi - c), \sigma^t(e_i, e_i)\rangle,$$

where $\{e_1, \ldots, e_{n-1}\}$ is an orthonormal reference frame on $\{t\} \times N$. Using (1), we have that $\delta\alpha = 0$. So, α is a harmonic 1-form, which can be extended on the whole round sphere. As $n - 1 \geq 2$, we get that $\alpha = 0$. Thus the tangent component of $J(\phi - c)$ is parallel to X. Hence, from (3) we obtain

$$\phi - c = aX + \rho JX \tag{5}$$

and

$$\sigma(X, X) = (n-1)\rho JX, \tag{6}$$

for a certain smooth function ρ on M. As $|X| = r$ we remark that $a^2 + \rho^2 = 1$.

Differentiating equation (5) with respect to any vector $v \in \mathcal{D}$ and using (4) and (6), it is easy to obtain $v(a) = v(\rho) = 0$, and so a and ρ are functions of t. Differentiating again the expression with respect to X, taking tangent and normal components to ϕ, and using elementary properties of ϕ, we reach

$$X - (c')^{\top} = a'X + a\nabla_X X - (n-1)\rho^2 X, \tag{7}$$

$$J(c')^{\perp} = (n-1)a\rho X + \rho' X + \rho\nabla_X X, \tag{8}$$

where $\top$ and $\perp$ stand for tangent and normal components to ϕ and where $'$ denotes the derivative with respect to $X \equiv \partial/\partial t$. From these equations it is easy to derive

$$\rho J(c')^{\perp} - a(c')^{\top} = \nabla_X X - aX.$$

Now, we write $\nabla_X X = hX + W$, where h is the function $h = X(|X|^2)/2r^2 = r'/r$ and W is the projection of $\nabla_X X$ on $\mathcal{D}$. Given any vector field V in $\mathcal{D}$, differentiating the last expression with respect to V yields

$$0 = a(h-a)V - r^{-2}a\langle V, W\rangle X + (\nabla_V W)^{\mathcal{D}},$$

$$0 = \rho(h-a)JV + r^{-2}\rho\langle V, W\rangle JX,$$

where $(\cdot)^{\mathcal{D}}$ stands for the component on the foliation. From the second equation we deduce that $\rho W = 0$ and $\rho(h-a) = 0$. Also, the first equation says that $aW = 0$. As $a^2 + \rho^2 = 1$, we get $W = 0$, and hence the first equation leads to $a(h-a) = 0$. Using again that $a^2 + \rho^2 = 1$, we finally get $\nabla_X X = aX$. Together with (4), this implies that $\nabla_w X = aw$ for any vector w tangent to M, which means that X is a closed and conformal vector field on M. Putting this information in (7) and (8) yields

$$(c')^{\top} = (1 - a' - a^2 + (n-1)\rho^2)X,$$

$$J(c')^{\perp} = (\rho' + na\rho)X.$$

The first member depends on t, so differentiating with respect to any $v \in \mathcal{D}$ and using (4) and the fact that $a^2 + \rho^2 = 1$ yields

$$\rho' + na\rho = 0, \quad 1 - a' - a^2 + (n-1)\rho^2 = 0, \quad \text{and} \quad c' = 0,$$

since div X does not vanish in a dense subset of M. Solving these ODEs we obtain that $c(t)$ is constant and then, up to a translation, we can take $c(t) = 0$. Using that $a^2 + \rho^2 = 1$, we have that a satisfies the equation $a' = n(1 - a^2)$. The trivial solutions $a(t) = \pm 1$ imply that $\rho = 0$, and then the normal component of our immersion ϕ vanishes. In this case, it is not difficult to see that ϕ is a cone over a minimal submanifold of a hypersphere of $\mathbb{C}^n$. As ϕ is foliated by round $(n-1)$-spheres of $\mathbb{C}^n$, the minimal submanifold of the hypersphere must be totally geodesic and so the cone is a hyperplane, which is impossible by the assumptions.

Up to reparameterizations, $a(t) = \tanh(nt)$ is the nontrivial solution; then $\rho(t) = \pm 1/\cosh(nt)$. Equation (5) says that

$$\phi(t,x) = \tanh(nt)\frac{\partial\phi}{\partial t} \pm \frac{1}{\cosh(nt)} J\frac{\partial\phi}{\partial t}. \tag{9}$$

Given $p \in N$, let $\gamma_p\colon (-\delta,\delta) \to \mathbb{C}^n$ be the curve defined by

$$\gamma_p(t) = \phi(t,p).$$

Then, (9) says that γ_p satisfies the ODE

$$\gamma_p(t) = \tanh(nt)\gamma_p'(t) \pm \frac{1}{\cosh(nt)} J\gamma_p'(t),$$

which is equivalent to

$$\gamma_p'(t) = \left(\tanh(nt) \mp \frac{i}{\cosh(nt)}\right)\gamma_p(t).$$

A standard integration for this equation leads to

$$\gamma_p(t) = \cosh^{1/n}(nt)e^{\mp i f(t)}\gamma_p(0),$$

where $f(t) = \frac{2}{n}\arctan\left(\tanh\frac{nt}{2}\right)$. Thus, up to a rotation in the plane where the curve γ_p lies, γ_p is exactly

$$\gamma_p(t) = \cosh^{1/n}(nt)e^{\mp i\beta(t)}\gamma_p(0).$$

(see Proposition 1). Using the definition of γ_p, we get that our immersion ϕ is given by

$$\phi(t,p) = \cosh^{1/n}(nt)e^{\mp i\beta(t)}\phi(0,p).$$

It is clear that the two possible solutions are congruent, so we consider the second one corresponding to the sign $+$.

Now $\phi(0,-)\colon N \to \mathbb{C}^n$ is a Lagrangian embedding of a round $(n-1)$-sphere. We can parameterize it by $\phi(0,x) = (x,0) \in \mathbb{R}^n \times \mathbb{R}^n \equiv \mathbb{C}^n$ and, in this way, we finish the proof. $\square$

Proof of Theorem 2. Because the method used to prove this theorem is very similar to that used for Theorem 1, we will omit some explanations.

First, we recall some elementary properties of complex immersions. For any vectors v, w tangent to M and for any normal vector ξ, it is well known that

$$\sigma(v, Jw) = J\sigma(v,w), \tag{10}$$

$$A_\xi Jv = -A_{J\xi}v = -JA_\xi v. \tag{11}$$

It is clear, following the same reasoning of Theorem 1, that

$$\phi - c = aX + \frac{1}{2n-1}\sigma(X,X) \tag{12}$$

and

$$\nabla_v X = av, \qquad (2n-1)\sigma(v,w) = -r^{-2}\langle v,w\rangle\sigma(X,X) \tag{13}$$

for any vectors v, w tangent to the leaf $\{t\} \times N$, where X is a vector field chosen in the same way as in Theorem 1.

Since the tangent space to M at a point p is complex, we define a complex subspace of T_pM of complex dimension $n-1$ by $H_p = \{v \in \mathcal{D}_p;\ \langle v, JX\rangle = 0\}$. For any vector $v \in H_p$ and any $p \in M$ we then obtain, taking into account (13), that $\sigma(v, Jv) = 0$. So (10) and (13) say that $\sigma(v, v) = 0$, $\sigma(X, X) = 0$, and $\sigma(v, X) = 0$, whence the immersion ϕ is totally geodesic, which is impossible by assumption. We therefore have $H_p = 0$ and thus $n = 1$.

The complex surface is not totally geodesic, so from now on we will work outside of the isolated zeroes of its second fundamental form. Now differentiating (12) with respect to JX, using (10), and comparing tangent and normal components, we obtain that

$$A_{\sigma(X,X)}X = (1-a^2)X, \qquad JX(a) = 0, \tag{14}$$

$$(\nabla\sigma)(X, X, X) = -3a\sigma(X, X). \tag{15}$$

Differentiating again the expression with respect to X, taking tangent and normal components, and using (14) and (15), we have

$$c'^{\top} = (2 - a^2 - a')X - a\nabla_X X, \tag{16}$$

$$c'^{\perp} = 2a\sigma(X, X) - 2\sigma(\nabla_X X, X), \tag{17}$$

where, as before, $\top$ and $\perp$ stand for tangent and normal components to ϕ and $'$ denotes the derivative with respect to $X \equiv \partial/\partial t$. As in the proof of Theorem 1, we write $\nabla_X X = hX + \alpha JX$, where $h = X(|X|^2)/2r^2 = r'/r$. Taking the tangent component of the derivative of (17) with respect to JX yields

$$0 = 2(1-a^2)\{(a-h)JX - \alpha X\},$$

which implies, taking into acount that σ has no zeroes (i.e. $a^2 < 1$), that $h = a$ and $\alpha = 0$. Now (16) and (17) can be written as

$$c'^{\top} = (2(1-a^2) - a')X, \qquad c'^{\perp} = 0.$$

Differentiating again the first expression of the foregoing equation with respect to JX and taking the normal component, we obtain that $a' = 2(1-a^2)$, which implies $c'^{\top} = 0$. Hence the curve $c(t)$ is constant and, up to a translation in $\mathbb{C}^n$, we can take $c = 0$. Also, we have proved that $\nabla_v X = (\log|X|)'v$ for any vector v tangent to M, which means that X is a holomorphic vector field. Now we can choose a complex parameter $z = x + iy$ on our complex surface in such a way that $X \equiv \partial/\partial x$ and the metric is $\langle\cdot,\cdot\rangle = |X|^2|dz|^2$. Equation (12) can therefore be written as

$$\phi = (\log|\phi_x|)_x\phi_x + \sigma(\partial/\partial x, \partial/\partial x) = \phi_{xx}.$$

Since ϕ is a holomorphic or antiholomorphic immersion, this equation can be written as $\phi_{zz} = \phi$ or $\phi_{\bar z\bar z} = \phi$. Integrating this equation, it is very easy to see that, up to congruences and dilations, the solution is $\phi(z) = (e^z, e^{-z})$, which finishes the proof. $\square$

Proof of Theorem 3. It is very easy to see that the compactification by the inversion of the Lagrangian catenoid is Lagrangian.

Conversely, let $\phi\colon M \to \mathbb{R}^{2n}$ be a (proper) minimal immersion with finite total scalar curvature of a complete n-dimensional manifold M. Let

$$\psi = F \circ \phi = \frac{\phi}{|\phi|^2}\colon \bar{M} \to \mathbb{R}^{2n}$$

be its compactification by the inversion, where $\bar{M}$ is a compact manifold such that $M = \bar{M} \setminus \psi^{-1}(0)$. It is easy to see that the induced metric g on M by the immersion ψ is given by

$$g = |\phi|^{-4}\langle \cdot, \cdot \rangle$$

where, as always, $\langle \cdot, \cdot \rangle$ denotes the induced metric by ϕ.

On the other hand, ϕ is also given by $\phi = \psi/|\psi|^2$. Hence, making easy computations (see e.g. [5, Lemma 2.1]) and using that ϕ is a minimal immersion, we obtain

$$d\phi(v) = |\psi|^{-2} d\psi(v) - 2|\psi|^{-4}\langle d\psi(v), \psi\rangle \psi \tag{18}$$

and

$$0 = H + 2|\psi|^{-2}\psi^{\perp}, \tag{19}$$

where H is the mean curvature of ψ and $\perp$ denotes the normal component to ψ.

Let J be the orthogonal complex structure on $\mathbb{R}^{2n}$ such that ψ is Lagrangian with respect to J. Then, by differentiating (19) and using elementary properties of the Lagrangian immersions, we obtain

$$0 = g(\nabla'_v JH, w) + 2g(\nabla'|\psi|^{-2}, v) g(J\psi^{\perp}, w) + 2|\psi|^{-2} g(\nabla'_v J\psi^{\perp}, w),$$

where ∇' is the Levi–Civita connection of the induced metric g. Since the Maslov 1-form is closed, we have that the first term is symmetric. Also, differentiating the expression $\psi = \psi^{\top} + \psi^{\perp}$ and using elemetary properties of the Lagrangian immersions yields that $g(\nabla'_v J\psi^{\perp}, w) = g(\sigma'(v, w), J\psi^{\top})$ and, in particular, that the third term is also symmetric. As a consequence, we obtain that the second term, too, is symmetric, and so

$$\alpha' \wedge \beta' = 0,$$

where α' and β' are the 1-forms on the submanifold M given by

$$\alpha'(v) = \langle d\psi(v), \psi\rangle, \qquad \beta'(v) = \langle d\psi(v), J\psi\rangle$$

for any vector v tangent to M.

Now using again (18) and that ψ is a Lagrangian immersion, we get

$$\langle d\phi(v), Jd\phi(w)\rangle = 2|\psi|^{-6}(\alpha' \wedge \beta')(v, w) = 0,$$

which means that ϕ is also a Lagrangian immersion with respect to J. Moreover, if α and β are the 1-forms on M defined by

$$\alpha(v) = \langle d\phi(v), \phi\rangle, \qquad \beta(v) = \langle d\phi(v), J\phi\rangle$$

then it is easy to see that $\alpha = -|\psi|^{-4}\alpha'$ and $\beta = |\psi|^{-4}\beta'$. So our immersion ϕ not only is Lagrangian but also satisfies the following property:

$$\alpha \wedge \beta = 0. \tag{20}$$

Let $A = \{p \in M;\ \alpha_p = 0\}$ and $B = \{p \in M;\ \beta_p = 0\}$. If $A = M$ then $|\phi|^2$ is constant, which is impossible by the minimality of ϕ. So A is a proper subset of M. Similarly, if $B = M$ then ϕ is tangent to the submanifold and, using a similar reasoning as in the proof of Theorem 1, ϕ is a cone over a minimal submanifold of a hypersphere of $\mathbb{R}^{2n}$, which is not complete unless M was a cone over a totally geodesic submanifold of a hypersphere. In this case M is a Lagrangian subspace, which is impossible by the assumptions. Hence, as $|\phi|$ has no zeroes, we have two disjoint closed subsets A and B of M such that $M \backslash (A \cup B)$ is a nonempty open subset of M. Now, (20) says that on $M \backslash A$ we can write $\beta = f\alpha$ for a certain smooth function f. So, taking $X = \sqrt{1+f^2}\,\phi^\top$, our immersion ϕ is given on $M \backslash A$ by $\phi = aX + bJX$, where a and b are smooth functions on $M \backslash A$ satisfying $a^2 + b^2 = 1$. Making a similar argument for B, we write ϕ on $M \backslash B$ as $\phi = a'X' + b'JX'$ with $a'^2 + b'^2 = 1$. It is clear that, on the nonempty subset $M \backslash (A \cup B)$, we can take $X' = X$, $a' = a$, and $b' = b$. In this way, we have obtained a vector field X, without zeroes and functions a and b defined on the whole M, such that

$$\phi = aX + bJX \quad \text{with } a^2 + b^2 = 1.$$

Let $\pi : \tilde{M} \to M$ be the universal covering of M, and let $\tilde{\phi} = \phi \circ \pi$ be the corresponding immersion in $\mathbb{R}^{2n}$. It is clear that the vector field X can be lifted to a vector field $\tilde{X}$ on $\tilde{M}$ in such a way that

$$\tilde{\phi} = \tilde{a}\tilde{X} + \tilde{b}J\tilde{X}, \tag{21}$$

where $\tilde{a} = a \circ \pi$ and $\tilde{b} = b \circ \pi$.

Making similar reasonings as in the proof of Theorem 1, we obtain that

$$\tilde{\nabla}_w X = \tilde{a}w, \quad \tilde{\sigma}(\tilde{X}, \tilde{X}) = (n-1)\tilde{b}J\tilde{X}, \quad \tilde{\sigma}(\tilde{X}, v) = -\tilde{b}v, \tag{22}$$

and that $v(\tilde{a}) = 0$ and $v(\tilde{b}) = 0$ for any vector w tangent to M and any vector v orthogonal to X, with $\tilde{\sigma}$ the second fundamental form of $\tilde{\phi}$ and $\tilde{\nabla}$ the Levi–Civita connection of the induced metric. Differentiating (21) with respect to $\tilde{X}$, using (22), and comparing tangent and normal components, we obtain

$$1 - \tilde{a}' - \tilde{a}^2 + (n-1)\tilde{b} = 0, \qquad \tilde{b}' + n\tilde{a}\tilde{b} = 0.$$

Thus, we arrive at the same situation as in Theorem 1. Proceeding in a similar way, we obtain

$$\tilde{\phi}(t, p) = \cosh^{1/n}(nt)e^{i\beta(t)}\tilde{\phi}(0, p).$$

Now we will study $\tilde{\phi}_0 : \tilde{N} \to \mathbb{C}^n$ defined by

$$\tilde{\phi}_0(p) = \tilde{\phi}(0, p).$$

First, as $\tilde{a}(t) = \tanh(nt)$, it is easy to get that $|\tilde{X}| = |\partial\tilde{\phi}/\partial t| = \cosh^{1/n}(nt)$. Because (9) is true here, we thus have $\tilde{\phi}_0(p) = \pm J(\partial\tilde{\phi}/\partial t)(0, p)$ and so can deduce that

$$|\tilde{\phi}_0|^2 = 1, \qquad \langle d\tilde{\phi}_0(v), J\tilde{\phi}_0\rangle = 0 \tag{23}$$

for any vector v tangent to $\tilde{N}$. If $\Pi\colon \mathbb{S}^{2n-1}(1) \to \mathbb{CP}^{n-1}$ is the Hopf fibration of the unit sphere over the complex projective space, then (23) says that $\tilde{\phi}_0$ is a horizontal immersion of $\tilde{N}$ in $\mathbb{S}^{2n-1}$, and so $\Pi \circ \tilde{\phi}_0\colon \tilde{N} \to \mathbb{CP}^{n-1}$ is a Lagrangian immersion. Using (22) and the fact that $\operatorname{span}\{\tilde{\phi}, J\tilde{\phi}\} = \operatorname{span}\{\partial\tilde{\phi}/\partial t, J(\partial\tilde{\phi}/\partial t)\}$, it is clear that $\Pi \circ \tilde{\phi}$ is also a minimal immersion.

We can now study the universal covering $\pi : \mathbb{R} \times \tilde{N} \to M$. Let

$$F = (F_1, F_2)\colon \mathbb{R} \times \tilde{N} \to \mathbb{R} \times \tilde{N}$$

be a deck transformation of this covering. Since $\pi(F(t, p)) = \pi((t, p))$, it follows that $\tilde{\phi}(F(t, p)) = \tilde{\phi}((t, p))$ and, in particular, that they have the same length. This means that $\cosh(nt) = \cosh(nF_1(t, p))$ for any $t \in \mathbb{R}$ and $p \in \tilde{N}$, and so $F_1(t, p) = t$. Also, as F is an isometry of the metric $\cosh^{2/n}(nt)(dt^2 + \tilde{g})$, we have that

$$\cosh^{2/n}(nt) = \cosh^{2/n}(nt)(1 + |\partial F_2/\partial t|^2)$$

and so $\partial F_2/\partial t \equiv 0$.

Thus we have proved that our deck transformation F is given by

$$F(t, p) = (t, F_2(p)).$$

In this way, our original Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$ is isometric to $\big(\mathbb{R} \times N, \cosh^{2/n}(nt)(dt^2 + g)\big)$, $\tilde{N}$ is the universal covering of N, ϕ is given by

$$\phi(t, p) = \cosh^{1/n}(nt)e^{i\beta(t)}\phi(0, p),$$

and $\pi \circ \phi_0\colon (N, g) \to \mathbb{CP}^{n-1}$ is a minimal Lagrangian isometric immersion.

To finish the proof we need only show that $f = \pi \circ \phi_0$ is totally geodesic. In order to prove this, we shall use that our Lagrangian submanifold has finite total scalar curvature. In fact, it is easy to see that

$$|\sigma|^2(t, p) = \frac{(n+2)(n-1)}{\cosh^{2(n+1)/n}(nt)} + |\hat{\sigma}|^2(p),$$

where $\hat{\sigma}$ is the second fundamental form of the immersion f. So, as $|\sigma|^2(t, p) = |\sigma|^2(-t, p)$, using the Fubini theorem and that $dV = \cosh(nt)(dt + dV_g)$ we obtain

$$\int_M |\sigma|^n\, dV = 2\int_0^\infty \cosh(nt)h(t)\, dt,$$

where

$$h(t) = \int_N \left(\frac{(n+2)(n-1)}{\cosh^{2(n+1)/n}(nt)} + |\hat{\sigma}|^2(p)\right)^{n/2} dV_g.$$

Since $h(t) \geq 0$ and $\int_M |\sigma|^n\, dV < \infty$, we have $\lim_{t\to\infty} h(t) = 0$. This means that $\int_N |\hat{\sigma}|^n\, dV_g = 0$ and so $|\hat{\sigma}| = 0$. This finishes the proof. $\square$

REMARK 1. The preceding description of the Lagrangian catenoid allows us to show a method of construction of minimal Lagrangian submanifolds in $\mathbb{C}^n$. In fact, given a minimal Lagrangian submanifold

$$\psi\colon N \to \mathbb{CP}^{n-1}$$

of an $(n-1)$-dimensional simply connected manifold N, we define

$$\phi\colon \mathbb{R} \times N \to \mathbb{C}^n$$

by

$$\phi(t,p) = \cosh^{1/n}(nt)e^{i\beta(t)}\tilde{\psi}(p),$$

where $\tilde{\psi}\colon N \to \mathbb{S}^{2n-1}(1)$ is the horizontal lift with respect to the Hopf fibration (which is unique up to rotations of $\mathbb{S}^{2n-1}(1)$) of ψ to $\mathbb{S}^{2n-1}(1)$. Then it is easy to check that ϕ is a minimal Lagrangian immersion of $\mathbb{R} \times N$ into $\mathbb{C}^n$.

This family of examples has a common property: any of these submanifolds admits a closed and conformal vector field X satisfying $\sigma(X,X) = \rho JX$ for a certain smooth function ρ. Following ideas developed in [11], it can be proved that this property characterizes the foregoing family of examples.

References

[1] M. T. Anderson, *The compactification of a minimal submanifold in Euclidean space by the Gauss map,* preprint.

[2] A. Borisenko, *Ruled special Lagrangian surfaces,* Adv. Soviet Math., 15, pp. 269–285, Amer. Math. Soc., Providence, RI.

[3] B. Y. Chen and J. M. Morvan, *Geometrie des surfaces lagrangiennes de* $\mathbb{C}^2$, J. Math. Pures Appl. (9) 66 (1987), 321–325.

[4] A. Enneper, *Die cyklischen Flächen*, Z. Math. Phys. 14 (1869), 393–421.

[5] K. Enomoto, *Compactification of submanifolds in Euclidean space by the inversion*, Adv. Stud. Pure Math., 22, pp. 1–11, Academic Press, Boston, 1993.

[6] R. Harvey and H. B. Lawson, *Calibrated geometries*, Acta Math. 148 (1982), 47–157.

[7] D. A. Hoffman and R. Osserman, *The geometry of the generalized Gauss map,* Mem. Amer. Math. Soc. 236 (1980).

[8] W. C. Jagy, *Minimal hypersurfaces foliated by spheres*, Michigan Math. J. 38 (1991), 255–270.

[9] G. Lawlor, *The angle criterion*, Invent. Math. 95 (1989), 437–446.

[10] H. Moore, *Minimal submanifolds with finite total scalar curvature,* Indiana Univ. Math. J. 45 (1996), 1021–1043.

[11] A. Ros and F. Urbano, *Lagrangian submanifolds of* $\mathbb{C}^n$ *with conformal Maslov form and the Whitney sphere,* J. Math. Soc. Japan 50 (1998), 203–226.

I. Castro
Departamento de Matemáticas
EPS
Universidad de Jaén
23071 Jaén
Spain

icastro@ujaen.es

F. Urbano
Departamento de Geometría
y Topología
Universidad de Granada
18071 Granada
Spain

furbano@goliat.ugr.es