p-Groups of Symmetries of Surfaces

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1. Introduction

Let Σ_g denote a closed orientable surface of genus $g \geq 2$. Let G be a nontrivial finite group. If G can be embedded in the group of orientation-preserving self-homeomorphisms of Σ_g, then we say that G *acts on* Σ_g. In this case, Σ_g can be realized as a Riemann surface and G as a subgroup of its automorphism group.

For each fixed g, there can be only finitely many finite groups G that act on Σ_g, since by a famous result of Hurwitz [11] the order of G is bounded above by $84(g-1)$. For some small values of g, complete listings of those groups which act on Σ_g have been obtained (see e.g. [15; 16; 19; 27]).

On the other hand, for each G there is an infinite sequence of values of g such that G acts on Σ_g [13; 22]. The determination of this sequence, here called the *genus spectrum* of G, is termed the *Hurwitz problem* in [22]. The genus spectrum for a cyclic group of prime order was determined in [12; 14; 22] and can be deduced from earlier results [17; 8] (see also [5]).

Much effort has gone in to determining the smallest member of this genus spectrum for various classes of groups [3; 6; 7; 19]. Indeed, moving beyond the restrictions imposed here—that is, to consider nonclosed and/or nonorientable surfaces or allowing the self-homeomorphisms to be orientation reversing—the corresponding smallest numbers have been widely investigated (see [1] and the references there).

To describe the results of this paper, we use the notation that evolved from [13; 14] as follows: For each finite group G, there is an integer $n_0(G)$, easily computed from the Sylow subgroup structure of G, such that if G acts on Σ_g then $g = 1 + n_0(G)g_0$ for some $g_0 \geq 1$. The integer g_0 is called a *reduced genus for* G. Let $\mu_0 = \mu_0(G)$ denote the *minimum reduced genus* for G and let $\sigma_0 = \sigma_0(G)$ denote the *minimum stable reduced genus* for G, that is, minimal with the property that all $g_0 \geq \sigma_0$ are reduced genera. In addition, the integers in the interval $[\mu_0, \sigma_0]$ that do not occur as reduced genera of G will constitute the (*reduced*) *gap sequence* of G. The infinite sequence of integers

$$\{\, g \geq 2 \mid G \text{ acts on } \Sigma_g \,\}$$

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will be called the *genus spectrum* of G. This is obviously determined by n_0, μ_0, σ_0, and the gap sequence of G.

When G is a p-group with p odd, $n_0(G) = p^{n-e}$ where G had order p^n and exponent p^e. The integer $n - e$ is the *cyclic p-deficiency* of G. In [14], these invariants were determined for cyclic p-groups. Formulas were obtained for $\sigma_0(G)$ and $\mu_0(G)$ and necessary and sufficient conditions were obtained for an integer to be a reduced genus for such a group G.

In this paper, these results are extended to consider p-groups of cyclic p-deficiency ≤ 2, but also to a class of p-groups satisfying a condition known as the *maximal exponent property* (MEP) (see Section 2). For MEP groups, a lower bound for $\sigma_0(G)$ in terms of the group exponent is obtained (Theorem 4.2). Not surprisingly, the minimum number of generators of G, the rank of G, also plays a role in the determination of $\sigma_0(G)$, and it is shown that $\sigma_0(G)$ is constant on all p-groups with MEP, rank 2, and the same exponent (Theorem 4.7). On the other hand, for elementary abelian groups, $\sigma_0(G)$ grows with the rank of G (Corollary 7.3).

The necessary and sufficient conditions for an integer to be a reduced genus of G, which are obtained in the cases where G has cyclic p-deficiency 2, show that for each $p \geq 5$ and each $e \geq 3$ there exist pairs of nonisomorphic p-groups of the same order and exponent p^e that have identical genus spectra (Theorem 6.1).

The situation for $p = 2$ will not be considered here. It is considerably more complicated than for odd p, as the analysis of 2-groups of 2-cyclic deficiency 1 in [26] shows.

A different approach to determining the genus spectrum is taken in [24]. It is shown there that a suitable translation of the genus spectrum is a subsemigroup of the positive integers. In [24], generators for the semigroup are obtained when the group G is the simple group of order 168. Related ideas are also widely applied in [21].

Finite groups acting on 3-dimensional handlebodies were intensively investigated in [23], wherein machinery was established allowing the methods and results discussed in this paper to be extended to that situation (cf. [20]).

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2. Preliminaries

It is assumed throughout that G is a p-group and that p is an odd prime. Let G have order p^n and exponent p^e, so that it has cyclic p-deficiency $n - e$. Then $n_0 = n_0(G) = p^{n-e}$ and, as shown in [13], our first result follows.

THEOREM 2.1. *If G acts on Σ_g $(g \geq 2)$, then $g - 1 \in n_0\mathbf{N}$ and furthermore, for all but a finite number of integers g where $g \in 1 + n_0\mathbf{N}$, G acts on Σ_g.*

The general approach is to regard these groups of symmetries as quotient groups of Fuchsian groups as follows. If G acts on the compact surface Σ_g $(g \geq 2)$, then

Σ_g can be endowed with a conformal structure such that G becomes a group of automorphisms of the compact Riemann surface Σ_g. The universal cover of Σ_g is then U, the upper half-plane, and Σ_g is conformally equivalent to U/K where K is a torsion-free Fuchsian group such that $K \cong \Pi_1(\Sigma_g)$. The action of G can then be lifted to conformal automorphisms of U, and we find it convenient to use the language of Fuchsian groups.

THEOREM 2.2. *G acts on a compact surface Σ_g of genus $g \geq 2$ if and only if there exists a Fuchsian group Γ and an epimorphism $\phi\colon \Gamma \to G$ such that the kernel of ϕ is isomorphic to $\Pi_1(\Sigma_g)$.*

If the kernel of ϕ is torsion-free, then ϕ is called a *smooth* epimorphism.

Since G is a p-group of order p^n and exponent p^e, in order for ϕ to be smooth, the periods of Γ can only be of the form p^i where $1 \leq i \leq e$. Let Γ have x_i conjugacy classes of maximal cyclic subgroups of order p^i so that Γ has presentation of the form

$$\begin{aligned} \text{Generators:} \quad & c_{11}, c_{12}, \ldots, c_{1x_1}, c_{21}, \ldots, c_{ex_e}, a_1, b_1, \ldots, a_h, b_h; \\ \text{Relations:} \quad & c_{ij}^{p^i} = 1 \ \text{ for } i = 1, 2, \ldots, e,\ j = 1, 2, \ldots, x_i; \\ & \prod_{k=1}^{h} [a_k, b_k] \prod_{i,j} c_{ij} = 1. \end{aligned} \tag{1}$$

The last relation is frequently referred to as the *long relation*. This group has *signature* $(h; p^{(x_1)}, p^{2(x_2)}, \ldots, p^{e(x_e)})$, where $n^{(r)}$ indicates that the period n is repeated r times.

If $\mu(\Gamma)$ denotes the area of a fundamental region for Γ and K the kernel of ϕ, then the equation $\mu(K) = \circ(G)\mu(\Gamma)$ yields the Riemann–Hurwitz relation

$$2(g-1) = p^n\left[2(h-1) + \sum_{i=1}^{e} x_i\left(1 - \frac{1}{p^i}\right)\right]. \tag{2}$$

Thus to determine if G acts on Σ_g, one must determine integers $h \geq 0$ and $x_i \geq 0$—*the data* $\{h; x_1, x_2, \ldots, x_e\}$—such that (2) holds and such that there exists a smooth epimorphism ϕ from the group Γ, determined by the data, onto G. Sometimes the abbreviated notation $\{h; x_i\}$ will be used for the data.

From (2), it is immediately clear that for p odd, $g \equiv 1 \pmod{p^{n-e}}$ (see Theorem 2.1), and it remains to consider solutions of the diophantine equation

$$N = p^e h + \sum_{i=1}^{e} x_i \frac{p^e - p^{e-i}}{2}. \tag{3}$$

Let $\Omega_e = \Omega_e(p)$ denote all the solutions N of (3) for which $h \geq 0$ and $x_i \geq 0$ for all i. The set Ω_e can be usefully expressed using a truncated p-adic expansion of $2N$, and the results below follow from this [14]. Let

$$2N = a_0 + a_1 p + a_2 p^2 + \cdots + a_e p^e, \tag{4}$$

where $0 \leq a_i < p$ for $i = 1, 2, \ldots, e-1$ and $a_e \geq 0$. Define

$$S_e(2N) = \sum_{k=0}^{e} a_k.$$

THEOREM 2.3.

$$\Omega_e = \{ N \in \mathbf{N} \mid S_e(2N) \geq (e-i)(p-1) \},$$

where a_i is the first nonzero coefficient in the expansion of $2N$ at (4).

Since the coefficients on the right-hand side of (3) have greatest common divisor 1, the equation will have a solution for all large enough N.

DEFINITION. Let $\sigma_e(p)$ denote the minimum stable solution in $\Omega_e(p)$; that is, $\sigma_e(p)$ is minimal with the property that all $N \geq \sigma_e(p)$ lie in $\Omega_e(p)$.

COROLLARY 2.4.

$$\sigma_e(p) = \tfrac{1}{2}[(e(p-1)-3)p^e + 3].$$

3. Groups with MEP

Recall that G is a group of order p^n and exponent p^e where p is an odd prime.

DEFINITIONS.

(1) For each N with $1 \leq N \leq e$, define

$$\Lambda^N(G) = \langle x \in G \mid x^{p^N} = 1 \rangle.$$

(2) G is said to have property M_N if all elements of order $\geq p^N$ lie in $G \setminus G'\Lambda^{N-1}(G)$.

(3) The group G is said to have the *maximal exponent property* (MEP) if G has exponent p^e and property M_e. This property can be stated more positively since it is equivalent to requiring that the set of elements of order less than p^e form a subgroup containing G'.

The formulation of the maximal exponent property is prompted by the problem under investigation, but it is worth noting how groups with MEP relate to other (more familiar) classes of p-groups.

First note that, if G is a regular p-group, then

$$\Lambda^m(G) = \{ x \in G \mid o(x) | p^m \}.$$

(For results on regular p-groups, see [10; 25]). Thus the set of elements of order less than p^e form a subgroup. This subgroup, however, may fail to contain G'. Indeed, any non-abelian p-group of exponent p must fail to have MEP. There are non-abelian p-groups of order p^3, so necessarily regular, and of exponent p, so such groups do not have MEP. There are also groups with MEP that fail to be regular (see below).

Perhaps more closely related are "powerful" p-groups, where G is powerful if G/G^p is abelian. (For results on powerful p-groups, see [4].)

LEMMA 3.1. *If G is a finite powerful p-group, then the set of elements of order less than p^e form a subgroup.*

Proof. Define inductively $G_1 = G$, $G_{i+1} = G_i^p[G_i, G]$. When G is powerful, $G_{i+1} = G_i^p$ so that $[G_i, G] \subset G_{i+1}$. It thus follows that $\gamma_3(G)$, the third term in the lower central series, is a subgroup of G_3.

Now, for any p-group,

$$(xy)^p \equiv x^p y^p [y, x]^{p(p-1)/2} \pmod{\gamma_3(G)}.$$

For a powerful p-group, $[y, x] \in G_2$ and so $[y, x]^{p(p-1)/2} \in G_3$. Thus $(xy)^p \equiv x^p y^p \pmod{G_3}$.

Thus the mapping $x \mapsto x^p G_3$ induces a homomorphism $\phi\colon G/G_2 \to G_2/G_3$. Now every normal subgroup of a powerful p-group is itself powerful, so we can repeat this. If G has exponent p^e, then G_{e+1} is trivial so that the composition $x \mapsto x^{p^{e-1}}$ is an endomorphism of G. The kernel consists of all elements of order less than p^e. □

COROLLARY 3.2. *If G is a powerful p-group, then G has MEP.*

Proof. If G is powerful, then $G' \subset G^p \subset \Lambda^{e-1}(G) = \{x \in G \mid o(x) | p^{e-1}\}$. Thus G has MEP. □

However, not all groups with MEP are powerful. Consider the following group, the details of which are discussed in Section 6.

$$G_6 = \langle x, y, z \mid x^{p^e} = y^p = z^p = 1,\ [x, y] = z,\ [x, z] = 1,\ [y, z] = x^{p^{e-1}} \rangle.$$

The elements $z, x^{p^{e-1}}$ lie in G_6', while $G_6^p = \langle x^p \rangle$ so that G_6 is not powerful. Note also that $G_6' \Lambda^{e-1}(G_6) = \langle x^p, y, z \rangle$ so that G_6 has MEP. In the case $p = 3$, this group also fails to be regular, as any 2-generator 3-group must have a cyclic commutator subgroup.

LEMMA 3.3. *The following classes of p-groups have the Maximal Exponent Property.*

(1) *Powerful p-groups, and hence abelian and metacyclic p-groups.*
(2) *Groups of orders p^3, p^4 apart from those that have exponent p and are non-abelian.*
(3) *Groups of exponent $\geq p^3$ and cyclic p-deficiency ≤ 2.*
(4) *Direct sums of MEP groups.*

Proof.

(1) This has already been proved (Corollary 3.2).
(2) This follows by checking the lists of such groups [2].

(3) Groups of cyclic deficiency 0 are cyclic and those of cyclic deficiency 1 or 2 (see [9]) are described in Sections 5 and 6, from which the condition may be verified.
(4) If G, H have MEP—treating separately the cases where G, H have the same exponent and different exponents—the result follows easily. □

4. The Invariant $\sigma_0(G)$ When G Has MEP

The main results of this section obtain information on $\sigma_0(G)$ for groups G with MEP.

LEMMA 4.1. *Let G act on Σ_g $(g \geq 2)$ with corresponding Fuchsian group Γ, whose associated data is $\{h; x_1, x_2, \ldots, x_N\}$ for some $N \leq e$. If G has property M_N and $x_N \neq 0$, then $x_N \geq 2$.*

Proof. Let $\phi\colon \Gamma \to G$ be a smooth epimorphism defining the action and suppose that $x_N = 1$. The long relation in Γ implies that $\phi(c_{N1}) \in G'\Lambda^{N-1}(G)$. This contradicts the property M_N. □

THEOREM 4.2. *Let G have exponent p^e. If G has MEP, then $\sigma_0(G) \geq \sigma_e(p) - 1$.*

Proof. Suppose that $g_0 = \sigma_e(p) - 2 = \frac{1}{2}[(e(p-1) - 3)p^e - 1]$ is a reduced genus for G. Then $g = n_0 g_0 + 1$ satisfies (2) and so

$$2g_0 = 2p^e(h-1) + \sum_{i=1}^{e} x_i(p^e - p^{e-i}). \tag{5}$$

Now $2g_0 \equiv -1 \pmod p$ and from (5) we have $x_e \equiv 1 \pmod p$. Thus, by Lemma 4.1, $x_e = x'_e p + p + 1$ with $x'_e \geq 0$. When $e = 1$, there is no possible solution to (5) with $x'_1 \geq 0$ and $h \geq 0$. From (5), since $e \geq 2$ we obtain

$$\frac{1}{2p}[2g_0 + 2p^e - (p+1)(p^e - 1)] = p^{e-1}h + \sum_{i=1}^{e-1} \frac{y_i}{2}(p^{e-1} - p^{e-1-i}), \tag{6}$$

where $y_1 = x_1 + px'_e$, $y_i = x_i$ for $i = 2, \ldots, e-2$, and $y_{e-1} = x_{e-1} + x'_e$. The integer defined by (6) lies in $\Omega_{e-1}(p)$. But this integer is equal to

$$\tfrac{1}{2}[((e-1)(p-1) - 3)p^{e-1} + 1].$$

If $e = 2$ and $p = 3$ then this expression is negative; otherwise, it is equal to $\sigma_{e-1}(p) - 1$. Either case gives a contradiction, since by its definition $\sigma_{e-1}(p) - 1 \notin \Omega_{e-1}(p)$. □

This result certainly does not hold for all p-groups G. We have already noted that a non-abelian group of order p^3 and exponent p fails to have MEP. A straightforward calculation for such a group G (cf. group G_4 in Theorem 6.1) shows that

$$\sigma_0(G) = \tfrac{1}{2}(p^2 - 6p + 3) \quad \text{for} \quad p \geq 7,$$

which is less than $\sigma_1(p) - 1$.

Now $\sigma_0(G)$ will be evaluated for the cases where G has *rank* 2—that is, when the minimal number of generators is 2. For a finite p-group, the Frattini subgroup $\Phi(G) = G'G^p$ and so

$$\Phi(G) \subset G'\Lambda^{e-1}(G).$$

When G has rank 2,

$$G/\Phi(G) \cong \mathbf{Z}_p \oplus \mathbf{Z}_p.$$

Thus, if G has rank 2 and MEP then

$$G/G'\Lambda^{e-1}(G) \cong \mathbf{Z}_p \text{ or } \mathbf{Z}_p \oplus \mathbf{Z}_p. \tag{7}$$

Note that, in the case where $e = 1$, G itself must be abelian and isomorphic to $\mathbf{Z}_p \oplus \mathbf{Z}_p$.

Lemma 4.3. *Let G have MEP and rank* 2*. Then G can be generated by two elements of order p^e whose product has order p^e. In the first of the two options at* (7)*, G can also be generated by two elements of order p^e whose product has order less than p^e. This cannot happen in the second option.*

Proof. Let $G = \langle a, b\rangle$ and $\Pi\colon G \to G/G'\Lambda^{e-1}(G)$. Then $\Pi(a)$ and $\Pi(b)$ cannot both be trivial. Suppose that $\Pi(a)$ is nontrivial. If $\Pi(b)$ is trivial then $\Pi(ab)$ is nontrivial and $G = \langle a, ab\rangle$. Thus assume that $\Pi(b)$ is nontrivial. Since p is odd, at least one of $\Pi(ab)$ and $\Pi(a^{-1}b)$ must be nontrivial.

Suppose that the first option at (7) holds. Suppose, as before, that $\Pi(a)$ and $\Pi(b)$ are nontrivial. Then there exists an integer i with $\Pi(b)^i = \Pi(a)$ and $(i, p) = 1$. Now $G = \langle a, b^{-i}\rangle$ and $\Pi(ab^{-i})$ is trivial. The last part is clear. □

If g_0 is a reduced genus for G and if $x_e \neq 0$ in equation (5), then $x_e \geq 2$ if G has MEP. This equation then yields

$$2(g_0 + 1) = 2p^e h + \sum_{i=1}^{e} y_i(p^e - p^{e-i}) \tag{8}$$

with $y_i = x_i$ for $1 \leq i \leq e - 1$ and $y_e = x_e - 2$. Thus $g_0 + 1 \in \Omega_e(p)$.

Theorem 4.4. *If $g_0 \geq 1$ satisfies* (8) *with either $h > 0$ or $y_e > 0$, then g_0 is a reduced genus for G when G has rank* 2 *and MEP.*

Proof. Under these conditions, the data $\{h; x_i\}$ where $x_i = y_i$ for $1 \leq i \leq e - 1$ and $x_e = y_e + 2$ defines a Fuchsian group Γ as at (1). By Lemma 4.3, we can let $G = \langle a, b\rangle$ where $\circ(a) = \circ(b) = \circ(ab) = p^e$.

If $y_e > 0$, and so $x_e > 2$, define a smooth epimorphism $\phi\colon \Gamma \to G$ as follows: $\phi(a_k) = \phi(b_k) = 1$ for all k; $\phi(c_{e1}) = a$, $\phi(c_{e2}) = b$ and $\phi(c_{ej}) = (ab)^{i_j}$ for $3 \leq j \leq x_e$. Map all other generators c_{ij} onto powers of ab such that ϕ is smooth, and adjust the i_j such that $(i_j, p) = 1$ and the long relation is satisfied. If $h > 0$, define $\phi(a_1) = \phi(b_1) = a$, map other hyperbolic generators trivially ($\phi(c_{e1}) = b^i$, $\phi(c_{e2}) = b^j$), and map all other elliptic generators onto powers of b such that the orders are preserved. Now adjust i, j such that the long relation holds. □

LEMMA 4.5.

(1) *If* $p \geq 5$ *and* $g_0 \geq \sigma_e(p) - 1$, *then there is a solution to* (8) *with either* $h > 0$ *or* $y_e > 0$.
(2) *If* $p = 3$ *and* $g_0 \geq \sigma_e(3)$, *then there is a solution to* (8) *with either* $h > 0$ *or* $y_e > 0$.

Proof. If $g_0 \geq \sigma_e(p) - 1$, then $g_0 + 1 \in \Omega_e(p)$ and so there exist $h' \geq 0$ and $s_i \geq 0$ such that

$$2(g_0 + 1) = 2p^e h' + \sum_{i=1}^{e} s_i(p^e - p^{e-i}). \tag{9}$$

Suppose $h' = s_e = 0$. Then

$$2(g_0 + 1) = \sum_{i=1}^{e-1} s_i(p^e - p^{e-i}) \geq 2\sigma_e(p).$$

Now, except for the case $p = 3$, at least one $s_i \geq p$. For otherwise

$$2(g_0 + 1) \leq [e(p-1) - p]p^e + p \leq [e(p-1) - 3]p^e + 3 = 2\sigma_e(p),$$

with equality only in the case $p = 3$. Thus for $p \geq 5$ and $g_0 \geq \sigma_e(p) - 1$, or for $p = 3$ and $g_0 \geq \sigma_e(3)$, we can assume that at least one $s_j \geq p$. Let $s_j = ps'_j + t$, where $0 \leq t < p$ and $s'_j \geq 1$. Then (9) can be rewritten as

$$2(g_0 + 1) = 2p^e h + \sum_{i=1}^{e} y_i(p^e - p^{e-i}),$$

where $h = s'_j(p-1)/2$ and $y_i = s_i$ for $1 \leq i \leq e-1$ and $i \neq j, j-1$. We thus have $y_{j-1} = s_{j-1} + s'_j$, $y_j = t$, and $y_e = 0$, so there is a solution of (8) with $h > 0$. $\square$

The exceptional case occurs when $p = 3$ and $g_0 = \sigma_e(3) - 1$, in which case there is a solution to (9) with $h' = s_e = 0$ and all other $s_i = 2$. This turns out to be the only solution of interest.

LEMMA 4.6. *The only solution to*

$$(2e - 3)3^e + 3 = 2h3^e + \sum_{i=1}^{e} y_i(3^e - 3^{e-i})$$

with $h, y_i \geq 0$ *occurs when* $h = y_e = 0$ *and all other* $y_i = 2$.

Proof. The proof proceeds by induction on e. The result is trivial for $e = 1$. When $e = 2$, the equation reduces to

$$18h + 6y_1 + 8y_2 = 12,$$

which clearly has only the solution $h = y_2 = 0$ and $y_1 = 2$. Now suppose the result is true for $e \geq 2$, and consider

$$(2(e+1)-3)3^{e+1}+3=2h3^{e+1}+\sum_{i=1}^{e+1} y_i(3^{e+1}-3^{e+1-i}).$$

Clearly $y_{e+1}=3y'_{e+1}$. Using $3^{e+1}-1=3(3^e-3^{e-1})+(3^e-1)$ and dividing by 3 yields

$$\begin{aligned}(2e-3)3^e+3+2(3^e-1) &= 2h3^e+(y_1+3y'_{e+1})(3^e-3^{e-1})\\ &\quad+\sum_{i=2}^{e-1} y_i(3^e-3^{e-i})+(y_e+y'_{e+1})(3^e-1).\end{aligned} \tag{10}$$

Now $y_e+y'_{e+1}\equiv 2 \pmod 3$ and so $y_e+y'_{e+1}\geq 2$. Thus, by applying the inductive assumption to (10), we obtain $h=0$, $y_1+3y'_{e+1}=2$, $y_i=2$ for $i=2,3,\ldots,e-1$ and $y_e+y'_{e+1}-2=0$. The result now follows for the case $e+1$. □

The preceding results are gathered together in the second main result of this section as follows.

THEOREM 4.7. *Let G be a finite p-group of exponent p^e. Let G have MEP and rank* 2.

(1) *If $p\geq 5$ then $\sigma_0(G)=\sigma_e(p)-1$.*
(2) *If $p=3$ and $G/G'\Lambda^{e-1}(G)\cong \mathbf{Z}_3$, then $\sigma_0(G)=\sigma_e(3)-1$.*
(3) *If $p=3$ and $G/G'\Lambda^{e-1}(G)\cong \mathbf{Z}_3\oplus\mathbf{Z}_3$, then $\sigma_0(G)=\sigma_e(3)$.*

Proof. If $p\geq 5$, the result follows from Theorems 4.2 and 4.4 and Lemma 4.5. Now assume that $p=3$ and $e\geq 2$, so that $\sigma_e(3)-1\leq\sigma_0(G)\leq\sigma_e(3)$. Suppose then that $g_0=\sigma_e(3)-1$. Then the only solution to (8) has $h=0$, $y_e=0$, and $y_i=2$ for other i, so that the only Fuchsian group Γ that could correspond to an action has signature $(0; 3^{(2)}, 3^{2(2)}, \ldots, 3^{e(2)})$. Let Γ have presentation as at (1).

If $G/G'\Lambda^{e-1}(G)\cong\mathbf{Z}_3$ then, by Lemma 4.3, G can be generated by two elements a, b of order 3^e whose product has order 3^m with $m<e$. We can thus define a smooth epimorphism $\psi\colon\Gamma\to G$ by $\psi(c_{i1})=\psi(c_{i2})^{-1}=a^{3^{e-i}}$ for $i\neq m, e$, $\psi(c_{m1})=ab$, $\psi(c_{m2})=(ab)^{-2}$, $\psi(c_{e1})=a$, and $\psi(c_{e2})=b$.

If $G/G'\Lambda^{e-1}(G)\cong\mathbf{Z}_3\oplus\mathbf{Z}_3$, then any two elements of order 3^e in G that generate G will have their product of order 3^e, by Lemma 4.3. In this case there is clearly no smooth epimorphism from Γ onto G.

If $p=3$ and $e=1$ then $G=\mathbf{Z}_3\oplus\mathbf{Z}_3$ and $\sigma_1(3)=0$. All the preceding discussion assumes that we are considering only $g\geq 2$ and so $g_0\geq 1$. However, note that in this case G acts on Σ_1, so the result remains true for $p=3$ and $e=1$. □

5. *p*-Groups of Cyclic Deficiency 1

Cyclic p-groups of automorphisms of compact surfaces were studied in detail in [14]. In this section, necessary and sufficient conditions are given for g_0 to be a reduced genus for the two classes of p-groups of p-cyclic deficiency 1. Details are given for the non-abelian case, the abelian case being similar and easier. In the next section, the same method will be employed on more complex groups, and there many details will be omitted (see [26]).

Let G have exponent p^e so that $\circ(G) = p^{e+1}$. For p odd, there are two classes of p-groups of cyclic deficiency 1:

(1) $\mathbf{Z}_{p^e} \oplus \mathbf{Z}_p$;

(2) $$\langle x, y \mid x^{p^e} = y^p = 1,\ y^{-1}xy = x^{1+p^{e-1}} \rangle \quad (e \geq 2) \tag{11}$$

(see e.g. [25]). For these groups $n_0 = p$ and so, if G acts on Σ_g, $g - 1 = pg_0$. Recall the general method to determine which g_0 are reduced genera for G.

From the Riemann–Hurwitz relation (2), we must have

$$2g_0 = 2p^e(h-1) + \sum_{i=1}^{e} x_i(p^e - p^{e-i}). \tag{12}$$

From the resulting data $\{h; x_i\}$, construct a Fuchsian group Γ and then show that there is a smooth epimorphism $\phi\colon \Gamma \to G$.

For any such data $\{h; x_1, x_2, \ldots, x_e\}$, the following notation is introduced and will be standard throughout the remainder of the paper:

$$M = \begin{cases} 0 & \text{if all } x_i = 0, \\ e & \text{if } x_e \neq 0, \\ m & \text{if } x_m \neq 0 \text{ but } x_{m+1} = \cdots = x_e = 0. \end{cases} \tag{13}$$

Note that, from (12), we would then have $g_0 = p^{e-M} g_0'$.

Let G be defined at (11). It is not difficult to show that every element of G can be uniquely expressed as $x^i y^j$ with $0 \leq i < p^e$ and $0 \leq j < p$. Also, $\circ(x^i y^j) = \circ(x^i)$ when $i \neq 0$. It follows that $\Lambda^N(G) = \langle x^{p^{e-N}}, y \rangle$, which is abelian for $1 \leq N < e$. Furthermore, the commutator subgroup is generated by $x^{p^{e-1}}$ so that G has property M_N for $2 \leq N \leq e$. Thus, from Theorem 4.7,

$$\sigma_0(G) = \sigma_e(p) - 1. \tag{14}$$

THEOREM 5.1. *Let $g_0 \geq 1$. Then g_0 is a reduced genus for G if and only if g_0 satisfies* (12) *and the data $\{h; x_i\}$ satisfies at least one of the conditions in the following table*:

	h	M	x_M
(1)	≥ 2	$= 0$	
(2)	≥ 1	$= 1$	≥ 1
(3)	≥ 1	$2 \leq M \leq e-1$	≥ 2
(4)	≥ 0	$= e$	≥ 2 *and the RHS of* (12) *is positive.*

Proof. Assume first that g_0 is a reduced genus for G. Clearly from (12), if $M = 0$ then $h \geq 2$. Since elements of order p^e in G cannot be products of elements of smaller order, if $1 \leq M < e$ then $h \geq 1$ since ϕ is an epimorphism. Since G has property M_N for $2 \leq N \leq e$, it follows from Lemma 4.1 that $x_M \geq 2$ for $2 \leq M \leq e$.

Now suppose conversely that g_0 satisfies (12) and that the data satisfies one of the conditions stated. We use the notation given at (1) for Γ and construct a smooth epimorphism $\phi\colon \Gamma \to G$ in each case.

(1) $\phi(a_1) = \phi(b_1) = x$, $\phi(a_2) = \phi(b_2) = y$, and $\phi(a_l) = \phi(b_l) = 1$ for $l \geq 2$.

(2) $\phi(a_1) = x$, $\phi(b_1) = y$, and $\phi(a_l) = \phi(b_l) = 1$ for $l > 1$. Also, $\phi(c_{1t}) = x^{j_t p^{e-1}}$ for $t = 1, 2, \ldots, x_1$ with j_t chosen such that $(j_t, p) = 1$ and $\sum_{i=1}^{x_1} j_t \equiv -1 \pmod{p}$.

(3) $\phi(a_1) = x$, $\phi(b_1) = y$, and $\phi(a_l) = \phi(b_l) = 1$ for $l > 1$. Note that

$$\phi\left(\prod_{j=1}^{h}[a_j, b_j]\right) = x^{p^{e-1}}.$$

For all $(i, j) \neq (M, 1), (M, 2)$, let $\phi(c_{ij})$ be a power of x of order p^i. Finally, let $\phi(c_{Mi}) = x^{j_i p^{e-M}}$ $(i = 1, 2)$, where j_1, j_2 are chosen such that $(j_i, p) = 1$ and the long relation is satisfied.

(4) If the RHS of (12) is positive, then either $h > 0$ or Γ has a period different from those generated by c_{e1}, c_{e2}. In the first case we can define ϕ as in case (3). Suppose then that $h = 0$ and that the data gives a generator c_{kl} different from c_{e1}, c_{e2}. For all c_{ij} different from these three, let $\phi(c_{ij})$ be a power of x of order p^i. Let $\phi(c_{kl}) = x^{j_1 p^{e-k}} y^{-1}$, $\phi(c_{e1}) = x^{j_2} y$, and $\phi(c_{e2}) = x^{j_3}$, where j_1, j_2, j_3 are relatively prime to p and chosen such that the long relation is satisfied. □

The four conditions given in Theorem 5.1 can now be converted into conditions on g_0 in terms of its p-adic expansion using the notation introduced in Section 2.

THEOREM 5.2. *Let $g_0 \geq 1$. Then g_0 is a reduced genus for G if and only if either*

(1) $g_0 = p^e g_0'$ *for some* $g_0' \geq 1$, *or*

(2) $g_0 = p^{e-1} g_0'$ *and* $g_0' - \frac{1}{2}(p-1) \in \Omega_1(p)$, *or*

(3) *for some* $1 < M < e$, $g_0 = p^{e-M} g_0'$ *and* $g_0' - p^M + 1 \in \Omega_M(p)$, *or*

(4) $g_0 + 1 \in \Omega_e(p) \setminus \{0\}$.

Proof.

(1) This is immediate from (12).

(2) Let $h' = h - 1$ and $x_1' = x_1 - 1$, so that $h' \geq 0$ and $x_1' \geq 0$. From (12) we have $g_0 = p^{e-1} g_0'$, where

$$2g_0' - (p-1) = 2ph' + x_1'(p-1) \in \Omega_1(p).$$

Conversely, if these conditions are satisfied then we can construct the data $\{h; x_i\}$ satisfying (12).

(3) A similar argument to that just given applies with $h' = h - 1$ and $x_M' = x_M - 2$.

(4) In this case, take $x_e' = x_e - 2$ to obtain

$$2g_0 + 2p^e - 2(p^e - 1) = 2hp^e + \sum_{i=1}^{e-1} x_i(p^e - p^{e-i}) + x_e'(p^e - 1).$$

The condition that the RHS of (12) is positive gives that not all the coefficients on the RHS of this last equation are zero. □

Using the description of the sets $\Omega_N(p)$ given in Theorem 2.3, one can determine the genus spectrum for G. In particular, we have the following corollary.

COROLLARY 5.3. $\mu_0(G) = \frac{1}{2}(p^e - p^{e-1} - 2)$.

Proof. Note that the smallest value in $\Omega_N(p)$ is 0, while the smallest nonzero value is $\frac{1}{2}(p^N - p^{N-1})$. The corollary then follows easily from Theorem 5.2. □

For the abelian groups defined previously, a similar analysis yields our next result.

THEOREM 5.4. *Let $g_0 \geq 1$. All such g_0 are reduced genera for $\mathbf{Z}_3 \oplus \mathbf{Z}_3$. If $p^e \neq 3$, then g_0 is a reduced genus for $\mathbf{Z}_{p^e} \oplus \mathbf{Z}_p$ if and only if either*

(1) *$g_0 = p^e g_0'$ for some $g_0' \geq 1$, or*
(2) *for some $1 \leq M \leq e-1$, $g_0 = p^{e-M} g_0'$ and $g_0' - p^M + 1 \in \Omega_M(p)$, or*
(3) *$g_0 + 1 \in \Omega_e(p) \setminus \{0\}$.*

We can readily deduce that

$$\sigma_0(\mathbf{Z}_{p^e} \oplus \mathbf{Z}_p) = \sigma_e(p) - 1,$$
$$\mu_0(\mathbf{Z}_{p^e} \oplus \mathbf{Z}_p) = \tfrac{1}{2}(p^e - p^{e-1} - 2).$$

A simple calculation then yields the following.

COROLLARY 5.5. *If $G_1 = \mathbf{Z}_9 \oplus \mathbf{Z}_3$ and $G_2 = \langle x, y \mid x^9 = y^3 = 1,\ y^{-1}xy = x^4\rangle$, then G_1 and G_2 have the same genus spectrum. It is $\{1+3g_0 \mid g_0 \geq 2$ except $g_0 = 4\}$.*

This result shows that the genus spectrum does not determine the isomorphism class of the group.

6. *p*-Groups of *p*-Cyclic Deficiency 2, *p* Odd

At the end of the preceding section, two nonisomorphic groups with identical genus spectra were exhibited. In this section we show that this is not an isolated example by producing, for each $p \geq 5$ and $e \geq 3$, four nonisomorphic groups of p-cyclic deficiency 2 with identical genus spectra (Theorem 6.1). In fact, in the twelve isomorphism classes of such groups there exist further nonisomorphic groups with identical genus spectra. The methods are as in the preceding section and information on the subgroup structure of these four nonisomorphic groups is given, but other details are omitted (cf. [26]).

If $e \geq 3$ and $p \geq 5$ then there are exactly twelve isomorphism classes of p-groups of exponent p^e and order p^{e+2} [9]. This classification will be detailed next.

Each such group G contains a normal subgroup H of index p that is a group of p-cyclic deficiency 1.

Case A: G is a split extension of H by $\mathbf{Z}_p$. In all cases, G can be generated by three elements x, y, z of orders p^e, p, p respectively. Every element of G can be uniquely expressed in the form

$$x^i y^j z^k \text{ with } 0 \le i \le p^e - 1 \text{ and } 0 \le j, k \le p - 1,$$

and the orders of these elements are the same as the orders of the corresponding elements of the abelian group $\mathbf{Z}_{p^e} \oplus \mathbf{Z}_p \oplus \mathbf{Z}_p$. There are seven classes of groups; the additional relations that the generators must satisfy, in addition to having orders p^e, p, p, are

(G_1) $[x, y] = [y, z] = [z, x] = 1$;
(G_2) $[x, y] = x^{p^{e-1}}$, $[x, z] = [y, z] = 1$;
(G_3) $[x, y] = [x, z] = 1$, $[y, z] = x^{p^{e-1}}$;
(G_4) $[x, y] = z$, $[x, z] = [y, z] = 1$;
(G_5) $[x, y] = z$, $[x, z] = x^{p^{e-1}}$, $[y, z] = 1$;
(G_6) $[x, y] = z$, $[x, z] = 1$, $[y, z] = x^{p^{e-1}}$;
(G_7) $[x, y] = z$, $[x, z] = 1$, $[y, z] = x^{rp^{e-1}}$, where r is a quadratic nonresidue mod p.

Note that G_1, G_2, G_3 have rank 3, while the remainder have rank 2.

Case B: G is a nonsplit extension of H by $\mathbf{Z}_p$. In this case G can be generated by two elements x, y of orders p^e, p^2 respectively. Every element of G can be uniquely expressed in the form

$$x^i y^j \text{ with } 0 \le i \le p^e - 1 \text{ and } 0 \le j \le p^2 - 1,$$

and the orders of the elements are the same as the orders of the corresponding elements in the abelian group $\mathbf{Z}_{p^e} \oplus \mathbf{Z}_{p^2}$.

There are five such classes of groups, which satisfy the additional relations:

(G_8) $[x, y] = 1$;
(G_9) $[x, y] = x^{p^{e-1}}$;
(G_{10}) $[x, y] = x^{p^{e-2}}$;
(G_{11}) $[x, y] = y^p$;
(G_{12}) $[x, y] = x^{p^{e-2}} y^p$, $[x, y^p] = x^{p^{e-1}}$.

NOTES. (1) For $e = 3$, G_{10} is isomorphic to G_{12}.

(2) In the presentation of G_{12} given in [9], the second listed equality is omitted. That relation does not seem to be a consequence of the others.

The underlying method of obtaining the necessary and sufficient conditions for g_0 to be a reduced genus for each of these groups is similar to that given in Section 5 for the non-abelian p-group, p odd, of cyclic deficiency 1. Thus one must determine the subgroups G' and $\Lambda^N(G)$ for each N in order to determine for which values of N the group has property M_N, $1 \le N \le e$ (see Section 4). All these groups have MEP and so satisfy $\sigma_0(G) \ge \sigma_e(p) - 1$ by Theorem 4.2; for those of rank 2, $\sigma_0(G) = \sigma_e(p) - 1$ by Theorem 4.7 $(p \ge 5)$.

For each of these groups, $n_0 = p^2$ and so $g - 1 = p^2 g_0$ with

$$2g_0 = 2p^e(h-1) + \sum_{i=1}^{e} x_i(p^e - p^{e-i}). \tag{15}$$

As in Section 5, the necessary and sufficient conditions that g_0 be a reduced genus for G are that g_0 satisfy (15) for some data $\{h; x_i\}$ satisfying certain conditions (cf. Theorem 5.1) expressed in terms of inequalities on h, x_i, M. These conditions can, in turn, be translated into number-theoretic conditions on g_0 in terms of the divisibility by powers of p and its truncated p-adic expansion via the sets $\Omega_M(p)$ (see Section 2 and Theorems 5.2 and 5.4).

THEOREM 6.1. *The groups G_4, G_5, G_6, G_7 have the same genus spectrum.*

Proof. As noted previously, each element in each of these groups can be uniquely expressed in the form $x^i y^j z^k$, with $0 \le i \le p^e - 1$, $0 \le j, k \le p-1$, and $\circ(x^i y^j z^k) = \circ(x'^i y'^j z'^k)$ where x', y', z' are the canonical generators of $\mathbf{Z}_{p^e} \oplus \mathbf{Z}_p \oplus \mathbf{Z}_p$. From the foregoing presentations, it is straightforward to obtain that

$$G_4' = \langle z \rangle, \quad G_i' = \langle x^{p^{e-1}}, z \rangle, \quad i = 5, 6, 7;$$
$$\Lambda^N(G) = \langle x^{p^{e-N}}, y, z \rangle \quad \text{for } 1 \le N \le e-1.$$

It thus follows that each G has property M_N for $2 \le N \le e$. Thus, as in Theorem 5.1, one deduces that g_0 is a reduced genus for G if and only if g_0 satisfies (15) and the data $\{h; x_i\}$ satisfies at least one of the following conditions:

	h	M	x_M	
(1)	≥ 2	$= 0$		
(2)	≥ 1	$= 1$	≥ 1	
(3)	≥ 1	$2 \le M \le e-1$	≥ 2	
(4)	≥ 0	$= e$	≥ 2	and the RHS of (15) is positive.

Because they are identical for each of the four groups and each group has the same exponent, these conditions translate into identical number-theoretic conditions on g_0 in order that g_0 be a reduced genus (cf. Theorem 5.2). The result follows. □

REMARK. A similar analysis on the other groups (cf. [26]) yields

(i) G_2 and G_3 have the same genus spectrum,
(ii) G_9 and G_{11} have the same genus spectrum,
(iii) G_{10} and G_{12} have the same genus spectrum.

7. Elementary Abelian *p*-Groups, *p* Odd

All groups of p-cyclic deficiency ≤ 2 have MEP. Hence, for these groups that also have rank 2, it follows from Theorem 4.7 that $\sigma_0(G) = \sigma_e(p) - 1$. It can also be shown that even when the rank is 3, $\sigma_0(G) = \sigma_e(p) - 1$, so that in all these cases the minimum stable reduced genus depends only on the exponent. In this section it will be shown that, for elementary abelian p-groups, $\sigma_0(G)$ grows with the rank of G.

Let $G = \mathbf{Z}_p^k$. Thus G has cyclic p-deficiency $k-1$, and the signature of a candidiate Fuchsian group has the form $(h;\, p^{(x)})$. Such a group is denoted by $\Gamma(x, h)$ with (x, h) belonging to the lattice $L = \mathbf{N} \times \mathbf{N}$. Let $f(x, h)$ be the function defined by

$$f(x, h) = \left(\frac{p-1}{2}\right)x + ph$$

so that, if there is a smooth epimorphism $\phi\colon \Gamma(x, h) \to \mathbf{Z}_p^k$, then the corresponding reduced genus $g_0 = f(x, h) - p$. This holds for all $k \geq 1$.

We use a geometric approach to investigate the genus spectrum of these groups and introduce some new notation:

$$S(G) = \{\, (x, h) \in L \mid \exists \text{ a smooth epimorphism } \phi\colon \Gamma(x, h) \to G \,\}.$$

Now, if $\Gamma = \Gamma(x, h)$ then

$$\frac{\Gamma}{\Gamma^p \Gamma'} \cong \begin{cases} \mathbf{Z}_p^{2h} & \text{if } x = 0, \\ \mathbf{Z}_p^{2h+x-1} & \text{if } x \neq 0. \end{cases}$$

Also, $\Gamma^p\Gamma'$ is torsion-free if and only if $x \neq 1$. The next lemma follows readily.

LEMMA 7.1. *Let $G = \mathbf{Z}_p^k$. Then $(x, h) \in S(G)$ if and only if:*

(a) *k is odd, and $x + 2h \geq k + 1$ with $x \neq 1$; or*

(b) *k is even, and $x + 2h \geq k + 1$ with $x \neq 1$ or $(x, h) = (0, k/2)$.*

REMARK. Note that the cyclic groups $\mathbf{Z}_p$ act on the sphere and the groups $\mathbf{Z}_p \oplus \mathbf{Z}_p$ act on the torus. However, our initial assumption that g exceed unity rules these cases out, so that small modifications must be made to Lemma 7.1 for these cases: If $k = 1$ then we require that $x + 2h \geq 3$ with $x \neq 1$, and if $k = 2$ then we exclude the possibility $(0, 1)$. Corresponding modifications are required in some of the following statements also.

Thus $S(G)$ consists of all lattice points above L1 with the exception of $\{\, (1, h) : h \geq 0 \,\}$, together with the point $(0, k/2)$ when k is even (see Figure 1). The level curves of the function f have slope $-(p-1)/2p$, and L2 and L3 are level curves. Thus

$$\mu_0(\mathbf{Z}_p^k) = \begin{cases} f(k+1, 0) - p = (k+1)(p-1)/2 - p & k \text{ odd}, \\ \min\{f(k+1, 0) - p,\ f(0, k/2) - p\} & k \text{ even}. \end{cases} \tag{16}$$

From this we obtain

$$\mu_0(G) = \begin{cases} (k+1)(p-1)/2 - p & \text{if } p \leq k+1, \\ kp/2 - p & \text{if } p > k+1. \end{cases} \tag{17}$$

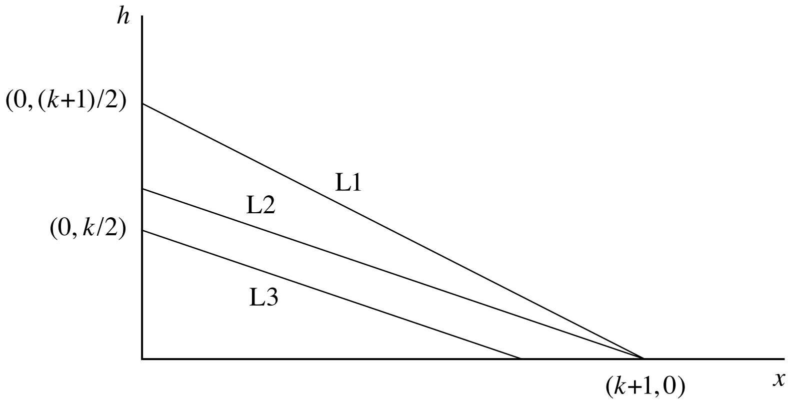

Figure 1

Let $I(G)$ denote the gap sequence for G.

THEOREM 7.2. $I(\mathbf{Z}_p^k) = I(\mathbf{Z}_p) \cap [\mu_0(\mathbf{Z}_p^k), \infty)$.

Proof. Consider first the case where $p \leq k+1$. Thus $I(\mathbf{Z}_p^k) = \{ g_0 \mid g_0 + p$ is not the value of a level curve through a point of $S(\mathbf{Z}_p^k) \}$. Now, if U is the set of lattice points above L2 with the exception of $\{(1, h) : h \geq 0\}$, then $I(\mathbf{Z}_p) \cap [\mu_0(G), \infty) = \{ g_0 \mid g_0 + p$ is not the value of a level curve through a point of $U \}$. Thus $I(\mathbf{Z}_p^k) \supset I(\mathbf{Z}_p) \cap [\mu_0(\mathbf{Z}_p^k), \infty)$.

Now consider the triangle bounded by the lines L1, L2 and the h-axis. Suppose (x, h) lies inside this triangle. Then $x + 2h < k + 1$ and $x + 2h + 2h/(p-1) > k + 1$. Thus $h > (p-1)/2$. Let $h = q(p-1)/2 + r$, where $0 \leq r < (p-1)/2$. Then $f(x, h) = f(x+q, r)$ and $(x+q, r) \in S(\mathbf{Z}_p^k)$. Thus

$$I(\mathbf{Z}_p^k) = I(\mathbf{Z}_p) \cap [\mu_0(\mathbf{Z}_p^k), \infty).$$

For the cases where $p > k + 1$, a similar argument applies. We need to consider the triangle bounded by the lines L1, L3 and the h-axis. Thus consider (x, h) such that $x + 2h \leq k$ and $x(p-1)/2 + ph \geq pk/2$. It follows that $pk = x \geq (p-1)x + 2ph \geq pk$. The only solution is $x = 0$, $h = k/2$, and the result again follows. □

COROLLARY 7.3.

(1) $I(\mathbf{Z}_p^k) = \phi$ *if and only if* $k \geq p - 1$. *In these cases,* $\sigma_0(\mathbf{Z}_p^k) = \mu_0(\mathbf{Z}_p^k) = (k+1)(p-1)/2 - p$.
(2) *If* $k < p - 1$ *then* $\sigma_0(\mathbf{Z}_p^k) = \sigma_0(\mathbf{Z}_p) = \frac{1}{2}[p(p-4) + 1]$.

Proof. By Theorem 7.2, $I(\mathbf{Z}_p^k) = \phi$ if and only if $\sigma_0(\mathbf{Z}_p) \leq \mu_0(\mathbf{Z}_p^k)$. Both parts of the corollary then follow by a direct computation. □

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