Immiscible Fluid Clusters in $\mathbf{R}^2$ and $\mathbf{R}^3$

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1. Introduction

Immiscible fluids $F_1, \dots, F_m$ in $\mathbf{R}^n$ (with ambient F_0) have an energy proportional to surface area, where the constant of proportionality $a_{ij} > 0$ depends on which fluids the surface separates. To prevent degeneracies, we assume strict triangle inequalities $a_{ik} < a_{ij} + a_{jk}$. B.White [W1; W4, Sec. 11] has announced that energy-minimizing clusters of prescribed volumes are smooth surfaces of constant mean curvature that meet along a singular set of Hausdorff dimension at most $n-2$. In $\mathbf{R}^2$ it would follow that an energy-minimizing cluster consists of arcs of circles meeting at isolated points. Our Regularity Theorem 4.3 gives a simple proof special to $\mathbf{R}^2$.

The special case of planar soap-bubble clusters ($a_{ij} = 1$) was treated in [M2]. That simple analysis generalizes immediately to the case of $m = 2$ immiscible fluids and to the case where each $a_{ij} \approx 1$.

1.1. The Proof of Regularity Theorem 4.3. Proposition 4.2 shows that if C is, in a small ball $\mathbf{B}(a, r)$, weakly close to a diameter separating (say) fluid F_1 from $F_0 = 0$, then C is a circular arc in a shrunken ball $\mathbf{B}(a, 0.9r)$. Its proof first uses projection in the space of coefficients to replace C inside $\mathbf{B}(a, r)$ with a cluster C' whose coefficients are all real multiples of F_1. It follows from the strict triangle inequality that this reduces cost at least $\varepsilon|C - C'|$. A circular arc is even cheaper. Second, lost amounts of other fluids, on the order of $|C - C'|^2$ by the isoperimetric inequality, may be restored elsewhere at cost $K|C - C'|^2$. At a small scale, $K|C - C'|^2 < \varepsilon|C - C'|$ and the circular arc is better.

To deduce Regularity Theorem 4.3, note that in a small ball about any point, C is weakly close to a tangent cone, which must consist of finitely many rays. By the previous Proposition 4.2, C consists of nearly radial circular arcs that meet at the point.

1.2. Clusters in $\mathbf{R}^3$. Section 5 generalizes Taylor's classification of soap-bubble cluster singularities to clusters of immiscible fluids with interface energies near unity.

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2. Norms

In Section 3 we will use Proposition 2.1 to define a norm on possible interface types and thus to define the energy of clusters. A similar proposition appears in [W4, Sec. 7].

2.1. PROPOSITION. *Consider* $\mathbf{R}^m$ *with basis* $\{ F_i : 1 \le i \le m \}$, *and let* $F_0 = 0$. *For* $i \neq j$, *let* $a_{ij} = a_{ji}$ *be positive constants satisfying the triangle inequalities*

$$a_{ik} \le a_{ij} + a_{jk}. \tag{1}$$

Define the norm on $\mathbf{R}^m$ *as*

$$\|g\| = \min\Big\{ \sum a_{ij}|\gamma_{ij}| : g = \sum \gamma_{ij}(F_i - F_j),\ 0 \le i, j \le m,\ \gamma_{ij} \in \mathbf{R} \Big\}.$$

(*We may assume* $\gamma_{ij} \ge 0$.) *Then*

$$\|F_i - F_j\| = a_{ij}. \tag{2}$$

If strict inequality holds in (1) *for fixed* i *and* k *for all* j, *then there exist a linear functional* L *on* $\mathbf{R}^m$ *with* $L(F_i - F_k) = \|F_i - F_k\|$ *and an* $\varepsilon > 0$ *such that, in terms of the associated projection* π *defined by*

$$\pi g = g - L(g)(F_i - F_k)/\|F_i - F_k\|,$$

we have

$$L(g) \le \|g\| - \varepsilon\|\pi g\|. \tag{3}$$

Proof. Of course $\|F_0 - F_1\| \le a_{01}$. A decomposition $F_0 - F_1 = \sum \beta_{ij}(F_i - F_j)$ may be interpreted as a flow on the complete graph with vertices F_i from F_0 to F_1, which may be decomposed as a sum of flows along paths, each of which has (by the triangle inequality) cost at least a_{01}. Therefore $\|F_1 - F_0\| \ge a_{01}$. In general, $\|F_i - F_j\| = a_{ij}$. (Alternatively, one may use an algebraic proof by induction as in [W4, Sec. 7].)

The norm is just the largest norm satisfying $\|F_i - F_j\| \le a_{ij}$, with unit ball the polyhedral convex hull of $\{(F_i - F_j)/a_{ij}\}$. Equation (2) states that each $(F_i - F_j)/a_{ij}$ lies on the boundary. Hence, if strict inequality holds in (1) when $\{i, k\} = \{0, 1\}$, then F_1/a_{10} lies outside the convex hull of the rest and there is a supporting hyperplane $L(g) = 1$ away from which the unit ball falls at a positive rate. Inequality (3) follows. □

REMARK. If strict inequality holds in (1), the norm may be made uniformly convex as follows. Let $c > 0$ so that

$$a'_{ij} = \begin{cases} a_{0j} - c & \text{if } 0 = i < j, \\ a_{ij} - c\sqrt{2} & \text{if } 0 < i < j, \end{cases}$$

still satisfy

$$a'_{ik} < a'_{ij} + a'_{jk}. \tag{4}$$

As in the proposition, define an associated norm $\|g\|'$. Finally, let

$$\|g\| = \|g\|' + c|g|.$$

Conditions (2) and (3) still hold.

3. Immiscible Fluid Clusters

This section parallels White's [W4] treatment of fluid clusters as chains with coefficients representing the various fluids.

3.1. DEFINITIONS. As a *cluster* C of m immiscible fluids F_i in $\mathbf{R}^n$, we wish to consider the perimeter (exterior boundary and interfaces) of m disjoint regions R_i ($1 \leq i \leq m$) of finite perimeter and prescribed volume $0 \leq V_i < \infty$, with complement R_0. The *energy* of an interface between distinct R_i and R_j is a given positive constant $a_{ij} = a_{ji}$ times its area. We assume triangle inequalities $a_{ik} \leq a_{ij} + a_{jk}$; otherwise, an interface between fluids F_i and F_k could be replaced by an infinitesimal layer of F_j with the same effect.

Technically we work in the space $\mathcal{F}_{n-1}(\mathbf{R}^n, G)$ of real flat chains with coefficients in the free abelian group G with generators $\{ F_i : 1 \leq i \leq m \}$ contained in $\mathbf{R} \otimes G$, with norm as in Proposition 2.1. (For convenience put $F_0 = 0$.) Then $C = \partial R$, where

$$R = \sum F_i \otimes R_i.$$

An interface from R_i to R_j has coefficient $F_i - F_j$, and the energy of C is given by the mass norm derived from the norm on G.

More generally we consider rectifiable chains C with coefficients in G, in competition with such of the form $C + \partial W$ with $\int_W dV = 0$ (the generalized volume constraints). Away from their boundaries, such chains locally bound regions with coefficients $\sum \gamma_i F_i$ ($\gamma_i \in \mathbf{Z}$) rather than merely F_i.

The spaces $\mathcal{F}_k(\mathbf{R}^n, G)$ and $\mathcal{F}_k(\mathbf{R}^n, \mathbf{R} \otimes G)$ are variations on the classical $G = \mathbf{Z}$ cases of geometric measure theory [M3] generalized to any normed abelian group G by Fleming [Fl], with further recent improvements by White [W2; W3; W4]. (In [W3] it is shown for which groups the flat chains of finite mass are rectifiable, as is well known to hold for our case of boundaries of regions of finite perimeter; see [M2, Lemma 2.1] or [Fe, 4.5.12, 2.10.6]).

3.2. MONOTONICITY. Alternatively, a $C \in \mathcal{F}_{n-1}(\mathbf{R}^n, G)$ may be viewed as a rectifiable varifold with multiplicities a_{ij}. If C is energy-minimizing, then away from ∂C the varifold has mean curvature weakly bounded by some $M > 0$, by a lemma of Almgren (see [M3, Lemma 13.5]; see also [M1, Sec. 3] and Proposition 4.2 herein). Hence, for any $a \in \operatorname{spt} C$,

$$\mathbf{M}(C \llcorner \mathbf{B}(a, r))e^{Mr} \tag{1}$$

is a monotonically nondecreasing function of r [A, 5.1(3), p. 446]. It follows that any minimizer must have compact support (assuming compactness for the given boundary, if any). It also follows by a compactness argument that a minimizer has at each interior point at least one oriented tangent cone T. The truncated cone $T \llcorner \mathbf{B}(0, 1)$ is minimizing (without area constraint) among chains with coefficients in $G_1 = G \cap V$, where V is the real vectorspace spanned by the coefficients occurring in C. The proof, a standard limit argument, considers $C + \partial W$ and requires small adjustments v in $\int_W dV$, which by a lemma of Almgren (see

[M3, Lemma 13.5]) may be accomplished at cost at most $K\|v\|$, provided that $v \in V$.

3.3. EXISTENCE. For a nice compact domain $D \subset \mathbf{R}^n$, the existence of an energy-minimizing cluster with prescribed volumes and boundary in $\mathcal{F}_{n-1}(D, G)$ follows by compactness [Fl, Cor. 7.5] and the lower semicontinuity of energy [Fl, Thm. 2.3, Sec. 3]. In $\mathbf{R}^2$ (our main concern), the reduction to the case of a compact domain $D \subset \mathbf{R}^2$ is an easy argument. Indeed, any cluster of perimeter p is contained in disjoint balls of radius r_i with $\sum r_i \leq p$. Those outside D may be translated inside a single ball of radius p. In $\mathbf{R}^n$, the existence and compactness of an energy-minimizing cluster in $\mathcal{F}_{n-1}(\mathbf{R}^n, G)$ requires the arguments of [M1, Sec. 4].

4. Regularity

The main Regularity Theorem 4.5 states that energy-minimizing clusters in $\mathbf{R}^2$ consist of circular arcs meeting at finitely many points. First Lemma 4.1 shows that classical minimizers (such as small pieces of hypersurfaces with constant mean curvature in $\mathbf{R}^n$) remain minimizing in comparison with very general surfaces (perhaps with bubbles of other fluids) enclosing the same volume.

4.1. LEMMA. *Suppose interface costs* $a_{ij} = a_{ji} > 0$ $(i \neq j)$ *per unit area in* $\mathbf{R}^n$ *satisfy the triangle inequalities*

$$a_{ik} \leq a_{ij} + a_{jk},$$

with strict inequality when $\{i, k\} = \{0, 1\}$. *By Proposition 2.1, there exist a linear functional* L *on* G *with* $L(F_1) = \|F_1\|$ *and an* $\varepsilon > 0$ *such that, if*

$$\pi(g) = g - L(g)F_1/\|F_1\|,$$

then

$$|L(g)| \leq \|g\| - \varepsilon\|\pi g\|. \tag{1}$$

Suppose C_1 *is the portion of a hyperspace having constant mean curvature inside a small ball* $\mathbf{B}(a, r) \subset \mathbf{R}^n$; *or, more generally, suppose* C_1 *minimizes area among* $(n-1)$*-dimensional real rectifiable currents* $C_1 + \partial W_1$ *in a ball* $\mathbf{B}(a, r) \subset \mathbf{R}^n$ *with* $\int_{W_1} dA = 0$. *Consider* $C = F_1 \otimes C_1$ *and* $C' = C + \partial W$ *to be rectifiable currents in* $\mathbf{B}(a, r)$ *with coefficients in* $R \otimes G$, *with*

$$L\left(\int_W dV\right) = 0. \tag{2}$$

Then

$$\mathbf{M}(C) \leq \mathbf{M}(C') - \varepsilon\mathbf{M}(\pi C').$$

Proof. A hypersurface of constant mean curvature is locally minimizing as claimed among real currents by a calibration argument [M3, Rmk., p. 76]. In particular, $\mathbf{M}(\|F_1\|C_1) \leq \mathbf{M}(L(C'))$. Therefore

$$\mathbf{M}(C) = \mathbf{M}(\|F_1\|C_1) \leq \mathbf{M}(L(C')) \leq \mathbf{M}(C') - \varepsilon\mathbf{M}(\pi C')$$

by (1). □

The following propositon contains the main regularity argument that a minimizing cluster weakly close (in a small ball) to a straight line is (in a smaller ball) a circular arc.

4.2. Proposition. *Suppose interface energies $a_{ij} = a_{ji} > 0$ $(i \neq j)$ per unit length in $\mathbf{R}^2$ satisfy the triangle inequalities*

$$a_{ik} \leq a_{ij} + a_{jk}, \tag{1}$$

with strict inequality when $\{i, k\} = \{0, 1\}$. Let C be an energy-minimizing planar cluster of prescribed areas $A_i \geq 0$ or a minimizer among rectifiable chains with coefficients in G of the form $C + \partial W$ with $\int_W dA = 0$. Suppose that, in a small ball $\mathbf{B}(a, r)$ and away from ∂C, C is weakly close to a diameter with coefficient F_1. Then, in a shrunken ball $\mathbf{B}(a, 0.9r)$, C is a circular arc (or straight line segment) with coefficient F_1. At all such points, the arcs have the same curvature.

Proof. Real linear combinations of the coefficients that occur in C away from ∂C constitute a vector subspace $V \subset \mathbf{R} \otimes G$. By a lemma of Almgren (see [M3, Lemma 13.5]), there exist $K_1, \delta_2 > 0$ such that, outside any ball of radius less than δ_2, arbitrary small adjustments v in $\int_W dA$ contained in V can be made at cost at most

$$K_1\|v\|. \tag{2}$$

By Proposition 2.1, there exist a linear functional L on $R \otimes G$ with $L(F_1) = \|F_1\|$ and $\varepsilon > 0$ such that, if

$$\pi(g) = g - L(g)F_1/\|F_1\|,$$

then

$$L(g) \leq \|g\| - \varepsilon\|\pi g\|. \tag{3}$$

Note that $\ker \pi = \{\text{multiples of } F_1\}$. We may assume that $L(F_i)/\|F_1\|$ is rational and hence πG is discrete. For any 2-dimensional flat chain X with coefficients in πG, by the isoperimetric inequality ([Fl, 7.6], which follows from the standard isoperimetric inequality [M3, 5.3]),

$$\mathbf{M}(X) \leq K_2\mathbf{M}(\partial X)^2. \tag{4}$$

For a nontrivial decomposition $F_1 = \sum g_i$ with $g_i \in G_1$,

$$\|F_1\| = L(F_1) = \sum L(g_i) \leq \sum \|g_i\| - \varepsilon \sum \|\pi g_i\|$$

by (3), so that

$$(1 + \delta_1)\|F_1\| \leq \sum \|g_i\|, \tag{5}$$

where $\delta_1 = \min\{\|\pi g\| > 0\}/\varepsilon\|F_1\|$. Let $K_3 > 0$ such that, for all $g \in G$,

$$\|\pi g\| \leq C_3\|g\|/6. \tag{6}$$

Let

$$\delta = \min\{0.04\delta_1, \delta_2, \varepsilon/K_1K_2K_3, 0.1\}. \tag{7}$$

Now for convenience we assume that $a = 0$ and that the diameter is the x-axis. Inside $\mathbf{B}(a, 0.95r)$, C projects onto the x-axis with coefficient F_1 and—by monotonicity (3.2), for example—lies inside the cylinder $\{|y| \leq \delta r\}$. Also, for $s \leq 0.95r$ we have

$$2\|F_1\|s \leq \mathbf{M}(S \llcorner \mathbf{B}(a, s)) \leq (2\|F_1\| + \delta)s.$$

By slicing theory, for some $0.9r < s < 0.95r$, the circle $\mathbf{S}(a, s)$ slices C in finitely many points with coefficients in G and total mass at most

$$\left(2 + \frac{\delta}{0.05}\right)\|F_1\| < (2 + \delta_1)\|F_1\|.$$

By (5), the slice is just two points with coefficient $\pm F_1$.

To construct a comparison cluster inside $\{ (x, y) \in \mathbf{B}(0, r) : |x| \leq s \}$, replace C by $C + \partial W_1$, the circular arc with coefficient F_1, so that $L\left(\int_{W_1} dA\right) = 0$. (If we used a straight line then $\int_{W_1} dA$ would be small because C is weakly close to a straight line. By using a slightly curved circular arc, we can make $L\left(\int_{W_1} dA\right) = 0$.) By Lemma 4.1, this reduces cost by at least

$$\varepsilon\mathbf{M}(\pi C \llcorner \mathbf{B}(0, s)) = \varepsilon\mathbf{M}(\pi\partial W_1),$$

because $\pi(C + \partial W_1) = 0$ inside $\mathbf{B}(0, s)$.

Since $F_1 \in V$, it follows that $\pi V \subset V$. Hence, by (2), $\int_W dA$ can be adjusted to zero at cost at most

$$K_1\|\pi W_2\| \leq K_1K_2\|\pi\partial W_1\|^2$$

by the isoperimetric inequality (4). Since the original cluster is minimizing,

$$\varepsilon\mathbf{M}(\pi\partial W_1) \leq K_1K_2\mathbf{M}(\pi\partial W_1)^2. \tag{8}$$

On the other hand, since C is weakly close to a straight line,

$$\mathbf{M}(\partial W_1) < 3\|F_1\|(2r) < \delta$$

because r is small compared to δ; hence, by (6),

$$\mathbf{M}(\pi\partial W_1) \leq K_3\delta. \tag{9}$$

By (7), (8), and (9), $\mathbf{M}(\pi\partial W_1) = 0$. Inside $\mathbf{B}(0, s)$, C has coefficient F_1. Hence, by a decomposition argument (cf. [M3, p. 98]) and the standard isoperimetric theorem, C must be a circular arc (with coefficient F_1). A variational argument implies that at all such points the arcs have the same curvature. $\square$

REMARK. White has shown me an elegant alternative argument for eliminating the possibility of other fluids. Suppose other fluids occurred in a shrunken ball in a sequence C_i of minimizers converging in the unit ball to two fluids separated by the diameter, with uniform bounds on the cost of area adjustments. By translation we may assume the origin lies on the boundary of unwanted fluids. Choose dilations $\mu_{r_i\#}C_i$ by just enough to be a fixed small flat distance $\varepsilon > 0$ from the closest diameter D in the open unit ball $\mathbf{U}(0, 1)$:

$$\inf\{\,\mathbf{M}(A) + \mathbf{M}(\partial B) : \operatorname{spt}(\mu_{r_i\#}C_i - D - A - \partial B) \cap \mathbf{U}(0, 1) = \emptyset\,\} = \varepsilon.$$

The amount of dilation goes to infinity, and we may assume convergence to a limit C in all of $\mathbf{R}^2$, through the origin, at distance at least ε from every diameter in the unit ball. Since the cost of area adjustments decreases under dilations, C is minimizing even without area constraints. The process is continuous in r_i because close dilations are flat close.

Inside a large ball $\mathbf{B}(0, R)$, each C_i and hence C has mass very near $2a_{01}R$; therefore, many large circles slice C in two points. Since C is minimizing, it must be a line (through the origin), in contradiction to being distance at least ε from any diameter in the unit ball.

This approach allows a weakening of the hypotheses of Regularity Theorem 4.3 from strictness in the triangle inequalities to uniqueness of norm decompositions

$$a_{i_0 i_k} = a_{i_0 i_1} + \cdots + a_{i_{k-1} i_k}$$

(see [W4, Sec. 11]).

The following regularity theorem is the main result of this paper.

4.3. Regularity Theorem. *Suppose interface energies a_{ij} in $\mathbf{R}^2$ satisfy strict triangle inequalities*: $a_{ik} < a_{ij} + a_{jk}$. *Let $C = \partial R$ be an energy-minimizing cluster of prescribed areas $A_i \geq 0$. Then C consists of circular arcs meeting at finitely many points.*

Proof. Any oriented tangent cone T to R at a must consist of regions with coefficients F_i separated by finitely many rays emanating from the origin with coefficients of the form $F_j - F_i$. If ∂T is a line, then C is a circular arc at a by Proposition 4.2. On the other hand, if a is a singular point then, inside a small ball $\mathbf{B}(a, 3r)$, C is weakly close to its tangent cone. By Proposition 4.2, inside $\mathbf{B}(a, 2r) - \mathbf{B}(a, r)$, C consists of several nearly radial circular arcs of bounded curvature. It follows that, at a, C consists of several circular arcs meeting at a point. □

4.4. Remarks. A variational argument shows that, at a vertex in an energy-minimizing planar cluster, the sum of the unit tangent vectors must vanish when weighted by the appropriate costs a_{ij}. In addition, adjacent vectors $v_1, \ldots, v_k$ with coefficients $F_{i_0} - F_{i_1}, \ldots, F_{i_{k-1}} - F_{i_k}$ must satisfy

$$\left|\sum a_{i_{j-1} i_j} v_j\right| \leq a_{i_0 i_k}.$$

Conversely these conditions imply that the tangent vectors are energy-minimizing for given boundary even without area constraints, assuming no other fluid is added [LM, Thm. 2.5].

Likewise, a variational argument shows that any two arcs separating the same two fluids have the same curvature, and that around any vertex, the weighted curvatures sum to zero.

Conversely, in a small neighborhood of a vertex where circular arcs meet with weighted unit tangents summing to zero and with weighted curvatures summing to zero, the cluster is energy-minimizing in comparison to clusters of the same combinatorial type. This follows by a strict calibration argument as in [M4, 2.1] and a limit argument to show that the vertex hardly moves. The restriction on combinatorial type is probably unnecessary if the tangent cone is "strictly minimizing"; see [M4, Conj. 2.4].

4.5. REMARK. The Regularity Theorem 4.3 and its proof apply to more general flat chains C with coefficients in G minimizing mass among such of the form $C + \partial W$ with $\int_W dA = 0$ at interior and isolated boundary points near which the coefficients lie in $\{F_i - F_j\}$. Where the coefficients lie outside $\{F_i - F_j\}$, regularity remains conjectural.

If the area constraints are dropped, then the regularity generalizes to mass-minimizing 1-dimensional flat chains in $\mathbf{R}^n$ with coefficients in a locally compact, finitely generated abelian group with strictly convex norm (so that, e.g., $\|2g\| < 2\|g\|$) or in a finitely generated free abelian group with strictly convex $\mathbf{Z}^+$-linear norm (so that, e.g., $\|2g\| = 2\|g\|$). Without the area constraints, the proofs reduce mainly to slicing arguments.

4.6. COROLLARY. *Divide the unit circle into finitely many intervals, assign a* (*not necessarily distinct*) *fluid to each interval, and* (*optionally*) *assign an area* A_i *to each fluid such that* $\sum A_i = \pi$ (*possibly including fluids not associated with boundary intervals*). *Suppose C is minimizing among rectifiable chains with coefficients in G of the form $C + \partial W$, with or without area constraints* $\int_W dA = 0$. *Then C consists of finitely many arcs of constant curvature meeting at isolated points, including points separating fluids at the boundary.*

This problem has also been studied in [E] and in [FMMP].

4.7. GENERAL NORMS. The more general case, where each interface energy is given by a norm $\Phi_{ij}(T)$ of the tangent direction, is much more subtle. Triangle inequalities $\Phi_{ik} < \Phi_{ij} + \Phi_{jk}$ no longer imply even the existence of a solution [M5].

If the norms are all multiples of a single norm, $\Phi_{ij} = a_{ij}\Phi$, then triangle inequalities do imply existence. Proposition 4.2 still holds. With the help of a linear transformation, one may assume that vertical projection does not increase cost. In the argument that the minimizer lies close to the diameter, monotonicity may be replaced by a simpler argument that there are no small components disconnected from the rest of the cluster. Theorem 4.3 does not follow without monotonicity to guarantee oriented tangent cones.

For norms that are all equal, $\Phi_{ij} = \Phi$ (the direct analog of soap-bubble clusters), existence and regularity are proved in [MFG].

4.8. Prescribed Combinatorial Type. The results of [M2, Sec. 3] on planar soap bubbles of prescribed combinatorial type (allowing infinitesimal layers) extend to planar immiscible fluid clusters with costs $a_{ij} > 0$, even without assuming triangle inequalities. At vertices [M2, 3.2(1)], the sum of the unit tangent vectors, weighted by the appropriate a_{ij}, sum to zero. (In the minimizer among clusters with connected regions, at a vertex [M2, 3.3(1)] the arcs need not meet in threes.) Figure 1 illustrates how, for the minimizer, combinatorial type A with $a_{01} = a_{12} = 1$ and $a_{02} = 3$ degenerates into one disc inside another.

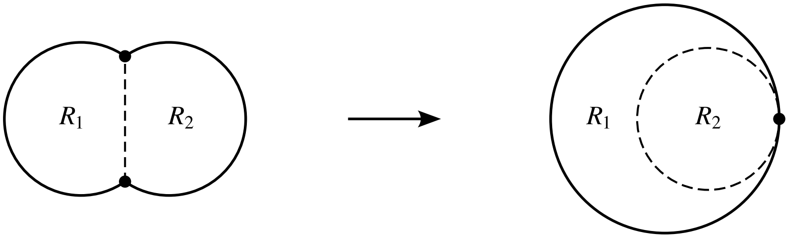

Figure 1 If $a_{01} = a_{12} = 1$ and $a_{02} = 3$, then combinatorial type A degenerates into one disc inside another, the minimizer.

5. Clusters in $\mathbf{R}^3$

In [T], Taylor proved that soap-bubble clusters in $\mathbf{R}^3$ consist of surfaces with constant mean curvature meeting in threes at angles of 120 degrees along curves, which in turn meet in faces at angles of about 109 degrees. She remarked (p. 492) that classification of stationary nets on the sphere with minimizing cones for the immiscible fluids problem would yield classification of singularities. Our Proposition 5.2 gives such a classification for interface energies a_{ij} near unity and therefore yields the following theorem.

5.1. Theorem. *For m immiscible fluids with interface energies sufficiently close to* 1, *an energy-minimizing cluster consists of surfaces with constant mean curvature meeting in threes at angles of about* 120 *degrees along curves, which in turn meet in fours at angles of about* 109 *degrees. The angles are precisely determined by the relevant interface costs.*

5.2. Proposition. *For m immiscible fluids with interface weights a_{ij} sufficiently close to* 1, *and for any assignment of distinct fluids to the regions of the two standard Y and tetrahedral cones, there is a unique nearby stationary net, which has a minimizing cone. There are no other stationary nets.*

Proof. By compactness, for weights sufficiently close to 1, any stationary net with energy-minimizing cone must be close to the standard Y or tetrahedral net. Because they are stationary, weighted normals to the associated planes satisfy simple linear relations, which means that these normals are the edges of a triangle or tetrahedron, unique up to rotation, so that the net is unique and furthermore the cone is minimizing by [LM, Thm. 2.5]. To prove existence of the Y or tetrahedral net, take a triangle or tetrahedron with the given weights as edge lengths; planes normal to the vertices determine the desired geodesics. $\square$

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