Invertibility Preserving Maps Preserve Idempotents

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Introduction and Statement of Main Results

Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras. A linear map $\phi\colon \mathcal{A} \to \mathcal{B}$ is called *unital* if $\phi(1) = 1$ and is called *invertibility preserving* if $\phi(a)$ is invertible in $\mathcal{B}$ for every invertible element $a \in \mathcal{A}$. Similarly, ϕ preserves idempotents if $\phi(p)$ is an idempotent whenever $p \in \mathcal{A}$ is an idempotent; it is called a *Jordan homomorphism* if $\phi(a^2) = (\phi(a))^2$ for every $a \in \mathcal{A}$.

In [14, Sec. 9] Kaplansky asked: When must unital surjective linear invertibility preserving maps be Jordan homomorphisms? This problem was motivated by the famous Gleason–Kahane–Żelazko theorem [9; 13; 17], which states that every unital invertibility preserving linear map from a Banach algebra to a semisimple commutative Banach algebra is multiplicative, as well as by results of Dieudonné [8] and Marcus and Purves [15] stating that every unital invertibility or singularity preserving linear map on a matrix algebra is either multiplicative or antimultiplicative. The case of a nonunital invertibility preserving mapping can be reduced to the unital case by considering θ defined by $\theta(a) = \phi(1)^{-1}\phi(a)$.

The answer to Kaplansky's question is not always affirmative. Some historical remarks on this problem can be found in [1, pp. 27–31], where the first noncommutative extensions of the Gleason–Kahane–Żelazko theorem were mentioned. Having in mind all known results and counterexamples, it is tempting to conjecture that the answer to Kaplansky's question is affirmative if $\mathcal{A}$ and $\mathcal{B}$ are semisimple Banach algebras [3; 10; 16].

Let X be a Banach space, and let $\mathcal{B}(X)$ be the algebra of all bounded linear operators on X. By $\mathcal{F}(X)$ we denote the ideal of all finite rank operators. For every $x \in X$ and every bounded linear functional f on X, we denote by $x \otimes f$ a rank-1 operator defined by $(x \otimes f)y = f(y)x$. Following Chernoff [7], we call a subalgebra $\mathcal{A} \subset \mathcal{B}(X)$ a *unital standard operator algebra* on X if it is closed and contains I and $\mathcal{F}(X)$. We will prove that the problem of characterizing linear invertibility preserving mappings can be reduced to the problem of characterizing linear maps preserving idempotents if the codomain is a unital standard operator algebra.

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THEOREM 1. *Let $\mathcal{A}$ be a unital Banach algebra and let $\mathcal{B}$ be a unital standard operator algebra on a Banach space X. Assume that $\phi\colon \mathcal{A} \to \mathcal{B}$ is a unital surjective linear mapping preserving invertibility. Then ϕ preserves idempotents.*

One must be careful when considering the invertibility preserving assumption in Theorem 1. Namely, it is possible that $B \in \mathcal{B} \subset \mathcal{B}(X)$ is an invertible bounded linear operator on X but is not invertible in the algebra $\mathcal{B}$.

In [16], Sourour proved that if H is a separable Hilbert space and Y a Banach space then every unital surjective invertibility preserving linear mapping $\phi\colon \mathcal{B}(H) \to \mathcal{B}(Y)$ is a Jordan homomorphism. Using Theorem 1, we will not only extend this result but also provide a considerably shorter proof.

COROLLARY 1. *Let $\mathcal{A}$ be a von Neumann algebra and $\mathcal{B}$ a unital standard operator algebra on a Banach space X. Assume that $\phi\colon \mathcal{A} \to \mathcal{B}$ is a unital surjective linear mapping preserving invertibility. Then ϕ is a Jordan homomorphism.*

Using Theorem 1, we can also obtain a new proof of the following result of Sourour, which is the main object of [16] (see also [12]).

COROLLARY 2 [16]. *Let X and Y be Banach spaces and let $\phi\colon \mathcal{B}(X) \to \mathcal{B}(Y)$ be a unital bijective linear mapping preserving invertibility. Then ϕ is either an isomorphism or an anti-isomorphism.*

Proofs

For the proof of Theorem 1 we will need several lemmas. The first one is an immediate consequence of [4, Lemma 7].

LEMMA 1. *Let $\mathcal{A}$ be a unital Banach algebra and $p \in \mathcal{A}$ a nonzero idempotent. Then there exists a norm $\|\cdot\|_p$ on $\mathcal{A}$ such that $(\mathcal{A}, \|\cdot\|_p)$ is a Banach algebra with $\|p\|_p = 1$.*

In the next three lemmas we will assume that $\mathcal{A}$ and $\mathcal{B}$ are unital Banach algebras and that $\phi\colon \mathcal{A} \to \mathcal{B}$ is a surjective unital linear mapping preserving invertibility. Therefore, $\sigma(\phi(a)) \subset \sigma(a)$ for every $a \in \mathcal{A}$, where $\sigma(a)$ denotes the spectrum of a. In each of these three lemmas we will also assume that $e \in \mathcal{A}$ is an idempotent and that $\phi(e) = E + Q$ with $E, Q \in \mathcal{B}$ satisfying $E^2 = E$, $EQ = QE$, and $\sigma(Q) = \{0\}$.

LEMMA 2. *Suppose that $A \in \mathcal{B}$ satisfies $A^2 = A(E+Q)A = 0$. Then*

$$\sigma(Q^2 + 2EQ - Q + A(E+Q) + (E+Q)A - A) = \{0\}.$$

Proof. Because of the surjectivity of ϕ we can find $a \in \mathcal{A}$ such that $\phi(a) = A$. For an arbitrary $\alpha \in \mathbf{C}$ we have

$$\begin{aligned}&\sigma(\alpha(ea + ae - a) + \alpha^2 a^2)\\&\qquad = \sigma((e + \alpha a)^2 - (e + \alpha a))\end{aligned}$$

$$
\begin{aligned}
&= \{\,\lambda^2-\lambda : \lambda \in \sigma(e+\alpha a)\,\} \supset \{\,\lambda^2-\lambda : \lambda\in\sigma(E+Q+\alpha A)\,\}\\
&= \sigma((E+Q+\alpha A)^2-(E+Q+\alpha A))\\
&= \sigma(Q^2+2EQ-Q+\alpha[A(E+Q)+(E+Q)A-A]).
\end{aligned}
$$

It follows that $r(B_1+\alpha B_2)\le r(\alpha a_1+\alpha^2 a_2)$, where r denotes the spectral radius and $B_1=Q^2+2EQ-Q$, $B_2=A(E+Q)+(E+Q)A-A$, $a_1=ea+ae-a$, and $a_2=a^2$.

For every complex number λ satisfying $|\lambda|\le 1$ we have

$$r(\lambda B_1+B_2)\le \|B_1\|+\|B_2\|.$$

If $|\lambda|>1$ then $r(\lambda B_1+B_2)=|\lambda| r(B_1+(1/\lambda)B_2)\le|\lambda| r((1/\lambda)a_1+(1/\lambda)^2a_2)= r(a_1+(1/\lambda)a_2)\le\|a_1\|+\|a_2\|$. Hence the function $\lambda\mapsto r(\lambda B_1+B_2)$ is bounded. Because $\lambda\mapsto r(\lambda B_1+B_2)$ is a subharmonic function [2, p. 52], it follows by the Liouville theorem for subharmonic functions that it is constant. This yields together with $B_2^3=0$ that $r(\lambda B_1+B_2)=r(B_2)=0$ for every complex λ. In particular, $r(B_1+B_2)=0$, which is the desired conclusion. □

Let us mention that a similar subharmonicity argument has also proved to be useful in the study of linear mappings preserving the spectral radius [6].

LEMMA 3. *Let $\mathcal{B}$ be a unital standard operator algebra on a Banach space X. Assume further that $x\in X$ satisfies $Ex=x$. Then $Q^3x=0$.*

Proof. Assume on the contrary that there exists $x\in X$ such that $Ex=x$ and $Q^3x\ne 0$. Since Q is a quasinilpotent, the vectors x, Qx, Q^2x, Q^3x are linearly independent. Indeed, if this were not true then the linear span of x, Qx, Q^2x, Q^3x would be invariant for Q. Then the restriction of Q to this subspace would be nilpotent, which would yield $Q^3x=0$—a contradiction. Thus, we can find a bounded linear functional f on X such that $f(Qx)=1$ and $f(x)=f(x-Qx-Q^2x)=f(x-2Q^2x-Q^3x)=0$. Obviously, $A=(x-Qx-Q^2x)\otimes f\in\mathcal{F}(X)\subset\mathcal{B}$ satisfies $A^2=0$. Applying $Ex=x$ and $EQ=QE$ one gets that $EQ^jx=Q^jx$ for $j=1,2,\dots$. It follows easily that $A(E+Q)A=0$. Hence, by Lemma 2 we have

$$\sigma(Q^2+2EQ-Q+A(E+Q)+(E+Q)A-A)=\{0\}.$$

Note that here σ denotes the spectrum of an operator with respect to $\mathcal{B}$ (which in general is not the same as the spectrum with respect to the whole algebra $\mathcal{B}(X)$). On the other hand, a straightforward computation gives

$$(Q^2+2EQ-Q+A(E+Q)+(E+Q)A-A)x=x,$$

which obviously yields

$$1\in\sigma(Q^2+2EQ-Q+A(E+Q)+(E+Q)A-A).$$

This contradiction completes the proof. □

LEMMA 4. *Let $\mathcal{B}$ be unital standard operator algebra on a Banach space X. Suppose that $\|e\|=1$ and that $x\in X$ satisfies $Ex=x$ and $Qx\ne 0$. Then $\{\,Q^nx : n=0,1,2,\dots\,\}$ is a linearly independent set.*

Proof. Assume on the contrary that $\{ Q^n x : n = 0, 1, 2, \ldots \}$ is a linearly dependent set. Since Q is a quasinilpotent, there is an integer $n > 1$ such that $Q^n x = 0$ and the set $\{x, Qx, \ldots, Q^{n-1}x\}$ is linearly independent. Because $\mathcal{B}$ contains all finite rank operators in $\mathcal{B}(X)$, we can find $A \in \mathcal{B}$ such that $AQx = x$ and $Ax = AQ^2 x = \cdots = AQ^{n-1}x = 0$. Choose $a \in \mathcal{A}$ such that $\phi(a) = A$. For every $\lambda \in \mathbf{C}$ we have

$$\begin{aligned}\sigma(A + \lambda^2(E + Q)) &\subset \sigma(a + \lambda^2 e) \\ &\subset D(0, \|a\| + |\lambda|^2).\end{aligned}$$

Here, $D(0, t)$ denotes the closed disk in the complex plane of radius t centered at the origin.

On the other hand,

$$\begin{aligned}&(A + \lambda^2(E + Q))(x + \lambda Qx + \cdots + \lambda^{n-1}Q^{n-1}x) \\ &\qquad = (\lambda^2 + \lambda)(x + \lambda Qx + \cdots + \lambda^{n-1}Q^{n-1}x),\end{aligned}$$

which yields that $\lambda^2 + \lambda \in \sigma(A + \lambda^2(E + Q))$ for every complex number λ. Hence, for every $\lambda \in \mathbf{C}$ we have $|\lambda^2 + \lambda| \le \|a\| + |\lambda|^2$. This contradiction completes the proof. □

Note that Lemma 4 is true not only for unital standard algebras but for all dense algebras of linear operators.

We are now in a position to prove our main result.

Proof of Theorem 1. Since ϕ is a unital linear mapping preserving invertibility, we have $\sigma(\phi(a)) \subset \sigma(a)$ for every $a \in \mathcal{A}$. Let e be any nonzero idempotent in $\mathcal{A}$. Because of Lemma 1, there is no loss of generality in assuming that $\|e\| = 1$. We know that $\sigma(\phi(e)) \subset \{0, 1\}$. Hence, by [5, p. 36] we have $\phi(e) = E + Q$ with $E^2 = E$, $EQ = QE$, and $\sigma(Q) = \{0\}$. We have to prove that $Q = 0$. Assume on the contrary that $Q \neq 0$. Then we can assume with no loss of generality that $EQ \neq 0$, since otherwise we would consider $1 - e$ instead of e. Hence, there exists $y \in X$ such that $EQy \neq 0$. Set $x = Ey$. Then $Ex = x$ and $Qx = QEx = QEy = EQy \neq 0$. It is now easy to obtain a contradiction by applying Lemmas 3 and 4. □

Proof of Corollary 1. Let $P_1, P_2 \in \mathcal{A}$ be orthogonal Hermitian idempotents. Since $P_1 + P_2$ is a projection, we have $(\phi(P_1) + \phi(P_2))^2 = \phi(P_1) + \phi(P_2)$. This yields $\phi(P_1)\phi(P_2) + \phi(P_2)\phi(P_1) = 0$. It follows that if $H \in \mathcal{A}$ is of the form $H = \sum_{j=1}^n t_j P_j$, where $t_j \in \mathbf{R}$ and P_j are Hermitian idempotents such that $P_i P_j = 0$ if $i \neq j$, then $\phi(H^2) = \phi(H)^2$. By [2, Thm. 5.5.2], ϕ is continuous. The set of all Hermitian elements that can be represented as finite real-linear combinations of mutually orthogonal projections is dense in the set of all Hermitian elements in $\mathcal{A}$. Therefore, we have $\phi(H^2) = (\phi(H))^2$ for every Hermitian element H in $\mathcal{A}$. Now, replacing H by $H + K$ where H and K are both Hermitian, we get $\phi(HK + KH) = \phi(H)\phi(K) + \phi(K)\phi(H)$. Since an arbitrary $A \in \mathcal{A}$ can be written in the form $A = H + iK$ with H, K Hermitian, the last two relations imply that $\phi(A^2) = (\phi(A))^2$. $\square$

Proof of Corollary 2. We will prove that ϕ maps every operator of rank 1 into an operator of rank 1. Every operator of rank 1 is either a scalar multiple of an idempotent or a square-zero operator. Assume first that $P \in \mathcal{B}(X)$ is an idempotent of rank 1 and that its image has rank greater than 1. By Theorem 1, $\phi(P)$ is an idempotent. Therefore, it has the following matrix representation:

$$\phi(P) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & Q \end{bmatrix},$$

where Q is an idempotent. Define $B \in \mathcal{B}(Y)$ by

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and find $A \in \mathcal{B}(X)$ such that $\phi(A) = B$. We can also find a complex number λ such that $\lambda + A$ is invertible. The operator $\lambda + A + \alpha P$ is invertible if and only if $I + \alpha(\lambda + A)^{-1}P$ is invertible. As $(\lambda + A)^{-1}P$ has rank 1, this operator is invertible for all but at most one complex number α. But $\phi(\lambda + A + \alpha P) = \lambda + B + \alpha\phi(P)$ is singular for $\alpha = -\lambda$ and $\alpha = -\lambda - 1$. This contradiction shows that $\phi(P)$ must be of rank 1.

Almost the same argument as in the proof of Corollary 1 shows that the restriction of ϕ to $\mathcal{F}(X)$ is a Jordan homomorphism. Since $\mathcal{F}(X)$ is a locally matrix algebra, a result of Jacobson and Rickart [11, Thm. 8] tells us that $\phi|_{\mathcal{F}(X)} = \varphi + \theta$, where $\varphi\colon \mathcal{F}(X) \to \mathcal{B}(Y)$ is a homomorphism and $\theta\colon \mathcal{F}(X) \to \mathcal{B}(Y)$ is an antihomomorphism. Pick an idempotent $P \in \mathcal{B}(X)$ of rank 1. Then $\phi(P)$ is the sum of idempotents $\varphi(P)$ and $\theta(P)$; therefore, as $\phi(P)$ also has rank 1, it follows that either $\varphi(P) = 0$ or $\theta(P) = 0$. Thus, at least one of φ and θ has a nonzero kernel. Since the kernels of homomorphisms and antihomomorphisms are ideals, and since the only nonzero ideal of $\mathcal{F}(X)$ is $\mathcal{F}(X)$ itself, we have $\varphi = 0$ or $\theta = 0$. Thus, the restriction of ϕ to $\mathcal{F}(X)$ is either a homomorphism or an antihomomorphism.

Take now an arbitrary square-zero operator N of rank 1. Then one can find an idempotent P of rank 1 such that $N = PN$. If the restriction of ϕ to $\mathcal{F}(X)$ is a homomorphism then $\phi(N) = \phi(P)\phi(N)$ and hence $\operatorname{rank}\phi(N) = 1$. Similarly we prove that $\phi(N)$ has rank 1 in the case that the restriction of ϕ to $\mathcal{F}(X)$ is an antihomomorphism.

Sourour's proof of Corollary 2 has two steps. The first and more difficult one shows that ϕ maps operators of rank 1 into operators of rank 1. In the second step this fact is used to conclude that ϕ is a Jordan isomorphism. Hence, we have obtained a new proof of the first step of Sourour's proof. To complete the proof of Corollary 2 we can now follow the second step of his proof, which can even be slightly shortened using our observation that the restriction of ϕ to $\mathcal{F}(X)$ is either a homomorphism or an antihomomorphism. We will omit the details since the main idea remains unchanged.

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