Holomorphic Sections of Prequantum Line Bundles on G/N

MENG-KIAT CHUAH

1. Introduction

Let K be a compact connected semisimple Lie group, let G be its complexification, and let $G = KAN$ be an Iwasawa decomposition. Let T be the centralizer of A in K, so that $H = TA$ is a Cartan subgroup of G. Since G and N are complex, G/N is a complex manifold. Besides the left G-action on G/N, there is also a right H-action because H normalizes N.

In [10], Schwarz suggests the following scheme of geometric quantization on the space G/N: Equip G/N with a K-invariant Kähler structure ω, and consider the corresponding prequantum line bundle $\mathbf{L}$ [6; 9]. Namely, the Chern class of $\mathbf{L}$ is the cohomology class $[\omega]$, and $\mathbf{L}$ has a connection ∇ whose curvature is ω. In fact, we shall see that if ω is Kähler then it is exact, so $\mathbf{L}$ is just a trivial bundle. However, the geometry arising from the connection is interesting. Given a section s of $\mathbf{L}$, we say that s is holomorphic if $\nabla_\xi s = 0$ for every antiholomorphic vector field ξ. Let $H(\mathbf{L})$ denote the holomorphic sections of $\mathbf{L}$. The K-action on G/N lifts to a K-representation on $H(\mathbf{L})$. Let $\mathfrak{k}$ be the Lie algebra of K. Then the infinitesimal representation on $H(\mathbf{L})$ is given by

$$\xi \cdot s = \nabla_{\xi^\sharp} s + \sqrt{-1}\phi^\xi s, \quad \xi \in \mathfrak{k},\ s \in H(\mathbf{L}) \tag{1.1}$$

[6, (3.1)], where $\xi^\sharp$ is the infinitesimal vector field on G/N induced by the left K-action and $\xi \mapsto \phi^\xi$ is the moment map $\mathfrak{k} \longrightarrow C^\infty(G/N)$ corresponding to the K-action preserving ω. Note that the moment map exists, since K is semisimple [7]. A K-invariant Kähler structure on G/N has potential function if and only if it is invariant under the right T-action [3]. In joint work with Guillemin [4], we carry out the foregoing construction for such Kähler structures and prove the following theorem.

THEOREM. *Let ω be a K-invariant Kähler structure on G/N. If it is right T-invariant, then $H(\mathbf{L})$ contains every finite-dimensional irreducible K-representation with multiplicity* 1.

Such a representation is called a *model* if it is equipped with a unitary structure—a term due to Gelfand and Zelevinski [5]. The preceding theorem is an analog of

Received April 23, 1997.
This work is supported in part by research grant NSC86-2115-M-009-012 of Taiwan.
Michigan Math. J. 45 (1998).

the Borel–Weil theorem [1], and holds even when ω is not positive definite. The main result of this paper is a converse to the previous theorem.

THEOREM 1. *Let ω be a K-invariant Kähler structure on G/N. If it is not right T-invariant, then $H(\mathbf{L}) = 0$.*

Hence, for a K-invariant Kähler structure ω, the multiplicity-free K-space $H(\mathbf{L})$ occurs on two extremes. Namely, $H(\mathbf{L})$ is either zero or contains every finite-dimensional irreducible K-representation, depending on whether ω is invariant under the right T-action (or equivalently whether ω has potential function).

A partial result of Theorem 1 is obtained in [3]. There, we show that if ω has no potential function then the trivial K-representation is missing in $H(\mathbf{L})$.

In Section 2, we review some results from [3] and [4] that will be needed in later sections. In Section 3, we construct an example where $K = \mathrm{SU}(2)$ and a Kähler structure on G/N whose prequantum line bundle has no holomorphic section other than the zero section. In Section 4, we use this example to prove Theorem 1 for the case where K has rank 1; in Section 5 we prove Theorem 1 for K of higher rank.

ACKNOWLEDGMENT. The author would like to thank So-Chin Chen, Victor Guillemin, and David Vogan for many helpful suggestions.

2. Preliminaries

In this section, we recall some results in [3] and [4] that will be needed later. Recall that $G = KAN$ is the Iwasawa decomposition and that $H = TA$ a Cartan subgroup of G. Let $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}, \mathfrak{n}, \mathfrak{t}, \mathfrak{h}$ be the Lie algebras of G, K, A, N, T, H respectively.

Let n be the rank of K, and let $\lambda_1, \ldots, \lambda_n \in \mathfrak{h}^*$ be the positive simple roots. Let $\mathbf{C}^\times$ be the multiplicative group of nonzero complex numbers, so that $\chi_i : H \to \mathbf{C}^\times$ is the character corresponding to λ_i. Thus $\exp(\lambda_i, v) = \chi_i(\exp v)$ for all $v \in \mathfrak{h}$. Given a K-invariant Kähler structure ω on G/N, it can be written as

$$\begin{aligned} \omega &= \sqrt{-1}\partial\bar{\partial}F + \sum_1^n \omega_i \\ &= \sqrt{-1}\partial\bar{\partial}F + \sum_1^n (\partial\alpha_i + \bar{\partial}\bar{\alpha}_i). \end{aligned} \tag{2.1}$$

This structure satisfies the following: $\sqrt{-1}\partial\bar{\partial}F$ is $K \times T$-invariant and Kähler; and, for $i = 1, \ldots, n$, each $\omega_i = \partial\alpha_i + \bar{\partial}\bar{\alpha}_i$ is a K-invariant $(1, 1)$-form. Also, ω_i is not right T-invariant and has no potential function unless it vanishes. In particular, for ω to be right T-invariant or to have potential function, a necessary and sufficient condition is that all the ω_i vanish. Each α_i is a $(0, 1)$-form with $\bar{\partial}\alpha_i = 0$. In [3] we show that α_i transforms by the character χ_i under the right T-action. This means that, for all $t \in T$ and its right action R_t,

$$R_t^*\alpha_i = \chi_i(t)\alpha_i.$$

We shall see that α_i also transforms by χ_i under the right A-action.

The possible values of each α_i is 1-dimensional in the sense that, if

$$\omega' = \sqrt{-1}\partial\bar{\partial}F' + \sum_1^n (\partial\alpha_i' + \bar{\partial}\bar{\alpha}_i')$$

is another Kähler form, then $a_i\alpha_i + b_i\alpha_i' = 0$ for some $a_i, b_i \in \mathbf{C}$. This result holds even when ω is merely a closed K-invariant real $(1, 1)$-form, which may not be positive-definite.

Since the possible values of α_i in (2.1) are 1-dimensional based on ω, we can find out more about α_i. Because K is compact semisimple, the Killing form on $\mathfrak{k}$ is negative-definite. Let $V \subset \mathfrak{k}$ be the orthocomplement of $\mathfrak{t} \subset \mathfrak{k}$ with respect to the Killing form. Thus we have a vector space direct sum

$$\mathfrak{k} = \mathfrak{t} + V. \tag{2.2}$$

The real vector space V has dimension $2m$, where m is the number of positive roots of G. Since G is semisimple, $n \leq m$. We may arrange the positive roots $\lambda_1, \ldots, \lambda_m$ so that the first n of them are simple. There exists a basis of V [8, p. 421],

$$\zeta_1, \gamma_1, \ldots, \zeta_m, \gamma_m \in V, \tag{2.3}$$

such that, for all $\xi \in \mathfrak{t}$,

$$[\xi, \zeta_i] = -\sqrt{-1}(\lambda_i, \xi)\gamma_i, \qquad [\xi, \gamma_i] = \sqrt{-1}(\lambda_i, \xi)\zeta_i. \tag{2.4}$$

Further, up to a constant scalar, $[\zeta_i, \gamma_i] \in \mathfrak{t}$ is dual to the restricted root $\lambda_i \in \mathfrak{t}^*$ via the Killing form. Let $\{\zeta_i^*, \gamma_i^*\} \subset V^*$ be the dual basis of (2.3), which we extend to $\{\zeta_i^*, \gamma_i^*\} \subset \mathfrak{k}^*$ by annihilating $\mathfrak{t}$. The Iwasawa decomposition allows us to imbed V into $\mathfrak{g}/\mathfrak{n}$ as a complex subspace via

$$V \hookrightarrow \mathfrak{k} \hookrightarrow \mathfrak{k} + \mathfrak{a} = \mathfrak{g}/\mathfrak{n}.$$

In fact, the almost-complex structure of $\mathfrak{g}/\mathfrak{n}$ sends ζ_i to γ_i and sends γ_i to $-\zeta_i$. It follows that $\zeta_i^* - \sqrt{-1}\gamma_i^* \in \wedge^{0,1}(\mathfrak{g}/\mathfrak{n})^*$. By Iwasawa, $G/N = KA$, so $K \times A$ acts transitively on G/N. Therefore, we may identify ζ_i^*, γ_i^* with the $K \times A$-invariant 1-forms whose values at $e \in G/N$ are exactly ζ_i^*, γ_i^*. Here $e \in G/N = KA$ denotes the Cartesian product of identity elements of K, A.

Consider the $K \times A$-invariant $(0, 1)$-form

$$v_i = \zeta_i^* - \sqrt{-1}\gamma_i^* \tag{2.5}$$

on G/N. From (2.4) we have that, for all $\xi \in \mathfrak{t}$,

$$ad_\xi^*\zeta_i^* = -\sqrt{-1}(\lambda_i, \xi)\gamma_i^*, \qquad ad_\xi^*\gamma_i^* = \sqrt{-1}(\lambda_i, \xi)\zeta_i^*.$$

This means that v_i of (2.5) satisfies

$$ad_\xi^* v_i = (\lambda_i, \xi)v_i \tag{2.6}$$

for all $\xi \in \mathfrak{t}$. Let $t \in T$, and let L_t, R_t denote (respectively) its left and right actions. In particular, the $(0, 1)$-form v_i is left T-invariant, so (2.6) means that

$$R_t^* v_i = L_t^* R_t^* v_i = Ad_t^* v_i = \chi_i(t)v_i.$$

There exists a unique $f_i \in C^\infty(A)$, which can be identified with a K-invariant function on G/N, so that

$$f_i v_i = \alpha_i.$$

Since $G/N = KA$, such an f_i is automatically right T-invariant. On the other hand, in the construction of $f_i v_i = \alpha_i$ [3, Prop. 2.2] we see that, up to a nonzero scalar, f_i is given by $f_i(ka) = \chi_i(a)^{-1}$ for all $ka \in KA = G/N$. Thus f_i transforms by χ_i under the right A-action. From the behaviors of f_i and v_i under the right actions of T and A, we obtain the following result for $\alpha_i = f_i v_i$.

PROPOSITION 2.1. *For $i = 1, \ldots, n$, α_i of* (2.1) *is a K-invariant* $(0, 1)$*-form that transforms by $\chi_i : H \to \mathbf{C}^\times$ under the right H-action. Namely, $R_h^* \alpha_i = \chi_i(h)\alpha_i$ for all right H-actions of $h \in H$. Its value at $e \in G/N$ is $c(\zeta_i^* - \sqrt{-1}\gamma_i^*)$ for some $c \in \mathbf{C}$.*

It can be checked from (2.1) that ω is exact, though this also follows from the Whitehead lemma [7, p. 417]:

$$H^2(G/N, \mathbf{R}) = H^2(KA, \mathbf{R}) = H^2(K, \mathbf{R}) = H^2(\mathfrak{k}) = 0.$$

Therefore, since ω is closed, it must be exact.

3. Example

In this section we construct an example of a Kähler structure ω on G/N, where $G = \mathrm{SL}(2, \mathbf{C})$, such that its prequantum line bundle $\mathbf{L}$ has no global holomorphic section other than the zero section. In later sections, our proof of Theorem 1 for arbitrary ω on G/N is based on this example.

Throughout this section, let $\mathbf{C}_0^2$ denote $\mathbf{C}^2$ with origin removed. For our example, let $K = \mathrm{SU}(2)$ and $G = \mathrm{SL}(2, \mathbf{C})$. Recall that $G = KAN$ is the Iwasawa decomposition, T is the centralizer of A in K, and $H = TA$ is a Cartan subgroup of G. In this case, we can have T, A, H to be diagonal matrices given by

$$\begin{gathered} T = \{\mathrm{diag}(e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}\theta})\},\ A = \{\mathrm{diag}(r, r^{-1});\ 0 < r \in \mathbf{R}\}, \\ H = \{\mathrm{diag}(z, z^{-1});\ 0 \neq z \in \mathbf{C}\}. \end{gathered} \tag{3.1}$$

Also, N is the complex upper triangular 2×2 matrix with 1 along the diagonal. Consider G acting on $\mathbf{C}^2$ in the standard manner. The G-orbit of the vector $(1, 0) \in \mathbf{C}^2$ is $\mathbf{C}_0^2$. The isotropy subgroup of $(1, 0)$ is N, so $G/N = \mathbf{C}_0^2$. In fact, since $K = \mathrm{SU}(2) = S^3$ and $A = \mathbf{R}^+$ as manifolds, the polar coordinates $\mathbf{C}_0^2 = S^3 \times \mathbf{R}^+$ is just the Iwasawa decomposition $G/N = KA$.

Let (z, u) be the standard coordinates on $\mathbf{C}_0^2$, and let r denote the length function

$$r = (z\bar{z} + u\bar{u})^{1/2}.$$

Fix a nonzero constant $c \in \mathbf{C}$, and consider the $(1, 1)$-form ω on $\mathbf{C}_0^2$ defined by

$$\alpha = \frac{c}{r^4}(\bar{z}\, d\bar{u} - \bar{u}\, d\bar{z}), \qquad \omega = \partial\alpha + \bar{\partial}\bar{\alpha}. \tag{3.2}$$

Note that c/r^4 is well-defined, for we ignore the origin here.

PROPOSITION 3.1. *The* $(1,1)$*-form* ω *in* (3.2) *is* $\mathrm{SU}(2)$*-invariant and closed*.

Proof. We first check that α in (3.2) is $\mathrm{SU}(2)$-invariant, and this will imply that ω is also $\mathrm{SU}(2)$-invariant. Because the function c/r^4 is clearly $\mathrm{SU}(2)$-invariant, it suffices to check $\bar{z}\,d\bar{u} - \bar{u}\,d\bar{z}$. Pick

$$k = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathrm{SU}(2)$$

satisfying $a\bar{a} + b\bar{b} = 1$, and let L_k denote the left action by k. Then

$$\begin{aligned} L_k^*(\bar{z}\,d\bar{u} - \bar{u}\,d\bar{z}) &= (L_k^*\bar{z})(L_k^*\,d\bar{u}) - (L_k^*\bar{u})(L_k^*\,d\bar{z}) \\ &= (\bar{a}\bar{z} + \bar{b}\bar{u})(-b\,d\bar{z} + a\,d\bar{u}) - (-b\bar{z} + a\bar{u})(\bar{a}\,d\bar{z} + \bar{b}\,d\bar{u}) \\ &= -\bar{u}\,d\bar{z} + \bar{z}\,d\bar{u}. \end{aligned}$$

It follows that α is $\mathrm{SU}(2)$-invariant, and so is ω.

To check that ω is closed, we note that

$$\begin{aligned} \frac{1}{c}\bar{\partial}\alpha &= \bar{\partial}(z\bar{z} + u\bar{u})^{-2} \wedge (\bar{z}\,d\bar{u} - \bar{u}\,d\bar{z}) + (z\bar{z} + u\bar{u})^{-2}\,\bar{\partial}(\bar{z}\,d\bar{u} - \bar{u}\,d\bar{z}) \\ &= -2(z\bar{z} + u\bar{u})^{-3}(z\,d\bar{z} + u\,d\bar{u}) \wedge (\bar{z}\,d\bar{u} - \bar{u}\,d\bar{z}) + 2(z\bar{z} + u\bar{u})^{-2} d\bar{z} \wedge d\bar{u} \\ &= 0. \end{aligned}$$

Hence $\bar{\partial}\alpha = \partial\bar{\alpha} = 0$ and so

$$d\omega = d(\partial\alpha + \bar{\partial}\bar{\alpha}) = (\partial + \bar{\partial})(\partial\alpha + \bar{\partial}\bar{\alpha}) = 0.$$

This proves the proposition. □

Let $\mathbf{L}$ be the prequantum line bundle associated to ω of (3.2). Namely, the Chern class of $\mathbf{L}$ is the cohomology class $[\omega]$, and $\mathbf{L}$ is equipped with a connection ∇ whose curvature is ω. Since ω is exact, $\mathbf{L}$ is a trivial bundle. Given a section s, we say that s is *holomorphic* if $\nabla_\xi s = 0$ for every antiholomorphic vector field ξ. We claim that, for ω of (3.2), $\mathbf{L}$ has no global holomorphic section other than the zero section. Suppose otherwise; let $H(\mathbf{L}) \neq 0$ be the space of its holomorphic sections. The K-action on G/N lifts to a K-representation on $H(\mathbf{L})$. Let $\mathbf{C}^\times$ be the multiplicative group of nonzero complex numbers. Recall that the Cartan subgroup of G is $H = TA$, where $H \cong \mathbf{C}^\times$ by (3.1). Let $\mathfrak{h}$ be its Lie algebra. Pick a nonzero element s of the weight space

$$H(\mathbf{L})_\lambda = \{s \in H(\mathbf{L});\ \xi \cdot s = \lambda(\xi)s \ \text{ for all } \xi \in \mathfrak{h}\}, \tag{3.3}$$

where $\lambda \in \mathfrak{h}^*$. For $\xi \in \mathfrak{t}$, $\xi \cdot s$ in (3.3) is the infinitesimal representation arising from the group action and is given in (1.1). Since $\mathfrak{a} = \sqrt{-1}\mathfrak{t}$, if $\xi \in \mathfrak{t}$ then [6, (5.2)] $\eta = \sqrt{-1}\xi \in \mathfrak{a}$ acts on s in (3.3) by $\eta \cdot s = \sqrt{-1}(\xi \cdot s)$.

From the section $0 \neq s \in H(\mathbf{L})_\lambda$, we define the domain $D = D_s$ by

$$D = \{p \in \mathbf{C}_0^2;\ s_p \neq 0\}. \tag{3.4}$$

Let Z and U be the z- and u-axes on $\mathbf{C}_0^2$, respectively:

$$Z = \{(z, 0) \in \mathbf{C}_0^2\}, \qquad U = \{(0, u) \in \mathbf{C}_0^2\}. \tag{3.5}$$

Because s is nonzero and holomorphic, D is a dense open set in $\mathbf{C}_0^2$. Choosing another weight space or holomorphic section if necessary, we may assume that D intersects Z and U. Let $\chi : H \to \mathbf{C}^\times$ be the character corresponding to $\lambda \in \mathfrak{h}^*$, and let L_h^* be the representation arising from the left action of $h \in H$. Since $L_h^* s = \chi(h)s$, if $\mathcal{O} \subset \mathbf{C}^2$ is an H-orbit then

$$\mathcal{O} \cap D = \emptyset \quad \text{or} \quad \mathcal{O} \subset D. \tag{3.6}$$

Since Z and U are H-orbits that intersect D, it follows that $Z, U \subset D$.

PROPOSITION 3.2. *Suppose that* $0 \neq s \in H(\mathbf{L})_\lambda$, *and that the domain D defined in* (3.4) *intersects Z and U. Then there exists a neighborhood B of the origin such that* $(B \backslash \{0\}) \subset D$.

Proof. Suppose otherwise, so that the origin is a limit point of $\mathbf{C}_0^2 \backslash D$. There exists a sequence $\{(z_i, u_i)\} \subset \mathbf{C}_0^2 \backslash D$ that converges to the origin. Since $Z, U \subset D$, we have $(z_i, u_i) \notin (Z \cup U)$ and therefore $z_i, u_i \neq 0$. By (3.1), we obtain $h_i \in H$ by $h_i = \operatorname{diag}(u_i, u_i^{-1})$. Because $(z_i, u_i) \notin D$, (3.6) implies that $h_i(z_i, u_i) \notin D$.

On the other hand, since (z_i, u_i) converges to the origin, the sequence $\{z_i u_i\} \subset \mathbf{C}$ converges to 0. It follows that the sequence $\{h_i(z_i, u_i) = (z_i u_i, 1)\}$, not contained in D, converges to $(0, 1) \in U \subset D$. But D is open, so we get a contradiction here. This proves the proposition. □

Recall that $0 \neq s \in H(\mathbf{L})_\lambda$ and the domain D defined in (3.4) contains the standard axes Z and U in (3.5). Since s is holomorphic, ∇s annihilates all antiholomorphic vector fields. Therefore, there exist complex-valued functions $f, g \in C^\infty(D)$ such that

$$\sqrt{-1}\nabla s = \gamma s = (f\,dz + g\,du)s$$

for some $(1,0)$-form $\gamma = f\,dz + g\,du$ on D. Here z and u are the standard coordinate functions on $\mathbf{C}_0^2$. By the definition of curvature, $d\gamma = \omega$ on D. We now derive a contradiction, which arises from the above assumption that $0 \neq s \in H(\mathbf{L})$ exists. In what follows, we compute the function f. From (3.2),

$$\begin{aligned} \omega &= \partial\left(\frac{c}{r^4}(\bar{z}\,d\bar{u} - \bar{u}\,d\bar{z})\right) + \bar{\partial}\left(\frac{\bar{c}}{r^4}(z\,du - u\,dz)\right) \\ &= \frac{-2}{r^6}\{(c\bar{z}^2 + \bar{c}u^2)\,dz \wedge d\bar{u} + (-c\bar{z}\bar{u} + \bar{c}zu)\,dz \wedge d\bar{z} \\ &\qquad + (c\bar{z}\bar{u} - \bar{c}zu)\,du \wedge d\bar{u} + (-c\bar{u}^2 - \bar{c}z^2)\,du \wedge d\bar{z}\}. \end{aligned} \tag{3.7}$$

Because $\omega = d\gamma$ is a $(1,1)$-form and $\gamma = f\,dz + g\,du$ is a $(1,0)$-form,

$$\begin{aligned} \omega &= d\gamma = \bar{\partial}\gamma \\ &= \frac{\partial f}{\partial \bar{z}}\,d\bar{z} \wedge dz + \frac{\partial f}{\partial \bar{u}}\,d\bar{u} \wedge dz + \frac{\partial g}{\partial \bar{z}}\,d\bar{z} \wedge du + \frac{\partial g}{\partial \bar{u}}\,d\bar{u} \wedge du. \end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we obtain

$$\frac{\partial f}{\partial \bar{z}} = 2(\bar{c}zu - c\bar{z}\bar{u})(z\bar{z} + u\bar{u})^{-3}, \qquad \frac{\partial f}{\partial \bar{u}} = 2(c\bar{z}^2 + \bar{c}u^2)(z\bar{z} + u\bar{u})^{-3}. \tag{3.9}$$

Taking antiderivatives with respect to $\bar{z}, \bar{u}$ in (3.9) yields

$$f = (z\bar{z} + u\bar{u})^{-2}(2cz^{-1}\bar{z}\bar{u} + cz^{-2}u\bar{u}^2 - \bar{c}u) + j(z, u, \bar{u}) \tag{3.10}$$

and

$$f = -(z\bar{z} + u\bar{u})^{-2}(c\bar{z}^2u^{-1} + \bar{c}u) + h(z, \bar{z}, u), \tag{3.11}$$

where j, h are independent of $\bar{z}, \bar{u}$ respectively. Let B be the neighborhood of the origin given by Proposition 3.2, and let $B_0 = B\backslash\{0\}$. Since $B_0 \subset D$, f is smooth on B_0. By (3.11), we know that uh is smooth on B_0. Further, by (3.10) and (3.11),

$$\begin{aligned} h &= (z\bar{z} + u\bar{u})^{-2}(c\bar{z}^2u^{-1} + \bar{c}u) + f \\ &= (z\bar{z} + u\bar{u})^{-2}(c\bar{z}^2u^{-1} + \bar{c}u + 2cz^{-1}\bar{z}\bar{u} + cz^{-2}u\bar{u}^2 - \bar{c}u) + j \\ &= (z\bar{z} + u\bar{u})^{-2}(cz^{-2}u^{-1}(z^2\bar{z}^2 + 2z\bar{z}u\bar{u} + u^2\bar{u}^2)) + j \\ &= cz^{-2}u^{-1} + j. \end{aligned}$$

Therefore h, and hence uh, are independent of $\bar{z}$. We conclude that uh is a holomorphic function on B_0. By Hartog's theorem, uh is holomorphic on B.

Consider the function z^2uh, which is holomorphic on B. Define $B_1 \subset B$ by

$$B_1 = \{(z, u) \in B;\ u = 0\}.$$

We claim that the restriction of z^2uh to B_1 is not constant.

Suppose that $z^2uh \equiv b \in \mathbf{C}$ on B_1; then $uh = bz^{-2}$ on B_1. But uh, being holomorphic on B, restricts to B_1 as a holomorphic function there. This gives a contradiction, since bz^{-2} blows up on $(0, 0) \in B_1$. Hence the restriction of the holomorphic function z^2uh to B_1 is not constant, as claimed.

Let c be the nonzero constant in (3.11). Since $(z^2uh)|_{B_1}$ is not a constant function, there exists $z_0 \neq 0$ such that $(z_0, 0) \in B_1 \subset B$ and

$$(z^2uh)|_{(z_0,0)} \neq c. \tag{3.12}$$

Then (3.11) says that, on $B_0\backslash Z$,

$$\begin{aligned} (z\bar{z} + u\bar{u})^2 f &= -c\bar{z}^2u^{-1} - \bar{c}u + (z\bar{z} + u\bar{u})^2 h \\ &= -c\bar{z}^2u^{-1} - \bar{c}u + \frac{(z\bar{z} + u\bar{u})^2}{u}(uh) \\ &= \bar{z}^2u^{-1}(-c + z^2uh) - \bar{c}u + (2z\bar{z}\bar{u} + u\bar{u}^2)uh. \end{aligned} \tag{3.13}$$

Fix $z_0 \neq 0$ given by (3.12), and consider $(z_0, u) \in B_0\backslash Z$. We evaluate (3.13) at (z_0, u) and take its limit as $u \to 0$. Then the limit of the LHS converges because f is smooth at $(z_0, 0) \in B_0 \subset D$. In the RHS of (3.13), we recall that uh is holomorphic near $(z_0, 0)$. Therefore,

$$\lim_{u\to 0}(-c + z_0^2uh)$$

converges and equals a nonzero constant, owing to (3.12). Also,

$$\lim_{u\to 0}(-\bar{c}u + (2z_0\bar{z}_0\bar{u} + u\bar{u}^2)uh) = 0.$$

Therefore, in (3.13),

$$\lim_{(z_0,u)\to(z_0,0)} \text{RHS}$$

blows up due to the term $\bar{z}_0^2 u^{-1}$.

This contradiction arises from our assumption that $\mathbf{L}$ has global holomorphic sections other than the zero section. We therefore conclude that for the example in this section where $K = \mathrm{SU}(2)$ and ω is the $(1, 1)$-form in (3.2), the only holomorphic section of $\mathbf{L}$ is the zero section. We shall use this example to prove Theorem 1 in the following two sections.

4. Groups of Rank 1

Recall that K is a compact connected semisimple Lie group. In this section, we prove Theorem 1 for the case where K has rank 1. In this case there are two possibilities for K, namely SU(2) or SO(3) [2, p. 185].

We first consider $K = \mathrm{SU}(2)$. From Secion 3 we know that $G = \mathrm{SL}(2, \mathbf{C})$ and $G/N = \mathbf{C}^2_0$, where $\mathbf{C}^2_0$ is $\mathbf{C}^2$ with origin removed. Given a closed SU(2)-invariant $(1, 1)$-form ω on $\mathbf{C}^2_0$, let $\mathbf{L}_\omega$ be its corresponding prequantum line bundle. The Chern class of $\mathbf{L}_\omega$ is the cohomology class $[\omega]$, and the curvature of the connection on $\mathbf{L}_\omega$ is ω. As observed in Section 2, ω is exact, so $\mathbf{L}_\omega$ is a trivial bundle. Hence, given any two such ω, ω', their prequantum line bundles $\mathbf{L}_\omega, \mathbf{L}_{\omega'}$ are topologically equivalent; however, the connections ∇, ∇' can give rise to distinct geometric properties.

Given an arbitrary closed SU(2)-invariant $(1, 1)$-form ω on $\mathbf{C}^2_0$, we apply (2.1) and express it canonically as

$$\omega = \omega_0 + \omega_1 = \omega_0 + (\partial\alpha_1 + \bar{\partial}\bar{\alpha}_1),$$

where ω_0 is right T-invariant. Suppose that ω is not right T-invariant, so that $\omega_1 = \partial\alpha_1 + \bar{\partial}\bar{\alpha}_1$ does not vanish. Let $\mathbf{L}_\omega, \mathbf{L}_{\omega_0}, \mathbf{L}_{\omega_1}$ be their corresponding prequantum line bundles. Because ω_0 is right T-invariant, there exist plenty of holomorphic sections on $\mathbf{L}_{\omega_0}$. In particular, $\mathbf{L}_{\omega_0}$ contains nonvanishing global holomorphic sections [4, Prop. 3.1]. Since $\mathbf{L}_\omega = \mathbf{L}_{\omega_0} \otimes \mathbf{L}_{\omega_1}$, such a nonvanishing section of $\mathbf{L}_{\omega_0}$ defines an isomorphism

$$H(\mathbf{L}_\omega) \cong H(\mathbf{L}_{\omega_1}). \tag{4.1}$$

Now let ω be the specific SU(2)-invariant $(1, 1)$-form given in (3.2), and let $\mathbf{L}_\omega$ be its prequantum line bundle. We write $\omega = \omega_0 + \omega_1$ as described in (2.1), where $\omega_1 = \partial\alpha_1 + \bar{\partial}\bar{\alpha}_1$. In Section 3 we saw that $H(\mathbf{L}_\omega) = 0$. It follows from (4.1) that $H(\mathbf{L}_{\omega_1}) = 0$. This means that $\omega_1 \neq 0$, for otherwise the prequantum line bundle corresponding to $\omega = \omega_0$ has plenty of holomorphic sections. Hence, in particular, $\alpha_1 \neq 0$.

Let ω' be another closed K-invariant $(1, 1)$-form on $\mathbf{C}^2_0$. We again apply (2.1) and write $\omega' = \omega'_0 + \omega'_1$, where $\omega'_1 = \partial\alpha'_1 + \bar{\partial}\bar{\alpha}'_1$. Suppose that ω is not right

T-invariant, so that $\alpha_1' \neq 0$. From Section 2, we know that the possible values of α_1 and α_1' are 1-dimensional. Therefore, choosing the correct constant $c \in \mathbf{C}$ in (3.2), we get $\alpha_1 = \alpha_1'$. It follows that $\omega_1 = \omega_1'$, so $H(\mathbf{L}_{\omega_1'}) = 0$. Applying (4.1) to $\omega' = \omega_0' + \omega_1'$, this implies that $H(\mathbf{L}_{\omega'}) = 0$. Thus Theorem 1 is proved for the case of $K = \mathrm{SU}(2)$.

We now consider the case $K = \mathrm{SO}(3)$, whose complexification is $G = \mathrm{SO}(3, \mathbf{C})$. The Iwasawa decomposition of $\mathrm{SO}(3, \mathbf{C})$ gives unipotent subgroup N_1, as well as a maximal torus T_1 of $\mathrm{SO}(3)$. The double covering $\mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$ extends to the covering

$$\pi : \mathrm{SL}(2, \mathbf{C})/N \to \mathrm{SO}(3, \mathbf{C})/N_1.$$

Here $\pi(T) = T_1$ is a double covering of the circle onto itself.

Because T_1 normalizes N_1, it acts on $\mathrm{SO}(3, \mathbf{C})/N_1$ on the right. Let ω be an $\mathrm{SO}(3)$-invariant Kähler structure on $\mathrm{SO}(3, \mathbf{C})/N_1$, and suppose that it is not right T_1-invariant. Then $\pi^*\omega$ is an $\mathrm{SU}(2)$-invariant Kähler structure on $\mathrm{SL}(2, \mathbf{C})/N$ and is not right T-invariant. If $\mathbf{L}_\omega$ has any nonzero holomorphic section then it induces a nonzero holomorphic section on $\pi^*\mathbf{L}_\omega$, which is the prequantum line bundle corresponding to $\pi^*\omega$. This is impossible, so $H(\mathbf{L}_\omega) = 0$.

This proves Theorem 1 for K of rank 1.

5. Groups of Higher Rank

In this section, we consider the case where the rank of the Lie group K may be greater than 1. Recall that $G = KAN$ is the Iwasawa decomposition and that $H = TA$ is a Cartan subgroup of G. Let

$$n = \operatorname{rank} K = \dim_{\mathbf{C}} H.$$

Let ω be a K-invariant Kähler structure on G/N. It has the form

$$\omega = \sum_0^n \omega_i = \sqrt{-1}\partial\bar{\partial}F + \sum_1^n (\partial\alpha_i + \bar{\partial}\bar{\alpha}_i), \tag{5.1}$$

as described in (2.1), where $\omega_0 = \sqrt{-1}\partial\bar{\partial}F$ is itself Kähler and has potential function. Suppose that ω is not right T-invariant, so that $\omega_i = \partial\alpha_i + \bar{\partial}\bar{\alpha}_i \neq 0$ for some $i = 1, \ldots, n$. Without loss of generality, we may assume that $\alpha_1 \neq 0$. Recall from Section 2 that α_1 is indexed by the simple root $\lambda_1 \in \mathfrak{h}^*$. Namely, under the right H-action, it transforms by the character $\chi_1 \colon H \to \mathbf{C}^\times$ associated to the root $\lambda_1 \in \mathfrak{h}^*$. This means that χ_1 satisfies $\chi_1(\exp v) = \exp(\lambda_1, v)$ for all $v \in \mathfrak{h}$, and that $R_h^*\alpha_1 = \chi_1(h)\alpha_1$ under the right action R_h of $h \in H$.

Let $\sigma \subset \mathfrak{t}$ be the hyperplane annihilated by λ_1;

$$\sigma = \{v \in \mathfrak{t};\ (\lambda_1, v) = 0\}.$$

Let $\mathfrak{k}^\sigma$ be the centralizer of σ in $\mathfrak{k}$, consisting of $\xi \in \mathfrak{k}$ such that $[\xi, v] = 0$ whenever $v \in \sigma$. We define the semisimple Lie algebra $\mathfrak{k}^\sigma_{\mathrm{ss}}$ by

$$\mathfrak{k}^\sigma_{\mathrm{ss}} = [\mathfrak{k}^\sigma, \mathfrak{k}^\sigma] \subset \mathfrak{k}.$$

Let $\mathfrak{g}^\sigma_{\mathrm{ss}} = \mathfrak{k}^\sigma_{\mathrm{ss}} \otimes \mathbf{C}$, and let a Cartan subalgebra of $\mathfrak{g}^\sigma_{\mathrm{ss}}$ be given by

$$\mathfrak{h}^\sigma = \{v \in \mathfrak{h};\ (v, \sigma) = 0\},$$

where the pairing used is the Killing form. Let $\mathfrak{n}^\sigma = \mathfrak{g}^\sigma_{ss} \cap \mathfrak{n}$; then we have an Iwasawa decomposition

$$\mathfrak{g}^\sigma_{ss} = \mathfrak{k}^\sigma_{ss} \oplus \mathfrak{a}^\sigma \oplus \mathfrak{n}^\sigma. \tag{5.2}$$

Here $\mathfrak{k}^\sigma_{ss}$ is a rank-1 semisimple Lie algebra, and a maximal toral subalgebra of $\mathfrak{k}^\sigma_{ss}$ is given by $\mathfrak{t}^\sigma = \mathfrak{t} \cap \mathfrak{h}^\sigma$. From the Lie algebras in (5.2), we have the connected subgroups G^σ_{ss}, K^σ_{ss}, A^σ, N^σ of G. Also, T^σ is the subgroup corresponding to $\mathfrak{t}^\sigma$ and $H^\sigma = T^\sigma A^\sigma$ is a Cartan subgroup of G^σ_{ss}. Consider the complex manifold $G^\sigma_{ss}/N^\sigma = K^\sigma_{ss}A^\sigma$. Since H^σ normalizes N^σ, it acts on G^σ_{ss}/N^σ on the right. The space G^σ_{ss}/N^σ imbeds naturally into G/N,

$$j : G^\sigma_{ss}/N^\sigma \hookrightarrow G/N. \tag{5.3}$$

This is a holomorphic $K^\sigma_{ss} \times H^\sigma$-equivariant imbedding. Since ω and ω_0 of (5.1) are K-invariant Kähler forms, it follows that $j^*\omega$ and $j^*\omega_0$ are K^σ_{ss}-invariant Kähler forms on G^σ_{ss}/N^σ. But since ω_1 is not Kähler, some work is still needed to ensure that it does not vanish on G^σ_{ss}/N^σ.

PROPOSITION 5.1. *Let j be the imbedding* (5.3)*, and let $\omega_1 \neq 0$ be the K-invariant* (1, 1)*-form in* (5.1)*. Then $j^*\omega_1 \neq 0$.*

Proof. Recall the elements $\zeta_1, \gamma_1 \in V \subset \mathfrak{k}$ in (2.3) and their dual $\zeta_1^*, \gamma_1^* \in \mathfrak{k}^*$. By Proposition 2.1, $\omega_1 = \partial\alpha_1 + \bar\partial\bar\alpha_1$ satisfies $(\alpha_1)_e = c(\zeta_1^* - \sqrt{-1}\gamma_1^*)$ for some nonzero constant $c \in \mathbf{C}$. Here $e \in G^\sigma_{ss}/N^\sigma = K^\sigma_{ss}A^\sigma \hookrightarrow KA = G/N$ is the product of identity elements of K and A. Since j is $K^\sigma_{ss} \times H^\sigma$-equivariant, $j^*\alpha_1$ is K^σ_{ss}-invariant and transforms by $\chi_1 : A^\sigma \to \mathbf{R}^+$ under the right A^σ-action.

Because $\mathfrak{k}^\sigma$ centralizes σ, (2.4) implies that $\zeta_1, \gamma_1 \in \mathfrak{k}^\sigma$. Also, up to a constant scalar, $[\zeta_1, \gamma_1] \in \mathfrak{t}$ is the vector dual to the restricted root $\lambda_1 \in \mathfrak{t}^*$ via Killing form. Thus $[\zeta_1, \gamma_1] \in \mathfrak{t}^\sigma \subset \mathfrak{k}^\sigma_{ss}$. In fact, taking the real span of these two vectors, we have a vector space direct sum

$$\mathfrak{k}^\sigma_{ss} = \mathfrak{t}^\sigma + \mathbf{R}(\zeta_1, \gamma_1). \tag{5.4}$$

Here λ_1 is the unique positive root of this rank-1 Lie algebra. We compare (5.4) with (2.2) and apply Proposition 2.1 to G^σ_{ss}/N^σ. It says that every K^σ_{ss}-invariant Kähler structure ω' on G^σ_{ss}/N^σ can be expressed uniquely as $\omega' = \omega'_0 + \omega'_1$, where $\omega'_1 = \partial\alpha'_1 + \bar\partial\bar\alpha'_1$. Further, the K^σ_{ss}-invariant α'_1 transforms by χ_1 under the right A^σ-action, and $(\alpha'_1)_e = c'(\zeta_1^* - \sqrt{-1}\gamma_1^*)$ for some $c' \in \mathbf{C}$. If $c' \neq 0$ then $\omega'_1 \neq 0$. Set $c = c'$, so that $(\alpha_1)_e = (\alpha'_1)_e$. Both α_1 and α'_1 are K^σ_{ss}-invariant and transform by $\chi_1 : A^\sigma \to \mathbf{R}^+$ under the right A^σ-action. Therefore, since $K^\sigma_{ss} \times A^\sigma$ acts transitively on G^σ_{ss}/N^σ, $(\alpha_1)_e = (\alpha'_1)_e$ implies that $j^*\alpha_1 = \alpha'_1$. Then

$$\begin{aligned} j^*\omega_1 &= j^*(\partial\alpha_1 + \bar\partial\bar\alpha_1) \\ &= \partial j^*\alpha_1 + \bar\partial j^*\bar\alpha_1 \\ &= \partial\alpha'_1 + \bar\partial\bar\alpha'_1 \\ &= \omega'_1 \neq 0. \end{aligned}$$

This proves the proposition. □

Recall that $\chi_1 : T \to S^1$ is the character corresponding to the restricted root $\lambda_1 \in \mathfrak{t}^*$. Since $(\lambda_1, \mathfrak{t}^\sigma) \neq 0$, there are many $t \in T^\sigma$ such that $\chi_1(t) \neq 1$. For such t, let R_t denote its right action. Then, since j is $K_{\rm ss}^\sigma \times H^\sigma$-equivariant,

$$R_t^* j^* \omega_1 = j^* R_t^* \omega_1 = j^* \chi_1(t) \omega_1 = \chi_1(t) j^* \omega_1 \neq j^* \omega_1.$$

It follows that $j^*\omega$ is not invariant under the right T^σ-action.

As observed in Section 2, ω is exact, so there exists a complex line bundle $\mathbf{L}$ whose Chern class is $[\omega] = 0$. It is equipped with a connection whose curvature is ω. Suppose that $s \neq 0$ is a global holomorphic section of $\mathbf{L}$. We derive a contradiction from here. Since G acts transitively on G/N, we may assume that $s_p \neq 0$ for some $p \in G_{\rm ss}^\sigma/N^\sigma \hookrightarrow G/N$. Then j^*s is a holomorphic section of the line bundle $j^*\mathbf{L}$ on $G_{\rm ss}^\sigma/N^\sigma$, and it is not the zero section. But $j^*\mathbf{L}$ is the prequantum line bundle corresponding to Kähler form $j^*\omega$. Since $K_{\rm ss}^\sigma$ has rank 1, this contradicts the result of Section 4. We therefore conclude that the only global holomorphic section of $\mathbf{L}$ is the zero section. This completes the proof of Theorem 1. □

References

[1] A. Borel and A. Weil, *Représentations linéaires et espaces homogènes Kählerians des groupes de Lie compacts,* Séminaire Bourbaki, 1954.

[2] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups,* Springer, New York, 1985.

[3] M. K. Chuah, *K-invariant Kaehler structure on $K_{\mathbf{C}}/N$ and the associated line bundles,* Proc. Amer. Math. Soc. 124 (1996), 3481–3491.

[4] M. K. Chuah and V. Guillemin, *Kaehler structures on $K_{\mathbf{C}}/N$,* Contemp. Math., 154, pp. 181–195, Amer. Math. Soc., Providence, RI, 1993.

[5] I. M. Gelfand and A. Zelevinski, *Models of representations of classical groups and their hidden symmetries,* Funct. Anal. Appl. 18 (1984), 183–198.

[6] V. Guillemin and S. Sternberg, *Geometric quantization and multiplicities of group representations,* Invent. Math. 67 (1982), 515–538.

[7] V. Guillemin and S. Sternberg, *Symplectic techniques in physics,* Cambridge Univ. Press, 1984.

[8] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces,* Academic Press, New York, 1978.

[9] B. Kostant, *Quantization and unitary representations,* Lecture Notes in Math., 170, pp. 87–208, Springer, Berlin, 1970.

[10] H. S. La, P. Nelson, and A. S. Schwarz, *Virasoro model space,* Comm. Math. Phys. 134 (1990), 539–554.

Department of Applied Mathematics
National Chiao Tung University
Hsinshu, Taiwan

chuah@math.nctu.edu.tw